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Full length article

Moduli of smoothness and growth properties of Fourier transforms: Two-sided estimates[☆]

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Abstract

We prove two-sided inequalities between the integral moduli of smoothness of a function on $\mathbb{R}^d/\mathbb{T}^d$ and the weighted tail-type integrals of its Fourier transform/series. Sharpness of obtained results in particular is given by the equivalence results for functions satisfying certain regular conditions. Applications include a quantitative form of the Riemann–Lebesgue lemma as well as several other questions in approximation theory and the theory of function spaces.

Keywords: Fourier transforms; Moduli of smoothness; Two-sided estimates; Pitt's inequality

1. Introduction

This paper studies the interrelation between the smoothness of a function and the growth properties of Fourier transforms/coefficients. Let us first recall the classical Riemann–Lebesgue lemma: $|\widehat{f_n}| \to 0$ as $|n| \to \infty$, where $f \in L^1(\mathbb{T}^d)$. Its quantitative version, the Lebesgue type estimate for the Fourier coefficients, is well known [34, Volume I, Chapter 4, Section 4] and

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given by

$$|\widehat{f}_n| \lesssim \omega_l \left(f, \frac{1}{|n|} \right)_1, \quad f \in L^1(\mathbb{T}^d), \tag{1.1}$$

where the modulus of smoothness $\omega_l(f, \delta)_p$ of a function $f \in L^p(X)$ is defined by

$$\omega_l\,(f, \delta)_p = \sup_{|h| \le \delta} \left\| \Delta_h^l f(x) \right\|_{L^p(X)}, \quad 1 \le p \le \infty, \tag{1.2}$$

and

$$\Delta_h^l f(x) = \Delta_h^{l-1} \left(\Delta_h f(x) \right), \qquad \Delta_h f(x) = f(x+h) - f(x).$$

As usual, $F \lesssim G$ means that $F \le C\,G$; by C we denote positive constants that may be different on different occasions. Also, $F \asymp G$ means that $F \lesssim G \lesssim F$.

For the Fourier transform, the estimate similar to (1.1) can be found in, e.g., [31]

$$|\widehat{f}(\xi)| \lesssim \omega_l \left(f, \frac{1}{|\xi|} \right)_1, \quad f \in L^1(\mathbb{R}^d), \tag{1.3}$$

where the Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\xi x}\, dx, \quad \xi \in \mathbb{R}^d. \tag{1.4}$$

However, unlike (1.1) the inequality (1.3) cannot be extended for the range $p > 1$ (see Section 7.2).

Very recently, Bray and Pinsky [6,7] and Ditzian [13] (see also Gioev's paper [17]) have extended the Lebesgue type estimate for the Fourier transform/coefficients. We will need the following average function. For a locally integrable function f the average on a sphere in $\mathbb{R}^d$ of radius $t > 0$ is given by

$$V_t f(x) := \frac{1}{m_t} \int_{|y-x|=t} f(y)\, dy \quad \text{with } V_t 1 = 1,\ d \ge 2.$$

For $l \in \mathbb{N}$ we define

$$V_{l,t} f(x) := \frac{-2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j \binom{2l}{l-j} V_{jt} f(x).$$

Theorem A. *Let $f \in L^p(\mathbb{R}^d)$, $d \ge 2$, and $1 \le p \le 2$, $1/p + 1/p' = 1$. Then for $t > 0$, $l \in \mathbb{N}$,*

$$\left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2l} |\widehat{f}(\xi)| \right]^{p'} d\xi \right)^{1/p'} \lesssim \|f - V_{l,t} f\|_p, \quad 1 < p \le 2, \tag{1.5}$$

and

$$\sup_{\xi \in \mathbb{R}^d} \left[\min(1, t|\xi|)^{2l} |\widehat{f}(\xi)| \right] \lesssim \|f - V_{l,t} f\|_1. \tag{1.6}$$

Similar results were also proved for moduli of smoothness of functions on $\mathbb{R}$ and $\mathbb{T}^d$ (see [13]). In the rest of the paper we will assume that $t > 0, l \in \mathbb{N}$, and

$$\Omega_l(f,t)_p = \|f - V_{l,t} f\|_p, \qquad \theta = 2, \tag{1.7}$$

if $d \geq 2$ and

$$\Omega_l(f,t)_p = \omega_l(f,t)_p, \qquad \theta = 1 \tag{1.8}$$

if $d = 1$.

The main goal of this paper is to extend inequalities (1.5) and (1.6) in the following sense. First, we prove sharper estimates by considering the weighted L^q norm of $\min(1, t|\xi|)^{\theta l} |\widehat{f}(\xi)|$, that is,

$$\left\| \min(1, t|\xi|)^{\theta l} |\widehat{f}(\xi)| \right\|_{L^q(u)} \lesssim \Omega_l(f,t)_p, \quad p \leq q \tag{1.9}$$

with the certain weight function u. Then varying the parameter q gives us the better bound from below of $\Omega_l(f,t)_p$. In particular, if $q = p'$ we arrive at (1.5) and (1.6).

Second, we prove the reverse inequalities showing how smoothness of a function depends on the average decay of its Fourier transform:

$$\Omega_l(f,t)_p \lesssim \left\| \min(1, t|\xi|)^{\theta l} |\widehat{f}(\xi)| \right\|_{L^q(u)}, \quad q \leq p. \tag{1.10}$$

Third, we define the class of general monotone functions and prove that for this class the equivalence result holds:

$$\Omega_l(f,t)_p \asymp \left\| \min(1, t|\xi|)^{\theta l} |\widehat{f}(\xi)| \right\|_{L^p(u)}. \tag{1.11}$$

Note that for $p = 2$, this follows from (1.9) and (1.10) in the general case (see also [6,17]).

The paper is organized as follows. In Section 2, we prove inequalities (1.9) and (1.10) when $1 < p \leq 2$ and $p \geq 2$ respectively. In Section 3 we study inequalities (1.9) and (1.10) in the case of radial functions and we show that, with a fixed p, the range of the parameter q is extended. In Section 4 we deal with the general monotone functions. Again, we prove inequalities (1.9) and (1.10) under wider range of the parameter q than in the case of radial functions. Moreover, we show equivalence (1.11) in this case. Section 5 studies inequalities (1.9) and (1.10) for functions on $\mathbb{T}^d$, $d \geq 1$. In Section 6 we obtain the equivalence result of type (1.11) for periodic functions whose sequence of Fourier coefficients is general monotone. Section 7 considers several applications of obtained results in approximation theory (sharp relations between best approximations and moduli of smoothness) and functional analysis (embedding theorems, characterization of the Lipschitz/Besov spaces in terms of the Fourier transforms).

Finally, we remark that inequalities between moduli of smoothness and the Fourier transform in the Lebesgue and Lorentz spaces were studied earlier in [8,16].

2. Growth of Fourier transforms via moduli of smoothness: the general case

The following theorem is the main result of this section.

Theorem 2.1. *Let $f \in L^p(\mathbb{R}^d)$, $d \geq 1$.*

(A) *Let* $1 < p \le 2$. *Then for* $p \le q \le p'$ *we have* $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$, *and*

$$\left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{\theta l}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \lesssim \Omega_l(f,t)_p. \tag{2.1}$$

(B) *Let* $2 \le p < \infty$, $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$, $q > 1$, *and* $\max\{q,q'\} \le p$. *Then*

$$\left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{\theta l}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \gtrsim \Omega_l(f,t)_p. \tag{2.2}$$

Remark. Theorem A follows from Theorem 2.1(A) (take $q = p'$). In part (B) we assume that for $f \in L^p(\mathbb{R}^d)$ the Fourier transform $\widehat{f}$ is well defined and such that $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$ for a certain $q > 1$ satisfying $\max\{q,q'\} \le p$.

Proof of Theorem 2.1. We will use the following Pitt's inequality [3] (see also [18]):

$$\left(\int_{\mathbb{R}^d}\left(|\xi|^{-\gamma}|\widehat{g}(\xi)|\right)^q d\xi\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d}\left(|x|^{\beta}|g(x)|\right)^p dx\right)^{1/p}, \tag{2.3}$$

where

$$\beta-\gamma = d\left(1-\frac{1}{p}-\frac{1}{q}\right), \qquad \max\left\{0, d\left(\frac{1}{p}+\frac{1}{q}-1\right)\right\} \le \gamma < \frac{d}{q},$$
$$1 < p \le q < \infty. \tag{2.4}$$

Here the Fourier transform $\widehat{g}$ is understood in the usual sense of weighted Fourier inequality (2.3); see, e.g., [4, Sections 1,2].

Let us write inequality (2.3) with change of parameters $\widehat{g} \leftrightarrow f, p \leftrightarrow q, \beta \leftrightarrow -\gamma$. Let $|\xi|^{-\gamma}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$, then

$$\left(\int_{\mathbb{R}^d}\left(|\xi|^{-\gamma}|\widehat{f}(\xi)|\right)^q d\xi\right)^{1/q} \gtrsim \left(\int_{\mathbb{R}^d}\left(|x|^{\beta}|f(x)|\right)^p dx\right)^{1/p}, \tag{2.5}$$

where

$$\beta-\gamma = d\left(1-\frac{1}{p}-\frac{1}{q}\right), \qquad \max\left\{0, d\left(\frac{1}{p}+\frac{1}{q}-1\right)\right\} \le -\beta < \frac{d}{p},$$
$$1 < q \le p < \infty. \tag{2.6}$$

The case of $d \ge 2$

Then by (1.7), $\Omega_l(f,t)_p = \|f - V_{l,t}f\|_p$, $\theta = 2$. Let us write the left-hand side in (2.1) and (2.2) as

$$I := \left\|\min(1,t|\xi|)^{2l}h(\xi)\right\|_q, \qquad h(\xi) = |\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|.$$

In [9, Corollary 2.3, Theorem 3.1], it is shown that for $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, $t > 0$, and integer l,

$$\|f - V_{l,t}f\|_p \asymp K_l(f,\Delta,t^{2l})_p \asymp R_l(f,\Delta,t^{2l})_p, \tag{2.7}$$

where

$$K_l(f, \Delta, t^{2l})_p := \inf\bigl\{\|f - g\|_p + t^{2l}\|\Delta^l g\|_p : \Delta^l g \in L^p(\mathbb{R}^d)\bigr\},$$

the Laplacian is given by $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$,

$$\begin{aligned} &R_l(f, \Delta, t^{2l})_p := \|f - R_{\lambda,l,b}(f)\|_p + t^{2l}\|\Delta^l R_{\lambda,l,b}(f)\|_p, \\ &\lambda = 1/t, \quad b \ge d + 2. \end{aligned} \tag{2.8}$$

Here (see [9, Section 2])

$$\begin{aligned} &R_{\lambda,l,b}(f)(x) = (G_{\lambda,l,b} * f)(x), \qquad G_{\lambda,l,b}(x) = \lambda^d G_{l,b}(\lambda x), \\ &\widehat{G_{l,b}}(\xi) = \eta_{l,b}(|\xi|), \end{aligned}$$

where

$$\eta_{l,b}(s) = (1 - s^{2l})_+^b, \quad s = |\xi| \ge 0, \tag{2.9}$$

and

$$\begin{aligned} &\bigl[R_{\lambda,l,b}(f)\bigr]\widehat{\ }(\xi) = \eta_{l,b}(t|\xi|)\widehat{f}(\xi), \\ &\quad \bigl[f - R_{\lambda,l,b}(f)\bigr]\widehat{\ }(\xi) = \bigl[1 - \eta_{l,b}(t|\xi|)\bigr]\widehat{f}(\xi), \\ &\Bigl[\Delta^l R_{\lambda,l,b}(f)\Bigr]\widehat{\ }(\xi) = (-1)^l|\xi|^{2l}\bigl[R_{\lambda,l,b}(f)\bigr]\widehat{\ }(\xi) = (-1)^l|\xi|^{2l}\eta_{l,b}(t|\xi|)\widehat{f}(\xi). \end{aligned} \tag{2.10}$$

Also,

$$\|G_{\lambda,l,b}(x)\|_1 = \|G_{l,b}\|_1 < \infty. \tag{2.11}$$

Taking into account that, for $b > 0$,

$$\eta_{l,b}(s) \sim 1 - bs^{2l}, \quad s \to 0, \qquad \eta_{l,b}(s) = 0, \quad s \ge 1,$$

we obtain

$$1 - \eta_{l,b}(s) \asymp \min(1, s)^{2l}, \quad s \ge 0. \tag{2.12}$$

Changing variables $b \leftrightarrow b + 1$ gives

$$\min(1, s)^{2l} \asymp 1 - \eta_{l,b+1}(s) = 1 - (1 - s^{2l})\eta_{l,b}(s) = 1 - \eta_{l,b}(s) + s^{2l}\eta_{l,b}(s).$$

Therefore,

$$I = \Bigl\|\min(1, t|\xi|)^{2l} h(\xi)\Bigr\|_q \asymp \Bigl\|\Bigl[1 - \eta_{l,b}(t|\xi|) + (t|\xi|)^{2l}\eta_{l,b}(t|\xi|)\Bigr] h(\xi)\Bigr\|_q. \tag{2.13}$$

Define

$$h_1(\xi) = \bigl[1 - \eta_{l,b}(t|\xi|)\bigr] h(\xi), \qquad h_2(\xi) = (t|\xi|)^{2l}\eta_{l,b}(t|\xi|)h(\xi). \tag{2.14}$$

Note that both h_1 and h_2 are non-negative. For non-negative functions we have

$$\|h_1 + h_2\|_q \asymp \|h_1\|_q + \|h_2\|_q, \quad 1 \le q \le \infty. \tag{2.15}$$

This, (2.13), and (2.14) yield

$$I \asymp \left\| |\xi|^{d(1-1/p-1/q)} \left[1-\eta_{l,b}(t|\xi|)\right] |\widehat{f}(\xi)| \right\|_q + \left\| |\xi|^{d(1-1/p-1/q)} (t|\xi|)^{2l} \eta_{l,b}(t|\xi|) |\widehat{f}(\xi)| \right\|_q,$$

or, by (2.10),

$$I \asymp \left\| |\xi|^{d(1-1/p-1/q)} \left|\left[f - R_{\lambda,l,b}(f)\right]^\frown(\xi)\right| \right\|_q + t^{2l} \left\| |\xi|^{d(1-1/p-1/q)} \left|\left[\Delta^l R_{\lambda,l,b}(f)\right]^\frown(\xi)\right| \right\|_q. \tag{2.16}$$

Now to prove (A), we assume that $p \le q$ and we use (2.16) and Pitt's inequality (2.3) with $\beta = 0$. In this case $\gamma = d\left(\frac{1}{p}+\frac{1}{q}-1\right)$ and $\gamma \ge 0$ (see (2.4)). The latter is ensured by $q \le p'$. Then $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$ and

$$I \lesssim \left\| f - R_{\lambda,l,b}(f) \right\|_p + t^{2l} \left\| \Delta^l R_{\lambda,l,b}(f) \right\|_p.$$

Combining this with (2.7), and (2.8) we get (A).

In part (B) we assume that $q \le p$. Inequality (2.2) follows from (2.16) and inequality (2.5) for $\beta = 0$. In this case, by (2.6), $\gamma = d\left(\frac{1}{p}+\frac{1}{q}-1\right)$ and $\max\{0,\gamma\} \le 0$, i.e., $\gamma \le 0$. The latter is $q \ge p'$ or, equivalently, $q' \le p$.

The case of $d = 1$

According to (1.8), we have $\Omega_l(f,t)_p = \omega_l(f,t)_p$ and $\theta = 1$. The proof of key steps is similar to the proof in the case of $d \ge 2$. The only difference is the realization result [14] given by

$$\omega_l(f,t)_p \asymp \inf\left(\|f-g\|_p + t^l \|g^{(l)}\|_p : g^{(l)} \in \mathrm{E}_\lambda \cap L^p(\mathbb{R})\right)$$
$$\asymp \|f-g_\lambda\|_p + t^l \|g_\lambda^{(l)}\|_p, \quad \lambda = 1/t,$$

where E_λ is the collection of all entire functions of exponential type λ and $g_\lambda \in \mathrm{E}_\lambda$ is such that

$$\|f-g_\lambda\|_p \lesssim E_\lambda(f)_p := \inf_{g \in \mathrm{E}_\lambda} \|f-g\|_p.$$

Since $\|g_\lambda^{(l)}\|_p \asymp \|Hg_\lambda^{(l)}\|_p$, $1<p<\infty$, where H is the Hilbert transform [29, Chapter 5], then $\omega_l(f,t)_p \asymp \|f-g_\lambda\|_p + t^l\|D_l g_\lambda\|_p$, where $D_l = (id/dx)^l$ for even l and $D_l = -iH(id/dx)^l$ for odd l.

Let $\chi_\lambda := \chi_{[0,\lambda]}$. As Hille and Tamarkin [20] showed, if $S_\lambda(f)$ is the partial Fourier integral of f, i.e.,

$$[S_\lambda(f)]^\frown(\xi) = \chi_\lambda(|\xi|)\widehat{f}(\xi), \tag{2.17}$$

we have

$$\|S_\lambda(f)\|_p \lesssim \|f\|_p, \quad 1<p<\infty.$$

Then (see also [28]) g_λ can be taken as $S_\lambda(f)$, that is, $\|f - S_\lambda(f)\|_p \lesssim E_\lambda(f)_p$. Therefore, for $1 < p < \infty$,

$$\omega_l(f,t)_p \asymp \|f - S_\lambda(f)\|_p + t^l \left\|S_\lambda^{(l)}(f)\right\|_p \asymp \|f - S_\lambda(f)\|_p + t^l \left\|D_l S_\lambda(f)\right\|_p, \tag{2.18}$$

where

$$\left[S_\lambda^{(l)}(f)\right]\widehat{\ }(\xi) = (-i\xi)^l \chi_\lambda(|\xi|)\widehat{f}(\xi), \qquad [D_l S_\lambda(f)]\widehat{\ }(\xi) = |\xi|^l \chi_\lambda(|\xi|)\widehat{f}(\xi). \tag{2.19}$$

For $s \ge 0$ we have $\min(1,s)^l = 1 - \chi_1(s) + s^l\chi_1(s)$ and $\chi_1(ts) = \chi_\lambda(s)$, which gives

$$\min(1,ts)^l = 1 - \chi_\lambda(s) + (ts)^l \chi_\lambda(s). \tag{2.20}$$

This, (2.15), (2.17), and (2.19) imply

$$\begin{aligned} I &:= \left\|\min(1,t|\xi|)^l |\xi|^{1-1/p-1/q}|\widehat{f}(\xi)|\right\|_q \\ &= \left\|\left[1 - \chi_\lambda(|\xi|) + (t|\xi|)^l\chi_\lambda(|\xi|)\right]|\xi|^{1-1/p-1/q}|\widehat{f}(\xi)|\right\|_q \\ &\asymp \left\||\xi|^{1-1/p-1/q}[1-\chi_\lambda(|\xi|)]|\widehat{f}(\xi)|\right\|_q + \left\||\xi|^{1-1/p-1/q}(t|\xi|)^l\chi_\lambda(|\xi|)|\widehat{f}(\xi)|\right\|_q \\ &= \left\||\xi|^{1-1/p-1/q}\,|[f - S_\lambda(f)]\widehat{\ }(\xi)|\right\|_q + t^l \left\||\xi|^{1-1/p-1/q}\,|[D_l S_\lambda(f)]\widehat{\ }(\xi)|\right\|_q, \end{aligned} \tag{2.21}$$

which is an analogue of (2.16). Then as in the case of $d \ge 2$ we continue by using Pitt's inequality (2.3) and its corollary (2.5) with $\beta = 0$ and $d = 1$. This concludes the proof of the case $d = 1$. $\square$

3. Growth of Fourier transforms via moduli of smoothness: the case of radial functions

Theorem 2.1 was proved under the condition $1 < p \le q \le p' < \infty$ (A) and $1 < \max\{q, q'\} \le p < \infty$ (B). When $d \ge 2$ these conditions can be extended if we restrict ourselves to radial functions

$$f(x) = f_0(|x|).$$

The Fourier transform of a radial function is also radial, $\widehat{f}(\xi) = F_0(|\xi|)$ (see [25, Chapter 4]) and it can be written as the Fourier–Hankel transform

$$F_0(s) = |S^{d-1}| \int_0^\infty f_0(t) j_{d/2-1}(st) t^{d-1}\, dt,$$

where $j_\alpha(t) = \Gamma(\alpha+1)(t/2)^{-\alpha} J_\alpha(t)$ is the normalized Bessel function ($j_\alpha(0) = 1$), $\alpha \ge -1/2$. Useful properties of J_α can be found in, e.g., [1, Chapter 9]; see also [18] for some properties of j_α.

Theorem 3.1. *Let $f \in L^p(\mathbb{R}^d)$ be a radial function and $d \ge 2$.*

(A) *Let $1 < p \le q < \infty$. Then, for $p \le \frac{2d}{d+1}$, $q < \infty$ or $\frac{2d}{d+1} < p \le 2$, $p \le q \le \left(\frac{d+1}{2} - \frac{d}{p}\right)^{-1}$,*

$$\left(\int_{\mathbb{R}^d} \left[\min(1,t|\xi|)^{2l} |\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \lesssim \|f - V_{l,t} f\|_p.$$

(B) *Let* $2 \le p < \infty$, $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$, $q > 1$ *and* $\max\left\{q, d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}\right\} \le p$. *Then*

$$\left(\int_{\mathbb{R}^d}\left[\min(1, t|\xi|)^{2l}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \gtrsim \|f - V_{l,t}f\|_p.$$

Remark. 1. Formally, when $d = 1$ conditions in Theorems 3.1 and 2.1 coincide. However, note that no regularity condition was assumed in Theorem 2.1.

2. The range of conditions on p and q in Theorem 3.1 is wider than the corresponding range in Theorem 2.1 for $d \ge 2$.

Indeed, in Theorem 2.1(A) we assume the following conditions: $1 < p \le 2$ and $p \le q \le p'$. If $p \le \frac{2d}{d+1}$, in Theorem 3.1(A) conditions are $p \le q < \infty$. If $\frac{2d}{d+1} < p \le 2$, then $\left(\frac{d+1}{2} - \frac{d}{p}\right)^{-1} \ge p'$. Thus, the conditions $p \le q \le \left(\frac{d+1}{2} - \frac{d}{p}\right)^{-1}$ are less restrictive than $p \le q \le p'$.

In its turn, in Theorem 2.1(B) we assume that $2 \le p < \infty$ and $\max\{q, q'\} \le p$. If $q < 2$, then $p \ge q'$ and $\max\left\{q, d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}\right\} = d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1} < q'$. If $2 \le q$, then $\max\left\{q, d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}\right\} = q$. Hence, we get $\max\left\{q, d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}\right\} \le \max\{q, q'\}$.

Proof of Theorem 3.1. The proof is similar to the proof of Theorem 2.1 but we use Pitt's inequality for radial functions. We also remark that for a radial function f, functions $f - R_{\lambda,l,b}(f)$ and $\Delta^l R_{\lambda,l,b}(f)$ are radial as well.

De Carli [11] proved Pitt's inequality for the Hankel transform. In particular, this gives inequality (2.3) for radial functions. As it was shown in [11], in this case the condition on γ is as follows

$$\frac{d}{q} - \frac{d+1}{2} + \max\left\{\frac{1}{p}, \frac{1}{q'}\right\} \le \gamma < \frac{d}{q}, \quad 1 < p \le q < \infty. \tag{3.1}$$

Therefore, (2.5) for radial functions holds under the condition

$$\frac{d}{p} - \frac{d+1}{2} + \max\left\{\frac{1}{q}, \frac{1}{p'}\right\} \le -\beta < \frac{d}{p}, \quad 1 < q \le p < \infty. \tag{3.2}$$

We will use (3.1) and (3.2) with $\beta = 0$ and $\gamma = d\left(\frac{1}{p} + \frac{1}{q} - 1\right)$.

To show (A), we assume (3.1), that is, the following two conditions hold simultaneously

$$\frac{d-1}{2} + \frac{1}{p} \le \frac{d}{p}, \qquad \frac{d-1}{2} + \frac{1}{q'} \le \frac{d}{p}.$$

If $d \ge 2$, the first condition is equivalent to $p \le 2$. If $p \le \frac{2d}{d+1}$, then the second condition is $q < \infty$. If $\frac{2d}{d+1} < p \le 2$, then respectively $q \le \left(\frac{d+1}{2} - \frac{d}{p}\right)^{-1}$.

Let us verify all conditions in (B). We assume (3.2), or, equivalently,

$$\frac{d}{p} - \frac{d+1}{2} + \frac{1}{q} \le 0, \qquad \frac{d}{p} - \frac{d+1}{2} + \frac{1}{p'} \le 0.$$

If $d \geq 2$, the second inequality is equivalent to the condition $p \geq 2$. The first inequality can be rewritten as $p \geq d\left(\frac{d+1}{2}-\frac{1}{q}\right)^{-1}$. Since also $p \geq q$, we finally arrive at condition $\max\left\{q, d\left(\frac{d+1}{2}-\frac{1}{q}\right)^{-1}\right\} \leq p$, under which needed Pitt's inequality holds. □

4. Growth of Fourier transforms via moduli of smoothness: the case of general monotone functions

The following equivalence holds for $p = 2$ (see [6,13,17] and Theorem 2.1(A), (B)):

$$\left(\int_{\mathbb{R}^d}\left[\min(1, t|\xi|)^{\theta l}|\widehat{f}(\xi)|\right]^p d\xi\right)^{1/p} \asymp \Omega_l(f, t)_p, \tag{4.1}$$

where $\Omega_l(f, t)_p$ and θ are given by (1.7) and (1.8).

In this section we show that similar two sided inequalities also hold for $\frac{2d}{d+1} < p < \infty$ provided $\widehat{f}$ is radial, nonnegative and regular in a certain sense.

4.1. General monotone functions and the $\widehat{GM}^d$ class

A function $\varphi(z), z > 0$, is called *general monotone* ($\varphi \in GM$), if it is locally of bounded variation on $(0, \infty)$, vanishes at infinity, and for some constant $c > 1$ depending on φ, the following is true

$$\int_z^\infty |d\varphi(u)| \lesssim \int_{z/c}^\infty \frac{|\varphi(u)|}{u}\, du < \infty, \quad z > 0 \tag{4.2}$$

(see [18]). Any monotone function vanishing at infinity satisfies the GM-condition. Note also that (4.2) implies

$$|\varphi(z)| \lesssim \int_{z/c}^\infty \frac{|\varphi(u)|}{u}\, du. \tag{4.3}$$

In particular, the latter gives, for any $b > 1$,

$$|\varphi(z)| \lesssim \int_{z/(bc)}^\infty u^{-1}\left(\int_{u/b}^{bu} \frac{|\varphi(v)|}{v}\, dv\right) du. \tag{4.4}$$

We will also use the following result on multipliers of general monotone functions.

Lemma 4.1. *Let $\varphi \in GM$ and a function $\alpha(z)$ be locally of bounded variation on $(0, \infty)$ such that $\lim_{z\to 0}\alpha(z) = 0$ and*

$$\int_0^{cu} |d\alpha(v)| \lesssim |\alpha(u)|, \quad u > 0.$$

Then $\varphi_1 = \alpha\varphi \in GM$.

Proof. By definition of GM, it is sufficient to verify

$$I := \int_z^\infty |d\varphi_1(u)| \lesssim \int_{z/c}^\infty \frac{|\varphi_1(u)|}{u}\, du, \quad z > 0. \tag{4.5}$$

First,

$$I \lesssim \int_z^\infty |\varphi(u)|\,|d\alpha(u)| + \int_z^\infty |\alpha(u)|\,|d\varphi(u)| =: I_1 + I_2,$$

and, by (4.3), we get

$$\begin{aligned} I_1 &= \int_z^\infty |\varphi(u)|\,|d\alpha(u)| \lesssim \int_z^\infty \left(\int_{u/c}^\infty \frac{|\varphi(v)|}{v}\,dv\right) |d\alpha(u)| \\ &= \int_{z/c}^\infty \left(\int_z^{cv} |d\alpha(u)|\right) \frac{|\varphi(v)|}{v}\,dv. \end{aligned}$$

To estimate I_2, using

$$|\alpha(u)| = \left|\alpha(z) + \int_z^u d\alpha(v)\right| \lesssim |\alpha(z)| + \int_z^u |d\alpha(v)|, \quad u > z,$$

and condition (4.2), we have

$$\begin{aligned} I_2 &\lesssim |\alpha(z)| \int_z^\infty |d\varphi(u)| + \int_z^\infty \left(\int_z^u |d\alpha(v)|\right) |d\varphi(u)| \\ &\lesssim |\alpha(z)| \int_{z/c}^\infty \frac{|\varphi(v)|}{v}\,dv + \int_z^\infty \left(\int_v^\infty |d\varphi(u)|\right) |d\alpha(v)| \\ &\lesssim |\alpha(z)| \int_{z/c}^\infty \frac{|\varphi(v)|}{v}\,dv + \int_z^\infty \left(\int_{v/c}^\infty \frac{|\varphi(u)|}{u}\,du\right) |d\alpha(v)| \\ &= |\alpha(z)| \int_{z/c}^\infty \frac{|\varphi(v)|}{v}\,dv + \int_{z/c}^\infty \left(\int_z^{cu} |d\alpha(v)|\right) \frac{|\varphi(u)|}{u}\,du. \end{aligned}$$

Therefore, since

$$|\alpha(z)| = \left|\int_0^z d\alpha(v)\right| \le \int_0^z |d\alpha(v)|,$$

we arrive at

$$\begin{aligned} I &\lesssim I_1 + I_2 \lesssim \int_{z/c}^\infty \left(|\alpha(z)| + \int_z^{cu} |d\alpha(v)|\right) \frac{|\varphi(v)|}{v}\,dv \\ &\le \int_{z/c}^\infty \left(\int_0^{cu} |d\alpha(v)|\right) \frac{|\varphi(v)|}{v}\,dv. \end{aligned}$$

Finally, the integral condition on α concludes the proof of (4.5). $\square$

Let $\widehat{GM}^d$, $d \ge 1$, be the collection of all radial functions $f(x) = f_0(|x|)$, $x \in \mathbb{R}^d$, which are defined in terms of the inverse Fourier–Hankel transform

$$f_0(z) = \frac{|S^{d-1}|}{(2\pi)^d} \int_0^\infty F_0(s)\, j_{d/2-1}(zs) s^{d-1}\,ds, \tag{4.6}$$

where the function $F_0 \in GM$ and satisfies the following condition

$$\int_0^1 s^{d-1}|F_0(s)|\,ds + \int_1^\infty s^{(d-1)/2}\,|dF_0(s)| < \infty. \tag{4.7}$$

Applying Lemma 1 from the paper [18] to F_0, we obtain that the integral in (4.6) converges in the improper sense and therefore $f_0(z)$ is continuous for $z > 0$. In addition, F_0 is a radial component of the Fourier transform of the function f, that is, $\widehat{f}(\xi) = F_0(|\xi|)$, $\xi \in \mathbb{R}^d$.

Let us give some examples of functions from the class $\widehat{GM}^d$.

Example 1. Let $f \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $1 \le p < 2d/(d+1)$ for $d \ge 2$ and $p = 1$ for $d = 1$, be a radial positive-definite function such that $F_0 \in GM$. Then $f \in \widehat{GM}^d$. Indeed, $\widehat{f}$ is a continuous function vanishing at infinity and $\widehat{f} \ge 0$ [25, Chapter 1]. From continuity of f at zero we get $\widehat{f} \in L^1(\mathbb{R}^d)$ [25, Corollary 1.26], i.e., $\int_0^\infty s^{d-1}|F_0(s)|\,ds < \infty$. Since any GM-function F_0 satisfies ([18, p. 111])

$$\int_1^\infty s^\sigma\,|dF_0(s)| \lesssim \int_{1/c}^\infty s^{\sigma-1}|F_0(s)|\,ds, \quad \sigma \ge 0,$$

then, using $(d-1)/2 - 1 < d - 1$, we get

$$\int_0^1 s^{d-1}|F_0(s)|\,ds + \int_1^\infty s^{(d-1)/2}\,|dF_0(s)| \lesssim \int_0^\infty s^{d-1}|F_0(s)|\,ds < \infty.$$

Therefore, condition (4.7) holds, that is, $f \in \widehat{GM}^d$. As an example of such function we can take $f(x) = (1 + |x|^2)^{-(d+1)/2}$ and the corresponding $F_0(s) = c_d e^{-s}$.

Example 2. Take $f(x) = j_{d/2}(|x|)$ (for $d = 1$, $f(x) = \frac{\sin x}{x}$). Then $F_0(s) = c\chi_1(s) \in GM$ and condition (4.7) holds, i.e., $f \in \widehat{GM}^d$. Moreover, we have (see, e.g., [18])

$$j_{d/2}(z) \asymp 1, \quad 0 \le z \le 1, \qquad |j_{d/2}(z)| \lesssim z^{-(d+1)/2}, \quad z \ge 1,$$

and

$$|j_{d/2}(z)| \gtrsim z^{-(d+1)/2}, \quad z \in \bigcup_{k=1}^\infty \left[\rho_{d/2,k} + \varepsilon, \rho_{d/2,k+1} - \varepsilon\right],$$

where $\rho_{\alpha,k}$ denote positive zeros of the Bessel function J_α, $\inf_{k\ge1}\left(\rho_{d/2,k+1} - \rho_{d/2,k}\right) \ge 3\varepsilon > 0$. This implies $f \in L^p(\mathbb{R}^d)$ if $p > \frac{2d}{d+1}$.

Example 3. Let $F_0(s) \in GM$ and $|\xi|^{d(1-1/p-1/q)}F_0(|\xi|) \in L^q(\mathbb{R}^d)$, $1 < q \le p < \infty$, $\frac{2d}{d+1} < p$. Then, using statement (A.1) below, condition (4.7) for F_0 holds, f is defined by (4.6), and $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$. The fact that $f \in L^p(\mathbb{R}^d)$ follows from Pitt's inequality (4.8) (take $\beta = 0$).

4.2. Two-sided inequalities

Theorem 4.1. *Let $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$, $d \ge 1$.*

(A) *If $\widehat{f} \ge 0$ and $1 < p \le q < \infty$, then*

$$\left(\int_{\mathbb{R}^d}\left[\min(1, t|\xi|)^{\theta l}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \lesssim \Omega_l(f,t)_p.$$

(B) *If $|\xi|^{d(1-1/p-1/q)}\widehat{f}(\xi) \in L^q(\mathbb{R}^d)$, $1 < q \le p < \infty$, $\frac{2d}{d+1} < p$, then*

$$\left(\int_{\mathbb{R}^d}\left[\min(1, t|\xi|)^{\theta l}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \gtrsim \Omega_l(f,t)_p.$$

Remark. Conditions on p and q in Theorem 4.1(A), (B) are less restrictive than corresponding conditions in Theorem 3.1. It is clear for (A). Since $\frac{2d}{d+1} \le d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}$, conditions $q \le p$ and $\frac{2d}{d+1} < p$ in Theorem 4.1(B) are weaker than $\max\left\{2, q, d\left(\frac{d+1}{2} - \frac{1}{q}\right)^{-1}\right\} \le p$, which is the corresponding condition in Theorem 3.1(B).

In the case of $p = q$ Theorem 4.1 gives the following equivalence result.

Corollary 4.1. *If* $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$, $d \ge 1$, $\widehat{f} \ge 0$, $\frac{2d}{d+1} < p < \infty$, *then*

$$\left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{\theta l} |\xi|^{d(1-2/p)} |\widehat{f}(\xi)|\right]^p d\xi\right)^{1/p} \asymp \Omega_l(f,t)_p.$$

Example. Take $f(x) = j_{d/2}(|x|)$ (see Example 2). By Corollary 4.1, for $0 < t < 1$ and $\frac{2d}{d+1} < p < \infty$, we have

$$\Omega_l(f,t)_p \asymp \left\| \min(1, t|\xi|)^{\theta l} |\xi|^{d(1-2/p)} \chi_1(|\xi|) \right\|_p \asymp t^{\theta l}.$$

4.3. Weighted Fourier inequalities

To prove Theorem 4.1, we will use several auxiliary results from the paper [18].

Let $d \ge 1$, $1 < p, q < \infty$, $\beta - \gamma = d\left(1 - \frac{1}{p} - \frac{1}{q}\right)$, $g(x) = g_0(|x|)$, and $\widehat{g}(\xi) = G_0(|\xi|)$.

(A.1) If $g_0 \in GM$, $p \le q$, and

$$\frac{d}{q} - \frac{d+1}{2} < \gamma < \frac{d}{q},$$

then the following Pitt's inequality holds [18, Theorem 2 (A)]

$$\left\| |\xi|^{-\gamma} \widehat{g}(\xi) \right\|_q \lesssim \left\| |x|^{\beta} g(x) \right\|_p.$$

Then changing variables $g \leftrightarrow \widehat{f}$, $p \leftrightarrow q$, and $\beta \leftrightarrow -\gamma$, we get

$$\left\| |x|^{\beta} f(x) \right\|_p \lesssim \left\| |\xi|^{-\gamma} \widehat{f}(\xi) \right\|_q, \quad \frac{d}{p} - \frac{d+1}{2} < -\beta < \frac{d}{p}, \ q \le p. \tag{4.8}$$

Here $\widehat{f}(\xi) = F_0(|\xi|)$ and $F_0 \in GM$. Note [18, Section 5.1] that the condition $|\xi|^{-\gamma} \widehat{f}(\xi) \in L^q(\mathbb{R}^d)$ implies condition (4.7).

(A.2) Let $g_0 \in GM$, $g_0 \ge 0$ and g_0 satisfy condition (4.7). Then if $q \le p$ and

$$\frac{d}{q} - \frac{d+1}{2} < \gamma,$$

then [18, Theorem 2(B)]

$$\left\| |\xi|^{-\gamma} \widehat{g}(\xi) \right\|_q \gtrsim \left\| |x|^{\beta} g(x) \right\|_p.$$

Again, changing variables $g \leftrightarrow \widehat{f}$, $p \leftrightarrow q$, and $\beta \leftrightarrow -\gamma$, we arrive at

$$\left\||x|^{\beta} f(x)\right\|_p \gtrsim \left\||\xi|^{-\gamma} \widehat{f}(\xi)\right\|_q, \quad \frac{d}{p} - \frac{d+1}{2} < -\beta,\ p \le q. \tag{4.9}$$

Here $\widehat{f}(\xi) = F_0(|\xi|) \ge 0$ and $F_0 \in GM$.

From (A.1) and (A.2) (see also [18, Theorem 1]), for a non-negative GM-function F_0 satisfying condition (4.7), we have

$$\left\||\xi|^{d(1-2/p)} \widehat{f}(\xi)\right\|_p \asymp \|f(x)\|_p, \quad \frac{2d}{d+1} < p < \infty. \tag{4.10}$$

(A.3) Let $g_0 \ge 0$. For $z > 0$ we get (see [18, formula (53)])

$$\begin{aligned} \int_{z/(bc)}^{\infty} u^{-1} \left(\int_{u/b}^{bu} \frac{g_0(v)}{v}\, dv\right) du \lesssim \int_0^{2bc/z} & u^{(d-1)/2-1} \\ & \times \left(\int_0^u v^{(d-1)/2} |G_0(v)|\, dv\right) du, \end{aligned} \tag{4.11}$$

where $1 < b < \rho_{d/2,1}$.

(A.4) The following inequality was shown in [18, pp. 115–116]

$$\begin{aligned} &\left[\int_0^{\infty} u^{-\gamma p + dp/q - dp - 1} \left(\int_0^u v^{(d-1)/2-1} \left(\int_0^v z^{(d-1)/2} |G_0(z)|\, dz\right) dv\right)^p du\right]^{1/p} \\ &\quad \lesssim \left(\int_{\mathbb{R}^d} \left[|x|^{-\gamma} |\widehat{g}(x)|\right]^q dx\right)^{1/q}, \quad \frac{d}{q} - \frac{d+1}{2} < \gamma,\ q \le p. \end{aligned}$$

Noting $u^{-\gamma p + dp/q - dp - 1} = u^{-p\beta - d - 1}$ and changing variables $\widehat{g} \leftrightarrow f$, $p \leftrightarrow q$, $\beta \leftrightarrow -\gamma$, we obtain

$$\begin{aligned} &\left[\int_0^{\infty} u^{q\gamma - d - 1} \left(\int_0^u v^{(d-1)/2-1} \left(\int_0^v z^{(d-1)/2} |f_0(z)|\, dz\right) dv\right)^q du\right]^{1/q} \\ &\quad \lesssim \left(\int_{\mathbb{R}^d} \left[|x|^{\beta} |f(x)|\right]^p dx\right)^{1/p}, \quad \frac{d}{p} - \frac{d+1}{2} < -\beta,\ p \le q. \end{aligned} \tag{4.12}$$

4.4. *Proof of Theorem 4.1 in the case $d \ge 2$*

Let $t > 0$, $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$, $f(x) = f_0(|x|)$, and $\widehat{f}(\xi) = F_0(|\xi|)$. Note that $F_0 \in GM$. We use notations from the proof of Theorem 2.1.

First, we prove (B). Let $|\xi|^{d(1-1/p-1/q)} \widehat{f}(\xi) \in L^q(\mathbb{R}^d)$. We have

$$\begin{aligned} I &= \left\|\min(1, t|\xi|)^{2l} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)|\right\|_q \\ &\asymp \left\||\xi|^{d(1-1/p-1/q)} \left[1 - \eta_{l,b}(t|\xi|)\right] |\widehat{f}(\xi)|\right\|_q \\ &\quad + \left\||\xi|^{d(1-1/p-1/q)} (t|\xi|)^{2l} \eta_{l,b}(t|\xi|) |\widehat{f}(\xi)|\right\|_q =: I_1 + I_2. \end{aligned}$$

Then inequalities

$$\int_0^{cu} |d(1-\eta_{l,b}(tv))| \asymp t^{2l}\int_0^{cu} v^{2l-1}\eta_{l,b-1}(tv)\,dv \le t^{2l}\int_0^{\min(cu,1/t)} v^{2l-1}\,dv$$
$$\asymp \min(1,ctu)^{2l} \asymp 1-\eta_{l,b}(tu), \quad b>1,$$

and Lemma 4.1 imply that the function $\left[1-\eta_{l,b}(ts)\right]F_0(s) = \left[1-\eta_{l,b}(t|\xi|)\right]\widehat{f}(\xi)$ is a GM-function. Using Pitt's inequality (4.8) for $\beta=0$ and $\gamma = d\left(\frac{1}{p}+\frac{1}{q}-1\right)$ yields

$$I_1 = \left\| |\xi|^{d(1-1/p-1/q)}\left[f-R_{\lambda,l,b}(f)\right]^{\widehat{}}(\xi)\right\|_q \gtrsim \left\|f-R_{\lambda,l,b}(f)\right\|_p \tag{4.13}$$

for

$$p > \frac{2d}{d+1}, \quad q \le p. \tag{4.14}$$

Since $\eta_{l,b}(s)=0$ when $s\ge 1$, then $(ts)^{2l}\eta_{l,b}(ts) = \min(1,ts)^{2l}\eta_{l,b}(ts)$. This and (2.10) give

$$(-1)^l t^{2l}\left[\Delta^l R_{\lambda,l,b}(f)\right]^{\widehat{}}(\xi) = \eta_{l,b}(ts)\min(1,ts)^{2l}F_0(s), \quad s=|\xi|.$$

Also, since $\eta_{l,b}(t|\xi|) = \widehat{G_{l,\lambda,b}}(\xi)$, then

$$(-1)^l t^{2l}\Delta^l R_{\lambda,l,b}(f) = G_{\lambda,l,b} * h, \qquad \widehat{h}(\xi) = \min(1,t|\xi|)^{2l}F_0(|\xi|).$$

Using Young's convolution inequality, we obtain

$$\left\|t^{2l}\Delta^l R_{\lambda,l,b}(f)\right\|_p \le \|G_{\lambda,l,b}\|_1\|h\|_p = \|G_{l,b}\|_1\|h\|_p \lesssim \|h\|_p.$$

We remark that

$$\min(1,ts)^{2l}F_0(s) \in GM. \tag{4.15}$$

This follows from the estimate

$$\int_0^{cu} |d\min(1,tv)^{2l}| \asymp t^{2l}\int_0^{\min(cu,1/t)} v^{2l-1}\,dv \asymp \min[(ctu)^{2l},1] \asymp \min(1,tu)^{2l},$$

and Lemma 4.1.

Using again Pitt's inequality (4.8), we have

$$I = \left\||\xi|^{d(1-1/p-1/q)}\widehat{h}(\xi)\right\|_q \gtrsim \|h\|_p \gtrsim \left\|t^{2l}\Delta^l R_{\lambda,l,b}(f)\right\|_p. \tag{4.16}$$

Adding estimates (4.13) and (4.16), we get

$$\|f-V_{l,t}f\|_p \asymp \left\|f-R_{\lambda,l,b}(f)\right\|_p + t^{2l}\left\|\Delta^l R_{\lambda,l,b}(f)\right\|_p \lesssim I_1 + I \lesssim I.$$

This and (4.14) give the part (B) of the theorem.

Let us now prove the part (A). If $p \le \frac{2d}{d+1}$, the proof follows from Theorem 3.1. Suppose $\widehat{f}(\xi) = F_0(|\xi|) \ge 0$. By [9, Lemma 3.4],

$$\left[f-V_{l,t}f\right]^{\widehat{}}(\xi) = [1-m_l(t|\xi|)]\,\widehat{f}(\xi),$$

where the function $m_l(s)$ satisfies for $d \geq 2$ the following conditions

$$0 < C_1 s^{2l} \leq 1 - m_l(s) \leq C_2 s^{2l}, \quad 0 < s \leq \pi, \qquad 0 < m_l(s) \leq v_{d,l} < 1, \quad s \geq \pi.$$

This gives

$$1 - m_l(s) \asymp \min(1, s)^{2l}, \quad s \geq 0. \tag{4.17}$$

Define $h(x) = f(x) - V_{l,t} f(x)$ and its radial component by $h_0 := G_0$. Using (4.11) for the non-negative function $g_0(s) = [1 - m_l(ts)]\, F_0(s)$, we obtain

$$\begin{aligned} J(z) &:= \int_{z/(bc)}^{\infty} u^{-1} \left(\int_{u/b}^{bu} \frac{g_0(v)}{v}\, dv \right) du \\ &\lesssim \int_0^{2bc/z} u^{(d-1)/2-1} \left(\int_0^u v^{(d-1)/2} |h_0(v)|\, dv \right) du. \end{aligned} \tag{4.18}$$

Using (4.17), we get

$$J(z) \asymp \int_{z/(bc)}^{\infty} u^{-1} \left(\int_{u/b}^{bu} \frac{\min(1, tv)^{2l} F_0(v)}{v}\, dv \right) du, \quad z > 0$$

where, by (4.15), $\min(1, tv)^{2l} F_0(v) \in GM$. Therefore, (4.4) for $z > 0$ yields

$$\min(1, tz)^{2l} F_0(z) \lesssim J(z).$$

Further, the latter and (4.18) imply

$$\begin{aligned} I &= \left\| \min(1, t|\xi|)^{2l} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right\|_q \\ &\asymp \left(\int_0^{\infty} \left[z^{d(1-1/p-1/q)} \min(1, tz)^{2l} F_0(z) \right]^q z^{d-1}\, dz \right)^{1/q} \\ &\lesssim \left(\int_0^{\infty} \left[z^{d(1-1/p-1/q)} J(z) \right]^q z^{d-1}\, dz \right)^{1/q} \\ &\lesssim \left(\int_0^{\infty} \left[z^{d(1-1/p-1/q)} \left(\int_0^{2bc/z} u^{(d-1)/2-1} \right.\right.\right. \\ &\qquad \left.\left.\left. \left(\int_0^u v^{(d-1)/2} |h_0(v)|\, dv \right) du \right) \right]^q z^{d-1}\, dz \right)^{1/q}. \end{aligned}$$

Changing variables $2bc/z \to z$, we obtain

$$\begin{aligned} I &\lesssim \left(\int_0^{\infty} z^{-qd(1-1/p-1/q)-d-1} \left[\int_0^z u^{(d-1)/2-1} \right.\right. \\ &\qquad \left.\left. \left(\int_0^u v^{(d-1)/2} |h_0(v)|\, dv \right) du \right]^q dz \right)^{1/q}. \end{aligned} \tag{4.19}$$

Let us now use (4.12) for $\beta = 0$ and $\gamma = d\left(\frac{1}{p} + \frac{1}{q} - 1\right)$. Since, in this case

$$z^{-qd(1-1/p-1/q)-d-1} = z^{q\gamma-d-1}$$

inequalities (4.12) and (4.19) give

$$I \lesssim \left(\int_{\mathbb{R}^d} |h(x)|^p \, dx\right)^{1/p} = \left\| f - V_{l,t} f \right\|_p \tag{4.20}$$

when $\frac{d}{p} - \frac{d+1}{2} < 0$ and $p \le q$. The latter is $\frac{2d}{d+1} < p \le q$. The proof of (A) is now complete. $\square$

4.5. Proof of Theorem 4.1 in the case $d = 1$

We follow the proof of Theorem 2.1. We have

$$\omega_l(f,t)_p \asymp \inf\left(\|f - g\|_p + t^l \|g^{(l)}\|_p : g^{(l)} \in \mathrm{E}_\lambda \cap L^p(\mathbb{R})\right), \quad \lambda = 1/t.$$

To show the estimate of $\omega_l(f,t)_p$ from above, that is, to prove (B), we take $g_\lambda(x)$ such that

$$\widehat{g}_\lambda(\xi) = \left[1 - (t|\xi|)^l\right]_+^b \widehat{f}(\xi), \quad b \ge 3.$$

Note that the function g_λ is analogous to the Riesz-type means $R_{\lambda,l,b}(f)$ and satisfies all required properties (2.9)–(2.12) with l in place of $2l$. In particular, $1 - \left[1 - (ts)^l\right]_+^b \asymp \min(1, ts)^l$. Proceeding similarly to the proof of (B) in the case $d \ge 2$, we arrive at the statement (B) in the case $d = 1$.

Let us now show (A). Let $\frac{2d}{d+1} < p \le q < \infty$ and $\widehat{f} \ge 0$. Equivalence (2.18) gives

$$\omega_l(f,t)_p \asymp \|f - S_\lambda(f)\|_p + t^l \|D_l S_\lambda(f)\|_p \ge \|h\|_p,$$

where $h = f - S_\lambda(f) + t^l D_l S_\lambda(f)$. Moreover, $\widehat{h}(\xi) = \left[1 - \chi_\lambda(|\xi|) + (t|\xi|)^l \chi_\lambda(|\xi|)\right] \widehat{f}(\xi)$. Because of (2.20) and (4.15) with $s \ge 0$, we have $\widehat{h}(\xi) = \min(1, ts)^l F_0(s) \in GM$. Using then (4.9) with $\beta = 0$, we obtain

$$\omega_l(f,t)_p \gtrsim \|h\|_p \gtrsim \left\| |\xi|^{1-1/p-1/q} \widehat{h}(\xi) \right\|_p = \left\| \min(1, t|\xi|)^l |\xi|^{1-1/p-1/q} \widehat{f}(\xi) \right\|_p. \quad \square$$

5. Growth of Fourier coefficients via moduli of smoothness: the case of functions on $\mathbb{T}^d$

Let $f \in L^p(\mathbb{T}^d)$, $1 < p < \infty$, and

$$\widehat{f}_n = \int_{\mathbb{T}^d} f(x) e^{inx} \, dx, \quad n \in \mathbb{Z}^d, \qquad \|\widehat{f}_n\|_{l^q(\mathbb{Z}^d)} := \left(\sum_{n \in \mathbb{Z}^d} |\widehat{f}_n|^q\right)^{1/q}.$$

In the paper [13, Theorem 4.1] the following was proved

$$\left\| \min(1, t|n|)^{\theta l} |\widehat{f}_n| \right\|_{l^{p'}(\mathbb{Z}^d)} \lesssim \Omega_l(f,t)_p, \quad 1 < p \le 2,$$

where $\Omega_l(f,t)_p$ is given by (1.7) and (1.8) with $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{T}^d)}$.

The goal of the section is to obtain the generalization of this result which is a periodic analogue of inequalities (2.1)–(2.2).

Theorem 5.1. *Let $f \in L^p(\mathbb{T}^d)$, $d \ge 1$, $1 < q < \infty$ and $\gamma = d\left(\frac{1}{p} + \frac{1}{q} - 1\right)$.*

(A) *Let* $1 < p \le 2$. *Then for* $p \le q \le p'$ *we have* $\{(1+|n|)^{-\gamma}\widehat{f}_n\} \in l^q(\mathbb{Z}^d)$, *and*

$$\left\| \min(1, t|n|)^{\theta l}(1+|n|)^{-\gamma}\left|\widehat{f}_n\right| \right\|_{l^q(\mathbb{Z}^d)} \lesssim \Omega_l(f,t)_p. \tag{5.1}$$

(B) *Let* $2 \le p < \infty$, $\{(1+|n|)^{-\gamma}\widehat{f}_n\} \in l^q(\mathbb{Z}^d)$, *and* $\max\{q, q'\} \le p$. *Then*

$$\left\| \min(1, t|n|)^{\theta l}(1+|n|)^{-\gamma}\left|\widehat{f}_n\right| \right\|_{l^q(\mathbb{Z}^d)} \gtrsim \Omega_l(f,t)_p. \tag{5.2}$$

The proof of this theorem is similar to the proof of estimates (2.1)–(2.2) from Theorem 2.1. The key points are Pitt's inequalities of form

$$\|\widehat{f}_n(1+|n|)^{-\gamma}\|_{l^q(\mathbb{Z}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}, \quad 1 < p \le 2 \tag{5.3}$$

and

$$\|\widehat{f}_n(1+|n|)^{-\gamma}\|_{l^q(\mathbb{Z}^d)} \gtrsim \|f\|_{L^p(\mathbb{T}^d)}, \quad p \ge 2, \tag{5.4}$$

under the corresponding conditions on q, as well as the realization results for the K-functionals in the periodic case (see [13,14]).

Proof of (5.3). Let us show that the proof of (5.3) follows from Pitt's inequality for functions on $\mathbb{R}^d$. Note that $\gamma \ge 0$. Let f_* be the function on $\mathbb{R}^d$ such that $f_* = f$ on $(-\pi,\pi]^d$ and $f_* = 0$ outside $(-\pi,\pi]^d$. Then

$$\|f_*\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{T}^d)}, \qquad \widehat{f_*}(\xi) = \int_{\mathbb{T}^d} f(x)e^{i\xi x}\,dx, \quad \xi \in \mathbb{R}^d,$$
$$\widehat{f_*}(n) = \widehat{f}_n, \quad n \in \mathbb{Z}^d.$$

Further, we use the results from [21, Chapter 3]. For an entire function g of exponential type $\sigma e, \sigma > 0$, we have

$$\|g\|_{l^q(\mathbb{Z}^d)} \le (1+\sigma)^d \|g\|_{L^q(\mathbb{R}^d)}, \quad q \ge 1. \tag{5.5}$$

Note that the function $\widehat{f_*}$ is an entire function of exponential type $\pi\overline{e}$, where $\overline{e} = (1,\dots,1) \in \mathbb{R}^d$. We cannot use (5.5) since the weight function $|\xi|^{-\gamma}, \gamma \ge 0$, is not an entire function. However, it is possible to construct a positive radial entire function of exponential (spherical) type such that for $|\xi| \ge 1$ this function is equivalent to $|\xi|^{-\gamma}$.

We consider

$$\psi_\gamma(u) = j_\nu\left(\frac{u+i}{2}\right) j_\nu\left(\frac{u-i}{2}\right), \quad u \in \mathbb{C},\ 2\nu+1 = \gamma \ge 0,$$

where j_ν is the normalized Bessel function. The function ψ_γ is an even positive entire function of type 1. Positivity of ψ_γ follows from the fact that all its zeros lie on lines $t \pm i, t \in \mathbb{R}$. The asymptotic expansion of Bessel functions [1, formula 9.2.1] yields, for $|z| \to \infty$,

$$j_\nu(z) = \frac{C_\nu}{z^{\nu+1/2}}\left(\cos(z - c_\nu) + O(|z|^{-1})\right), \qquad \operatorname{Re} z \ge 0, \qquad |\operatorname{Im} z| \lesssim 1.$$

This and $\psi_\gamma(0) > 0$ give $\psi_\gamma(u) \asymp (1+|u|)^{-\gamma}, u \in \mathbb{R}$.

Let us now consider the radial function $\psi_\gamma(|\xi|), \xi \in \mathbb{R}^d$, which is an entire function of (spherical) type 1, and therefore, of type $\overline{e}$. Also,

$$\psi_\gamma(|\xi|) \asymp (1+|\xi|)^{-\gamma}, \quad \xi \in \mathbb{R}^d. \tag{5.6}$$

Define $g(\xi) = \widehat{f_*}(\xi)\psi_\gamma(|\xi|)$, which is an entire function of type $(\pi + 1)\bar{e}$. Using (5.6), we get

$$\|g\|_{l^q(\mathbb{Z}^d)} = \left(\sum_{n\in\mathbb{Z}^d} \left|\widehat{f_*}(n)\psi_\gamma(|n|)\right|^q\right)^{1/q} \asymp \left(\sum_{n\in\mathbb{Z}^d} \left|\widehat{f_n}(1+|n|)^{-\gamma}\right|^q\right)^{1/q},$$

$$\|g\|_{L^q(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left|\widehat{f_*}(\xi)\psi_\gamma(|\xi|)\right|^q\, d\xi\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d} \left|\widehat{f_*}(\xi)|\xi|^{-\gamma}\right|^q\, d\xi\right)^{1/q}.$$

Then by (5.5) and Pitt's inequality for function on $\mathbb{R}^d$, we have

$$\begin{aligned}\left\|\widehat{f_n}(1+|n|)^{-\gamma}\right\|_{l^q(\mathbb{Z}^d)} &\asymp \|g\|_{l^q(\mathbb{Z}^d)} \le (\pi+2)^d\|g\|_{L^q(\mathbb{R}^d)} \lesssim \left\|\widehat{f_*}(\xi)|\xi|^{-\gamma}\right\|_{L^q(\mathbb{R}^d)}\\ &\lesssim \|f_*\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{T}^d)}.\end{aligned}$$

Thus we have proved Pitt's inequality (5.3) for the function on $\mathbb{T}^d$. $\square$

Proof of (5.4). The following inequality is a consequence of [22, Theorem 7] and Hardy's inequality for rearrangements:

$$\|f\|_{L_p} \lesssim \left(\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}(|k_j|+1)^{q/p'-1}|\widehat{f_k}|^q\right)^{1/q}, \qquad \max\left\{q, q'\right\} \le p. \tag{5.7}$$

The latter immediately gives (5.4). We would like to thank Erlan Nursultanov for drawing our attention to his result (5.7), which simplifies the proof. $\square$

6. An equivalence result for periodic functions

A complex null-sequence $a = \{a_n\}_{n\in\mathbb{N}}$ is said to be *general monotone*, written $a \in GM$, if (see [15]) there exists $c > 1$ such that ($\Delta a_k = a_k - a_{k+1}$)

$$\sum_{k=n}^{\infty}|\Delta a_k| \lesssim \sum_{k=[n/c]}^{\infty}\frac{|a_k|}{k}, \quad n\in\mathbb{N}.$$

Theorem 6.1. *Let $f \in L^p(\mathbb{T})$, $1 < p < \infty$, and*

$$f(x) \sim \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx),$$

where nonnegative $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are general monotone sequences. Then

$$\omega_l(f,t)_p \asymp \left(\sum_{\nu=1}^{\infty}\min(1,\nu t)^{lp}\nu^{p-2}\left(a_\nu^p + b_\nu^p\right)\right)^{1/p}. \tag{6.1}$$

We will use the following lemma (see [2]).

Lemma 6.1. *Let $1 < p < \infty$ and let $\sum_{\nu=1}^{\infty} a_\nu \cos\nu x$ be the Fourier series of $f \in L^1(\mathbb{T})$.*
(A) *If the sequences $\{a_n\}$ and $\{\beta_n\}$ are such that*

$$\sum_{k=\nu}^{\infty}|\Delta a_k| \lesssim \beta_\nu, \quad \nu\in\mathbb{N}, \tag{6.2}$$

then

$$\|f\|_p^p \lesssim \sum_{\nu=1}^{\infty} \nu^{p-2}\beta_\nu^p. \tag{6.3}$$

(B) *If* $a=\{a_n\}$ *is a nonnegative sequence, then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=[n/2]}^{n} a_k\right)^p n^{-2} \lesssim \|f\|_p^p. \tag{6.4}$$

Proof of Theorem 6.1. First, we remark that since $1<p<\infty$ it is sufficient to prove that

$$\omega_l^p\left(f,\frac{1}{n}\right)_p \asymp I_1+I_2,$$

where

$$I_1 = n^{-lp}\sum_{\nu=1}^{n} a_\nu^p \nu^{(l+1)p-2}, \qquad I_2=\sum_{\nu=n+1}^{\infty} a_\nu^p\nu^{p-2},$$

$$f(x)\sim\sum_{n=1}^{\infty} a_n\cos nx, \quad \{a_n\}_{n\in\mathbb{N}}\in GM.$$

We will also use the realization result for the modulus of smoothness (see [14]), that is,

$$\omega_l^p\left(f,\frac{1}{n}\right)_p \asymp \left\|f(x)-T_n(x)\right\|_p^p + n^{-lp}\left\|T_n^{(l)}(x)\right\|_p^p, \tag{6.5}$$

where $T_n(f)$ is the n-th almost best approximant, i.e., $\|f(x)-T_n(x)\|_p \lesssim E_n(f)_p$. In particular we can take T_n as $S_n=S_n(f)$, i.e., the n-th partial sum of $\sum_{k=1}^{\infty} a_k\cos kx$.

Let us prove the estimate of I_1 and I_2 from above. Since $\{a_n\}\in GM$, we have

$$a_\nu \le \sum_{l=\nu}^{\infty}|\Delta a_l| \lesssim \sum_{l=[\nu/c]}^{\infty}\frac{a_l}{l}, \tag{6.6}$$

then Hölder's inequality yields

$$\begin{aligned} I_1 &\lesssim n^{-lp}\sum_{\nu=1}^{n}\left(\sum_{j=[\nu/c]}^{\infty}\frac{a_j}{j}\right)^p \nu^{(l+1)p-2} \\ &\lesssim n^{-lp}\sum_{\nu=1}^{n}\left(\sum_{j=[\nu/c]}^{n}\frac{a_j}{j}\right)^p \nu^{(l+1)p-2} + n^{p-1}\left(\sum_{j=n}^{\infty}\frac{a_j}{j}\right)^p \\ &\lesssim n^{-lp}\sum_{\nu=1}^{n}\left(\sum_{j=\nu}^{n}\frac{a_j}{j}\right)^p \nu^{(l+1)p-2} + \sum_{j=n+1}^{\infty} a_j^p j^{p-2} =: I_3+I_2. \end{aligned}$$

To estimate I_2 and I_3, we are going to use the following inequalities

$$\sum_{s=n}^{\infty} a_s \lesssim \sum_{s=n}^{\infty}\frac{1}{s}\sum_{m=[s/2]}^{s} a_m \quad \text{and} \quad \sum_{s=1}^{n} a_s \lesssim \sum_{s=1}^{2n}\frac{1}{s}\sum_{m=[s/2]}^{s} a_m. \tag{6.7}$$

Then by Hardy's inequality [19], we have

$$I_3 \lesssim n^{-lp} \sum_{\nu=1}^{n} \left(\sum_{j=\nu}^{2n} \frac{1}{j^2} \sum_{m=[j/2]}^{l} a_m \right)^p \nu^{(l+1)p-2}$$
$$\lesssim n^{-lp} \sum_{j=1}^{2n} \left(\sum_{m=[j/2]}^{j} a_m \right)^p j^{lp-2}.$$

Then Lemma 6.1 (B) and (6.5) yield

$$I_3 \lesssim n^{-lp} \left\| \sum_{\nu=1}^{2n} \nu^l a_\nu \cos \nu x \right\|_p^p \asymp n^{-lp} \left\| S_{2n}^{(l)}(f) \right\|_p^p \lesssim \omega_l^p \left(f, \frac{1}{2n} \right)_p \lesssim \omega_l^p \left(f, \frac{1}{n} \right)_p .$$

Further, using (6.6), (6.7), and Hardy's inequality, we have

$$I_2 \lesssim \sum_{j=n+1}^{\infty} j^{p-2} \left(\sum_{s=[j/c]}^{\infty} \frac{a_s}{s} \right)^p \lesssim \sum_{j=n+1}^{\infty} j^{p-2} \left(\sum_{s=[j/c]}^{\infty} \frac{1}{s^2} \sum_{m=[s/2]}^{s} a_m \right)^p$$
$$\lesssim \sum_{s=[n/c]}^{\infty} s^{-2} \left(\sum_{m=[s/2]}^{s} a_m \right)^p \lesssim \sum_{s=2n}^{\infty} s^{-2} \left(\sum_{m=[s/2]}^{s} a_m \right)^p$$
$$+ n^{-lp} \sum_{s=1}^{2n} s^{lp-2} \left(\sum_{m=[s/2]}^{s} a_m \right)^p .$$

The last sum was estimated above. Again, by Lemma 6.1(B) and (6.5),

$$\sum_{s=2n}^{\infty} s^{-2} \left(\sum_{m=[s/2]}^{s} a_m \right)^p \lesssim \left\| \sum_{\nu=n}^{\infty} a_\nu \cos \nu x \right\|_p^p \lesssim \omega_l^p \left(f, \frac{1}{n} \right)_p .$$

So, we showed that

$$I_1 + I_2 \lesssim \omega_l^p \left(f, \frac{1}{n} \right)_p .$$

To prove the reverse, we use Lemma 6.1(A), the definition of the GM class, Hölder's and Hardy's inequalities:

$$\|f - S_n\|_p^p \lesssim \sum_{j=1}^{\infty} \beta_j^{p-2} j^{p-2} \lesssim n^{p-1} \left(\sum_{s=n}^{\infty} |\Delta a_s| \right)^p + \sum_{j=n}^{\infty} j^{p-2} \left(\sum_{s=l}^{\infty} |\Delta a_s| \right)^p$$
$$\lesssim n^{p-1} \left(\sum_{s=[n/c]}^{\infty} \frac{a_s}{s} \right)^p + \sum_{j=n}^{\infty} j^{p-2} \left(\sum_{s=[j/c]}^{\infty} \frac{a_s}{s} \right)^p$$
$$\lesssim \sum_{j=[n/c]}^{\infty} a_j^p j^{p-2} \lesssim I_1 + I_2,$$

where $\beta_j = \sum_{s=\max(j,n)}^{\infty} |\Delta a_s|$. Similarly,

$$n^{-lp} \left\| S_n^{(l)}(f) \right\|_p^p \lesssim n^{-lp} \left\| \sum_{\nu=1}^{n} \nu^l a_\nu \cos \nu x \right\|_p^p \lesssim n^{-lp} \sum_{\nu=1}^{n} \nu^{p-2} \left(\sum_{s=\nu}^{n} |\Delta(s^l a_s)| \right)^p .$$

Further,

$$\sum_{s=\nu}^{n}|\Delta(s^l a_s)|\lesssim\sum_{s=\nu}^{n}s^{l-1}a_s+\sum_{s=\nu}^{n}s^l|\Delta a_s|\lesssim\sum_{s=\nu}^{n}s^{l-1}a_s+\sum_{s=\nu}^{n}|\Delta a_s|\left(\sum_{m=\nu}^{s}m^{l-1}+\nu^l\right),$$

and after routine calculations, we arrive at

$$\sum_{s=\nu}^{n}|\Delta(s^l a_s)|\lesssim\sum_{s=[\nu/c]}^{n}s^{l-1}a_s+n^l\sum_{m=n}^{\infty}\frac{a_m}{m}.$$

Using this and Hardy's inequality, we get $n^{-lp}\left\|S_n^{(l)}(f)\right\|_p^p\lesssim I_1+I_2$. Finally, by (6.5),

$$\omega_l^p\left(f,\frac{1}{n}\right)_p\lesssim I_1+I_2.\quad\square$$

Remark. The partial case of Theorem 6.1 was stated in [26]. Note also that Theorem 6.1 is an analogue of Corollary 4.1 for the case of $d=1$. It would be interesting to obtain a similar result for the periodic functions of several variables.

7. Discussion and applications

7.1. Riemann–Lebesgue-type results

From Theorem A and [13, Theorem 2.2], one has the following estimate of the Fourier transform

$$t^{\theta l}\left(\int_{|\xi|<1/t}|\xi|^{\theta l p'}|\widehat{f}(\xi)|^{p'}\,d\xi\right)^{1/p'}+\left(\int_{1/t\le|\xi|}|\widehat{f}(\xi)|^{p'}\,d\xi\right)^{1/p'}\lesssim\Omega_l(f,t)_p,\quad 1<p\le 2.\tag{7.1}$$

On the other hand, Theorem 2.1 gives ($p\le q\le p'$, $1<p\le 2$)

$$t^{\theta l}\left(\int_{|\xi|<1/t}|\xi|^{\theta l q+dq(1-1/p-1/q)}|\widehat{f}(\xi)|^{q}\,d\xi\right)^{1/q}+\left(\int_{1/t\le|\xi|}|\xi|^{dq(1-1/p-1/q)}|\widehat{f}(\xi)|^{q}\,d\xi\right)^{1/q}\lesssim\Omega_l(f,t)_p.\tag{7.2}$$

If $q=p'$ (7.2) reduces to (7.1). The following example shows that (7.2), in general, provides better estimates than (7.1).

Example. Let $\widehat{f}(\xi)=F_0(|\xi|)$,

$$F_0(s)=\frac{s^{-d/p'}}{\ln^{2/p}(2+s)},\qquad\frac{2d}{d+1}<p<\infty.$$

Note that F_0 is decreasing to zero and therefore $F_0\in GM$. Also, it is easy to see that $|\xi|^{d(1-2/p)}\widehat{f}(\xi)\in L^p(\mathbb{R}^d)$. Hence, as in Example 3 (for $q=p$) we get $f\in\widehat{GM}^d\cap L^p(\mathbb{R}^d)$.

We have

$$t^{\theta l}\left(\int_{|\xi|<1/t}|\xi|^{\theta lq+dq(1-1/p-1/q)}|\widehat{f}(\xi)|^q\,d\xi\right)^{1/q}$$
$$+\left(\int_{|\xi|\ge 1/t}|\xi|^{dq(1-1/p-1/q)}|\widehat{f}(\xi)|^q\,d\xi\right)^{1/q}\asymp\big[\ln(2+1/t)\big]^{-2/p+1/q}.$$

Then (7.1) gives

$$\big[\ln(2+1/t)\big]^{1-3/p}\lesssim\Omega_l(f,t)_p,\quad p\le 2,$$

and (7.2) implies (with $q=p$)

$$\big[\ln(2+1/t)\big]^{-1/p}\lesssim\Omega_l(f,t)_p,\quad p\le 2.$$

The latter estimate is stronger. Moreover, it is sharp since by Corollary 4.1 we in fact have

$$\big[\ln(2+1/t)\big]^{-1/p}\asymp\Omega_l(f,t)_p,\quad \frac{2d}{d+1}<p<\infty.\quad\square$$

7.2. *Pointwise Riemann–Lebesgue-type results*

For $f\in L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$, $1<p\le 2$, the Riemann–Lebesgue-type inequality

$$|\widehat{f}(\xi)|\lesssim|\xi|^{\varkappa}\Omega_l(f,1/|\xi|)_p \tag{7.3}$$

does not hold in general for sufficiently small $\varkappa\ge 0$.

Note that the case of $\varkappa=0$ easily follows from the fact that the Hausdorff–Young inequality $\|\widehat{f}\|_q\lesssim\|f\|_p$ holds for all $f\in L^p(\mathbb{R}^d)$ if and only if $1\le p\le 2$ and $q=p'$ (see e.g. [33]).

Let us consider the case of $l>d/2$ for $d\ge 2$ and $l>1$ for $d=1$. For $d\ge 1$ we define $\varphi=\widehat{\chi}^2$, where χ is the characteristic function of the unit ball. Then $\|\varphi\|_q<\infty$, $q\ge 1$, $\widehat{\varphi}=\chi*\chi$ and the support of $\widehat{\varphi}$ is contained in the ball of radius 2. We define

$$f(x)=\sum_{n\in\mathbb{Z}^d}a_n\psi_n(x),\qquad \widehat{f}(x)=\sum_{n\in\mathbb{Z}^d}a_n\widehat{\psi_n}(x),$$

where

$$\psi_n(x)=\varepsilon_n^d\varphi(\varepsilon_n x)e^{-inx},\qquad \widehat{\psi_n}(\xi)=\widehat{\varphi}(\varepsilon_n^{-1}(\xi-n))$$

and positive sequences a_n and ε_n vanish at infinity and $\|a_n\|_{l^1(\mathbb{Z}^d)}<\infty$. The support of the function $\widehat{\psi_n}$ is the ball of radius $2\,\varepsilon_n$ centered at a point n.

We get

$$\|\psi_n\|_q=\left(\int_{\mathbb{R}^d}\left|\varepsilon_n^d\varphi(\varepsilon_n x)e^{-inx}\right|^q\,dx\right)^{1/q}=\varepsilon_n^{d/q'}\|\varphi\|_q,$$

which implies

$$\|f\|_q\le\sum_{n\in\mathbb{Z}^d}a_n\varepsilon_n^{d/q'}\|\varphi\|_q\lesssim\sum_{n\in\mathbb{Z}^d}a_n<\infty$$

for any $q\ge 1$. Therefore, $f\in L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$.

Using (2.7) for $d \ge 2$ and (2.18) for $d = 1$, we get

$$\Omega_l(f,t)_p \lesssim \|f - S_\lambda(f)\|_p + t^{\theta l}\|D^l S_\lambda(f)\|_p, \quad \lambda = 1/t,$$

where $\theta = 2$ and $D = \Delta$ for $d \ge 2$ and $\theta = 1$ and $D = d/dx$ for $d = 1$, $\widehat{S_\lambda(f)} = \widehat{f}\chi_\lambda$.

Let $\varepsilon_n < 1/2$. Then supports of functions $\widehat{\psi_n}$ are disjoint and therefore $\widehat{S_\lambda(f)} = \sum_{|n|<N} a_n \widehat{\psi_n}$, where $N = \lambda + O(1)$. Hence,

$$f - S_\lambda(f) = \sum_{|n|\ge N} a_n \psi_n, \qquad D^l S_\lambda(f) = \sum_{|n|<N} a_n D^l \psi_n.$$

We have

$$\|f - S_\lambda(f)\|_p \le \|\varphi\|_p \sum_{|n|\ge N} a_n \varepsilon_n^{d/p'}$$

and, by Bernstein's inequality,

$$\|D^l S_\lambda(f)\|_p \lesssim \sum_{|n|<N} a_n (2\varepsilon_n + |n|)^{\theta l}\|\psi_n\|_p \lesssim \sum_{|n|<N} (1+|n|)^{\theta l} a_n \varepsilon_n^{d/p'}.$$

Thus,

$$\Omega_l(f,t)_p \lesssim \sum_{|n|\ge N} a_n \varepsilon_n^{d/p'} + t^{\theta l} \sum_{|n|<N} (1+|n|)^{\theta l} a_n \varepsilon_n^{d/p'}.$$

Let now $l > d/\theta$ and

$$a_n \asymp (1+|n|)^{-\alpha}, \qquad \varepsilon_n^{d/p'} \asymp (1+|n|)^{-\beta},$$

where $d < \alpha < \theta l$ and $d < \beta < \theta l - \alpha + d$. Then

$$\sum_{\mathbb{Z}^d} a_n \lesssim \int_1^\infty u^{-\alpha+d-1}\,du < \infty,$$

$$\sum_{|n|\ge N} a_n \varepsilon_n^{d/p'} \lesssim \int_N^\infty u^{-\alpha-\beta+d-1}\,du \lesssim N^{-\alpha-\beta+d},$$

$$\sum_{|n|<N} (1+|n|)^{\theta l} a_n \varepsilon_n^{d/p'} \lesssim \int_1^N u^{\theta l-\alpha-\beta+d-1}\,du \lesssim 1.$$

Therefore, since $N \asymp \lambda = 1/t$, we have

$$\Omega_l(f,t)_p \lesssim N^{-\alpha-\beta+d} + t^{\theta l} = N^{-\alpha}(N^{-\beta+d} + N^{\alpha-\theta l}) \asymp N^{-\alpha-\beta+d}.$$

Moreover, if $\xi \in \mathbb{Z}^d$, $|\xi| \asymp N$, then

$$\widehat{f}(\xi) = a_\xi \widehat{\varphi}(0) \asymp N^{-\alpha}.$$

Finally, we get

$$\Omega_l(f, 1/|\xi|)_p \lesssim (|\xi|^{-\beta+d} + |\xi|^{\alpha-\theta l})\widehat{f}(\xi) \asymp |\xi|^{-\beta+d}\widehat{f}(\xi)$$

and therefore inequality (7.3) does not hold for $0 \le \varkappa < \beta - d$. □

However, let us remark that for functions from the $\widehat{GM}^d$ class it is possible to obtain the pointwise bound of the Fourier transform.

Corollary 7.1. *Let* $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$, $d \geq 1$, $\widehat{f}(\xi) = F_0(|\xi|) \geq 0$, *and* $\frac{2d}{d+1} < p < \infty$. *Then*

$$F_0(t) \lesssim t^{-d/p'} \Omega_l(f, 1/t)_p. \tag{7.4}$$

Proof. Since $f \in \widehat{GM}^d$, using (4.3) and Hölder's inequality, we get

$$F_0(s) \lesssim \int_{s/c}^{\infty} \frac{F_0(u)}{u}\, du = \int_{s/c}^{\infty} F_0(u) u^{d-d/p-1/p} u^{-d+(d+1-p)/p}\, du$$
$$\lesssim s^{d/p-d} \left(\int_{s/c}^{\infty} F_0^p(u) u^{dp-d-1}\, du \right)^{1/p}. \tag{7.5}$$

Then using Corollary 4.1, we have

$$\Omega_l^p(f,t)_p \asymp t^{\theta l p} \int_0^{1/t} s^{\theta l p + dp - d - 1} F_0^p(s)\, ds + \int_{1/t}^{\infty} s^{dp-d-1} F_0^p(s)\, ds \tag{7.6}$$

and by (7.5), we finally get

$$F_0(t) \lesssim t^{d/p-d} \left(\int_{t/c}^{\infty} F_0^p(u) u^{dp-d-1}\, du \right)^{1/p} \lesssim t^{-d/p'} \Omega_l(f, 1/t)_p. \qquad \square$$

7.3. Moduli of smoothness and best approximations: sharp relations

The following direct and inverse theorems of trigonometric approximation are well known (see e.g. [12, p. 210], [10, Intr.]):

$$\frac{1}{n^l} \left(\sum_{\nu=0}^{n} (\nu+1)^{\tau l - 1} E_\nu^\tau(f)_p \right)^{1/\tau} \lesssim \omega_l\left(f, \frac{1}{n}\right)_p$$
$$\lesssim \frac{1}{n^l} \left(\sum_{\nu=0}^{n} (\nu+1)^{q l - 1} E_\nu^q(f)_p \right)^{1/q}, \tag{7.7}$$

where $f \in L^p(\mathbb{T})$, $1 < p < \infty$, $l, n \in \mathbf{N}$, $q = \min(2, p)$, $\tau = \max(2, p)$, $E_n(f)_p$ denotes the n-th best trigonometric approximation of f in L^p, and $\omega_l(f, \delta)_p$ is the L^p-modulus of smoothness, see (1.2) with $X = \mathbb{T}$.

We remark that (7.7) is the sharp version of classical Jackson and weak-type inequalities [12, p. 205,208] and it can be written equivalently as follows [10]:

$$t^l \left(\int_t^1 u^{-\tau l - 1} \omega_{l+1}^\tau(f,u)_p\, du \right)^{1/\tau} \lesssim \omega_l(f,t)_p$$
$$\lesssim t^l \left(\int_t^1 u^{-q l - 1} \omega_{l+1}^q(f,u)_p\, du \right)^{1/q}. \tag{7.8}$$

Constructing individual functions shows [10] that the parameters $q = \min(2, p)$ and $\tau = \max(2, p)$ are optimal in (7.7) and (7.8). For functions on $[-1, 1]$ inequalities of type (7.7) and (7.8) were obtained in [30,10].

For functions on $L^p(\mathbb{R}^d)$, similar results were also proved for $\Omega_k(f,t)_p$ and $E_n(f)_p$, i.e., the best L^p-approximation by functions of exponential type n (see [10]). For example, an analogue of (7.7) is given by

$$\frac{1}{2^{\theta ln}}\left(\sum_{\nu=0}^{n} 2^{\theta l\tau\nu} E_{2^\nu}^{\tau}(f)_p\right)^{1/\tau} \lesssim \Omega_l\left(f, \frac{1}{2^n}\right)_p \lesssim \frac{1}{2^{\theta ln}}\left(\sum_{\nu=0}^{n} 2^{\theta lq\nu} E_{2^\nu}^{q}(f)_p\right)^{1/q},$$
$$\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^d)}.$$

Below we show that for functions from the class $\widehat{GM}^d$ we can completely solve the problem of description of relationships between $\Omega_l(f,t)_p$ and $E_n(f)_p$ as well as $\Omega_l(f,t)_p$ and $\Omega_{l+1}(f,t)_p$.

Theorem 7.1. *If $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$, $d \geq 1$, $\widehat{f} \geq 0$, and $\frac{2d}{d+1} < p < \infty$, then*

$$\Omega_l(f,t)_p \asymp \left(t^{\theta lp}\int_t^1 u^{-\theta lp}\Omega_{l+1}^p(f,u)_p\frac{du}{u}\right)^{1/p} + t^{\theta l}A(f)_p, \quad 0 < t < \frac{1}{2}, \tag{7.9}$$

where $A(f)_p := \left\| |\xi|^{\theta l + d(1-2/p)}\chi_1(|\xi|)\widehat{f}(\xi)\right\|_p \lesssim \Omega_l(f,1)_p$. In particular, we have

$$\left(t^{\theta lp}\int_t^1 u^{-\theta lp}\Omega_{l+1}^p(f,u)_p\frac{du}{u}\right)^{1/p}$$
$$\lesssim \Omega_l(f,t)_p \lesssim \left(t^{\theta lp}\int_t^1 u^{-\theta lp}\Omega_{l+1}^p(f,u)_p\frac{du}{u}\right)^{1/p} + t^{\theta l}\|f\|_p$$

and

$$\frac{1}{2^{\theta ln}}\left(\sum_{\nu=0}^{n} 2^{\theta lp\nu} E_{2^\nu}^{p}(f)_p\right)^{1/p} \lesssim \Omega_l\left(f, \frac{1}{2^n}\right)_p \lesssim \frac{1}{2^{\theta ln}}\left(\sum_{\nu=0}^{n} 2^{\theta lp\nu} E_{2^\nu}^{p}(f)_p\right)^{1/p} + \frac{1}{2^{\theta ln}}\|f\|_p. \tag{7.10}$$

Remark 1. In (7.9) one cannot drop $t^{\theta l}A(f)_p$. Indeed, consider

$$F_0^p(s) = s^{-(dp-d-1)}\chi_{1/n}(s).$$

Then

$$\Omega_l^p(f,t)_p \asymp t^{\theta lp}\int_0^{1/n} s^{\theta lp+dp-d-1}F_0^p(s)\,ds \asymp t^{\theta lp}\int_0^{1/n} s^{\theta lp}\,ds \asymp t^{\theta lp}n^{-\theta lp-1}.$$

Using this,

$$t^{\theta lp}\int_t^1 u^{-\theta lp}\Omega_{l+1}^p(f,u)_p\frac{du}{u} \asymp t^{\theta lp}\int_t^1 u^{\theta p}n^{-\theta(l+1)p-1}\frac{du}{u} \asymp t^{\theta lp}n^{-\theta(l+1)p-1}.$$

Hence, writing

$$t^{\theta l}n^{-\theta l-1/p} \lesssim \Omega_l(f,t)_p \lesssim t^{\theta l}\left(\int_t^1 u^{-\theta lp}\Omega_{l+1}^p(f,u)_p\frac{du}{u}\right)^{1/p} \lesssim n^{-\theta}t^{\theta l}n^{-\theta l-1/p}$$

we arrive at a contradiction as $n \to \infty$.

Proof of Theorem 7.1. Using Corollary 4.1, we get

$$\begin{aligned}\Omega_l^p(f,t)_p &\asymp t^{\theta lp}\int_0^{1/t} s^{\theta lp+dp-d-1}F_0^p(s)\,ds+\int_{1/t}^\infty s^{dp-d-1}F_0^p(s)\,ds\\ &=: J_1(t)+J_2(t)\end{aligned}$$

and

$$\begin{aligned}&t^{\theta lp}\int_t^1 u^{-\theta lp}\,\Omega_{l+1}^p(f,u)_p\frac{du}{u}\\ &\asymp t^{\theta lp}\int_1^{1/t} u^{-\theta p-1}\left[\int_0^u s^{\theta(l+1)p+dp-d-1}F_0^p(s)\,ds\right]du\\ &\quad+t^{\theta lp}\int_1^{1/t} u^{\theta lp-1}\left[\int_u^\infty s^{dp-d-1}F_0^p(s)\,ds\right]du\\ &=: I_1(t)+I_2(t).\end{aligned}$$

Then

$$\begin{aligned}I_1(t) &= t^{\theta lp}\int_1^{1/t} u^{-\theta p-1}\left[\left(\int_0^1+\int_1^u\right)s^{\theta(l+1)p+dp-d-1}F_0^p(s)\,ds\right]du\\ &\asymp t^{\theta lp}\int_0^1 s^{\theta(l+1)p+dp-d-1}F_0^p(s)\,ds+t^{\theta lp}\int_1^{1/t} s^{\theta(l+1)p+dp-d-1}F_0^p(s)\\ &\quad\times\int_s^{1/t} u^{-\theta p-1}\,du\,ds\\ &\lesssim J_1(t)\end{aligned}$$

and

$$\begin{aligned}I_2(t) &= t^{\theta lp}\int_1^{1/t} u^{\theta lp-1}\left[\left(\int_u^{1/t}+\int_{1/t}^\infty\right)s^{dp-d-1}F_0^p(s)\,ds\right]du\\ &\asymp t^{\theta lp}\int_1^{1/t} s^{dp-d-1}F_0^p(s)\int_1^s u^{\theta lp-1}\,du\,ds+\int_{1/t}^\infty s^{dp-d-1}F_0^p(s)\,ds\\ &\lesssim J_1(t)+J_2(t).\end{aligned}$$

Using again Corollary 4.1,

$$\begin{aligned}A(f)_p &\asymp \left(\int_0^1 s^{\theta lp+dp-d-1}F_0^p(s)\,ds\right)^{1/p}\\ &\lesssim \left(\int_0^\infty s^{dp-d-1}\min(1,s)^{\theta lp}F_0^p(s)\,ds\right)^{1/p}\\ &\asymp \left\|\min(1,|\xi|)^{\theta l}|\xi|^{d(1-2/p)}\widehat{f}(\xi)\right\|_p \asymp \Omega_l(f,1)_p.\end{aligned}$$

Moreover, $A^p(f)_p\lesssim J_1(t)$. Thus,

$$I_1(t)+I_2(t)+t^{\theta lp}A^p(f)_p\lesssim J_1(t)+J_2(t).$$

To prove the inverse inequality, we first remark that $s^{-\theta p} \lesssim \int_s^{1/t} u^{-\theta p-1} du,\ 1 < s < 1/(2t)$ and therefore using (4.10),

$$
\begin{aligned}
J_1(2t) &\lesssim t^{\theta l p} \int_0^1 s^{\theta l p+dp-d-1} F_0^p(s)\, ds + t^{\theta l p} \int_1^{1/2t} s^{\theta(l+1)p+dp-d-1} F_0^p(s) \\
&\quad \times \left(\int_s^{1/t} u^{-\theta p-1}\, du \right) ds \\
&\lesssim t^{\theta l p} A^p(f)_p + I_1(t).
\end{aligned}
$$

Also,

$$
\begin{aligned}
J_2(2t) &\lesssim \int_{1/(2t)}^{\infty} s^{dp-d-1} F_0^p(s)\, ds \\
&\lesssim \int_{1/(2t)}^{1/t} s^{dp-d-1} F_0^p(s)\, ds + t^{\theta l p} \int_{1/(2t)}^{1/t} u^{\theta l p-1} \int_u^{\infty} s^{dp-d-1} F_0^p(s)\, ds\, du \\
&\lesssim I_1(t) + I_2(t).
\end{aligned}
$$

Finally, to verify (7.10), we apply [10, (5.7) and (5.8)]. □

Using (6.1), we state the analogous result for periodic functions; compare with (7.7) and (7.8).

Theorem 7.2. *Let $f \in L^p(\mathbb{T})$, $1 < p < \infty$, and*

$$
f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
$$

where non-negative $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are general monotone sequences. Then

$$
\omega_l(f,t)_p \asymp \left(t^{lp} \int_t^1 u^{-lp} \omega_{l+1}^p(f,u)_p \frac{du}{u} \right)^{1/p}, \quad 0 < t < \frac{1}{2}.
$$

In particular,

$$
\omega_l(f,1/n)_p \asymp \left(n^{-lp} \sum_{\nu=0}^{n} (\nu+1)^{lp-1} E_\nu^p(f)_p \right)^{1/p},
$$

where $E_\nu(f)_p$ is the best L^p-approximation of f by trigonometric polynomials of degree ν.

Note that similar equivalence results for continuous functions were obtained in [27, Theorems 5.1, 5.2].

7.4. A characterization of the Besov spaces

For $1 \le p \le \infty$ and $\tau, r > 0$, define the Besov space $B_{p,\tau}^r(\mathbb{R}^d)$ as the collection of functions $f \in L^p(\mathbb{R}^d)$ such that

$$
\|f\|_{B_{p,\tau}^r(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \left(\int_0^1 \left(\frac{\Omega_l(f,t)_p}{t^r} \right)^{\tau} \frac{dt}{t} \right)^{1/\tau} < \infty,
$$

where $0 < r < \theta l$. Similarly we define the Lipschitz space $\mathrm{Lip}_p^r(\mathbb{R}^d) \equiv B_{p,\infty}^r(\mathbb{R}^d)$, i.e.,

$$\|f\|_{\mathrm{Lip}_p^r(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \sup_t \frac{\Omega_l(f,t)_p}{t^r}, \quad 0 < r < \theta l.$$

It turns out that it is possible to characterize functions from the Besov space $B^r_{p,\tau}(\mathbb{R}^d)$ in terms of growth properties of their Fourier transforms.

Theorem 7.3. *Let $d \ge 1$, $1 < \tau \le \infty$, and $\frac{2d}{d+1} < p \le \tau$. If $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$ and $\widehat{f} \ge 0$, then a necessary and sufficient condition for $f \in B^r_{p,\tau}(\mathbb{R}^d)$ is*

$$\int_0^\infty s^{r\tau+d\tau-d\tau/p-1} F_0^\tau(s)\, ds < \infty \quad \text{if } 1 < \tau < \infty \tag{7.11}$$

and

$$\sup_s s^{r+d-d/p} F_0(s) < \infty \quad \text{if } \tau = \infty. \tag{7.12}$$

Proof. *The case of* $1 < \tau < \infty$. Let first (7.11) hold. By (7.6), we get

$$|f|_{B^r_{p,\tau}} \asymp K_1 + K_2 + K_3 := \int_0^1 t^{(\theta l - r)\tau - 1} \left(\int_0^1 s^{\theta l p + dp - d - 1} F_0^p(s)\, ds \right)^{\tau/p} dt$$
$$+ \int_1^\infty t^{(r-\theta l)\tau - 1} \left(\int_1^t s^{\theta l p + dp - d - 1} F_0^p(s)\, ds \right)^{\tau/p} dt$$
$$+ \int_0^1 t^{r\tau - 1} \left(\int_{1/t}^\infty s^{dp-d-1} F_0^p(s)\, ds \right)^{\tau/p} dt.$$

Then by Hölder's inequality with parameters $\alpha = \tau/p$ and α', we get

$$K_1 \lesssim \int_0^1 s^{r\tau+d\tau-d\tau/p-1} F_0^\tau(s)\, ds.$$

By Hardy's inequalities (see e.g. [5, p. 124]), we have

$$K_2 + K_3 \lesssim \int_1^\infty s^{r\tau+d\tau-d\tau/p-1} F_0^\tau(s)\, ds.$$

Hence, if (7.11) holds, $f \in B^r_{p,\tau}(\mathbb{R}^d)$.

Let $f \in B^r_{p,\tau}(\mathbb{R}^d)$. By (7.5),

$$F_0(s)^\tau \lesssim s^{d\tau/p - d\tau} \left(\int_{s/c}^\infty F_0^p(u) u^{dp-d-1}\, du \right)^{\tau/p}.$$

Therefore, making use of this, we have

$$\int_0^\infty s^{r\tau+d\tau-d\tau/p-1} F_0^\tau(s)\, ds \lesssim \int_1^\infty s^{r\tau-1} \left(\int_s^\infty F_0^p(u) u^{dp-d-1}\, du \right)^{\tau/p} ds$$
$$+ \int_0^1 s^{r\tau-1} \left(\int_s^\infty F_0^p(u) u^{dp-d-1}\, du \right)^{\tau/p} ds$$
$$\lesssim |f|_{B^r_{p,\tau}} + \left\| |\xi|^{d(1-2/p)} \widehat{f}(\xi) \right\|_p^\tau \int_0^1 s^{r\tau-1} ds.$$

Finally, since $\left\| |\xi|^{d(1-2/p)} \widehat{f}(\xi) \right\|_p \lesssim \|f\|_p$ (see (4.10)), (7.11) holds.

The case of $\tau = \infty$. Let first (7.12) hold. Then by (7.6), (7.12) yields

$$\Omega_l^p(f,t)_p \lesssim t^{\theta l p}\int_0^{1/t} s^{\theta l p - rp - 1}\,ds + \int_{1/t}^{\infty} s^{-rp-1}\,ds \lesssim t^{rp},$$

i.e., $f \in \mathrm{Lip}_p^r(\mathbb{R}^d)$.

On the other hand, if $f \in \mathrm{Lip}_p^r(\mathbb{R}^d)$, we use (7.5) and (7.6)

$$F_0^p(s) \lesssim s^{d-dp}\int_{s/c}^{\infty} F_0^p(u)u^{dp-d-1}\,du \lesssim s^{d-dp}\Omega_l^p(f,1/s)_p \lesssim s^{d-dp-rp},$$

which is (7.12). $\square$

7.5. Embedding theorems

The following Sobolev-type embedding result for the Besov space with the limiting smoothness parameter is well known: $B_{p,q}^r \hookrightarrow L^q$, $r = d\left(\frac{1}{p} - \frac{1}{q}\right)$ (see, e.g., [23, (8.2)]). Note that this embedding is closely related to the sharp Ul'yanov inequalities for moduli of smoothness in different metrics [24], [32, Theorem 2.4]. Theorem 7.3 gives the sharpness of the embedding result in the following sense.

Corollary 7.2. *Let* $d \ge 1$ *and* $\frac{2d}{d+1} < p < q < \infty$. *If* $f \in \widehat{GM}^d \cap L^p(\mathbb{R}^d)$ *and* $\widehat{f} \ge 0$, *then*

$$f \in B_{p,q}^r(\mathbb{R}^d), \quad r = d\left(\frac{1}{p} - \frac{1}{q}\right) \Longleftrightarrow f \in L^q(\mathbb{R}^d). \tag{7.13}$$

Proof. To show (7.13), we combine Theorem 7.3 and $\left\| |\xi|^{d(1-2/p)}\widehat{f}(\xi)\right\|_p \asymp \|f\|_p$, $\frac{2d}{d+1} < p < \infty$ (see (4.10)). $\square$

Feng Dai made us aware of the fact that inequalities from Theorems 2.1 and 5.1 for the case $q = p$ were also proved in the recent work by Zeev Ditzian.

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