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# On generalized trigonometric functions with two parameters

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**Abstract**

The generalized $p$-trigonometric and $(p, q)$-trigonometric functions were introduced by P. Lindqvist and S. Takeuchi, respectively. We prove some inequalities and present a few conjectures for the $(p, q)$-functions.


*Keywords:* Eigenfunctions of $p$-Laplacian; $\sin_{p,q}$; Generalized trigonometric function

## 1. Introduction

During the past decade, many authors have studied the generalized trigonometric functions introduced by Lindqvist in a highly cited paper [18]. These so called $p$-trigonometric functions $p > 1$, which agree for $p = 2$ with the familiar functions, have also been extended in various directions. The recent literature on these functions includes several dozens of papers; see the bibliographies of [7,10,17]. Most recently, Takeuchi [22] has taken one step further and investigated the $(p, q)$-trigonometric functions depending on two parameters instead of one, and which for $p = q$ reduce to the $p$-functions of Lindqvist. See also Edmunds et al. [11].

Drábek and Manásevich [10] considered the following $(p, q)$-eigenvalue problem with the Dirichlét boundary condition. Let $\phi_p(x) = |x|^{p-2}x$. For $T, \lambda > 0$ and $p, q > 1$

$$\begin{cases} (\phi_p(u'))' + \lambda\, \phi_q(u) = 0, & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases}$$

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They found the complete solution to this problem. This solution is also given in [22, Theorem 2.1]. In particular, for $T = \pi_{p,q}$ the function $u(t) \equiv \sin_{p,q}(t)$ is a solution to this problem with $\lambda = \frac{p}{q}(p-1)$ where

$$\pi_{p,q} = \int_0^1 (1-t^q)^{-1/p}\, dt = \frac{2}{q} B\left(1-\frac{1}{p}, \frac{1}{q}\right).$$

If $p = 2$, this eigenvalue-boundary value problem reduces to the familiar boundary value problem whose solution is the usual sin function. Next, we will give an alternative equivalent definition of the function $\sin_{p,q}$, which is carried out in two steps: in the first step we define the inverse function of $\sin_{p,q}$, denoted by $\arcsin_{p,q}$, and in the second step the function itself. For $x \in [0, 1]$, set

$$F_{p,q}(x) = \int_0^x (1-t^q)^{-1/p}\, dt.$$

Then $F_{p,q} : [0, 1] \to [0, \pi_{p,q}/2]$ is an increasing homeomorphism, denoted by $\arcsin_{p,q}$, and therefore its inverse

$$\sin_{p,q} \equiv F_{p,q}^{-1},$$

is defined on the interval $[0, \pi_{p,q}/2]$. Below we discuss also other related functions such as $\arccos_{p,q}$, and $\operatorname{arsinh}_{p,q}$.

For the expression of the function $\arcsin_{p,q}$ in terms of well-known special functions we introduce some notation. The *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbf{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) ={}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1,$$

for given complex numbers $a$, $b$ and $c$ with $c \neq 0, -1, -2, \ldots$. Here $(a, 0) = 1$ for $a \neq 0$, and $(a, n)$ is the *shifted factorial function* or the *Appell symbol*

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, \ldots$. The hypergeometric function has numerous special functions as its special or limiting cases; see [1].

For $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, we define the classical *gamma function* $\Gamma(x)$, the *psi function* $\psi(x)$ and the *beta function* $B(x, y)$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively.

For $x \in I = [0, 1]$ the function $\arcsin_{p,q}$ considered above can be expressed in terms of the hypergeometric function as follows

$$\arcsin_{p,q} x = \int_0^x (1-t^q)^{-1/p} dt = x\, F\left(\frac{1}{p}, \frac{1}{q}; 1+\frac{1}{q}; x^q\right).$$

We also define $\arccos_{p,q} x = \arcsin_{p,q}((1-x^p)^{1/q})$ (see [11, Proposition 3.1]), and

$$\operatorname{arsinh}_{p,q} x = \int_0^x (1+t^q)^{-1/p} dt = x\, F\left(\frac{1}{p}, \frac{1}{q}; 1+\frac{1}{q}; -x^q\right).$$

Their inverse functions are

$$\sin_{p,q} : (0, \pi_{p,q}/2) \to (0,1), \qquad \cos_{p,q} : (0, \pi_{p,q}/2) \to (0,1),$$
$$\sinh_{p,q} : (0, m_{p,q}) \to (0,1), \qquad m_{p,q} = \frac{1}{2^{1/p}} F\left(1, \frac{1}{p}; 1 + \frac{1}{q}; \frac{1}{2}\right).$$

The significance of these expressions for this paper lies in the fact that we can now apply the vast available information about the hypergeometric functions to the functions $\operatorname{arcsin}_{p,q}$ and $\sin_{p,q}$.

When $p = q$ these $(p,q)$-functions coincide with the $p$-functions studied in the extensive earlier literature such as in [7,10,17,6,8], and for $p = q = 2$ they coincide with familiar elementary functions.

The main result of this paper is the following theorem which refines our earlier results in [6].

**Theorem 1.1.** *For $p, q > 1$ and $x \in (0,1)$, we have*

(1) $x\left(1 + \frac{x^q}{p(1+q)}\right) < \operatorname{arcsin}_{p,q} x < \min\left\{\frac{\pi_{p,q}}{2} x, (1 - x^q)^{-1/(p(1+q))} x\right\}$,

(2) $\left(\frac{x^p}{1+x^q}\right)^{1/p} L(p,q,x) < \operatorname{arsinh}_{p,q} x < \left(\frac{x^p}{1+x^q}\right)^{1/p} U(p,q,x)$,

*where*

$$L(p,q,x) = \max\left\{\left(1 - \frac{q x^q}{p(1+q)(1+x^q)}\right)^{-1}, (x^q + 1)^{1/p}\left(\frac{pq + p + q x^q}{p(q+1)}\right)^{-1/q}\right\},$$

*and* $U(p,q,x) = \left(1 - \frac{x^q}{1+x^q}\right)^{-q/(p(q+1))}$.

**Theorem 1.2.** *For $p, q > 1$, we have*

(1) $\left(\frac{p}{p-1}\right)^{1/q} \alpha\left(\frac{1}{100}, q\right) < \pi_{p,q} < \left(\frac{pq+p-q}{q(p-1)}\right)^{1-1/q} \left(\frac{p}{p-1}\right)^{1/q} \alpha\left(\frac{1}{30}, q\right)$,

$$\alpha(c,q) = \frac{2\sqrt{\pi}}{(e\,q)^{1/q}} \sqrt[6]{\frac{q(q+4)+8}{q^3} + c},$$

(2) $2^{1-2/p}\sqrt{\frac{\pi}{p}(4+p)} < \pi_{p',p} < 2^{1-2/p}\sqrt{\frac{\pi}{p}(4+p) + \left(2\sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2}$,

(3) $2^{2/p}\sqrt{\pi}\sqrt{\frac{5}{4} - \frac{1}{p}} < \pi_{p,p'} < 2^{2/p}\sqrt{\pi}\,\frac{(2-1/p)^{3/2-1/p}}{\sqrt{e}(3/2-1/p)^{1-1/p}}$,

*where $p' = p/(p-1)$.*

The area enclosed by the so-called $p$-circle

$$|x|^p + |y|^p = 1$$

is $\pi_{p,p'}$; see [19]. In particular, $\pi_{2,2} = \pi = 3.14\cdots$.

## 2. Some relations for $(p,q)$-functions

In this section, we shall prove some inequalities for the functions defined in Section 1.

**Lemma 2.1.** *Fix $p, q > 1$ and $x \in (0,1)$.*

(1) *The functions*

$$(\mathrm{arcsin}_{p,q}(x^k))^{1/k}, \qquad (\mathrm{arsinh}_{p,q}(x^k))^{1/k}$$

*are decreasing and increasing, respectively in $k \in (0,\infty)$.*

(2) *The function*

$$k\ \mathrm{arcsin}_{p,q}(x/k)$$

*is decreasing on $k \in (1,\infty)$.*

(3) *In particular, for $k \geq 1$*

$$\sqrt[k]{\mathrm{arcsin}_{p,q}(x^k)} \leq \mathrm{arcsin}_{p,q}(x) \leq (\mathrm{arcsin}_{p,q}\sqrt[k]{x})^k,$$

$$(\mathrm{arsinh}_{p,q}\sqrt[k]{x})^k \leq \mathrm{arsinh}_{p,q}(x) \leq \sqrt[k]{\mathrm{arsinh}_{p,q}(x^k)},$$

$$\mathrm{arcsin}_{p,q}(x/k) \leq (\mathrm{arcsin}_{p,q}(x))/k.$$

**Proof.** Let

$$G(x) = \int_0^x g(t)\,dt, \qquad E = G(x^k), \qquad f(k) = (E)^{1/k}.$$

We get

$$\begin{aligned} f' &= -E^{1/k}\log E\frac{1}{k^2} + \frac{1}{k}E^{1/k-1}E'x^k\log x \\ &= \frac{E^{1/k}}{k^2}\left(-\log\frac{E}{x^k} - \left(x^k\frac{E'}{E} - 1\right)\log\frac{1}{x^k}\right). \end{aligned}$$

If $g \geq 1$, then

$$\frac{E}{x^k} = \frac{1}{x^k}\int_0^{x^k} g(t)\,dt \geq 1.$$

If $g$ is increasing, then

$$E' - \frac{E}{x^k} = g(x^k) - \frac{1}{x^k}\int_0^{x^k} g(t)\,dt \geq 0,$$

so that $x^k\frac{E'}{E} - 1 \geq 0$. Thus $f' \leq 0$ under these assumptions.

For the case of $\mathrm{arcsin}_{p,q}$, let $g(t) = (1-t^q)^{-1/p}$, so theconditions are clearly satisfied. Next, for $\mathrm{arsinh}_{p,q}$, we set $g(t) = (1+t^q)^{-1/p}$ and note that $g(t) \leq 1$ for all $t > 0$ and that $g$ is decreasing and thus conclude that $f' \geq 0$, and the claims in (1) follow. For (2), let

$$h(k) = k\ \mathrm{arcsin}_{p,q}\left(\frac{x}{k}\right) = x\,F\left(\frac{1}{p},\frac{1}{q};1+\frac{1}{q};\left(\frac{x}{k}\right)^q\right).$$

We get

$$h'(k) = -\frac{q\,x}{k\,p(1+q)}\left(\frac{x}{k}\right)^q\ F\left(\frac{1}{p},\frac{1}{q};1+\frac{1}{q};\left(\frac{x}{k}\right)^q\right) \leq 0,$$

and this completes the proof.

The proof of (3) follows from parts (1) and (2). □

**Theorem 2.2.** *For $p, q > 1$ and $r, s \in (0,1)$, the following inequalities hold:*

(1) $\mathrm{arcsin}_{p,q}(r\,s) \le \sqrt{\mathrm{arcsin}_{p,q}(r^2)\ \mathrm{arcsin}_{p,q}(s^2)} \le \mathrm{arcsin}_{p,q}(r)\ \mathrm{arcsin}_{p,q}(s)$,

(2) $\mathrm{arsinh}_{p,q}(r)\,\mathrm{arsinh}_{p,q}(s) \le \sqrt{\mathrm{arsinh}_{p,q}(r^2)\,\mathrm{arsinh}_{p,q}(s^2)} \le \mathrm{arsinh}_{p,q}(r\,s)$.

**Proof.** Let $h(x) = \log f(e^x)$ where $f(u) > 0$. Then $h$ is convex (in the $C^2$ case) when $h'' \ge 0$, i.e. iff

$$\frac{f}{y}(f' + yf'') \ge (f')^2,$$

where $y = e^x$ and the function is evaluated at $y$. If $f'' \ge 0$, then

$$\frac{f}{y} \ge f'(0),$$

so a sufficient condition for convexity is $f'(0)(f' + yf'') \ge (f')^2$. If $f'' \le 0$, the reverse holds, so a sufficient condition for concavity is $f'(0)(f' + yf'') \le (f')^2$. Suppose

$$f(x) = \int_0^x g(t)\,dt.$$

Then $f' = g$ and $f'' = g'$. One easily checks that $h$ is convex in case $g(t)$ is $(1 - t^p)^{-1/q}$, and concave for $g(t)$ equal to $(1 + t^p)^{-1/q}$. Now the proof follows easily from Lemma 2.1. $\square$

**Lemma 2.3** ([15, Theorem 1.7])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function and for $c \ne 0$ define*

$$g(x) = \frac{f(x^c)}{(f(x))^c}.$$

*We have the following*

1. *if $h(x) = \log(f(e^x))$ is a convex function, then $g(x)$ is monotone increasing for $c, x \in (0,1)$ and monotone decreasing for $c > 1, x \in (0,1)$ or $c < 0, x \in (0,1)$,*
2. *if $h(x)$ is a concave function, then $g(x)$ is monotone increasing for $c > 1, x \in (0,1)$ or $c < 0, x \in (0,1)$ and monotone decreasing for $c, x \in (0,1)$.*

We get the following lemma by the proof of Theorem 2.2 and applying Lemma 2.3.

**Lemma 2.4.** *Let $I = (0,1)$. For $p, q > 1$ the function*

$$g_1(x) = \frac{\mathrm{arcsin}_{p,q}(x^k)}{(\mathrm{arcsin}_{p,q}(x))^k}$$

*is increasing (decreasing) in $x \in I$ for $k \in I$ ($k \in \mathbb{R} \setminus [0,1]$), and*

$$g_2(x) = \frac{\mathrm{arsinh}_{p,q}(x^k)}{(\mathrm{arsinh}_{p,q}(x))^k}$$

*is increasing (decreasing) in $x \in I$ for $k \in \mathbb{R} \setminus I$ ($k \in [0,1]$). In particular, for $k \in I$,*

$$\left(\frac{\pi_{p,q}}{2}\right)^{1-1/k} \sqrt[k]{\mathrm{arcsin}_{p,q}(x^k)} \le \mathrm{arcsin}_{p,q}(x)$$

$$\left(m_{p,q}\right)^{1-1/k} \sqrt[k]{\mathrm{arsinh}_{p,q}(x^k)} \ge \mathrm{arsinh}_{p,q}(x).$$

*Both the inequalities reverse for $k \in \mathbb{R} \setminus [0,1]$.*

**Lemma 2.5** (*[20, Theorem 2.1]*). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable, log-convex function and let $a \geq 1$. Then $g(x) = (f(x))^a / f(a\,x)$ decreases on its domain. In particular, if $0 \leq x \leq y$, then the following inequalities*

$$\frac{(f(y))^a}{f(a\,y)} \leq \frac{(f(x))^a}{f(a\,x)} \leq (f(0))^{a-1}$$

*hold true. If $0 < a \leq 1$, then the function $g$ is an increasing function on $\mathbb{R}_+$ and inequalities are reversed.*

**Lemma 2.6.** *For $k, p, q > 1$ and $r, s \in (0, 1)$ with $r \geq s$, we have*

$$\left(\frac{\operatorname{arcsin}_{p,q}(s)}{\operatorname{arcsin}_{p,q}(r)}\right)^k \leq \frac{\operatorname{arcsin}_{p,q}(s^k)}{\operatorname{arcsin}_{p,q}(r^k)},$$
$$\frac{\operatorname{arsinh}_{p,q}(s^k)}{\operatorname{arsinh}_{p,q}(r^k)} \leq \left(\frac{\operatorname{arsinh}_{p,q}(s)}{\operatorname{arsinh}_{p,q}(r)}\right)^k.$$

**Proof.** For $x > 0$, the following functions

$$u(x) = \operatorname{arcsin}_{p,q}(e^{-x}), \qquad v(x) = 1/\operatorname{arsinh}_{p,q}(e^{-x})$$

are log-convex by the proof of Theorem 2.2. With the change of variables $e^{-x} = r$ the inequalities follow from Lemma 2.5. □

**Lemma 2.7** (*[16, Theorem 2, p. 151]*). *Let $J \subset \mathbb{R}$ be an open interval, and let $f : J \to \mathbb{R}$ be strictly monotonic function. Let $f^{-1} : f(J) \to J$ be the inverse to $f$ then*

(1) *if $f$ is convex and increasing, then $f^{-1}$ is concave,*
(2) *if $f$ is convex and decreasing, then $f^{-1}$ is convex,*
(3) *if $f$ is concave and increasing, then $f^{-1}$ is convex,*
(4) *if $f$ is concave and decreasing, then $f^{-1}$ is concave.*

**Lemma 2.8.** *For $k, p, q > 1$ and $r \geq s$, we have*

$$\left(\frac{\sin_{p,q}(r)}{\sin_{p,q}(s)}\right)^k \leq \frac{\sin_{p,q}(r^k)}{\sin_{p,q}(s^k)}, \quad r, s \in (0, 1),$$
$$\left(\frac{\sinh_{p,q}(r)}{\sinh_{p,q}(s)}\right)^k \geq \frac{\sinh_{p,q}(r^k)}{\sinh_{p,q}(s^k)}, \quad r, s \in (0, 1),$$

*inequalities reverse for $k \in (0, 1)$.*

**Proof.** It is clear from the proof of Theorem 2.2 that the functions

$$f(x) = \log(\operatorname{arcsin}_{p,q}(e^{-x})), \qquad h(x) = \log(1/\operatorname{arsinh}_{p,q}(e^{x}))$$

are convex and decreasing. Then Lemma 2.7(2) implies that

$$f^{-1}(y) = \log(1/\sin_{p,q}(e^{y})), \qquad h^{-1}(y) = \log(\sinh_{p,q}(e^{-y})),$$

are convex, now the result follows from Lemma 2.5. □

Let $f : I \to (0, \infty)$ be continuous, where $I$ is a subinterval of $(0, \infty)$. Let $M$ and $N$ be any two mean values. We say that $f$ is $MN$-convex (concave) if

$$f(M(x, y)) \le (\ge) N(f(x), f(y)) \quad \text{for all } x, y \in I.$$

For some properties of these functions, see [3]. If $A(x, y) = (x + y)/2$ is the arithmetic mean, then we see that convex functions are $AA$-convex.

**Lemma 2.9** (*[3, Theorem 2.4(1)]*). *Let $I = (0, b), 0 < b < \infty$, and let $f : I \to (0, \infty)$ be continuous. Then $f$ is $AA$-convex (concave) if and only if $f$ is convex (concave), where $A$ is the arithmetic mean.*

**Lemma 2.10.** *For $p, q > 1$, and $r, s \in (0, 1)$, we have*

(1) $\operatorname{arcsin}_{p,q} r + \operatorname{arcsin}_{p,q} s \le 2 \operatorname{arcsin}_{p,q} \left(\frac{r+s}{2}\right)$,
(2) $\sin_{p,q} r + \sin_{p,q} s \ge 2 \sin_{p,q} \left(\frac{r+s}{2}\right)$,
(3) $\operatorname{arsinh}_{p,q} r + \operatorname{arsinh}_{p,q} s \ge 2 \operatorname{arsinh}_{p,q} \left(\frac{r+s}{2}\right)$,
(4) $\sinh_{p,q} r + \sinh_{p,q} s \le 2 \sinh_{p,q} \left(\frac{r+s}{2}\right)$.

**Proof.** Let $f(x) = \operatorname{arcsin}_{p,q} x$ and $g(x) = \operatorname{arsinh}_{p,q} x$. Then

$$f'(x) = (1 - x^p)^{-1/p}, \qquad g'(x) = (1 + x^p)^{-1/p}$$

are increasing and decreasing, respectively. This implies that $f$ and $g$ are convex and concave. Now it follows from Lemma 2.7(1), (3) that $f^{-1}$ and $g^{-1}$ are concave and convex, respectively. The proof follows from Lemma 2.9. □

For the following inequalities, see [5, Corollary 1.26] and [3, Corollary 1.10]: for all $x, y \in (0, \infty)$,

$$\cosh(\sqrt{x\,y}) \le \sqrt{\cosh(x)\,\cosh(y)},$$
$$\sinh(\sqrt{x\,y}) \le \sqrt{\sinh(x)\,\sinh(y)},$$

with equality if and only if $x = y$.

On the basis of our computer experiments we have arrived at the following conjecture.

**Conjecture 2.11.** *For $p, q \in (1, \infty)$ and $r, s \in (0, 1)$, we have*

(1) $\sin_{p,q}(\sqrt{r\,s}) \le \sqrt{\sin_{p,q}(r)\sin_{p,q}(s)}$,
(2) $\sinh_{p,q}(\sqrt{r\,s}) \ge \sqrt{\sinh_{p,q}(r)\sinh_{p,q}(s)}$.

**Remark 2.12.** Edmunds et al. [11, Proposition 3.4] proved that for $x \in [0, \pi_{4/3,4}/4)$

$$\sin_{4/3,4}(2x) = \frac{2uv^{1/3}}{(1 + 4u^4v^{4/3})^{1/2}}, \quad u = \sin_{4/3,4}(x), v = \cos_{4/3,4}(x). \tag{2.13}$$

Note that in this case $q = p/(p - 1)$. The Edmunds–Gurka–Lang identity (2.13) suggests that in the particular case $q = p/(p - 1)$ some exceptional behavior might be expected for $\sin_{p,q}$. This special case might be worth of further investigation.

It seems to be a natural question to ask whether the addition formulas for the trigonometric functions have counterparts for the $(p, q)$-functions. Our next results give a subadditive inequality.

**Lemma 2.14.** *For $p, q > 1$, the following inequalities hold*

(1) $\sin_{p,q}(r+s) \le \sin_{p,q}(r) + \sin_{p,q}(s),\ r, s \in (0, \pi_{p,q}/4)$,
(2) $\sinh_{p,q}(r+s) \ge \sinh_{p,q}(r) + \sinh_{p,q}(s),\ r, s \in (0, \infty)$.

**Proof.** Let $f(x) = \operatorname{arcsin}_{p,q}(x)$, $x \in (0, 1)$. We get

$$f'(x) = (1 - x^q)^{-1/p},$$

which is increasing; hence $f$ is convex. Clearly, $f$ is increasing. Therefore

$$f_1 = f^{-1}(y) = \sin_{p,q}(y)$$

is concave by Lemma 2.7(1). This implies that $f_1'$ is decreasing. Clearly $f_1(0) = 0$, and by Anderson et al. [4, Theorem 1.25], $f_1(y)/y$ is decreasing. Now it follows from [4, Lemma 1.24] that

$$f_1(r+s) \le f_1(r) + f_1(s),$$

and (1) follows. The proofs of part (2) follow similarly. □

For $p, q > 1$, $x \in (0, 1)$ and $z \in (0, \pi_{p,q}/2)$, it follows from Theorem 1.1 that

$$\operatorname{arsinh}_{p,q} x < \operatorname{arcsin}_{p,q} x, \qquad \sin_{p,q} z < \sinh_{p,q} z.$$

**Lemma 2.15.** *For $p, q > 1$, $s \in (0, r]$ and $r \in (0, 1)$, we have*

(1) $\dfrac{\operatorname{arcsin}_{p,q} s}{s} \le \dfrac{\operatorname{arcsin}_{p,q} r}{r}$,

(2) $\dfrac{\operatorname{arsinh}_{p,q} s}{\sqrt[p]{s^p/(1+s^q)}} \le \dfrac{\operatorname{arsinh}_{p,q} r}{\sqrt[p]{r^p/(1+r^q)}}$,

(3) $\dfrac{\operatorname{arsinh}_{p,q} s}{s} \ge \dfrac{\operatorname{arsinh}_{p,q} r}{r}$.

**Proof.** By definition we get

$$\frac{\operatorname{arcsin}_{p,q} s}{\operatorname{arcsin}_{p,q} r} = \frac{s}{r}\,\frac{F(1/p, 1/q; 1+1/q; s^q)}{F(1/p, 1/q; 1+1/q; r^q)} \le \frac{s}{r}.$$

Similarly,

$$\frac{\operatorname{arsinh}_{p,q} s}{\operatorname{arsinh}_{p,q} r} = \frac{s/(1+s^q)^{1/p}}{r/(1+r^q)^{1/p}}\,\frac{F(1, 1/p; 1+1/q; s^q/(1+s^q))}{F(1, 1/p; 1+1/q; r^q/(1+r^q))} \le \left(\frac{s/(1+s^q)}{r/(1+r^q)}\right)^{1/p}$$

because $F(a, b, ; c; x)$ is increasing in $x$. Part (3) follows from [4, Theorem 1.25]. □

## 3. Proof of the main results

For the following lemma, see [4, Theorems 1.19(10) and 1.52(1), Lemmas 1.33 and 1.35].

**Lemma 3.1.** (1) *For $a, b, c > 0$, $c < a + b$, and $|x| < 1$,*

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x).$$

(2) *For $a, x \in (0, 1)$, and $b, c \in (0, \infty)$*

$$F(-a, b; c; x) < 1 - \frac{a\,b}{c}\,x.$$

(3) *For* $a, x \in (0,1)$*, and* $b, c \in (0,\infty)$

$$F(a,b;c;x) + F(-a,b;c;x) > 2.$$

(4) *Let* $a, b, c \in (0,\infty)$ *and* $c > a+b$*. Then for* $x \in [0,1]$,

$$F(a,b;c;x) \le \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

(5) *For* $a, b > 0$*, the following function*

$$f(x) = \frac{F(a,b;a+b;x)-1}{\log(1/(1-x))}$$

*is strictly increasing from* $(0,1)$ *onto* $(a\,b/(a+b), 1/B(a,b))$.

We will refer in our proofs to the following identity [1, 15.3.5]:

$$F(a,b;c;z) = (1-z)^{-b} F(b, c-a; c; -z/(1-z)). \tag{3.2}$$

**Lemma 3.3** (*[9, Theorem 2]*)**.** *For* $0 < a < c, -\infty < x < 1$ *and* $0 < b < c$*, the following inequality holds*

$$\max\left\{\left(1-\frac{b\,x}{c}\right)^{-a}, (1-x)^{c-a-b}\left(1-x+\frac{b\,x}{c}\right)^{a-c}\right\}$$
$$< F(a,b;c;x) < (1-x)^{-ab/c}.$$

**Proof of Theorem 1.1.** For (1), we get from Lemma 3.1(3), (2)

$$\begin{aligned}\operatorname{arcsin}_{p,q} x &= x\,F\left(\frac{1}{p},\frac{1}{q};1+\frac{1}{q};x^q\right)\\ &> \left(2-F\left(-\frac{1}{p},\frac{1}{q};1+\frac{1}{q};x^q\right)\right)x\\ &> x\left(1+\frac{x^q}{p(1+q)}\right).\end{aligned}$$

The second inequality of (1) follows easily from Lemmas 3.1 and 3.3(4).

For (2), if we replace $b = 1/q, c-a = 1/q, c = 1+1/q$ and $x^q = z/(1-z)$ in (3.2) then we get

$$\begin{aligned}\operatorname{arsinh}_{p,q} x &= x\,F\left(\frac{1}{p},\frac{1}{q};1+\frac{1}{q};-x^q\right)\\ &= \left(\frac{x^p}{1+x^q}\right)^{1/p} F\left(1,\frac{1}{p};1+\frac{1}{q};\frac{x^q}{1+x^q}\right),\end{aligned}$$

now the proof follows easily from Lemma 3.3. $\square$

For the following Lemma, see [2,12,14], [13, Theorem 1], [23], respectively.

**Lemma 3.4.** *The following relations hold,*

(1) $\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6}$
$< \Gamma(1+x) < \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6},\ x \ge 0,$

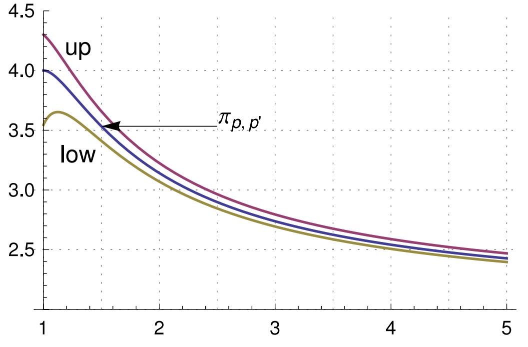


Fig. 1. We denote the lower and upper bounds of $\pi_{p,p'}$ by low and up.

(2) $\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\left(\frac{1}{4}+s\right)^{1/2}\right)^{1-s}$, $x>0, s\in(0,1)$,

(3) $\frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b}$, $b>a>0$.

(4) $\left(\frac{x}{x+s}\right)^{1-s} \le \frac{\Gamma(x+s)}{x^s\Gamma(x)} \le 1, x>0, s\in(0,1)$,

**Proof of Theorem 1.2.** If we let $x = 1-1/p$ and $s = 1/q$, then by definition

$$\pi_{p,q} = \frac{2\Gamma(x)\Gamma(1+s)}{\Gamma(s+x)}.$$

By Lemma 3.4(4) we get

$$\frac{2}{q}\,\Gamma(s)\left(\frac{p}{p-1}\right)^{1/q} < \pi_{p,q} < \frac{2}{q}\,\Gamma(s)\left(\frac{pq+p-q}{q(p-1)}\right)^{1-1/q}\left(\frac{p}{p-1}\right)^{1/q}.$$

Now (1) follows if we use $\Gamma(1+x) = x\,\Gamma(x)$ and Lemma 3.4(1). From [1, 6.1.18] we get

$$\pi_{p',p} = 2\frac{\Gamma(1/p)\Gamma(1/p)}{p\Gamma(2/p)} = 2\frac{\Gamma(1/p)\Gamma(1+1/p)}{\Gamma(2/p)}$$
$$= 2^{2-2/p}\sqrt{\pi}\,\frac{\Gamma(1+1/p)}{\Gamma(1/2+1/p)},$$

and (2) follows from Lemma 3.4(2) if we take $x = 1/p$ and $s = 1/2$.

For (3), we see that

$$\pi_{p,p'} = \frac{2x\Gamma(x)^2}{\Gamma(2x)} = \frac{2^{2-2x}\sqrt{\pi}x\Gamma(x)^2}{\Gamma(x)\Gamma(1/2+x)} = \frac{2^{2-2x}\sqrt{\pi}\Gamma(1+x)}{\Gamma(1/2+x)},$$

and the lower bound follows from Lemma 3.4(2), and the upper bound follows if we replace $b = x+1$ and $a = x+s$ with $s = 1/2$ in 3.4(3) (see Fig. 1). □

**Remark 3.5.** For the benefit of an interested reader we give an algorithm for the numerical computation of $\sin_{p,q}$ with the help of Mathematica® [21]. The same method also applies to $\sinh_{p,q}$ .

```
arcsinp[p_, q_, x_] := x * Hypergeometric2F1[1/p, 1/q, 1 + 1/q, x^p]
sinp[p_, q_, y_] := x /. FindRoot[arcsinp[p, q, x] == y, {x, 0.5 }].
```

In the following tables, we use the values of $p = 2.5$ and $q = 3$.

| $x$ | $\mathrm{arcsin}_{p,q}(x)$ | $\mathrm{arccos}_{p,q}(x)$ | $\mathrm{arsinh}_{p,q}(x)$ |
|---|---|---|---|
| 0.0000 | 0.0000 | 1.2748 | 0.0000 |
| 0.2500 | 0.2504 | 1.2048 | 0.2496 |
| 0.5000 | 0.5066 | 1.0688 | 0.4940 |
| 0.7500 | 0.7887 | 0.8536 | 0.7227 |
| 1.0000 | 1.2748 | 0.0000 | 0.9262 |

| $x$ | $\sin_{p,q}(x)$ | $\cos_{p,q}(x)$ | $\sinh_{p,q}(x)$ |
|---|---|---|---|
| 0.0000 | 0.0000 | 1.0000 | 0.0000 |
| 0.2500 | 0.2496 | 0.9937 | 0.2504 |
| 0.5000 | 0.4937 | 0.9500 | 0.5063 |
| 0.7500 | 0.7183 | 0.8309 | 0.7817 |
| 1.0000 | 0.8995 | 0.5943 | 0.1003 |

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