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Short communication

Short Communication: Determining the average attitude of a rigid body

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ABSTRACT

Three-dimensional angular kinematics exist on the surface of a unit hypersphere, therefore the average attitude cannot always be accurately computed by averaging Cardan angles. This study derives and evaluates a method for determining average body attitude, by exploiting the singular value decomposition of the average of a set of attitude matrices. To test the method 1000 criterion attitudes were determined, and for each attitude 10 noisy attitude matrices generated. The new method and the averaging of Cardan angles extracted from the 10 noisy attitude matrices were evaluated for their ability to estimate the criterion attitude. At low attitude variance the two approaches provided equivalent results, but with increasing attitude variance levels the new procedure was superior. The method provides superior estimates of average attitude compared with averaging Cardan angles, by accounting for the geometric distribution of rigid body attitudes on the surface of a unit hypersphere.

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1. Introduction

A basic step in many biomechanical studies is measuring three-dimensional kinematics. In many circumstances these kinematic data may be averaged, for example multiple measurements may be made so that averaging improves measurement accuracy, or repeat trials of the same task are averaged to produce a representative time series. Three-dimensional linear kinematics are defined in three-dimensional Euclidean space, so averaging is simply computed by summing the vectors and dividing by the number of samples. Three-dimensional angular kinematics are not defined in three-dimensional Euclidean space but exist on the surface of a unit hypersphere (McCarthy, 1990), as a consequence the average orientation is not simply a case of, for example, averaging a set of Cardan angles.

Errors can occur in, for example, the averaging of 3D angles to determine rigid body attitude as it is equivalent to taking a chord to the surface of the unit hypersphere, but appropriate averaging should take into account the contour of the surface described by the hypersphere. The purpose of this study is to derive a method for determining average body attitude, which revolves around taking the singular value decomposition of the average of attitude matrices. This method is compared numerically to another method to assess its utility.

2. The method

In this section the method will be presented, its derivation is presented in an Appendix. If a rigid body in three-dimensions undergoes translation and rotation then the new pose (position and orientation) of a point on that body can be described by,

$$y = Rx + \underline{v} \quad (1)$$

where y are points measured in pose 2, R is a 3×3 attitude matrix, x are points measured in pose 1, and \underline{v} is a 3×1 vector describing the translation from one pose to the other. The attitude matrix belongs to the special-orthogonal group of order three, $R \in SO(3)$. As a consequence being in this group the inverse of the attitude matrix also belongs to the special-orthogonal group, as does the product of any matrices in this group. The attitude matrix has the following properties,

$$R^T = R^{-1} \quad RR^T = R^T R = I \quad (2)$$

$$\det(R) = 1 \quad (3)$$

where I is the identity matrix, and $\det()$ refers to the determinant of a matrix.

If the average of a sequence of rigid body attitudes is to be computed from m measures the attitude matrices associated with each attitude are used (R_i), with the average matrix computed (E),

$$E = \frac{1}{m} \sum_{i=1}^m R_i \quad (4)$$

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The average matrix is not necessarily proper orthogonal, this is actually unlikely, therefore the singular value decomposition of the average matrix is then computed (Golub and Reinsch, 1971),

$$E = UDV^T \quad (5)$$

where U is a 3×3 orthogonal matrix, consisting of vectors u_1, u_2, u_3 , D is a 3×3 diagonal matrix, whose elements are non-negative real values (the singular values), and V is a 3×3 orthogonal matrix, consisting of vectors v_1, v_2, v_3 . Then the average attitude matrix (\bar{R}), from which average angles can be determined, is computed from,

$$\bar{R} = UV^T \quad (6)$$

3. Evaluation

To evaluate the new method for determining average rigid body attitude criterion data were generated and the new method performance evaluated and compared to the averaging of Cardan angles. In the following the method of error assessment is presented, followed by the methods evaluated.

3.1. Error assessment

One thousand criterion attitude matrices were generated, via exploiting a random number generator, for each criterion attitude matrix 10 noisy versions of the matrix were generated. The noisy perturbations to each criterion matrix (R_{crit}) were represented by an error angle (ϕ) and the unit vector (e) of the axis about which this rotation occurs. Each noisy attitude matrix (\hat{R}) was generated based an error model (Downs, 1972),

$$\hat{R} = R_{crit} \exp([\omega]_{\times}) \quad (7)$$

where $\omega = \phi e$, and $[\omega]_{\times}$ is a skew-symmetric matrix,

$$[\omega]_{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad (8)$$

The noise was varied from a standard deviation of zero to 30° . A low noise level, for example 0.035° , reflects the errors which might occur in Roentgen stereo-photogrammetry (e.g., Kibsgard et al., 2012), one of 3° reflecting the spread of performances which may occur if a subject performs the same task multiple times (e.g., Winter, 1991), and extreme condition values might occur if different subjects performed the same task. The noise levels can vary with task and joint examined (e.g., Begon et al., 2017). The noisy attitude matrices were generated with the unit vectors and error angles randomly generated.

To assess the error in computing the average attitude, the product of the attitude matrix estimate of the average (R_{Est}), and the transpose of the criterion matrix were computed,

$$R_{Err} = R_{Est} R_{crit}^T \quad (9)$$

The error matrix (R_{Err}) can be quantified by the error angle (θ), which can be computed from the error attitude matrix,

$$\theta = \cos^{-1} \left(\frac{\text{trace}(R_{Err}) - 1}{2} \right) \quad (10)$$

If the estimated average attitude matrix exactly equals the criterion matrix, then the error matrix would be the identity matrix, giving an error angle of zero. This error angle reflects the angle through which the rigid body attitude defined by the estimated average attitude matrix must be rotated so that it corresponds with

the attitude defined by the criterion attitude matrix (Chasles Theorem). The mean error angle for the estimation of all 1000 criterion matrices was computed, for each noise level.

3.2. Methods evaluated

For each of the 10 noisy attitude matrices, for each noise level, the average rigid body attitude was determined by two methods.

Method 1 - Rigid body attitude can be described by an ordered sequence of rotations about three coordinate axes, for example Cardan angles (Wittenburg, 1977). For an X, Y, Z Cardanic sequence, a set of rotation angles ($\alpha_i, \beta_i, \gamma_i, i = 1, 10$) can be extracted from the noisy attitude matrices. The average Cardanic angles can be computed ($\bar{\alpha}, \bar{\beta}, \bar{\gamma}$)

$$\bar{\alpha} = \frac{1}{m} \sum_{i=1}^m \alpha_i \quad (11)$$

$$\bar{\beta} = \frac{1}{m} \sum_{i=1}^m \beta_i \quad (12)$$

$$\bar{\gamma} = \frac{1}{m} \sum_{i=1}^m \gamma_i \quad (13)$$

From the average set of Cardanic angles the average attitude matrix was estimated.

With Cardan angles confusion can arise if an angle exceeds $\pm\pi$ radians, in which case the averaging of Cardan angles can produce spurious results (e.g., the mean of $-\pi$ and $+\pi$). In a similar fashion if the middle rotation in a Cardan sequence is equal to $\pm\frac{\pi}{2}$ radians the two terminal angle are undefined (Wittenburg, 1977). Therefore when generating the criterion and noisy data, potential data were rejected if they produced Cardan angles greater than π radians or less than $-\pi$ radians, or a middle rotation of $\pm\frac{\pi}{2}$ radians.

Method 2 - The average attitude matrix was estimated from the singular value decomposition of the average of 10 noisy attitude matrices (see Eqs. (4) and (6)).

4. Results

For both methods with increasing attitude noise variance there was a decrease in the accuracy of the estimate of true rigid body attitude (Fig. 1a). For the lowest variance level condition the methods produced very similar performances but for greater noise levels the average of Cardanic angles produced increasingly worse estimates compared with the singular value decomposition based procedure (Fig. 1b).

5. Discussion

Unlike linear vectors three-dimensional angular kinematics are not defined in three-dimensional Euclidean space, but exist on the surface of a non-linear manifold; as a consequence computing average orientation is not simply a case of, for example, averaging a set of Cardan angles. This study has shown that unless the variability of the attitudes to be averaged is small then taking the average of a set of angles should be avoided; problems akin to this exist when computing angular range of motion (Michaud et al., 2014) or interpolating between two attitudes (e.g., Begon et al., 2017). Three-dimensional attitudes exist on the surface of a unit hypersphere therefore taking the average of a set of Cardan, or Euler angles, takes the chord beneath the surface of the sphere which for small angle changes can provide a reasonable approximation to the surface but not for larger angle changes (Fig. 2). For average-

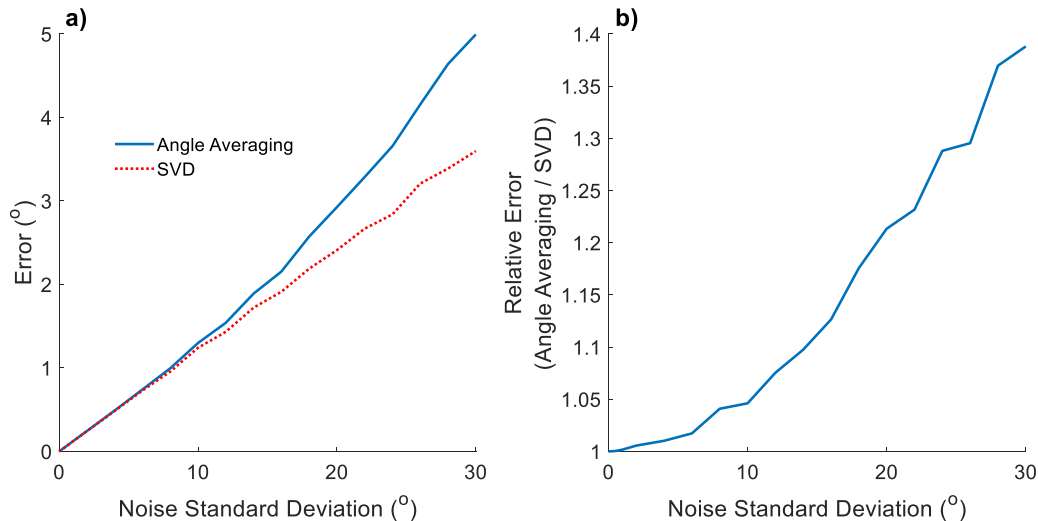


Fig. 1. (a) the mean error angle corresponding to the estimation of rigid body attitude by estimating the average of multiple measurements of rigid body attitude for different noise levels, and (b) the relative error between the two methods for different noise levels. Two methods are compared: the average of Cardan angles, and the average of the attitude matrix determined using the singular value decomposition (SVD).

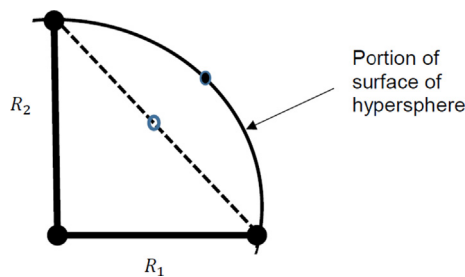


Fig. 2. A portion of the surface of a hypersphere, with two attitudes described by two matrices (R_1, R_2) identified. The dotted line represents the averaging of two sets of Cardan or Euler angles which take a chord to the surface of the hypersphere, the resulting average value is the mid-point on this chord (indicated by \bullet). In contrast the actual average value takes into account the shape of the surface of the hypersphere. The actual mid-point and therefore average is indicated by \bullet .

ing rigid body attitudes the singular value decomposition based method is recommended, as it can be more accurate than averaging Cardan angles and is not subject to the problems which occur when the angles to be averaged are around $\pm\pi$ radians. Rigid body averaging in biomechanics can occur in a number of circumstance, for example making multiple measurements so that averaging improves accuracy, or averaging repeat trials of the same task to produce a representative time series signal(s). In the former case the distribution of attitudes may be relatively small, but in the latter case can potentially be much larger.

Given multiple measurements which require averaging one option is to average the coordinates of the points measured in Euclidean space before computation of the angular orientation. With an image-based motion analysis system coordinate data are recorded which can then be used to compute angular kinematics so averaging at this stage of data processing is an option, but some measurement devices record the angular kinematics directly, for example goniometers and inertial measurement units, in which case to average angular kinematics the method presented is necessary.

The analysis here assumed the errors in the data were isotropic, that is the noise has the same variance for each angle. In reality it is likely that these errors will at best be mildly anisotropic. Based on previous analysis examining the determination of rigid body attitude (Challis, 1995), it is anticipated that anisotropy would

increase the errors in averaging rigid body attitude irrespective of the method employed for averaging.

As an alternative to Cardan or Euler angles three-dimensional attitude can be described using unit quaternions, it is feasible to average a set of quaternions. Unless complex numerical methods are used (e.g., Markley et al., 2007), the averaging of quaternions has two problems. The first is the averaging of a series of unit quaternions does not normally provide a unit quaternion, which is required for describing rigid body attitude. To obviate that problem after averaging the result can be normalized giving a unit quaternion. The other problem is that a rigid body attitude can be described by two quaternions, q and $-q$ (Hanson, 2006), such a 2:1 mapping means that averaging of quaternions does not necessarily reflect the actual average attitude. The method presented here, based around the use of the singular value decomposition, avoids these problems.

In conclusion, a method has been presented for determining average rigid body attitude. The same method can be used for computing average joint orientation. This method exploits some of the properties of the singular value decomposition, and provides superior estimates of the average attitude compared with averaging Cardan angles.

Declaration of Competing Interest

The authors declared that there is no conflict of interest.

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None.

Appendix

The derivation of the singular value decomposition based algorithm for averaging attitude matrices is presented. The matrix representing the average attitude (\bar{R}) can be determined by minimizing,

$$\min \sum_{i=1}^m \|I - R_i^T \bar{R}\|_F^2 \quad (\text{A.1})$$

where m is the number of attitude matrices (R_i) to average, I is the identity matrix, and $\|\cdot\|_F$ is the Frobenius norm. If, for example, the

two matrices (R_1, R_2) describe the same rigid body attitude then $\bar{R} = R_1 = R_2$ and $R_1^T \bar{R} = R_2^T \bar{R} = I$, in which case the metric described in Eq. (1) gives a value of zero as might be anticipated. Eq. (A.1) is a form of the orthogonal Procrustes problem (Golub and van Loan, 1983), it can be expanded by exploiting the trace (tr) property of matrix products,

$$\min \left[tr(I^T I) + tr\left(\sum R_i^T \sum R_i\right) - 2tr\left(\bar{R}^T \sum R_i^T\right) \right] \quad (A.2)$$

Ignoring constant factors in A.2, a solution to A.1 requires maximizing the following,

$$\max \left[tr\left(\bar{R}^T \sum R_i^T\right) \right] \quad (A.3)$$

The singular value decomposition of $\sum R_i^T$ can be computed,

$$\sum R_i^T = UDV^T \quad (A.4)$$

where U is a 3×3 orthogonal matrix, D is a 3×3 diagonal matrix comprising the singular values ($\sigma_1, \sigma_2, \sigma_3$), and V is a 3×3 orthogonal matrix. Eq. (A.3) can be re-organized to,

$$tr\left(\bar{R}^T \sum R_i^T\right) = tr\left(\bar{R}^T UDV^T\right) = tr(XD) \leq \sum_{i=1}^3 \sigma_i \quad (A.5)$$

where $X = V^T \bar{R}^T U$. Given the orthogonal properties of the matrices, V, \bar{R} , and U then Eq. (A.5) is maximized and therefore A.1 minimized

if $X = I$, which means that $\bar{R} = UV^T$, and thus the average attitude matrix is determined.

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