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A REMARK ON TWO GENERALIZED ORLICZ-MORREY SPACES

SADEK GALA, YOSHIHIRO SAWANO, AND HITOSHI TANAKA

Abstract. There have been known two generalized Orlicz-Morrey spaces. One is defineed earlier by Nakai and the other is by Sugano, the second and third authors. In this paper we investigate differences between these two spaces in some typical cases. The arguments rely upon property of the characteristic function of the Cantor set.

1. Introduction

There does exist two different scales whose names are both generalized Orlicz-Morrey spaces, which we shall establish in this paper. We first introduce two function spaces which are are originally considered to extend and supplement Lebesgue spaces. Orlicz spaces can describe the endpoint cases of the boundedness of classical operators such as the Hardy-Littlewood maximal operator, fractional integral operator and singular integral operator. Morrey spaces are mainly used to describe the precise property of the Riesz potential or the fractional integral operator.

A function $\Phi : [0, \infty) \to [0, \infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Let Φ be a Young function. We define the Orlicz space $L^\Phi(\mathbb{R}^n)$ to be the set of measurable functions such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty.$$

The space $L^\Phi(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$\|f\|_{L^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

If $\Phi(t) \equiv t^p$, $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Historically, there have been two different expressions of Morrey spaces. We use the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ to denote the volume of Q.

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- Let $\lambda \in [0,1)$ and $1 \le q < \infty$. We define the Morrey space $\mathcal{L}^{\lambda,q}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{L}^{\lambda,q}} := \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|^{\lambda}} \int_Q |f(x)|^q \, dx \right)^{1/q}.$$

- Let $1 \le q \le p < \infty$. We define the Morrey space $\mathcal{M}_{p,q}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{M}_{p,q}} := \sup_{Q \in \mathcal{Q}} |Q|^{1/p} \left(\frac{1}{|Q|} \int_Q |f(x)|^q \, dx \right)^{1/q}.$$

One has then

$$\mathcal{L}^{\lambda,q}(\mathbb{R}^n) = \mathcal{M}_{p,q}(\mathbb{R}^n), \text{ whenever } \lambda + \frac{q}{p} = 1 \tag{1.1}$$

with norm coincidence.

Recently, we are faced with the situation of mixing Orlicz spaces and Morrey spaces and considered generalized Orlicz-Morrey spaces. One definition of generalized Orlicz-Morrey spaces dates back to Nakai's paper [12]. The other one is from the paper [19] by Sugano and the authors. Now we describe those definitions.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that $0 < \Phi(t) < \infty$ for $0 < t < \infty$. If $\Phi \in \mathcal{Y}$, then Φ becomes absolutely continuous on any closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

Let $\mathcal{G}_1$ be the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t)$ is nondecreasing but that $\frac{\varphi(t)}{t}$ is nonincreasing. For $\Phi \in \mathcal{Y}$, we denote by $\Phi^{-1} : (0,\infty) \to (0,\infty)$ the inverse of $\Phi : (0,\infty) \to (0,\infty)$.

For $\Phi \in \mathcal{Y}$ let $\mathcal{G}_2$ be the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t)$ is nondecreasing but that, for any $s > 0$, $\dfrac{\varphi((s+t)^n)}{\Phi^{-1}(((s+t)/s)^n)}$ is nonincreasing.

Definition 1.1. Let $\Phi \in \mathcal{Y}$.

(1) Let $\varphi \in \mathcal{G}_1$. For a cube $Q \in \mathcal{Q}$ define the (φ, Φ)-average over Q of the measurable function f by

$$\|f\|_{(\varphi,\Phi);Q} := \inf \left\{ \lambda > 0 \,:\, \frac{\varphi(|Q|)}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

We define the generalized Orlicz-Morrey space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{L}^{\varphi,\Phi}} := \sup_{Q \in \mathcal{Q}} \|f\|_{(\varphi,\Phi);Q}.$$

(2) For a cube $Q \in \mathcal{Q}$ define the Φ-average over Q of the measurable function f by

$$\|f\|_{\Phi;Q} := \inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Let $\varphi \in \mathcal{G}_2$. We define the generalized Orlicz-Morrey space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{M}_{\varphi,\Phi}} := \sup_{Q \in \mathcal{Q}} \varphi(|Q|) \|f\|_{\Phi;Q}.$$

The space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ generalizes $\mathcal{L}^{\lambda,q}(\mathbb{R}^n)$ while the space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$ generalizes $\mathcal{M}_{p,q}(\mathbb{R}^n)$. Unlike (1.1), these two generalized Orlicz-Morrey spaces are different in some typical cases. We establish the difference between them, which is the focus of this paper.

Remark 1.2. By the definition we have, for any $t > 0$,

$$\|\chi_{[0,t]^n}\|_{\mathcal{L}^{\varphi,\Phi}} = \frac{1}{\Phi^{-1}(\varphi(t^n)^{-1})}$$

and

$$\|\chi_{[0,t]^n}\|_{\mathcal{M}_{\varphi,\Phi}} = \frac{\varphi(t^n)}{\Phi^{-1}(1)},$$

where χ_E stands for the characteristic function of a set $E \subset \mathbb{R}^n$. We see also that φ satisfies the doubling condition: $\varphi(2t) \le C_0\varphi(t)$ for $t > 0$.

We now verify a simple relation between these two scales.

Claim 1.3. *Suppose that $\Phi \in \mathcal{Y}$ and $\varphi \in \mathcal{G}_2$ satisfy $\Phi \circ \varphi \in \mathcal{G}_1$. If there exists a constant $C > 1$ such that, for any $a, t > 0$,*

$$\Phi(a)\Phi(C^{-1}t) \le \Phi(at) \le \Phi(a)\Phi(Ct), \tag{1.2}$$

then we have a norm equivalence

$$\|f\|_{\mathcal{L}^{\Phi\circ\varphi,\Phi}} \approx \|f\|_{\mathcal{M}_{\varphi,\Phi}}.$$

Proof. It is known that, for any Young function Φ,

$$\|f\|_{L^\Phi} \approx \inf\left\{\lambda + \lambda\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \,:\, \lambda > 0\right\}. \tag{1.3}$$

It follows from (1.2) and (1.3) that, for any $Q \in \mathcal{Q}$,

$$\begin{aligned}
\varphi(|Q|)\|f\|_{\Phi;Q} &\approx \varphi(|Q|)\inf\left\{\lambda + \frac{\lambda}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \,:\, \lambda > 0\right\}\\
&\approx \inf\left\{\lambda + \frac{\lambda}{|Q|}\int_Q \Phi\left(\varphi(|Q|)\frac{|f(x)|}{\lambda}\right) dx \,:\, \lambda > 0\right\}\\
&\approx \inf\left\{\lambda + \lambda\frac{\Phi(\varphi(|Q|))}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \,:\, \lambda > 0\right\}\\
&\approx \|f\|_{(\Phi\circ\varphi,\Phi);Q},
\end{aligned}$$

which yields the proof. □

Let $\varphi(t) \equiv t$ and $\Phi(t) \equiv t^2 + t^3$. Then the space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ is isomorphic to the Orlicz space $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$. It follows that, for any $a, t > 0$,

$$\Phi(at) = (at)^2 + (at)^3 \le (a^2 + a^3)(t^2 + t^3) = \Phi(a)\Phi(t).$$

This enables us by an argument similar to Claim 1.3 that

$$L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n) \subset \mathcal{M}_{\psi,\Phi}(\mathbb{R}^n), \tag{1.4}$$

when ψ satisfies $\psi(t)^2 + \psi(t)^3 = t$ for $t > 0$. The following theorem disproves that equality (1.4) holds.

Theorem 1.4. *There is no pair of functions $\varphi \in \mathcal{G}_2$ and $\Phi \in \mathcal{Y}$ such that $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$ is isomorphic to $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$.*

For a locally integrable function f on $\mathbb{R}^n$ the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

For $j = 1, 2, \ldots$, let $\Phi(t) \equiv t(\log \max(e,t))^j$ and M^j be the j-fold composition of M. It is known (see [21]) that one has the norm equivalence on a finite ball $B \subset \mathbb{R}^n$

$$\|f\|_{L^\Phi(B)} \approx \|M^j f\|_{L^1(B)}.$$

However, this equivalence does not hold on $\mathbb{R}^n$. By the use of the generalized Orlicz-Morrey space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$, we can prove the following a generalization:

Proposition 1.5 ([19, Lemma 3.5])**.** *Let $\varphi \in \mathcal{G}_1$, $\Phi(t) \equiv t(\log \max(e,t))^j$, $1(t) \equiv t$ and M^j be the j-fold composition of M. Then we have a norm equivalence*

$$\|f\|_{\mathcal{M}_{\varphi,\Phi}} \approx \|M^j f\|_{\mathcal{M}_{\varphi,1}}.$$

Let $p(t) \equiv t^{1/p}$, $p > 1$, and $\mathrm{LlogL}\,(t) \equiv t \log \max(e,t)$. A simple calculation shows that, for any $a, t > 0$,

$$\begin{aligned} \mathrm{LlogL}\,(at) &= (at) \log \max(e, at) \le a \log \max(e,a) \cdot (et) \log \max(e, et) \\ &= \mathrm{LlogL}\,(a)\, \mathrm{LlogL}\,(et). \end{aligned}$$

This enables us by an argument similar to Claim 1.3 that

$$\mathcal{L}^{\psi,\,\mathrm{LlogL}}(\mathbb{R}^n) \subset \mathcal{M}_{p,\,\mathrm{LlogL}}(\mathbb{R}^n), \tag{1.5}$$

when

$$\psi(t) \equiv t^{1/p} \log \max(e, t^{1/p}).$$

The following theorem disproves that the equality (1.5) holds.

Theorem 1.6. *There is no pair of functions $\varphi \in \mathcal{G}_1$ and $\Phi \in \mathcal{Y}$ such that $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ is isomorphic to $\mathcal{M}_{p,\,\mathrm{LlogL}}(\mathbb{R}^n)$. Here, $p(t) \equiv t^{1/p}$, $p > 1$, and $\mathrm{LlogL}\,(t) \equiv t \log \max(e,t)$.*

According to our best knowledge, it seems that the generalized Orlicz-Morrey space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ is more investigated than the generalized Orlicz-Morrey space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$. The space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ is investigated in [4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17], and, the space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$ is investigated in [1, 2, 3, 19].

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_0, C_1, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive constant c independent of appropriate quantities.

2. Proof of Theorem 1.4

Assume to the contrary that there exists such a pair. That is, for all $f \in L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ we have

$$\|f\|_{\mathcal{M}_{\varphi,\Phi}} \approx \|f\|_{L^2 \cap L^3}. \tag{2.1}$$

Then, by Remark 1.2, we must have for all $t > 0$

$$\varphi(t^n) \approx \|\chi_{[0,t]^n}\|_{\mathcal{M}_{\varphi,\Phi}} \approx \|\chi_{[0,t]^n}\|_{L^2 \cap L^3} \approx \max(t^{n/2}, t^{n/3}). \tag{2.2}$$

Let us now use the characteristic function of the Cantor set: Fix $0 < \kappa \le 1/10$. Define, inductively, a sequence of the sets $\{E_j\}_{j=0}^{\infty}$ by

$$E_0 := [0,1]^n, \quad E_j := \bigcup_{e \in \{0,1\}^n} ((1-\kappa)e + \kappa E_{j-1}).$$

Observe that (see, for example, [18]) E_j is made up of 2^{jn} cubes of sidelength κ^j. Let us set

$$F_j = \kappa^{-j} E_j, \quad j \in \mathbb{N}.$$

Observe now that F_j is made up of 2^{jn} cubes of sidelength 1.

We first verify that

$$\|\chi_{E_j}\|_{\Phi;[0,1]^n} = \|\chi_{F_j}\|_{\Phi;[0,\kappa^{-j}]^n}. \tag{2.3}$$

Indeed,

$$\begin{aligned}\|\chi_{E_j}\|_{\Phi;[0,1]^n} &= \inf\left\{\lambda > 0 \,:\, \int_{E_j} \Phi\left(\frac{1}{\lambda}\right) dx \le 1\right\} \\ &= \inf\left\{\lambda > 0 \,:\, \kappa^{jn}\int_{F_j} \Phi\left(\frac{1}{\lambda}\right) dx \le 1\right\} \\ &= \|\chi_{F_j}\|_{\Phi;[0,\kappa^{-j}]^n}.\end{aligned}$$

We now claim that, for any $j \in \mathbb{N}$,

$$\varphi(\kappa^{-nj})\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-j}]^n} \approx \|\chi_{F_j}\|_{\mathcal{M}_{\varphi,\Phi}} \tag{2.4}$$

and

$$\|\chi_{E_j}\|_{\Phi;[0,1]^n} \approx \|\chi_{E_j}\|_{\mathcal{M}_{\varphi,\Phi}}. \tag{2.5}$$

Indeed, by symmetry of the set E_j and the doubling property of φ, we see that

$$\max_{k=1,2,\ldots,j} \varphi(\kappa^{-nk})\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-k}]^n} \approx \|\chi_{F_j}\|_{\mathcal{M}_{\varphi,\Phi}},$$

and, hence, we can choose $k_j \le j$ so that

$$\|\chi_{F_j}\|_{\mathcal{M}_{\varphi,\Phi}} \approx \varphi(\kappa^{-nk_j})\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-k_j}]^n}.$$

A simple geometric observation shows that

$$\varphi(\kappa^{-nk_j})\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-k_j}]^n} \le \varphi(\kappa^{-nk_j})\|\chi_{F_{k_j}}\|_{\Phi;[0,\kappa^{-k_j}]^n} \le \|\chi_{F_{k_j}}\|_{\mathcal{M}_{\varphi,\Phi}}.$$

This yields together with (2.1)

$$\|\chi_{F_j}\|_{L^2\cap L^3} \le C\|\chi_{F_{k_j}}\|_{L^2\cap L^3}.$$

Thus, we must have $k_j \approx j$ and we have verified (2.4). Likewise (2.5) is achieved.

It follows from (2.1) and (2.4) that

$$\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-j}]^n} = \frac{\varphi(\kappa^{-nj})\|\chi_{F_j}\|_{\Phi;[0,\kappa^{-j}]^n}}{\varphi(\kappa^{-nj})} \lesssim \frac{\|\chi_{F_j}\|_{L^2\cap L^3}}{\varphi(\kappa^{-nj})} \approx (2\kappa)^{nj/2}, \tag{2.6}$$

where we have used $\|\chi_{F_j}\|_{L^2\cap L^3} = 2^{nj/2}$ and, by (2.2), $\varphi(\kappa^{-nj}) \approx \kappa^{-nj/2}$.

Likewise, it follows from (2.5) that

$$\|\chi_{E_j}\|_{\Phi;[0,1]^n} \approx \|\chi_{E_j}\|_{\mathcal{M}_{\varphi,\Phi}} \approx \|\chi_{E_j}\|_{L^2\cap L^3} \approx (2\kappa)^{jn/3}. \tag{2.7}$$

The equations (2.3), (2.6) and (2.7) contradict because $0 < \kappa < 1/10$.

3. Proof of Theorem 1.6

We need the following lemma.

Lemma 3.1 ([14, Theorem 5.1]). *Let $\Phi, \Psi \in \mathcal{Y}$ and $\varphi, \psi \in \mathcal{G}_1$. Then the following are equivalent:*

(i) *There exists a constant $A \geq 1$ such that*
$$\Phi^{-1}(\varphi(t)^{-1}) \leq A\Psi^{-1}(\psi(t)^{-1}), \quad t > 0,$$
and
$$\int_{\Psi^{-1}(\psi(t)^{-1})}^{s/A} \frac{\Psi(t)}{t^2}\,dt \leq A\frac{\Phi(s)}{s}\frac{\varphi(t)}{\psi(t)}, \quad (t,s) \in E,$$
where
$$E := \left\{(t,s) \in (0,\infty)^2 \,:\, 2A\Psi^{-1}(\psi(t)^{-1}) < s < \sup_{r>0}\Phi^{-1}(\varphi(r)^{-1})\right\}.$$

(ii) *The Hardy-Littlewood maximal operator M is bounded from $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ to $\mathcal{L}^{\psi,\Psi}(\mathbb{R}^n)$.*

Recall that $\varphi \in \mathcal{G}_1$, $\Phi \in \mathcal{Y}$, $p(t) \equiv t^{1/p}$, $p > 1$, and $\mathrm{LlogL}\,(t) \equiv t\log\max(e,t)$.

Assume to the contrary that there exists such a pair. That is, for all $f \in \mathcal{M}_{p,\,\mathrm{LlogL}}\,(\mathbb{R}^n)$,
$$\|f\|_{\mathcal{M}_{p,\,\mathrm{LlogL}}} \approx \|f\|_{\mathcal{L}^{\varphi,\Phi}}. \tag{3.1}$$
Then, by Remark 1.2, we must have for all $t > 0$
$$t^{-1/p} \approx \Phi^{-1}(\varphi(t)^{-1}),$$
and, hence,
$$\varphi(t)\Phi(t^{-1/p}) \approx 1. \tag{3.2}$$
By (3.1) and Proposition 1.5 we also have for all $f \in \mathcal{M}_{p,\,\mathrm{LlogL}}\,(\mathbb{R}^n)$
$$\|f\|_{\mathcal{L}^{\varphi,\Phi}} \approx \|f\|_{\mathcal{M}_{p,\,\mathrm{LlogL}}} \approx \|Mf\|_{\mathcal{M}_{p,1}}. \tag{3.3}$$

It follows by (3.2), (3.3) and Lemma 3.1 that
$$\Phi(t^{-1/p})t^{1/p}s\log(t^{1/p}s) \leq C\Phi(s)$$
as long as $t, s > 0$ satisfy $t^{1/p}s > 1$. Let us set $a = t^{1/p}s$. Then
$$\Phi(a^{-1}s)a\log a \leq C\Phi(s)$$
for all $s > 0$ and $a > 1$. Set $a^{-1}s = r$. Then
$$\Phi(r)a\log a \leq C\Phi(ar) \tag{3.4}$$
for any $a > 1$ and $r > 0$.

Letting $r = 1$ in (3.4), we have first $\mathrm{LlogL}\,(a) \leq C\Phi(a)$ for $a > 1$. Using this inequality with $a = \sqrt{t}, t > 1$, and using (3.4) again with $r = a = \sqrt{t}$, $t > 1$, we conclude that
$$\log t \cdot \mathrm{LlogL}\,(t) \leq C\Phi(t), \quad t > 1.$$
These observations allow us to assume that, for any $t > 1$,
$$\mathrm{LlogL}\,(t) \leq \Phi(t) \text{ and } \log t \cdot \mathrm{LlogL}\,(t) \leq \Phi(t). \tag{3.5}$$

Let us again use the characteristic function of the Cantor set: Let $0 < \kappa < 1$ be the solution to the equation

$$\kappa^{1/p} = 2\kappa. \tag{3.6}$$

Define, inductively, a sequence of the sets $\{E_j\}_{j=0}^{\infty}$ by

$$E_0 := [0,1]^n, \quad E_j := \bigcup_{e \in \{0,1\}^n} ((1-\kappa)e + \kappa E_{j-1}).$$

Observe that E_j is made up of 2^{jn} cubes of sidelength κ^j. Observe also that from (3.6) for all $0 \le k \le j$

$$\frac{|[0,\kappa^k]^n \cap E_j|}{|[0,\kappa^k]^n|} = \kappa^{n(j-k)/p}. \tag{3.7}$$

We now claim that the quantity

$$\kappa^{nk/p} \|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,\kappa^k]^n}$$

is a decreasing sequence of $k = 0, 1, \ldots, j$. Indeed, in the same manner as in the proof of Claim 1.3, using (1.3) and (3.7) we have

$$\begin{aligned} \kappa^{nk/p} \|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,\kappa^k]^n} &\approx \inf\left\{\lambda + \lambda \kappa^{n(j-k)/p}\,\mathrm{LlogL}\left(\frac{\kappa^{nk/p}}{\lambda}\right) \,:\, \lambda > 0\right\} \\ &= \inf\left\{\lambda + \kappa^{nj/p}\log\max\left(e, \frac{\kappa^{nk/p}}{\lambda}\right) \,:\, \lambda > 0\right\}. \end{aligned}$$

This quantity is the solution to the equation

$$\lambda e^{\kappa^{-nj/p}\lambda} = \kappa^{nk/p}$$

and becomes a decreasing sequence of k.

By this claim we see that

$$\|\chi_{E_j}\|_{\mathcal{M}_{p,\,\mathrm{LlogL}}} = \|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,1]^n},$$

which gives us that

$$\frac{\|\chi_{e_j}\|_{\mathcal{L}^{\varphi,\Phi}}}{\|\chi_{E_j}\|_{\mathcal{M}_{p,\,\mathrm{LlogL}}}} \ge \frac{\|\chi_{e_j}\|_{(\varphi,\Phi);[0,1]^n}}{\|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,1]^n}}. \tag{3.8}$$

If we can prove

$$\lim_{j\to\infty} \frac{\|\chi_{e_j}\|_{(\varphi,\Phi);[0,1]^n}}{\|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,1]^n}} = \infty, \tag{3.9}$$

then (3.8) contradicts (3.1) and proof of the theorem will be finished. Therefore, we need only verify (3.9).

Without loss of generality we may assume that $\varphi(1) = 1$. It follows then that

$$\|\chi_{e_j}\|_{(\varphi,\Phi);[0,1]^n} = \frac{1}{\Phi^{-1}(\kappa^{-nj/p})} \tag{3.10}$$

and

$$\|\chi_{E_j}\|_{\,\mathrm{LlogL}\,;[0,1]^n} = \frac{1}{(\,\mathrm{LlogL}\,)^{-1}(\kappa^{-nj/p})}. \tag{3.11}$$

For $t > 1$ let $\theta(t) := \dfrac{(\,\mathrm{LlogL}\,)^{-1}(t)}{\Phi^{-1}(t)}$. By (3.5) we see that $\theta(t) \geq 1$ and

$$\begin{aligned} t = &\ \mathrm{LlogL}\,((\,\mathrm{LlogL}\,)^{-1}(t)) = \mathrm{LlogL}\,(\theta(t)\Phi^{-1}(t)) \leq \theta(t)^2\,\mathrm{LlogL}\,(\Phi^{-1}(t)) \\ &\leq \frac{\theta(t)^2}{\log(\Phi^{-1}(t))}\Phi(\Phi^{-1}(t)) = \frac{\theta(t)^2}{\log(\Phi^{-1}(t))}t. \end{aligned}$$

This implies $\log(\Phi^{-1}(t)) \leq \theta(t)^2$ and

$$\lim_{t\to\infty}\theta(t) = \infty. \tag{3.12}$$

The equations (3.10)–(3.12) read (3.8). This completes the proof of the theorem.

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