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Journal of Approximation Theory 252 (2020) 105362

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

On the optimal relationships between L^P-norms for the Hardy operator and its dual for decreasing functions[☆]

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Received 16 October 2019; accepted 29 December 2019
Available online 7 January 2020

Communicated by D. Leviatan

Abstract

We prove sharp inequalities between L^p-norms $(1 < p < \infty)$ of functions Hf and H^*f, where H is the Hardy operator, H^* is its dual, and f is a nonnegative nonincreasing function on $(0, \infty)$. In particular, we extend one result obtained for integer $p \geq 2$ by Boza and Soria (2019), to the whole range of values $p \geq 2$.

MSC: primary 26D10; 26D15; secondary 46E30

Keywords: Hardy operator; Dual operator; Best constants

1. Introduction and main results

Denote by $\mathcal{M}^+(\mathbb{R}_+)$ the class of all nonnegative measurable functions on $\mathbb{R}_+ \equiv (0, +\infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}_+)$. Set

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

and

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

☆ This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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https://doi.org/10.1016/j.jat.2019.105362
0021-9045/

These equalities define the classical Hardy operator H and its dual operator H^*. By Hardy's inequalities [4, Ch. 9], these operators are bounded in $L^p(\mathbb{R}_+)$ for any $1 < p < \infty$. Furthermore, it is easy to show that for any $f \in \mathcal{M}^+(\mathbb{R}_+)$ and any $1 < p < \infty$

$$\frac{1}{p'}\|Hf\|_p \le \|H^*f\|_p \le p\|Hf\|_p \quad \text{for} \quad 1 < p < \infty \tag{1.1}$$

(as usual, $p' = p/(p-1)$).

However, the constants in (1.1) are not optimal. Sharp constants are contained in the following theorem.

Theorem 1.1. *Let $f \in \mathcal{M}^+(\mathbb{R}_+)$ and let $1 < p < \infty$. Then*

$$(p-1)\|Hf\|_p \le \|H^*f\|_p \le (p-1)^{1/p}\|Hf\|_p \tag{1.2}$$

if $1 < p \le 2$, and

$$(p-1)^{1/p}\|Hf\|_p \le \|H^*f\|_p \le (p-1)\|Hf\|_p \tag{1.3}$$

if $2 \le p < \infty$. All constants in (1.2) and (1.3) *are the best possible.*

The first inequality in (1.3) and the second inequality in (1.2) were obtained in [1, §21] (the proofs were given in the discrete case). In full, Theorem 1.1 was proved in the paper [6]. As it was observed in [6], the first inequality in (1.3) can be also derived from the results obtained in [2,7].

In the paper [3], the authors showed that for nonincreasing functions and integer $p \ge 2$ the first inequality in (1.3) can be improved. Namely, they proved the following theorem.

Theorem 1.2. *Let f be a nonincreasing and nonnegative function on $\mathbb{R}_+$ and let $2 \le p < +\infty$ be an integer number. Then*

$$\|Hf\|_p \le \left(\frac{p'}{(p+1)!}\right)^{1/p} \|H^*f\|_p, \tag{1.4}$$

and the constant is sharp.

It was conjectured in [3] that inequality (1.4) can be extended to all values $p \ge 2$ with the sharp constant

$$C(p) = \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}$$

at the right-hand side.

In the present note, it is proved that this conjecture is true. We prove also that for nonincreasing functions and $1 < p \le 2$ the second inequality in (1.2) can be improved in a similar way.

Thus, the main result of this paper is the following theorem.

Theorem 1.3. *Let f be a nonincreasing and nonnegative function on $\mathbb{R}_+$. Then for $2 \le p < +\infty$*

$$\|Hf\|_p \le \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} \|H^*f\|_p, \tag{1.5}$$

and for $1 < p \le 2$

$$\|H^* f\|_p \le \left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_p, \tag{1.6}$$

The constants in these inequalities are optimal.

We observe that for nonincreasing functions the first inequality in (1.2) and the second inequality in (1.3) cannot be improved. Indeed, the optimality of the first inequality in (1.2) was proved in [6] by considering the family of decreasing functions $f_\varepsilon(x) = x^{\varepsilon-1/p}\chi_{[0,1]}(x)$ $(0 < \varepsilon < 1/p)$. To show the optimality of the second inequality in (1.3), it is sufficient to consider the family of decreasing functions $f_\varepsilon(x) = \chi_{[0,1]}(x) + x^{-\varepsilon-1/p}\chi_{(1,+\infty)}(x)$ $(0 < \varepsilon < 1/p')$ (see [6]).

2. Proof of Theorem 1.3

Let $1 < p < \infty$. Taking into account (1.1), we may assume that Hf and H^*f belong to $L^p(\mathbb{R}_+)$. Denote

$$I_p = \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx.$$

Since $Hf \in L^p(\mathbb{R}_+)$, we have

$$Hf(x) = o(x^{-1/p}) \quad \text{as} \quad x \to 0+ \quad \text{or} \quad x \to +\infty.$$

Thus, integrating by parts, we obtain

$$I_p = p' \int_0^\infty x^{1-p} f(x) \left(\int_0^x f(t)\,dt\right)^{p-1} dx. \tag{2.1}$$

Further, set

$$I_p^* = \int_0^\infty \left(\int_t^\infty \frac{f(x)}{x}\,dx\right)^p dt,$$

$$\Phi(t,x) = \int_t^x \frac{f(u)}{u}\,du, \; 0 < t \le x,$$

and $G(t,x) = \Phi(t,x)^p$. Since $G(t,t) = 0$, we have

$$\left(\int_t^\infty \frac{f(x)}{x}\,dx\right)^p = \int_t^\infty G'_x(t,x)\,dx = p\int_t^\infty \frac{f(x)}{x}\Phi(t,x)^{p-1}\,dx.$$

Thus, by Fubini's theorem,

$$I_p^* = p\int_0^\infty \int_t^\infty \frac{f(x)}{x}\Phi(t,x)^{p-1}\,dx\,dt$$

$$= p\int_0^\infty \frac{f(x)}{x}\int_0^x \Phi(t,x)^{p-1}\,dt dx. \tag{2.2}$$

Set

$$g(x) = \int_0^x \Phi(t,x)^{p-1}\,dt \quad \text{and} \quad \Lambda(t,x) = \Phi(t,x)^{p-1}.$$

We shall estimate $g(x)$ from below. Applying differentiation under the integral sign (see, e.g., [5, §4.9], [8, 17.2]) and taking into account that $\Lambda(x,x)=0$, we obtain

$$g'(x)=\int_0^x \Lambda_x'(t,x)dt=(p-1)\frac{f(x)}{x}\int_0^x\left(\int_t^x \frac{f(u)}{u}du\right)^{p-2}dt \tag{2.3}$$

(observe that without loss of generality we may assume that f is continuous and bounded on $\mathbb{R}_+$). Since f is decreasing, we have

$$\int_t^x \frac{f(u)}{u}du \ge f(x)\ln\frac{x}{t}, \quad 0<t\le x. \tag{2.4}$$

Let now $p\ge 2$. Then (2.3) and (2.4) imply that

$$\begin{aligned} g'(x) &\ge (p-1)\frac{f(x)^{p-1}}{x}\int_0^x\left(\ln\frac{x}{t}\right)^{p-2}dt \\ &= (p-1)f(x)^{p-1}\int_0^1\left(\ln\frac{1}{y}\right)^{p-2}dy \\ &= (p-1)f(x)^{p-1}\Gamma(p-1)=\Gamma(p)f(x)^{p-1}. \end{aligned}$$

Since $g(0)=0$, we obtain

$$g(x)=\int_0^x g'(z)dz\ge\Gamma(p)\int_0^x f(z)^{p-1}dz.$$

By Hölder's inequality,

$$\left(\int_0^x f(z)dz\right)^{p-1}\le x^{p-2}\int_0^x f(z)^{p-1}dz.$$

Thus,

$$g(x)\ge\Gamma(p)x^{2-p}\left(\int_0^x f(z)dz\right)^{p-1}.$$

Using this inequality and (2.2), we get

$$I_p^*=p\int_0^\infty\frac{f(x)}{x}g(x)dx\ge p\Gamma(p)\int_0^\infty f(x)\left(\frac{1}{x}\int_0^x f(z)dz\right)^{p-1}dx.$$

Thus, by (2.1),

$$I_p^*\ge\frac{\Gamma(p+1)}{p'}I_p.$$

This implies (1.5).

Now, let $1<p\le 2$. We apply the same arguments. First, by (2.3) and (2.4), we have, as above

$$g'(x)\le(p-1)\frac{f(x)^{p-1}}{x}\int_0^x\left(\ln\frac{x}{t}\right)^{p-2}dt=\Gamma(p)f(x)^{p-1}.$$

This implies that

$$g(x)=\int_0^x g'(z)dz\le\Gamma(p)\int_0^x f(z)^{p-1}dz.$$

Applying Hölder's inequality with the exponent $1/(p-1)$, we obtain

$$g(x)\le\Gamma(p)x^{2-p}\left(\int_0^x f(z)dz\right)^{p-1}.$$

Using this inequality, (2.2), and (2.1), we get

$$I_p^* \le p\Gamma(p)\int_0^\infty f(x)\left(\frac{1}{x}\int_0^x f(z)dz\right)^{p-1}dx = \frac{\Gamma(p+1)}{p'}I_p.$$

This implies (1.6).

As it was observed in [3], for $f = \chi_{[0,1]}$ we have for any $1 < p < \infty$

$$\|Hf\|_p^p = p' \quad \text{and} \quad \|H^*f\|_p^p = \Gamma(p+1).$$

Thus, the constants in (1.5) and (1.6) are optimal. The proof is completed.

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