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Exceptional Meixner and Laguerre orthogonal polynomials☆

Antonio J. Durán

Departamento de Análisis Matemático, Universidad de Sevilla, Apdo (P. O. BOX) 1160, 41080 Sevilla, Spain

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Abstract

Using Casorati determinants of Meixner polynomials $(m_n^{a,c})_n$, we construct for each pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers a sequence of polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, which are eigenfunctions of a second order difference operator, where $\sigma_{\mathcal{F}}$ is certain infinite set of nonnegative integers, $\sigma_{\mathcal{F}} \subsetneq \mathbb{N}$. When c and $\mathcal{F}$ satisfy a suitable admissibility condition, we prove that the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are actually exceptional Meixner polynomials; that is, in addition, they are orthogonal and complete with respect to a positive measure. By passing to the limit, we transform the Casorati determinant of Meixner polynomials into a Wronskian type determinant of Laguerre polynomials $(L_n^{\alpha})_n$. Under the admissibility conditions for $\mathcal{F}$ and α, these Wronskian type determinants turn out to be exceptional Laguerre polynomials.

Keywords: Orthogonal polynomials; Exceptional orthogonal polynomial; Difference operators; Differential operators; Meixner polynomials; Krawtchouk polynomials; Laguerre polynomials

1. Introduction

In [12], we have introduced a systematic way of constructing exceptional discrete orthogonal polynomials using the concept of dual families of polynomials. Using Charlier polynomials, we

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E-mail address: duran@us.es.

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applied this procedure to construct exceptional Charlier polynomials and, passing to the limit, exceptional Hermite polynomials. The purpose of this paper is to extend this construction using Meixner and Laguerre polynomials.

Exceptional orthogonal polynomials $p_n, n \in X \subsetneq \mathbb{N}$, are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order differential operator. They extend the classical families of Hermite, Laguerre and Jacobi. The last few years have seen a great deal of activity in the area of exceptional orthogonal polynomials (see, for instance, [7,16,17] (where the adjective exceptional for this topic was introduced), [18,19,21–23,28,30,31,33,37,38,40] and the references therein).

In the same way, exceptional discrete orthogonal polynomials are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order difference operator, extending the discrete classical families of Charlier, Meixner, Krawtchouk and Hahn, or Wilson, Racah, etc., if orthogonal discrete polynomials on nonuniform lattices are considered [12,31,41]. One can also add to the list the exceptional q-orthogonal polynomials related to second order q-difference operators [31,32,34–36].

The most apparent difference between classical or classical discrete orthogonal polynomials and their exceptional counterparts is that the exceptional families have gaps in their degrees, in the sense that not all degrees are present in the sequence of polynomials (as it happens with the classical families) although they form a complete orthonormal set of the underlying L^2 space defined by the orthogonalizing positive measure. This means in particular that they are not covered by the hypotheses of Bochner's and Lancaster's classification theorems (see [4] or [26]) for classical and classical discrete orthogonal polynomials, respectively. Exceptional orthogonal polynomials have been applied to shape-invariant potentials [37], supersymmetric transformations [18], to discrete quantum mechanics [31], mass-dependent potentials [28], and to quasi-exact solvability [40].

As mentioned above, we use the concept of dual families of polynomials to construct exceptional discrete orthogonal polynomials (see [27]). One can then also construct examples of exceptional orthogonal polynomials by taking limits in some of the parameters in the same way as one goes from classical discrete polynomials to classical polynomials in the Askey tableau.

Definition 1.1. Given two sets of nonnegative integers $U, V \subset \mathbb{N}$, we say that the two sequences of polynomials $(p_u)_{u\in U}, (q_v)_{v\in V}$ are dual if there exists a couple of sequences of numbers $(\xi_u)_{u\in U}, (\zeta_v)_{v\in V}$ such that

$$\xi_u p_u(v) = \zeta_v q_v(u), \quad u \in U, v \in V. \tag{1.1}$$

Duality has shown to be a fruitful concept regarding discrete orthogonal polynomials, and its utility will be again manifested in the exceptional discrete polynomials world. Indeed, as we pointed out in [12], it turns out that duality interchanges exceptional discrete orthogonal polynomials with the so-called Krall discrete orthogonal polynomials. A Krall discrete orthogonal family is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$, p_n of degree n, orthogonal with respect to a positive measure which, in addition, are also eigenfunctions of a higher order difference operator. A huge amount of families of Krall discrete orthogonal polynomials have been recently introduced by the author by means of a certain Christoffel transform of the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see [8,9,13]). A Christoffel transform is a transformation which consists in multiplying a measure μ by a polynomial r. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E.B. Christoffel (see [6] and also [39]) studied it for the particular case $r(x) = x$.

The content of this paper is as follows. In Section 2, we include some preliminary results about Christoffel transforms and finite sets of positive integers.

In Section 3, using Casorati determinants of Meixner polynomials we associate to a pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers a sequence of polynomials which are eigenfunctions of a second order difference operator.

Denote by $\mathcal{F} = (F_1, F_2)$ a pair of finite sets of positive integers, and write k_i for the number of elements of F_i, $i = 1, 2$ and $k = k_1 + k_2$ for the number of elements of $\mathcal{F}$. One of the components of $\mathcal{F}$, but not both, can be the empty set. We define the nonnegative integer $u_{\mathcal{F}}$ by $u_{\mathcal{F}} = \sum_{f\in F_1} f + \sum_{f\in F_2} f - \binom{k_1+1}{2} - \binom{k_2}{2}$ and the infinite set of nonnegative integers $\sigma_{\mathcal{F}}$ by

$$\sigma_{\mathcal{F}} = \{u_{\mathcal{F}}, u_{\mathcal{F}}+1, u_{\mathcal{F}}+2, \ldots\} \setminus \{u_{\mathcal{F}} + f, f \in F_1\}.$$

Given $a, c \in \mathbb{R}$ with $a \neq 0, 1$ and $c \neq 0, -1, -2, \ldots,$ we then associate to the pair $\mathcal{F}$ the sequence of polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, defined by

$$m_n^{a,c;\mathcal{F}}(x) = \begin{vmatrix} m^{a,c}_{n-u_{\mathcal{F}}}(x) & m^{a,c}_{n-u_{\mathcal{F}}}(x+1) & \cdots & m^{a,c}_{n-u_{\mathcal{F}}}(x+k) \\ \begin{bmatrix} m_f^{a,c}(x) \\ f \in F_1 \end{bmatrix} & m_f^{a,c}(x+1) & \cdots & m_f^{a,c}(x+k) \\ \begin{bmatrix} m_f^{1/a,c}(x) \\ f \in F_2 \end{bmatrix} & m_f^{1/a,c}(x+1)/a & \cdots & m_f^{1/a,c}(x+k)/a^k \end{vmatrix} \tag{1.2}$$

where $(m_n^{a,c})_n$ are the Meixner polynomials (see (2.21)). Along this paper, we use the following notation: given a finite set of positive integers $F = \{f_1, \ldots, f_m\}$, the expression

$$\begin{bmatrix} z_{f,1} & z_{f,2} & \cdots & z_{f,m} \\ f \in F & & & \end{bmatrix} \tag{1.3}$$

inside of a matrix or a determinant will mean the submatrix defined by

$$\begin{pmatrix} z_{f_1,1} & z_{f_1,2} & \cdots & z_{f_1,m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{f_m,1} & z_{f_m,2} & \cdots & z_{f_m,m} \end{pmatrix}.$$

The determinant (1.2) should be understood in this form.

When $-1 < a < 1$ Meixner polynomials $(m_n^{a,c})_n$ are orthogonal with respect to the discrete measure

$$\rho_{a,c} = \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c)}{x!} \delta_x.$$

Consider now the measure

$$\rho_{a,c}^{\mathcal{F}} = \prod_{f\in F_1} (x-f) \prod_{f\in F_2} (x+c+f) \rho_{a,c}. \tag{1.4}$$

Orthogonal polynomials with respect to $\rho_{a,c}^{\mathcal{F}}$ are eigenfunctions of higher order difference operators (see [8,9,13]). It turns out that the sequence of polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, and the sequence of orthogonal polynomials with respect to the measure $\rho_{a,c}^{\mathcal{F}}$ are dual sequences (see

Lemma 3.2). As a consequence we get that the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are always eigenfunctions of a second order difference operator $D_{\mathcal{F}}$ (whose coefficients are rational functions); see Theorem 3.3.

The most interesting case appears when the measure $\rho_{a,c}^{\mathcal{F}}$ is positive. This gives rise to the concept of admissibility for the real number c and the pair $\mathcal{F}$, which we study in Section 2.4.

Definition 1.2. Let $\mathcal{F} = (F_1, F_2)$ be a pair of finite sets of positive integers. For a real number $c \neq 0, -1, -2, \ldots$, write $\hat{c} = \max\{-[c], 0\}$, where $[c]$ denotes the value of the floor function at c (i.e. $[c] = \max\{s \in \mathbb{Z} : s \leq c\}$). We say that c and $\mathcal{F}$ are admissible if for all $x \in \mathbb{N}$

$$\frac{\prod_{f\in F_1}(x-f)\prod_{f\in F_2}(x+c+f)}{(x+c)_{\hat{c}}} \geq 0. \tag{1.5}$$

As usual $(a)_j$ will denote the Pochhammer symbol defined by

$$(a)_0 = 1, \qquad (a)_j = a(a+1)\cdots(a+j-1), \quad \text{for } j \geq 1, a \in \mathbb{C}.$$

Let us remind that for Charlier and Hermite polynomials, the admissibility of a finite set F of positive integers is defined by $\prod_{f\in F}(x-f) \geq 0$, for $x \in \mathbb{N}$. The concept of admissibility defined in (1.5) is more involved than the corresponding one for exceptional Charlier and Hermite polynomials because of two reasons. On the one hand, we have now a pair $\mathcal{F}$ of finite sets instead of a single finite set F. On the other hand, the admissibility also depends on the parameter c of the Meixner polynomials (or on the parameter α of the Laguerre polynomials) while Charlier and Hermite admissibility only depends on the finite set F. The concept of admissibility for exceptional Charlier and Hermite polynomials has appeared several times in the literature (see, for instance, [25,1] or [41]); however, we have not found in the literature a definition as (1.5) for Meixner and Laguerre admissibility.

In Section 4, we prove (Theorems 4.3 and 4.4) that if c and $\mathcal{F}$ are admissible, then the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are orthogonal and complete with respect to the positive measure

$$\omega_{a,c}^{\mathcal{F}} = \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c+k)}{x!\,\Omega_{\mathcal{F}}^{a,c}(x)\Omega_{\mathcal{F}}^{a,c}(x+1)}\delta_x,$$

where $\Omega_{\mathcal{F}}^{a,c}$ is the polynomial defined by

$$\Omega_{\mathcal{F}}^{a,c}(x) = \left|\begin{matrix} \begin{bmatrix} m_f^{a,c}(x) & m_f^{a,c}(x+1) & \cdots & m_f^{a,c}(x+k-1) \\ f \in F_1 & & & \end{bmatrix} \\ \begin{bmatrix} m_f^{1/a,c}(x) & m_f^{1/a,c}(x+1)/a & \cdots & m_f^{1/a,c}(x+k-1)/a^{k-1} \\ f \in F_2 & & & \end{bmatrix} \end{matrix}\right|. \tag{1.6}$$

In particular we characterize the admissibility of c and $\mathcal{F}$ in terms of the positivity of $\Gamma(x+c+k)\Omega_{\mathcal{F}}^{a,c}(x)\Omega_{\mathcal{F}}^{a,c}(x+1)$ for $x \in \mathbb{N}$ (Lemma 4.2).

In Sections 5 and 6, we construct exceptional Laguerre polynomials by taking limit (in a suitable way) in the exceptional Meixner polynomials when $a \to 1$. We then get (see Theorem 5.2) that for each pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers, the polynomials

$$L_n^{\alpha;\mathcal{F}}(x)=\begin{vmatrix} L^\alpha_{n-u_\mathcal{F}}(x) & (L^\alpha_{n-u_\mathcal{F}})'(x) & \cdots & (L^\alpha_{n-u_\mathcal{F}})^{(k)}(x)\\ \begin{bmatrix} L^\alpha_f(x) & (L^\alpha_f)'(x) & \cdots & (L^\alpha_f)^{(k)}(x) \\ f\in F_1 & & & \end{bmatrix} \\ \begin{bmatrix} L^\alpha_f(-x) & L^{\alpha+1}_f(-x) & \cdots & L^{\alpha+k}_f(-x) \\ f\in F_2 & & & \end{bmatrix}\end{vmatrix}, \tag{1.7}$$

$n\in\sigma_\mathcal{F}$, are eigenfunctions of a second order differential operator.

When $\alpha+1$ and $\mathcal{F}$ are admissible, we prove that $\alpha+k>-1$ and that the determinant Ω^α_F defined by

$$\Omega^\alpha_\mathcal{F}(x)=\begin{vmatrix} \begin{bmatrix} L^\alpha_f(x) & (L^\alpha_f)'(x) & \cdots & (L^\alpha_f)^{(k-1)}(x) \\ f\in F_1 & & & \end{bmatrix} \\ \begin{bmatrix} L^\alpha_f(-x) & L^{\alpha+1}_f(-x) & \cdots & L^{\alpha+k-1}_f(-x) \\ f\in F_2 & & & \end{bmatrix}\end{vmatrix}, \tag{1.8}$$

does not vanish in $[0,+\infty)$. We guess that the converse is also true:

Conjecture 1. *Let $\alpha\neq -1,-2,\ldots$. If $\alpha+k>-1$ and $\Omega^\alpha_\mathcal{F}(x)\neq 0$, $x>0$, then $\alpha+1$ and $\mathcal{F}$ are admissible.*

We also prove that the polynomials $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_\mathcal{F}$, are orthogonal with respect to the positive weight

$$\omega_{\alpha;F}=\frac{x^{\alpha+k}e^{-x}}{(\Omega^\alpha_F(x))^2},\quad x>0.$$

Moreover, they form a complete orthogonal system in $L^2(\omega_{\alpha;\mathcal{F}})$ (see Theorem 6.3).

When c (or $\alpha+1$) and $\mathcal{F}$ are admissible, exceptional Meixner and Laguerre polynomials $m_n^{a,c;\mathcal{F}}$ and $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_\mathcal{F}$, can be constructed in an alternative way. Indeed, consider the involution I in the set of all finite sets of positive integers defined by

$$I(F)=\{1,2,\ldots,f_k\}\setminus\{f_k-f,\ f\in F\}.$$

The set $I(F)$ will be denoted by G: $G=I(F)$. We also write $G=\{g_1,\ldots,g_m\}$ with $g_i<g_{i+1}$ so that m is the number of elements of G and g_m the maximum element of G. We also need the nonnegative integer $v_\mathcal{F}$ defined by

$$v_\mathcal{F}=u_\mathcal{F}+M_{F_1}+1,$$

where M_{F_1} is the maximum element of F_1. For the exceptional Meixner polynomials, we then have ($n\geq v_\mathcal{F}$)

$$m_n^{a,c;\mathcal{F}}(x) = \beta_n\begin{vmatrix} r_0^{\tilde c}(x)m^{a,\tilde c}_{n-v_\mathcal{F}}(x) & r_1^{\tilde c}(x)m^{a,\tilde c}_{n-v_\mathcal{F}}(x-1) & \cdots & r_m^{\tilde c}(x)m^{a,\tilde c}_{n-v_\mathcal{F}}(x-m)\\ \begin{bmatrix} m_g^{a,2-\tilde c}(-x-1) & a m_g^{a,2-\tilde c}(-x) & \cdots & a^m m_g^{a,2-\tilde c}(-x+m-1) \\ g\in G_1 & & & \end{bmatrix} \\ \begin{bmatrix} m_g^{1/a,2-\tilde c}(-x-1) & m_g^{1/a,2-\tilde c}(-x) & \cdots & m_g^{1/a,2-\tilde c}(-x+m-1) \\ g\in G_2 & & & \end{bmatrix}\end{vmatrix}, \tag{1.9}$$

where $\tilde{c} = c + M_{F_1} + M_{F_2} + 2$, $r_j^c(x) = (c + x - m)_{m-j}(x - j + 1)_j$, $j = 0, \dots, m$, and β_n is certain normalization constant.

For the exceptional Laguerre polynomials we have ($n \geq v_{\mathcal{F}}$)

$$L_n^{\alpha;\mathcal{F}}(x) = \gamma_n \begin{vmatrix} x^m L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}}(x) & w_{n-v_{\mathcal{F}}}^{\tilde{\alpha},1} x^{m-1} L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}-1}(x) & \cdots & w_{n-v_{\mathcal{F}}}^{\tilde{\alpha},m} L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}-m}(x) \\ \begin{bmatrix} L_g^{-\tilde{\alpha}}(-x) \\ g \in G_1 \end{bmatrix} & L_g^{-\tilde{\alpha}+1}(-x) & \cdots & L_g^{-\tilde{\alpha}+m}(-x) \\ \begin{bmatrix} L_g^{-\tilde{\alpha}}(x) \\ g \in G_2 \end{bmatrix} & (L_g^{-\tilde{\alpha}})'(x) & \cdots & (L_g^{-\tilde{\alpha}})^{(m)}(x) \end{vmatrix}, \tag{1.10}$$

where $\tilde{\alpha} = \alpha + M_{F_1} + M_{F_2} + 2$, $w_n^{\alpha,j} = j!\binom{n+\alpha}{j}$, and γ_n is certain normalization constant.

We have however computational evidence that shows that both identities (1.9) and (1.10) are true for every pair $\mathcal{F}$ of finite sets of positive integers. Using a physics approach, similar formulas to (1.10) have been introduced by Grandati, Quesne–Grandati and Odake–Sasaki [22,23,33].

Both determinantal definitions (1.2) and (1.9) of the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, automatically imply a couple of factorizations of the second order difference operator $D_{\mathcal{F}}$ in two first order difference operators. Using these factorizations, we prove that the sequence $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, and the operator $D_{\mathcal{F}}$ can be constructed in two different ways using Darboux transforms (see Definition 2.1). The same happens with determinantal definitions of the exceptional Laguerre polynomials $L_n^{\alpha,\mathcal{F}}$ (1.7) and (1.10). This fact agrees with the Gómez–Ullate–Kamran–Milson conjecture and its corresponding discrete version (see [20]): every exceptional and exceptional discrete orthogonal polynomials can be obtained by applying a sequence of Darboux transforms to a classical or classical discrete orthogonal family, respectively.

We would like to include in this introduction other conjecture. There seems to be a very nice invariant property of the polynomial $\Omega_{\mathcal{F}}^{a,c}$ (1.6) underlying the fact that the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, admit both determinantal definitions (1.2) and (1.9): except for a constant (depending on a but neither on x nor on c), $\Omega_{\mathcal{F}}^{a,c}(x)$ remains invariant if we change $\mathcal{F} = (F_1, F_2)$ to $\mathcal{G} = (I(F_1), I(F_2))$, x to $-x$ and c to $-c - M_{F_1} - M_{F_2}$. More precisely:

Conjecture 2.

$$\Omega_{\mathcal{F}}^{a,c}(x) = (-1)^{u_{\mathcal{F}}+k_1} \frac{u_a(\mathcal{F})}{u_a(\mathcal{G})} \Omega_{\mathcal{G}}^{a,-c-M_{F_1}-M_{F_2}}(-x), \tag{1.11}$$

where $u_a(\mathcal{F}) = a^{\binom{k_2}{2} - k_2(k-1)}(1-a)^{k_1 k_2}$.

For the cases when F_1 is formed by consecutive integers and $F_2 = \emptyset$, or $F_1 = \emptyset$ and F_2 is formed by consecutive integers, the conjecture appeared by the first time in [10] and it was proved in [11].

Passing to the limit, the invariant property (1.11) gives

$$\Omega_{\mathcal{F}}^{\alpha}(x) = \epsilon \Omega_{\mathcal{G}}^{-\alpha - M_{F_1} - M_{F_2} - 2}(-x), \tag{1.12}$$

where ϵ is the sign $\epsilon = (-1)^{u_{\mathcal{F}} + k_1 + \sum_{f \in F_1} f + \sum_{g \in G_1} g}$.

Krawtchouk exceptional polynomials can be formally derived from the Meixner case taking into account that $k_n^{a,N}(x) = m_n^{-a,-N+1}(x)$. That is, by setting $a \to -a$, $c \to -N+1$ in the formulas for the polynomials, and changing

$$x \in \mathbb{N}, \quad \frac{a^x \Gamma(x+c)}{x!} \quad \text{to } x = 0, \ldots, N-1, \ \frac{a^x}{\Gamma(N-x)x!}$$

in the orthogonalizing measure.

We finish pointing out that, as explained above, the approach of this paper is the same as in [12] for Charlier and Hermite polynomials. Since we work here with a pair of finite sets of positive integers instead of only one set, and more parameters (two for Meixner and one for Laguerre instead of one for Charlier and zero for Hermite), the computations are technically more involved. Anyway, we will omit those proofs which are too similar to the corresponding ones in [12].

2. Preliminaries

Let μ be a Borel measure (positive or not) on the real line. The n-th moment of μ is defined by $\int_{\mathbb{R}} t^n d\mu(t)$. When μ has finite moments for any $n \in \mathbb{N}$, we can associate to it a bilinear form defined in the linear space of polynomials by

$$\langle p, q \rangle = \int pq d\mu. \tag{2.1}$$

Given an infinite set X of nonnegative integers, we say that the polynomials $p_n, n \in X$, are orthogonal with respect to μ if they are orthogonal with respect to the bilinear form defined by μ; that is, if they satisfy

$$\int p_n p_m d\mu = 0, \quad n \neq m, \ n, m \in X.$$

When $X = \mathbb{N}$ and the degree of p_n is $n, n \geq 0$, we get the usual definition of orthogonal polynomials with respect to a measure. When $X = \mathbb{N}$, orthogonal polynomials with respect to a measure are unique up to multiplication by non null constant. Let us remark that this property is not true when $X \neq \mathbb{N}$. Positive measures μ with finite moments of any order and infinitely many points in their support always have a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$, p_n of degree n (it is enough to apply the Gram–Schmidt orthogonalizing process to $1, x, x^2, \ldots$); in this case the orthogonal polynomials have positive norm: $\langle p_n, p_n \rangle > 0$. Moreover, given a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$ with respect to a measure μ (positive or not) the bilinear form (2.1) can be represented by a positive measure if and only if $\langle p_n, p_n \rangle > 0, n \geq 0$.

When $X = \mathbb{N}$, Favard's Theorem establishes that a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials, p_n of degree n, is orthogonal (with non null norm) with respect to a measure if and only if it satisfies a three term recurrence relation of the form ($p_{-1} = 0$)

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 0,$$

where $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are sequences of real numbers with $a_{n-1}c_n \neq 0, n \geq 1$. If, in addition, $a_{n-1}c_n > 0, n \geq 1$, then the polynomials $(p_n)_{n\in\mathbb{N}}$ are orthogonal with respect to a positive measure with infinitely many points in its support, and the reciprocal is also true. Again, Favard's Theorem is not true for a sequence of orthogonal polynomials $(p_n)_{n\in X}$ when $X \neq \mathbb{N}$. In fact, exceptional orthogonal polynomials do not satisfy a three-term recurrence relation.

Darboux transformations are an important tool for constructing exceptional orthogonal polynomials. We define them next for second order difference and differential operators.

Definition 2.1. Given a system $(T, (\phi_n)_n)$ formed by a second order difference or differential operator T and a sequence $(\phi_n)_n$ of eigenfunctions for T, $T(\phi_n) = \pi_n \phi_n$, by a Darboux transform of the system $(T, (\phi_n)_n)$ we mean the following. For a real number λ, we factorize $T - \lambda Id$ as the product of two first order difference or differential operators $T = BA + \lambda Id$ (Id denotes the identity operator). We then produce a new system consisting in the operator $\hat{T}$, obtained by reversing the order of the factors, $\hat{T} = AB + \lambda Id$, and the sequence of eigenfunctions $\hat{\phi}_n = A(\phi_n)$: $\hat{T}(\hat{\phi}_n) = \pi_n \hat{\phi}_n$. We say that the system $(\hat{T}, (\hat{\phi}_n)_n)$ has been obtained by applying a Darboux transformation with parameter λ to the system $(T, (\phi_n)_n)$.

We will also need the following straightforward lemma.

Lemma 2.2. *Let M be a $(s + 1) \times m$ matrix with $m \geq s + 1$. Write c_i, $i = 1, \ldots, m$, for the columns of M (from left to right). Assume that the consecutive columns c_i, $i = 1, \ldots, s$, of M are linearly independent while for $0 \leq j \leq m - s - 1$ the consecutive columns c_{j+i}, $i = 1, \ldots, s + 1$, are linearly dependent. Then* rank $M = s$.

Given a finite set of numbers $F = \{f_1, \ldots, f_k\}$ we denote by V_F the Vandermonde determinant defined by

$$V_F = \prod_{1=i<j=k} (f_j - f_i). \tag{2.2}$$

Along this paper, we will use some properties of determinate measures. A positive measure μ is determinate if there is not other measure with the same moments as those of μ (see, for instance, [2]). Using moment problem standard techniques, it is easy to prove that if the Fourier transform $H(z)$ of μ, defined by $H(z) = \int e^{-ixz} d\mu(x)$, is an analytic function in the half plane $\Im z < a$, with $a > 0$, then the measure μ is determinate. We also point out that for a determinate measure μ, the linear space of polynomials is always dense in $L^2(\mu)$.

2.1. Christoffel transform

Let μ be a measure (positive or not) and assume that μ has a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$, p_n with degree n and $\langle p_n, p_n \rangle \neq 0$ (as we mentioned above, that always happens if μ is positive, with finite moments and infinitely many points in its support).

Given a finite set F of real numbers, $F = \{f_1, \ldots, f_k\}$, $f_i < f_{i+1}$, we write Φ_n, $n \geq 0$, for the $k \times k$ determinant

$$\Phi_n = |p_{n+j-1}(f_i)|_{i,j=1,\ldots,k}. \tag{2.3}$$

Notice that Φ_n, $n \geq 0$, depends on both, the finite set F and the measure μ. In order to stress this dependence, we sometimes write in this section $\Phi_n^{\mu,F}$ for Φ_n.

Along this section we assume that the set $\Theta_\mu^F = \{n \in \mathbb{N} : \Phi_n^{\mu,F} = 0\}$ is finite. We denote $\theta_\mu^F = \max \Theta_\mu^F$. If $\Theta_\mu^F = \emptyset$ we take $\theta_\mu^F = -1$.

The Christoffel transform of μ associated to the annihilator polynomial $\mathfrak{p}_F$ of F,

$$\mathfrak{p}_F(x) = (x - f_1) \cdots (x - f_k),$$

is the measure defined by $\mu_F = \mathfrak{p}_F \mu$.

Orthogonal polynomials with respect to μ_F can be constructed by means of the formula

$$q_n(x) = \frac{1}{\mathfrak{p}_F(x)} \det \begin{pmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+k}(x) \\ p_n(f_1) & p_{n+1}(f_1) & \cdots & p_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(f_k) & p_{n+1}(f_k) & \cdots & p_{n+k}(f_k) \end{pmatrix}. \tag{2.4}$$

Notice that the degree of q_n is equal to n if and only if $n \notin \Theta_\mu^F$. In that case the leading coefficient λ_n^Q of q_n is equal to $(-1)^k \lambda_{n+k}^P \Phi_n$, where λ_n^P denotes the leading coefficient of p_n.

The next lemma follows easily using [39], Th. 2.5.

Lemma 2.3. *The measure μ_F has a sequence $(q_n)_{n=0}^\infty$, q_n of degree n, of orthogonal polynomials if and only if $\Theta_\mu^F = \emptyset$. In that case, an orthogonal polynomial of degree n with respect to μ_F is given by (2.4) and also $\langle q_n, q_n \rangle_{\mu_F} \neq 0, n \geq 0$. If $\Theta_\mu \neq \emptyset$, the polynomial q_n (2.4) has still degree n for $n \notin \Theta_\mu^F$, and satisfies $\langle q_n, r \rangle_{\mu_F} = 0$ for all polynomials r with degree less than n and $\langle q_n, q_n \rangle_{\mu_F} \neq 0$.*

From (2.4), one can also deduce (see Lemma 2.8 of [12])

$$\langle q_n, q_n \rangle_{\mu_F} = (-1)^k \frac{\lambda_{n+k}^P}{\lambda_n^P} \Phi_n \Phi_{n+1} \langle p_n, p_n \rangle_\mu, \quad n > \theta_\mu^F + 1. \tag{2.5}$$

This identity holds for $n \geq 0$ when $\Theta_\mu = \emptyset$.

2.2. Finite sets and pair of finite sets of positive integers.

For a finite set F of positive integers, we denote $M_F = \max F$, $m_F = \min F$; if $F = \emptyset$, we define $M_F = m_F = -1$.

Consider the set Υ formed by all finite sets of positive integers:

$$\Upsilon = \{F : F \text{ is a finite set of positive integers}\}.$$

We consider the involution I in Υ defined by

$$I(F) = \{1, 2, \ldots, M_F\} \setminus \{M_F - f, f \in F\}. \tag{2.6}$$

The definition of I implies that $I^2 = Id$.

The set $I(F)$ will be denoted by G: $G = I(F)$. Notice that

$$M_F = M_G, \qquad m = M_F - k + 1,$$

where k and m are the number of elements of F and G, respectively.

For a finite set $F = \{f_1, \ldots, f_k\}$, $f_i < f_{i+1}$, of positive integers, we define the number s_F by

$$s_F = \begin{cases} 1, & \text{if } F = \emptyset, \\ k+1, & \text{if } F = \{1, 2, \ldots, k\}, \\ \min\{s \geq 1 : s < f_s\}, & \text{if } F \neq \{1, 2, \ldots, k\}, \end{cases} \tag{2.7}$$

and the set $F_\Downarrow$ of positive integers by

$$F_\Downarrow = \begin{cases} \emptyset, & \text{if } F = \emptyset \text{ or } F = \{1, 2, \ldots, k\}, \\ \{f_{s_F} - s_F, \ldots, f_k - s_F\}, & \text{if } F \neq \{1, 2, \ldots, k\}. \end{cases} \tag{2.8}$$

Notice that if $F \neq \emptyset$ and $s_F > 1$ then

$$F = \{1, \ldots, s_F - 1\} \cup (s_F + F_{\Downarrow}). \tag{2.9}$$

From now on, $\mathcal{F} = (F_1, F_2)$ will denote a pair of finite sets of positive integers. We will write $F_1 = \{f_1^{1\rceil}, \ldots, f_{k_1}^{1\rceil}\}$, $F_2 = \{f_1^{2\rceil}, \ldots, f_{k_2}^{2\rceil}\}$, with $f_i^{j\rceil} < f_{i+1}^{j\rceil}$ (the use of f_i^2 to describe elements of F_2 is confusing because it looks like a square, this is the reason why we use the notation $f_i^{2\rceil}$). Hence k_j is the number of elements of F_j, $j = 1, 2$, and $k = k_1 + k_2$ is the number of elements of $\mathcal{F}$. One of the components of $\mathcal{F}$, but not both, can be the empty set.

We associate to $\mathcal{F}$ the nonnegative integers $u_{\mathcal{F}}$ and $v_{\mathcal{F}}$ and the infinite set of nonnegative integers $\sigma_{\mathcal{F}}$ defined by

$$u_{\mathcal{F}} = \sum_{f \in F_1} f + \sum_{f \in F_2} f - \binom{k_1 + 1}{2} - \binom{k_2}{2}, \tag{2.10}$$

$$v_{\mathcal{F}} = \sum_{f \in F_1} f + \sum_{f \in F_2} f - \binom{k_2}{2} + M_{F_1} - \frac{(k_1 - 1)(k_1 + 2)}{2}, \tag{2.11}$$

$$\sigma_{\mathcal{F}} = \{u_{\mathcal{F}}, u_{\mathcal{F}} + 1, u_{\mathcal{F}} + 2, \ldots\} \setminus \{u_{\mathcal{F}} + f, f \in F_1\}. \tag{2.12}$$

The infinite set $\sigma_{\mathcal{F}}$ will be the set of indices for the exceptional Meixner or Laguerre polynomials associated to $\mathcal{F}$.

Notice that $v_{\mathcal{F}} = u_{\mathcal{F}} + M_{F_1} + 1$; hence $\{v_{\mathcal{F}}, v_{\mathcal{F}} + 1, v_{\mathcal{F}} + 2, \ldots\} \subset \sigma_{\mathcal{F}}$.

For a pair $\mathcal{F} = (F_1, F_2)$ of positive integers we denote by $\mathcal{F}_{j,\{i\}}$, $i = 1, \ldots, k_j$, $j = 1, 2$, and $\mathcal{F}_{\Downarrow}$ the pair of finite sets of positive integers defined by

$$\mathcal{F}_{1,\{i\}} = (F_1 \setminus \{f_i^{1\rceil}\}, F_2), \tag{2.13}$$

$$\mathcal{F}_{2,\{i\}} = (F_1, F_2 \setminus \{f_i^{2\rceil}\}), \tag{2.14}$$

$$\mathcal{F}_{\Downarrow} = ((F_1)_{\Downarrow}, F_2), \tag{2.15}$$

where $(F_1)_{\Downarrow}$ is defined by (2.8). We also define

$$s_{\mathcal{F}} = s_{F_1} \tag{2.16}$$

where the number s_{F_1} is defined by (2.7).

2.3. Admissibility

Using the determinants (1.2) and (1.7), whose entries are Meixner $m_n^{a,c}$ or Laguerre polynomials L_n^{α}, respectively, we will associate to each pair $\mathcal{F}$ of finite sets of positive integers a sequence of polynomials which are eigenfunctions of a second order difference or differential operator, respectively. The more important of these examples are those which, in addition, are orthogonal and complete with respect to a positive measure. The key concept for the existence of such positive measure is that of admissibility.

Let us remind that for Charlier and Hermite polynomials, the admissibility of a finite set F of positive integers is defined as follows.

Definition 2.4. Let F be a finite set of positive integers. Split up the set F, $F = \bigcup_{i=1}^{K} Y_i$, in such a way that $Y_i \cap Y_j = \emptyset$, $i \neq j$, the elements of each Y_i are consecutive integers and $1 + \max(Y_i) <$

$\min Y_{i+1}$, $i = 1, \dots, K-1$. We then say that F is admissible if each Y_i, $i = 1, \dots, K$, has an even number of elements.

It is easy to see that this is equivalent to

$$\prod_{f\in F}(x-f) \geq 0, \quad x \in \mathbb{N}.$$

This implies that the measure $\rho_a^F = \sum_{x=0}^{\infty} \prod_{f\in F}(x-f)a^x/x!\delta_x$ is positive. As shown in [12], Charlier exceptional polynomials are dual of the orthogonal polynomials with respect to this measure.

Given a pair $\mathcal{F}$ of finite sets of positive integers, consider the measure $\rho_{a,c}^{\mathcal{F}}$ defined by (1.4). We show in the next section that Meixner exceptional polynomials are dual of the orthogonal polynomials with respect to this measure. Hence, the admissibility condition in the Meixner case should be equivalent to the positivity of the measure $\rho_{a,c}^{\mathcal{F}}$.

Definition 2.5. Let $\mathcal{F} = (F_1, F_2)$ be a pair of finite sets of positive integers. For a real number $c \neq 0, -1, -2, \dots$, write $\hat{c} = \max\{-[c], 0\}$, where $[c]$ denotes the value of the floor function at c (i.e. $[c] = \max\{s \in \mathbb{Z} : s \leq c\}$). We say that c and $\mathcal{F}$ are admissible if for all $x \in \mathbb{N}$

$$\frac{\prod_{f\in F_1}(x-f)\prod_{f\in F_2}(x+c+f)}{(x+c)_{\hat{c}}} \geq 0. \tag{2.17}$$

Notice that the condition $x \in \mathbb{N}$ can be changed to $x \in \{0, 1, \dots, \max(\max F_1, \hat{c})\}$.

As we wrote in the introduction, this admissibility concept is more involved than the corresponding for exceptional Charlier and Hermite polynomials. Indeed, on the one hand, we have now a pair $\mathcal{F}$ of finite sets instead of a single finite set F. On the other hand, the admissibility also depends on the parameter c of the Meixner polynomials (or on the parameter α of the Laguerre polynomials).

In the following lemma we include some important consequences derived from the admissibility.

Lemma 2.6. *Given a real number $c \neq 0, -1, -2, \dots$, and a pair $\mathcal{F}$ of finite sets of positive integers, we have*

1. *if c and $\mathcal{F}$ are admissible then $c + k > 0$.*
2. *If $c > 0$, then c and $\mathcal{F}$ are admissible if and only if F_1 is admissible (in the sense of Definition 2.4).*
3. *If $F_1 = \emptyset$, c and $\mathcal{F}$ are admissible if and only if $c > 0$.*
4. *If c and $\mathcal{F}$ are admissible then $c + s_{\mathcal{F}}$ and $\mathcal{F}_{\Downarrow}$ are admissible (where the positive integer $s_{\mathcal{F}}$ and the pair $\mathcal{F}_{\Downarrow}$ are defined by (2.16) and (2.15), respectively).*

Proof. Proof of (1). We first point out that

$$\operatorname{sign}((x+c)_{\hat{c}}) = \begin{cases} (-1)^{\hat{c}-x}, & 0 \leq x \leq \hat{c}, \\ 1, & x > \hat{c}. \end{cases} \tag{2.18}$$

Given an l-tuple $A = (a_1, \dots, a_l)$ of non null real numbers, we denote by $n_{\pm}(A)$ the number of sign changes along the elements of A (for instance, if $A = (-\pi, 2, 1, -\sqrt{2}, -1, 1, 1)$ then $n_{\pm}(A) = 3$).

We next prove that given a finite set I of nonnegative integers with elements ordered in increasing size, we have

$$n_{\pm}\left(\left(\prod_{f\in F_2}(x+c+f), x\in I\right)\right)\leq |F_2\cap(\hat{c}-I)|, \tag{2.19}$$

where $|X|$ denotes the number of elements of the finite set X and $\hat{c}-I$ denotes the set $\{\hat{c}-i : i\in I\}$. Indeed, for $a\in I$, write $A_a=\{f\in F_2 : a+c+f<0\}$. Notice that $\prod_{f\in F_2}(a+c+f)$ is positive or negative depending on whether $|A_a|$ is even or odd, respectively. Take now consecutive numbers $x=a, x=a+1\in I$ where $\prod_{f\in F_2}(x+c+f)$ changes its sign. Then $A_{a+1}\neq A_a$ because $|A_{a+1}|$ and $|A_a|$ have different parity. Since $A_{a+1}\subset A_a$, there exists $f_a\in A_a\setminus A_{a+1}$. That is, $a+c+f_a<0<a+c+1+f_a$, or $-a-f_a-1<c<-a-f_a$. In other words $-\hat{c}=-a-f_a$, and then $f_a\in\hat{c}-I$.

Take now $x=a, x=b\in I$, with $a+1<b$, and $i\notin I$ if $a<i<b$, and where $\prod_{f\in F_2}(x+c+f)$ changes its sign. Abusing of notation we write $f_a=\max A_a$. If $a+c+f_a+1>0$, proceeding as before we get $f_a\in\hat{c}-I$ and $f_a\notin A_b$. On the other hand, if $a+c+f_a+1<0$, by the definition of f_a, we conclude that $f_a+1\notin F_2$, and then $\prod_{f\in F_2}(x+c+f)$ does not change its sign from a to $a+1$. In the same way, we have that if $a+c+f_a+2>0$ then $f_a\in\hat{c}-I$ and $f_a\notin A_b$, while if $a+c+f_a+2<0$ then $\prod_{f\in F_2}(x+c+f)$ does not change its sign from a to $a+2$; in particular $b>a+2$. Since $\prod_{f\in F_2}(x+c+f)$ changes its sign from a to b proceeding in the same way, we can conclude that $f_a\in\hat{c}-I$ and $f_a\notin A_b$.

We have then proved that if $\prod_{f\in F_2}(x+c+f)$ changes its sign in two consecutive elements a and b of I then there exists $f_a\in F_2$ satisfying $f_a\in\hat{c}-I$, $f_a\in A_a$ and $f_a\notin A_b$. This implies that (2.19) holds.

Decompose now the set $\{0,1,\ldots,\hat{c}\}$ as follows:

$$\{0,1,\ldots,\hat{c}\}=X_1\cup Y_1\cup X_2\cup Y_2\cup\cdots\cup X_l\cup Y_l,$$

where each X_i, Y_i is formed by consecutive nonnegative integers, $1+\max X_i=\min Y_i$, $1+\max Y_i=\min X_{i+1}$, $X_i\cap F_1=\emptyset$, $Y_i\subset F_1$ and $Y_l=\emptyset$ if $f_{k_1}^{1\rceil}<\hat{c}$ (let us remind that k_1 is the number of elements of F_1 and that $f_{k_1}^{1\rceil}$ is the maximum element of F_1). Write $x_i=|X_i|$, $y_i=|Y_i|$, $i=1,\ldots,l$. Since X_i, Y_i, $i=1,\ldots,l$, are disjoint sets and $Y_i\subset F_1$, we get

$$\hat{c}+1=\sum_{i=1}^{l}(x_i+y_i),\qquad \sum_{i=1}^{l}y_i\leq k_1. \tag{2.20}$$

Notice that the sign of $\prod_{f\in F_1}(x-f)$ is constant in each X_i. Since $(x+c)_{\hat{c}}$ alternates its sign in consecutive numbers of $\{0,1,\ldots\hat{c}\}$ (see (2.18)), (2.17) implies that $\prod_{f\in F_2}(x+c+f)$ changes its sign (x_i-1)-times in X_i. On the other hand, one can carefully check that also $\prod_{f\in F_2}(x+c+f)$ changes its sign between the maximum element of X_i and the minimum element of X_{i+1}. That means that $\prod_{f\in F_2}(x+c+f)$ changes its sign $(-1+\sum_{i=1}^{l}x_i)$-times in $\cup_{i=1}^{l}X_i$. Hence, (2.19) gives

$$\sum_{i=1}^{l}x_i\leq|F_2\cap(\hat{c}-\cup_{i=1}^{l}X_i)|+1\leq k_2+1.$$

(2.20) gives then $\hat{c}+1\leq k_1+k_2+1$, from where we get $c+k>0$.

Proof of (2). It is a straightforward consequence of the following fact: if $c > 0$ then

$$\operatorname{sign}\left(\frac{\prod_{f\in F_1}(x-f)\prod_{f\in F_2}(x+c+f)}{(x+c)_{\hat c}}\right) = \operatorname{sign}\prod_{f\in F_1}(x-f).$$

Proof of (3). Assume $F_1 = \emptyset$, which is an admissible set (in the sense of Definition 2.4). If $c > 0$, using (2) of this lemma we deduce that c and $\mathcal{F}$ are admissible. On the other hand, if c and $\mathcal{F}$ are admissible we have ($F_1 = \emptyset$ implies that $\prod_{f\in F_1}(x-f) = 1, x \ge 0$) from (2.17) that

$$\operatorname{sign}\prod_{f\in F_2}(x+c+f) = \operatorname{sign}(x+c)_{\hat c}, \quad x \ge 0.$$

If $c < 0$, then $\hat c > 0$. Take $x_{\hat c} = \hat c - 1 \ge 0$. Hence $\operatorname{sign}(x_{\hat c}+c)_{\hat c} = -1$ and so $\operatorname{sign}\prod_{f\in F_2}(x_{\hat c} + c + f) = -1$. Since $-\hat c - 1 < c < \hat c$, we have $-1 < x_{\hat c} + c$ and then

$$\{f \in F_2 : x_{\hat c} + c + f < 0\} \subset \{f \in F_2 : -1 + f < 0\} = \emptyset.$$

Hence $\operatorname{sign}\prod_{f\in F_2}(x_{\hat c} + c + f) = 1$, which is a contradiction.
Proof of (4). Write

$$H^c_{\mathcal{F}}(x) = \frac{\prod_{f\in F_1}(x-f)\prod_{f\in F_2}(x+c+f)}{(x+c)_{\hat c}}.$$

Since c and $\mathcal{F}$ are admissible, we have $H^c_{\mathcal{F}}(x) \ge 0, x \ge 0$. On the other hand, since $\mathcal{F}_\Downarrow = \{(F_1)_\Downarrow, F_2\}$, we have to prove that

$$H^{c+s_{\mathcal{F}}}_{\mathcal{F}_\Downarrow}(x) = \frac{\prod_{f\in(F_1)_\Downarrow}(x-f)\prod_{f\in F_2}(x+c+s_{\mathcal{F}}+f)}{(x+c+s_{\mathcal{F}})_{\hat c+s_{\mathcal{F}}}} \ge 0, \quad x \ge 0.$$

Using the definition of $(F_1)_\Downarrow$ (2.8) and (2.9), we have for $x \ge 0$ that

$$H^{c+s_{\mathcal{F}}}_{\mathcal{F}_\Downarrow}(x) = \frac{H^c_{\mathcal{F}}(x+s_{\mathcal{F}})}{\prod_{j=1}^{s_{\mathcal{F}}-1}(x+s_{\mathcal{F}}-j)(x+s_{\mathcal{F}}+c+\hat c)_{s_{\mathcal{F}}}} \ge 0. \quad \square$$

2.4. Meixner and Laguerre polynomials

We include here basic definitions and facts about Meixner and Laguerre polynomials, which we will need in the following sections.

For $a \ne 0, 1$ we write $(m_n^{a,c})_n$ for the sequence of Meixner polynomials defined by

$$m_n^{a,c}(x) = \frac{a^n}{(1-a)^n}\sum_{j=0}^{n} a^{-j}\binom{x}{j}\binom{-x-c}{n-j} \tag{2.21}$$

(we have taken a slightly different normalization from the one used in [5], pp. 175–177 from where the next formulas can be easily derived; see also [24], pp, 234–7 or [29], ch. 2). Meixner polynomials are eigenfunctions of the following second order difference operator

$$D_{a,c} = \frac{x\mathfrak{s}_{-1} - [(1+a)x + ac]\mathfrak{s}_0 + a(x+c)\mathfrak{s}_1}{a-1}, \qquad D_{a,c}(m_n^{a,c}) = n m_n^{a,c}, \quad n \ge 0,$$

where $\mathfrak{s}_l$ denotes the shift operator $\mathfrak{s}_l(f) = f(x+l)$. When $a \neq 0, 1$, they satisfy the following three term recurrence formula ($m_{-1} = 0$)

$$xm_n = (n+1)m_{n+1} - \frac{(a+1)n + ac}{a-1} m_n + \frac{a(n+c-1)}{(a-1)^2} m_{n-1}, \quad n \geq 0 \tag{2.22}$$

(to simplify the notation we remove the parameters in some formulas). Hence, for $a \neq 0, 1$ and $c \neq 0, -1, -2, \ldots$, they are always orthogonal with respect to a moment functional $\rho_{a,c}$, which we normalize by taking $\langle \rho_{a,c}, 1 \rangle = \Gamma(c)/(1-a)^c$. For $0 < |a| < 1$ and $c \neq 0, -1, -2, \ldots$, we have

$$\rho_{a,c} = \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c)}{x!} \delta_x,$$

and

$$\langle m_n^{a,c}, m_n^{a,c} \rangle = \frac{a^n \Gamma(n+c)}{n!(1-a)^{2n+c}}. \tag{2.23}$$

The moment functional $\rho_{a,c}$ can be represented by a positive measure only when $0 < a < 1$ and $c > 0$.

Meixner polynomials satisfy the following identities ($n, m \in \mathbb{N}$, $x \in \mathbb{R}$)

$$m_n^{a,c}(x+1) - m_n^{a,c}(x) = m_{n-1}^{a,c+1}(x), \tag{2.24}$$

$$m_n^{1/a,c}(x+1) - am_n^{1/a,c}(x) = (1-a)m_n^{1/a,c+1}(x), \tag{2.25}$$

$$a^{m-n} n!(1+c)_{m-1} m_n^{a,c}(m) = (a-1)^{m-n} m!(1+c)_{n-1} m_m^{a,c}(n), \tag{2.26}$$

$$m_n^{a,c}(x) = (-1)^n m_n^{1/a,c}(-x-c). \tag{2.27}$$

For $\alpha \in \mathbb{R}$, we write $(L_n^\alpha)_n$ for the sequence of Laguerre polynomials

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(-x)^j}{j!} \binom{n+\alpha}{n-j} \tag{2.28}$$

(that and the next formulas can be found in [14], vol. II, pp. 188–192; see also [24], pp. 241–244).

They satisfy the three-term recurrence formula ($L_{-1}^\alpha = 0$)

$$xL_n^\alpha = -(n+1)L_{n+1}^\alpha + (2n+\alpha+1)L_n^\alpha - (n+\alpha)L_{n-1}^\alpha.$$

Hence, for $\alpha \neq -1, -2, \ldots$, they are orthogonal with respect to a measure $\mu_\alpha = \mu_\alpha(x)dx$. This measure is positive only when $\alpha > -1$ and then

$$\mu_\alpha(x) = x^\alpha e^{-x}, \quad x > 0.$$

The Laguerre polynomials are eigenfunctions of the following second-order differential operator

$$D_\alpha = -x\left(\frac{d}{dx}\right)^2 - (\alpha+1-x)\frac{d}{dx}, \qquad D_\alpha(L_n^\alpha) = nL_n^\alpha, \quad n \geq 0. \tag{2.29}$$

We will also use the following formulas

$$\left(L_n^{\alpha}\right)' = -L_{n-1}^{\alpha+1}, \tag{2.30}$$

$$L_n^{\alpha} = L_{n-1}^{\alpha} + L_n^{\alpha-1}. \tag{2.31}$$

One can obtain Laguerre polynomials from Meixner polynomials using the limit

$$\lim_{a\to 1}(a-1)^n m_n^{a,c}\left(\frac{x}{1-a}\right) = L_n^{c-1}(x) \tag{2.32}$$

see [24], p. 243 (take into account that we are using for the Meixner polynomials a different normalization to that in [24]). The previous limit is uniform in compact sets of $\mathbb{C}$.

3. Constructing polynomials which are eigenfunctions of second order difference operators

As in Section 2.2, $\mathcal{F} = (F_1, F_2)$ will denote a pair of finite sets of positive integers. We will write $F_i = \{f_1^{i\rceil}, \ldots, f_{k_i}^{i\rceil}\}$, with $f_j^{i\rceil} < f_{j+1}^{i\rceil}$, $i = 1, 2$. Hence k_i is the number of elements of F_i and $f_{k_i}^{i\rceil}$ is the maximum element of F_i.

For real numbers a, c, with $a \neq 0, 1$ and $c \neq 0, -1, -2, \ldots$, we associate to each pair $\mathcal{F}$ the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, displayed in the following definition. It turns out that these polynomials are always eigenfunctions of a second order difference operator with rational coefficients. We call them exceptional Meixner polynomials when, in addition, they are orthogonal and complete with respect to a positive measure (this will happen as long as c and the pair $\mathcal{F}$ are admissible; see definition (2.17) in the previous section).

Definition 3.1. Let $\mathcal{F} = (F_1, F_2)$ be a pair of finite sets of positive integers. We define the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, as

$$m_n^{a,c;\mathcal{F}}(x) = \begin{vmatrix} m_{n-u_{\mathcal{F}}}^{a,c}(x) & m_{n-u_{\mathcal{F}}}^{a,c}(x+1) & \cdots & m_{n-u_{\mathcal{F}}}^{a,c}(x+k) \\ \begin{bmatrix} m_f^{a,c}(x) \\ f \in F_1 \end{bmatrix} & m_f^{a,c}(x+1) & \cdots & \begin{matrix} m_f^{a,c}(x+k) \end{matrix} \\ \begin{bmatrix} m_f^{1/a,c}(x) \\ f \in F_2 \end{bmatrix} & m_f^{1/a,c}(x+1)/a & \cdots & m_f^{1/a,c}(x+k)/a^k \end{vmatrix} \tag{3.1}$$

where the number $u_{\mathcal{F}}$ and the infinite set of nonnegative integers $\sigma_{\mathcal{F}}$ are defined by (2.10) and (2.12), respectively.

The determinant (3.1) should be understood as explained in the Introduction (see (1.3)).

To simplify the notation, we will sometimes write $m_n^{\mathcal{F}} = m_n^{a,c;\mathcal{F}}$.

Using Lemma 3.4 of [13], we deduce that $m_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, is a polynomial of degree n with leading coefficient equal to

$$(-1)^{k_2(k_1+1)} \frac{(a-1)^{k_2(k_1+1)} V_{F_1} V_{F_2} \prod_{f\in F_1}(f-n+u_{\mathcal{F}})}{a^{k_2k_1+\binom{k_2+1}{2}}(n-u_{\mathcal{F}})! \prod_{f\in F_1} f! \prod_{f\in F_2} f!}, \tag{3.2}$$

where V_F is the Vandermonde determinant (2.2). With the convention that $m_n^{a,c} = 0$ for $n < 0$, the determinant (3.1) defines a polynomial for any $n \geq 0$, but for $n \notin \sigma_{\mathcal{F}}$ we have $m_n^{\mathcal{F}} = 0$.

Combining columns in (3.1) and taking into account (2.24) and (2.25), we have the alternative definition

$$m_n^{a,c;\mathcal{F}}(x) = \begin{vmatrix} m_{n-u_{\mathcal{F}}}^{a,c}(x) & m_{n-u_{\mathcal{F}}-1}^{a,c+1}(x) & \cdots & m_{n-u_{\mathcal{F}}-k}^{a,c+k}(x) \\ \begin{bmatrix} m_f^{a,c}(x) \\ f \in F_1 \end{bmatrix} & m_{f-1}^{a,c+1}(x) & \cdots & m_{f-k}^{a,c+k}(x) \\ \begin{bmatrix} m_f^{1/a,c}(x) \\ f \in F_2 \end{bmatrix} & \frac{1-a}{a} m_f^{1/a,c+1}(x) & \cdots & \frac{(1-a)^k}{a^k} m_f^{1/a,c+k}(x) \end{vmatrix}. \tag{3.3}$$

The polynomials $m_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are strongly related by duality with the polynomials $q_n^{\mathcal{F}}$, $n \geq 0$, defined by

$$q_n^{\mathcal{F}}(x) = \frac{\begin{vmatrix} m_n^{a,c}(x-u_{\mathcal{F}}) & m_{n+1}^{a,c}(x-u_{\mathcal{F}}) & \cdots & m_{n+k}^{a,c}(x-u_{\mathcal{F}}) \\ \begin{bmatrix} m_n^{a,c}(f) \\ f \in F_1 \end{bmatrix} & m_{n+1}^{a,c}(f) & \cdots & m_{n+k}^{a,c}(f) \\ \begin{bmatrix} m_n^{1/a,c}(f) \\ f \in F_2 \end{bmatrix} & -m_{n+1}^{1/a,c}(f) & \cdots & (-1)^k m_{n+k}^{1/a,c}(f) \end{vmatrix}}{(-1)^{nk_2} \prod_{f \in F_1}(x-f-u_{\mathcal{F}}) \prod_{f \in F_2}(x+c+f-u_{\mathcal{F}})}. \tag{3.4}$$

Lemma 3.2. *If u is a nonnegative integer and $v \in \sigma_{\mathcal{F}}$, then*

$$q_u^{\mathcal{F}}(v) = \kappa \xi_u \zeta_v m_v^{\mathcal{F}}(u), \tag{3.5}$$

where

$$\kappa = \frac{(-1)^{\sum_{f\in F_2} f} a^{k_2(k_1+1)+\sum_{f \in F_2} f} \prod_{f\in F_1} f! \prod_{f \in F_2} f!}{(a-1)^{k_2(k_1+1)} \prod_{f \in F_1}(1+c)_{f-1} \prod_{f\in F_2}(1+c)_{f-1}},$$

$$\xi_u = \frac{a^{(k_1+1)u} \prod_{i=0}^{k}(1+c)_{u+i-1}}{(a-1)^{(k+1)u} \prod_{i=0}^{k}(u+i)!},$$

$$\zeta_v = \frac{(a-1)^v (v-u_{\mathcal{F}})!}{a^v (1+c)_{v-u_{\mathcal{F}}-1} \prod_{f\in F_1}(v-f-u_{\mathcal{F}}) \prod_{f \in F_2}(v+c+f-u_{\mathcal{F}})}.$$

Proof. It is an easy consequence of the duality (2.26) for the Meixner polynomials. □

We now prove that the polynomials $m_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are eigenfunctions of a second order difference operator with rational coefficients. To establish the result in full, we need some more notations. We denote by $\Omega_{\mathcal{F}}^{a,c}(x)$ and $\Lambda_{\mathcal{F}}^{a,c}(x)$ the polynomials

$$\Omega_{\mathcal{F}}^{a,c}(x)=\left|\begin{matrix}\begin{bmatrix} m_f^{a,c}(x) & m_f^{a,c}(x+1) & \cdots & m_f^{a,c}(x+k-1)\\ f\in F_1 & & & \end{bmatrix}\\ \begin{bmatrix} m_f^{1/a,c}(x) & m_f^{1/a,c}(x+1)/a & \cdots & m_f^{1/a,c}(x+k-1)/a^{k-1}\\ f\in F_2 & & & \end{bmatrix}\end{matrix}\right|, \tag{3.6}$$

$$\Lambda_{\mathcal{F}}^{a,c}(x)=\left|\begin{matrix}\begin{bmatrix} m_f^{a,c}(x) & m_f^{a,c}(x+1) & \cdots & m_f^{a,c}(x+k-2) & m_f^{a,c}(x+k)\\ f\in F_1 & & & & \end{bmatrix}\\ \begin{bmatrix} m_f^{1/a,c}(x) & \dfrac{m_f^{1/a,c}(x+1)}{a} & \cdots & \dfrac{m_f^{1/a,c}(x+k-2)}{a^{k-2}} & \dfrac{m_f^{1/a,c}(x+k)}{a^k}\\ f\in F_2 & & & & \end{bmatrix}\end{matrix}\right|. \tag{3.7}$$

To simplify the notation we sometimes write $\Omega_{\mathcal{F}}=\Omega_{\mathcal{F}}^{a,c}$, $\Lambda_{\mathcal{F}}=\Lambda_{\mathcal{F}}^{a,c}$. Using Lemma 3.4 of [13], we deduce that the degree of both $\Omega_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$ is $u_{\mathcal{F}}+k_1$. Moreover, the leading coefficient of $\Omega_{\mathcal{F}}$ is

$$\frac{V_{F_1}V_{F_2}a^{\binom{k_2}{2}-k_2(k-1)}(a-1)^{k_1k_2}}{\prod_{f\in F_1}f!\prod_{f\in F_2}f!}.$$

As for $m_n^{\mathcal{F}}$ (see (3.3)), we have for $\Omega_{\mathcal{F}}$ the following alternative definition

$$\Omega_{\mathcal{F}}(x)=\left|\begin{matrix}\begin{bmatrix} m_f^{a,c}(x) & m_{f-1}^{a,c+1}(x) & \cdots & m_{f-k+1}^{a,c+k-1}(x)\\ f\in F_1 & & & \end{bmatrix}\\ \begin{bmatrix} m_f^{1/a,c}(x) & \frac{1-a}{a}m_f^{1/a,c+1}(x) & \cdots & \frac{(1-a)^k}{a^k}m_f^{1/a,c+k}(x)\\ f\in F_2 & & & \end{bmatrix}\end{matrix}\right|. \tag{3.8}$$

From here and (3.3), it is easy to deduce that

$$m_{u_{\mathcal{F}}}^{a,c;\mathcal{F}}(x)=\left(\frac{1-a}{a}\right)^{s_{\mathcal{F}}k_2}\Omega_{\mathcal{F}_{\Downarrow}}^{a,c+s_{\mathcal{F}}}(x), \tag{3.9}$$

where the positive integer $s_{\mathcal{F}}$ and the pair $\mathcal{F}_{\Downarrow}$ of finite sets of positive integers are defined by (2.16) and (2.15), respectively.

We also need the determinants $\Phi_n^{\mathcal{F}}$ and $\Psi_n^{\mathcal{F}}$, $n\geq 0$, defined by

$$\Phi_n^{\mathcal{F}}=(-1)^{nk_2}\left|\begin{matrix}\begin{bmatrix} m_n^{a,c}(f) & m_{n+1}^{a,c}(f) & \cdots & m_{n+k-1}^{a,c}(f)\\ f\in F_1 & & & \end{bmatrix}\\ \begin{bmatrix} m_n^{1/a,c}(f) & -m_{n+1}^{1/a,c}(f) & \cdots & (-1)^{k-1}m_{n+k-1}^{1/a,c}(f)\\ f\in F_2 & & & \end{bmatrix}\end{matrix}\right|, \tag{3.10}$$

$$\Psi_n^{F}=(-1)^{nk_2}\times\left|\begin{matrix}\begin{bmatrix} m_n^{a,c}(f) & m_{n+1}^{a,c}(f) & \cdots & m_{n+k-2}^{a,c}(f) & m_{n+k}^{a,c}(f)\\ f\in F_1 & & & & \end{bmatrix}\\ \begin{bmatrix} m_n^{1/a,c}(f) & -m_{n+1}^{1/a,c}(f) & \cdots & (-1)^{k-2}m_{n+k-2}^{1/a,c}(f) & (-1)^k m_{n+k}^{1/a,c}(f)\\ f\in F_2 & & & & \end{bmatrix}\end{matrix}\right|. \tag{3.11}$$

Using the duality (2.26), we have

$$\Omega_{\mathcal{F}}(n) = \left(\frac{a}{a-1}\right)^{u_{\mathcal{F}}+n+k} \frac{(1+c)_{n+k-1}}{(n+k)!\kappa\xi_n} \Phi_n^{\mathcal{F}}, \tag{3.12}$$

$$\Lambda_{\mathcal{F}}(n) = \left(\frac{a}{a-1}\right)^{u_{\mathcal{F}}+n+k-1} \frac{(1+c)_{n+k-2}}{(n+k-1)!\kappa\xi_n} \Psi_n^{\mathcal{F}}. \tag{3.13}$$

Taking into account (2.27) and according to Lemma 2.3, as long as $\Phi_n^{\mathcal{F}} \neq 0, n \geq 0$, the polynomials $q_n^{\mathcal{F}}$, $n \geq 0$, are orthogonal with respect to the measure

$$\rho_{a,c}^{\mathcal{F}} = \sum_{x=u_{\mathcal{F}}}^{\infty} \prod_{f\in F_1} (x-f-u_{\mathcal{F}}) \prod_{f\in F_2} (x+c+f-u_{\mathcal{F}}) \frac{a^{x-u_{\mathcal{F}}}\Gamma(x+c-u_{\mathcal{F}})}{(x-u_{\mathcal{F}})!}\delta_x. \tag{3.14}$$

The measure $\rho_{a,c}^{\mathcal{F}}$ is shifted by $u_{\mathcal{F}}$ so that the polynomials $m_n^{a,c;\mathcal{F}}$ (3.1) have degree n. Indeed, because of this shift in the measure, the Meixner polynomials in the definition of the orthogonal polynomials $q_n^{\mathcal{F}}$, $n \geq 0$, (see (3.4)) have also to be shifted by $u_{\mathcal{F}}$. The duality between $q_n^{\mathcal{F}}$ and $m_n^{a,c;\mathcal{F}}$ changes the shift from the variable x to the index n, and then the definition (3.1) produces polynomials of degree n.

Notice that the measure $\rho_{a,c}^{\mathcal{F}}$ is supported in the infinite set of nonnegative integers $\sigma_{\mathcal{F}}$ (2.12).

Theorem 3.3. *Let $\mathcal{F} = (F_1, F_2)$ be a pair of finite sets of positive integers. Then the polynomials $m_n^{\mathcal{F}}$ (3.1), $n \in \sigma_{\mathcal{F}}$, are common eigenfunctions of the second order difference operator*

$$D_{\mathcal{F}} = h_{-1}(x)\mathfrak{s}_{-1} + h_0(x)\mathfrak{s}_0 + h_1(x)\mathfrak{s}_1, \tag{3.15}$$

where

$$h_{-1}(x) = \frac{x\Omega_{\mathcal{F}}(x+1)}{(a-1)\Omega_{\mathcal{F}}(x)}, \tag{3.16}$$

$$h_0(x) = -\frac{(1+a)(x+k)+ac}{a-1} + u_{\mathcal{F}} + \Delta\left(\frac{a(x+c+k-1)\Lambda_{\mathcal{F}}(x)}{(a-1)\Omega_{\mathcal{F}}(x)}\right), \tag{3.17}$$

$$h_1(x) = \frac{a(x+c+k)\Omega_{\mathcal{F}}(x)}{(a-1)\Omega_{\mathcal{F}}(x+1)}, \tag{3.18}$$

and Δ denotes the first order difference operator $\Delta f = f(x+1) - f(x)$. Moreover $D_{\mathcal{F}}(m_n^{\mathcal{F}}) = nm_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$.

Proof. The proof is similar to that of Theorem 3.3 in [12] but using here the three term recurrence relation for the Meixner polynomials (2.22) and the dualities (3.5), (3.12) and (3.13). □

The determinantal definition (3.1) of the polynomials $m_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, automatically implies a factorization for the corresponding difference operator $D_{\mathcal{F}}$ (3.15) in two difference operators of order 1. This is a consequence of the Sylvester identity (see [15], p. 32, or [12], Lemma 2.1). This can be done by choosing one of the components of $\mathcal{F} = (F_1, F_2)$ and removing one element in the chosen component. An iteration of this procedure shows that the polynomials $m_n^{\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, and the corresponding difference operator $D_{\mathcal{F}}$ can be constructed by applying a sequence of at most k Darboux transforms (see Definition 2.1) to the Meixner system (where k is the number of elements of $\mathcal{F}$). We display the details in the following lemma, where we first remove one element of the component F_1 of $\mathcal{F}$, and then one element of the component F_2 of $\mathcal{F}$ (notice that the formulas cannot be obtained by a simple interchange of the F_1 and F_2 indices). The proof proceeds in the same way as the proof of Lemma 3.6 in [12] and it is omitted.

Lemma 3.4. *Let $\mathcal{F} = (F_1, F_2)$ be a pair of finite sets of positive integers.*

- *Assume $F_1 \neq \emptyset$. We define the first order difference operators $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ as*

$$A_{\mathcal{F}} = \frac{\Omega_{\mathcal{F}}(x+1)}{\Omega_{\mathcal{F}_{1,\{k_1\}}}(x+1)}\mathfrak{s}_0 - \frac{\Omega_{\mathcal{F}}(x)}{\Omega_{\mathcal{F}_{1,\{k_1\}}}(x+1)}\mathfrak{s}_1, \tag{3.19}$$

$$B_{\mathcal{F}} = \frac{x\Omega_{\mathcal{F}_{1,\{k_1\}}}(x+1)}{(a-1)\Omega_{\mathcal{F}}(x)}\mathfrak{s}_{-1} - \frac{a(x+c+k)\Omega_{\mathcal{F}_{1,\{k_1\}}}(x)}{(a-1)\Omega_{\mathcal{F}}(x)}\mathfrak{s}_0, \tag{3.20}$$

where k_1 is the number of elements of F_1 and the pair $\mathcal{F}_{1,\{k_1\}}$ is defined by (2.14). Then $m_n^{\mathcal{F}}(x) = A_{\mathcal{F}}(m^{\mathcal{F}_{1,\{k_1\}}}_{n-f^{1\rceil}_{k_1}+k_1})(x)$, $n \in \sigma_{\mathcal{F}}$. Moreover

$$D_{\mathcal{F}_{1,\{k_1\}}} = B_{\mathcal{F}}A_{\mathcal{F}} + (f^{1\rceil}_{k_1} + u_{\mathcal{F}_{1,\{k_1\}}})Id,$$

$$D_{\mathcal{F}} = A_{\mathcal{F}}B_{\mathcal{F}} + (f^{1\rceil}_{k_1} + u_{\mathcal{F}})Id.$$

In other words, the system $(D_F, (m_n^{\mathcal{F}})_{n\in\sigma_{\mathcal{F}}})$ can be obtained by applying a Darboux transform to the system $(D_{\mathcal{F}_{1,\{k_1\}}}, (m_n^{\mathcal{F}_{1,\{k_1\}}})_{n\in\sigma_{\mathcal{F}_{1,\{k_1\}}}})$.

- *Assume $F_2 \neq \emptyset$. We define the first order difference operators $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ as*

$$A_{\mathcal{F}} = \frac{\Omega_{\mathcal{F}}(x+1)}{a\Omega_{\mathcal{F}_{2,\{k_2\}}}(x+1)}\mathfrak{s}_0 - \frac{\Omega_{\mathcal{F}}(x)}{\Omega_{\mathcal{F}_{2,\{k_2\}}}(x+1)}\mathfrak{s}_1, \tag{3.21}$$

$$B_{\mathcal{F}} = \frac{ax\Omega_{\mathcal{F}_{2,\{k_2\}}}(x+1)}{(a-1)\Omega_{\mathcal{F}}(x)}\mathfrak{s}_{-1} - \frac{a(x+c+k)\Omega_{\mathcal{F}_{2,\{k_2\}}}(x)}{(a-1)\Omega_{\mathcal{F}}(x)}\mathfrak{s}_0, \tag{3.22}$$

where k_2 is the number of elements of F_2 and the pair $\mathcal{F}_{2,\{k_2\}}$ is defined by (2.14). Then $m_n^{\mathcal{F}}(x) = A_{\mathcal{F}}(m^{\mathcal{F}_{2,\{k_2\}}}_{n-f^{2\rceil}_{k_2}+k_2-1})(x)$, $n \in \sigma_{\mathcal{F}}$. Moreover

$$D_{\mathcal{F}_{2,\{k_2\}}} = B_{\mathcal{F}}A_{\mathcal{F}} - (c + f^{2\rceil}_{k_2} - u_{\mathcal{F}_{2,\{k_2\}}})Id,$$

$$D_{\mathcal{F}} = A_{\mathcal{F}}B_{\mathcal{F}} - (c + f^{2\rceil}_{k_2} - u_{\mathcal{F}})Id.$$

In other words, the system $(D_F, (m_n^{\mathcal{F}})_{n\in\sigma_{\mathcal{F}}})$ can be obtained by applying a Darboux transform to the system $(D_{\mathcal{F}_{2,\{k_2\}}}, (m_n^{\mathcal{F}_{2,\{k_2\}}})_{n\in\sigma_{\mathcal{F}_{2,\{k_2\}}}})$.

Analogous factorization can be obtained by removing instead of $f^{1\rceil}_{k_1}$ or $f^{2\rceil}_{k_2}$ any other element of F_1 or F_2, respectively.

4. Exceptional Meixner polynomials

Given real numbers a, c, with $a \neq 0, 1$ and $c \neq 0, -1, -2, \ldots,$ in the previous section we have associated to each pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, which are always eigenfunctions of a second order difference operator with rational coefficients. We are interested in the cases when, in addition, those polynomials are orthogonal and complete with respect to a positive measure.

Definition 4.1. The polynomials $m_n^{a,c;\mathcal{F}}, n \in \sigma_{\mathcal{F}}$, defined by (3.1) are called exceptional Meixner polynomials, if they are orthogonal and complete with respect to a positive measure.

As we point out in Section 2.3, the key concept for the construction of exceptional Meixner polynomials is that of admissibility (see Definition 2.5). The admissibility of c and $\mathcal{F}$ can also be characterized in terms of the measure $\rho_{a,c}^{\mathcal{F}}$ (3.14) and the sign of the Casorati polynomial $\Omega_{\mathcal{F}}^{a,c}$ in $\mathbb{N}$.

Lemma 4.2. *Given real numbers a, c, with $0 < a < 1$ and $c \neq 0, -1, -2, \ldots$, and a pair $\mathcal{F}$ of finite sets of positive integers, the following conditions are equivalent.*

1. *The measure $\rho_{a,c}^{\mathcal{F}}$ (3.14) is positive.*
2. *c and $\mathcal{F}$ are admissible.*
3. *$\Gamma(n+c+k)\Omega_{\mathcal{F}}^{a,c}(n)\Omega_{\mathcal{F}}^{a,c}(n+1) > 0$ for all nonnegative integer n, where the polynomial $\Omega_{\mathcal{F}}^{a,c}$ is defined by (3.6).*

Proof. As in Section 2.3, write $\hat{c} = \max\{-[c], 0\}$. We then have

$$\Gamma(x+c-u_{\mathcal{F}}) = \frac{\Gamma(x+c+\hat{c}-u_{\mathcal{F}})}{(x+c-u_{\mathcal{F}})_{\hat{c}}}.$$

Since $x+c+\hat{c}-u_{\mathcal{F}} > 0$, for $x \geq u_{\mathcal{F}}$, the equivalence between (1) and (2) is an easy consequence of the definitions of admissibility (2.17) and of the measure $\rho_{a,c}^{\mathcal{F}}$.

We now prove the equivalence between (1) and (3).

(1) $\Rightarrow$ (3). Since the measure $\rho_{a,c}^{\mathcal{F}}$ is positive, the polynomials $(q_n^{\mathcal{F}})_n$ (3.4) are orthogonal with respect to the measure $\rho_{a,c}^{\mathcal{F}}$ and have a positive L^2-norm. According to (2.5), we have

$$\langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}} \rangle = \frac{(-1)^k n!}{(n+k)!} \langle m_n^{a,c}, m_n^{a,c} \rangle \Phi_n^{\mathcal{F}} \Phi_{n+1}^{\mathcal{F}} = \frac{(-1)^k a^n \Gamma(n+c)}{(1-a)^{2n+c}(n+k)!} \Phi_n^{\mathcal{F}} \Phi_{n+1}^{\mathcal{F}}. \tag{4.1}$$

We deduce then that $(-1)^k \Gamma(n+c)\Phi_n^{\mathcal{F}}\Phi_{n+1}^{\mathcal{F}} > 0$ for all n. Using the duality (3.12) and the definition of ξ_n in Lemma 3.2, we conclude that the sign of $(-1)^k\Gamma(n+c)\Phi_n^{\mathcal{F}}\Phi_{n+1}^{\mathcal{F}}$ is equal to the sign of $\Gamma(n+c+k)\Omega_{\mathcal{F}}^{a,c}(n)\Omega_{\mathcal{F}}^{a,c}(n+1)$. This proves (3).

(3) $\Rightarrow$ (1). Using Lemma 2.3, the duality (3.12), the definition of ξ_n in Lemma 3.2 and proceeding as before, we conclude that the polynomials $(q_n^{\mathcal{F}})_n$ are orthogonal with respect to $\rho_{a,c}^{\mathcal{F}}$ and have a positive L^2-norm. This implies that there exists a positive measure μ with respect to which the polynomials $(q_n^{\mathcal{F}})_n$ are orthogonal. Taking into account that the Fourier transform $H(z)$ of $\rho_{a,c}^{\mathcal{F}}$, defined by $H(z) = \int e^{-ixz} d\rho_{a,c}^{\mathcal{F}}(x)$, is an analytic function in the half plane $\Im z < -\log a$, and using moment problem standard techniques (see the last comment in the Preliminaries of this paper), it is not difficult to prove that μ has to be equal to $\rho_{a,c}^{\mathcal{F}}$. Hence the measure $\rho_{a,c}^{\mathcal{F}}$ is positive. $\square$

According to part 1 of Lemmas 2.6 and 4.2, if c and $\mathcal{F}$ are admissible, we have $c+k > 0$ and $\Gamma(n+c+k)\Omega_{\mathcal{F}}^{a,c}(n)\Omega_{\mathcal{F}}^{a,c}(n+1) > 0$, for all $n \in \mathbb{N}$. One can then deduce that if c and $\mathcal{F}$ are admissible, then $\Omega_{\mathcal{F}}^{a,c}(n)\Omega_{\mathcal{F}}^{a,c}(n+1) > 0$, for all $n \in \mathbb{N}$. We point out that the converse is not true. Indeed, take $a = 1/2$, $c = -7/2$, $F_1 = \{1\}$, $F_2 = \emptyset$. A straightforward computation gives

$$\Omega_{\mathcal{F}}^{a,c}(n)\Omega_{\mathcal{F}}^{a,c}(n+1) = \frac{(2n+7)(2n+9)}{4} > 0, \quad n \in \mathbb{N}.$$

However, it is easy to see that c and $\mathcal{F}$ are not admissible ((2.17) is negative for $x = 0, 3$).

In the two following theorems we prove that when c and $\mathcal{F}$ are admissible the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are orthogonal and complete with respect to a positive measure.

Theorem 4.3. *Let $\mathcal{F}$ be a pair of finite sets of positive integers satisfying that $\Omega_{\mathcal{F}}^{a,c}(n) \neq 0$ for all nonnegative integers n. Assume $-1 < a < 1, a \neq 0$ and $c \neq 0, -1, -2, \ldots$. Then the polynomials $m_n^{a,c,\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are orthogonal with respect to the (possibly signed) discrete measure*

$$\omega_{a,c}^{\mathcal{F}} = \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c+k)}{x! \Omega_{\mathcal{F}}^{a,c}(x) \Omega_{\mathcal{F}}^{a,c}(x+1)} \delta_x. \tag{4.2}$$

Moreover, for $-1 < a < 0$ the measure $\omega_{a,c}^{\mathcal{F}}$ is not positive for any choice of c and $\mathcal{F}$, and for $0 < a < 1$ the measure $\omega_{a,c}^{\mathcal{F}}$ is positive if and only if c and $\mathcal{F}$ are admissible.

Proof. Write $\mathbb{A}$ for the linear space generated by the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$. Using Lemma 2.5 of [12], the definition of the measure $\omega_{a,c}^{\mathcal{F}}$ and the expressions for the difference coefficients of the operator $D_{\mathcal{F}}$ (see Theorem 3.3), it is straightforward to check that $D_{\mathcal{F}}$ is symmetric with respect to the pair $(\omega_{a,c}^{\mathcal{F}}, \mathbb{A})$. Since the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are eigenfunctions of $D_{\mathcal{F}}$ with different eigenvalues, Lemma 2.4 of [12] implies that they are orthogonal with respect to $\omega_{a,c}^{\mathcal{F}}$.

If $-1 < a < 0$ and the measure $\omega_{a,c}^{\mathcal{F}}$ is positive, since $\Gamma(n+c+k)$ is positive for n big enough, we conclude that $\Omega_F^{a,c}(2n+1)\Omega_F^{a,c}(2n+2) < 0$ for n big enough. But this would imply that $\Omega_{\mathcal{F}}^{a,c}$ has infinitely many real roots, which it is impossible since $\Omega_{\mathcal{F}}^{a,c}$ is a polynomial.

If $0 < a < 1$, according to Lemma 4.2, c and $\mathcal{F}$ are admissible if and only if $\Gamma(x+c+k)\Omega_F^{a,c}(x)\Omega_F^{a,c}(x+1) > 0$ for all nonnegative integers x. □

Theorem 4.4. *Given real numbers a, c, with $0 < a < 1$ and $c \neq 0, -1, -2, \ldots$, and a pair $\mathcal{F}$ of finite sets of positive integers, assume that c and $\mathcal{F}$ are admissible. Then the linear combinations of the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are dense in $L^2(\omega_{a,c}^{\mathcal{F}})$, where $\omega_{a,c}^{\mathcal{F}}$ is the positive measure (4.2). Hence $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are exceptional Meixner polynomials.*

Proof. Using Lemma 4.2 and taking into account that c and $\mathcal{F}$ are admissible, it follows that the measure $\rho_{a,c}^{\mathcal{F}}$ (3.14) is positive. As we pointed out above (see the proof of Lemma 4.2) this positive measure is also determinate. Since for determinate measures the polynomials are dense in the associated L^2 space (see the last comment in the Preliminaries of this paper), we deduce that the sequence $(q_n^{\mathcal{F}}/\|q_n^{\mathcal{F}}\|_2)_n$ (where $q_n^{\mathcal{F}}$ is the polynomial defined by (3.4)) is an orthonormal basis in $L^2(\rho_{a,c}^{\mathcal{F}})$.

For $s \in \sigma_{\mathcal{F}}$, consider the function $h_s(x) = \begin{cases} 1/\rho_{a,c}^{\mathcal{F}}(s), & x = s \\ 0, & x \neq s, \end{cases}$ where by $\rho_{a,c}^{\mathcal{F}}(s)$ we denote the mass of the discrete measure $\rho_{a,c}^{\mathcal{F}}$ at the point s. Since the support of $\rho_{a,c}^{\mathcal{F}}$ is $\sigma_{\mathcal{F}}$, we get that $h_s \in L^2(\rho_{a,c}^{\mathcal{F}})$. Its Fourier coefficients with respect to the orthonormal basis $(q_n^{\mathcal{F}}/\|q_n^{\mathcal{F}}\|_2)_n$ are $q_n^{\mathcal{F}}(s)/\|q_n^{\mathcal{F}}\|_2$, $n \geq 0$. Hence

$$\sum_{n=0}^{\infty} \frac{q_n^{\mathcal{F}}(s) q_n^{\mathcal{F}}(r)}{\|q_n^{\mathcal{F}}\|_2^2} = \langle h_s, h_r \rangle_{\rho_{a,c}^{\mathcal{F}}} = \frac{1}{\rho_{a,c}^{\mathcal{F}}(s)} \delta_{s,r}. \tag{4.3}$$

This is the dual orthogonality associated to the orthogonality

$$\sum_{u\in\sigma_{\mathcal{F}}} q_n^{\mathcal{F}}(u)q_m^{\mathcal{F}}(u)\rho_{a,c}^{\mathcal{F}}(u) = \langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}}\rangle\delta_{n,m}$$

of the polynomials $q_n^{\mathcal{F}}, n \geq 0$, with respect to the positive measure $\rho_{a,c}^{\mathcal{F}}$ (see, for instance, [3], Appendix III, or [24], Th. 3.8).

Using (4.1), (2.23) and the duality (3.12), we get

$$\frac{1}{\langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}}\rangle_{\rho_{a,c}^{\mathcal{F}}}} = \omega_{a,c}^{\mathcal{F}}(n)x_n, \tag{4.4}$$

where x_n is the positive number given by

$$x_n = \left(\frac{a}{a-1}\right)^{2u_{\mathcal{F}}+2k+1} \frac{(-1)^k(1-a)^c n!(1+c)_{n+k-1}(1+c)_{n+k}}{\kappa^2\Gamma(n+c)\Gamma(n+c+k)(n+k+1)!\xi_n\xi_{n+1}}, \tag{4.5}$$

and κ and ξ_n are defined in Lemma 3.2.

Using now the duality (3.5), we can rewrite (4.3) for $s = r$ as

$$\sum_{n=0}^{\infty} \omega_{a,c}^{\mathcal{F}}(n)(m_r^{\mathcal{F}}(n))^2\kappa^2 x_n\xi_n^2\zeta_r^2 = \frac{1}{\rho_{a,c}^{\mathcal{F}}(r)}. \tag{4.6}$$

A straightforward computation using (4.5) and the definitions of κ, ξ_n and ζ_r in Lemma 3.2 gives

$$\kappa^2 x_n\xi_n^2\zeta_r^2 = \frac{(1-a)^{c+2r-2u_{\mathcal{F}}-k}}{a^{k_1-2k}(\rho_{a,c}^{\mathcal{F}}(r))^2}. \tag{4.7}$$

Inserting it in (4.6), we get

$$\langle m_r^{\mathcal{F}}, m_r^{\mathcal{F}}\rangle_{\omega_{a,c}^{\mathcal{F}}} = \frac{a^{k_1-2k}}{(1-a)^{c+2r-2u_{\mathcal{F}}-k}}\rho_{a,c}^{\mathcal{F}}(r). \tag{4.8}$$

Consider now a function f in $L^2(\omega_{a,c}^{\mathcal{F}})$ and write $g(n) = f(n)/x_n^{1/2}$, where x_n is the positive number given by (4.5). Using (4.4), we get

$$\sum_{n=0}^{\infty} \frac{|g(n)|^2}{\langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}}\rangle_{\rho_{a,c}^{\mathcal{F}}}} = \sum_{n=0}^{\infty}\omega_{a,c}^{\mathcal{F}}(n)|f(n)|^2 = \|f\|_2^2 < \infty.$$

Define now

$$v_r = \sum_{n=0}^{\infty} \frac{g(n)q_n^{\mathcal{F}}(r)}{\langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}}\rangle_{\rho_{a,c}^{\mathcal{F}}}}.$$

Using Theorem III.2.1 of [3], we get

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \frac{|g(n)|^2}{\langle q_n^{\mathcal{F}}, q_n^{\mathcal{F}}\rangle_{\rho_{a,c}^{\mathcal{F}}}} = \sum_{r\in\sigma_{\mathcal{F}}} |v_r|^2\rho_{a,c}^{\mathcal{F}}(r). \tag{4.9}$$

On the other hand, using the dualities (3.12) and (3.5), and (4.5) and (4.6), we have

$$v_r = \frac{1}{(\rho_{a,c}^{\mathcal{F}}(r))^{1/2}} \sum_{n=0}^{\infty} f(n)\frac{m_r^{\mathcal{F}}(n)}{\|m_r^{\mathcal{F}}\|_2}\omega_{a,c}^{\mathcal{F}}(n).$$

This is saying that $(\rho_{a,c}^{\mathcal{F}}(r))^{1/2}v_r, r \in \sigma_{\mathcal{F}}$, are the Fourier coefficients of f with respect to the orthonormal system $(m_n^{\mathcal{F}}/\|m_n^{\mathcal{F}}\|_2)_n$. Hence, the identity (4.9) is Parseval's identity for the function f, from where we deduce that the orthonormal system $(m_n^{\mathcal{F}}/\|m_n^{\mathcal{F}}\|_2)_n$ is complete in $L^2(\omega_{a,c}^{\mathcal{F}})$. □

5. Constructing polynomials which are eigenfunctions of second order differential operators

One can construct exceptional Laguerre polynomials by taking limit in the exceptional Meixner polynomials. We use the basic limit (2.32).

Given a pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers, using the expression (3.3) for the polynomials $m_n^{a,c;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, setting $x \to x/(1-a)$ and $c = \alpha + 1$ and taking limit as $a \to 1$, we get (up to normalization constants) the polynomials, $n \in \sigma_{\mathcal{F}}$,

$$L_n^{\alpha;\mathcal{F}}(x) = \begin{vmatrix} L^{\alpha}_{n-u_{\mathcal{F}}}(x) & (L^{\alpha}_{n-u_{\mathcal{F}}})'(x) & \cdots & (L^{\alpha}_{n-u_{\mathcal{F}}})^{(k)}(x) \\ \begin{bmatrix} L^{\alpha}_{f}(x) \\ f \in F_1 \end{bmatrix} & (L^{\alpha}_{f})'(x) & \cdots & (L^{\alpha}_{f})^{(k)}(x) \\ \begin{bmatrix} L^{\alpha}_{f}(-x) \\ f \in F_2 \end{bmatrix} & L^{\alpha+1}_{f}(-x) & \cdots & L^{\alpha+k}_{f}(-x) \end{vmatrix}. \tag{5.1}$$

More precisely

$$\lim_{a\to 1}(a-1)^{n-(k_1+1)k_2} m_n^{a,c;\mathcal{F}}\left(\frac{x}{1-a}\right) = (-1)^{\binom{k+1}{2}+\sum_{f\in F_2} f} L_n^{\alpha;\mathcal{F}}(x) \tag{5.2}$$

uniformly in compact sets.

Notice that $L_n^{\alpha;\mathcal{F}}$ is a polynomial of degree n with leading coefficient equal to

$$(-1)^{n-u_{\mathcal{F}}+\sum_{f\in F_1} f} \frac{V_{F_1}V_{F_2}\prod_{f\in F_1}(f-n+u_F)}{(n-u_{\mathcal{F}})!\prod_{f\in F_1} f!\prod_{f\in F_2} f!},$$

where V_F is the Vandermonde determinant defined by (2.2).

We introduce the associated polynomial

$$\Omega^{\alpha}_{\mathcal{F}}(x) = \begin{vmatrix} \begin{bmatrix} L^{\alpha}_{f}(x) \\ f \in F_1 \end{bmatrix} & (L^{\alpha}_{f})'(x) & \cdots & (L^{\alpha}_{f})^{(k-1)}(x) \\ \begin{bmatrix} L^{\alpha}_{f}(-x) \\ f \in F_2 \end{bmatrix} & L^{\alpha+1}_{f}(-x) & \cdots & L^{\alpha+k-1}_{f}(-x) \end{vmatrix}. \tag{5.3}$$

Notice that $\Omega^{\alpha}_{\mathcal{F}}$ is a polynomial of degree $u_{\mathcal{F}} + k_1$. To simplify the notation we sometimes write $\Omega_{\mathcal{F}} = \Omega^{\alpha}_{\mathcal{F}}$.

When $F_1 = \emptyset$, using (2.31), we have for $\Omega^{\alpha}_{\mathcal{F}}$ the identity

$$\Omega^{\alpha}_{\mathcal{F}}(x) = \begin{vmatrix} \begin{bmatrix} L^{\alpha}_{f}(-x) \\ f \in F_2 \end{bmatrix} & -(L^{\alpha}_{f})'(-x) & \cdots & (-1)^{k-1}(L^{\alpha}_{f})^{(k-1)}(-x) \end{vmatrix}. \tag{5.4}$$

We also straightforwardly have

$$L_{u_F}^{\alpha;\mathcal{F}}(x) = (-1)^{\binom{s_\mathcal{F}}{2}+s_\mathcal{F}k_1}\Omega_{\mathcal{F}_\Downarrow}^{\alpha+s_\mathcal{F}}(x), \tag{5.5}$$

where the positive integer $s_\mathcal{F}$ and the pair $\mathcal{F}_\Downarrow$ are defined by (2.16) and (2.15), respectively.

We will need to know the value at 0 of the polynomial $\Omega_\mathcal{F}^\alpha$.

Lemma 5.1. *Let $\mathcal{F}$ be a pair of finite sets of positive integers, then $\Omega_\mathcal{F}^\alpha(0)$ is a polynomial in α of degree $u_\mathcal{F}+k_1$ which does not vanish in $\mathbb{R}\setminus\{-1,-2,\dots\}$. Moreover*

$$\begin{aligned}\Omega_\mathcal{F}^\alpha(0) &= (-1)^{\binom{k_1}{2}} \frac{\prod_{j=1}^{2} V_{F_j}\prod_{i=1}^{k_j}(\alpha+i)_{k_j-i+1}\prod_{f\in F_j}(\alpha+k_j+1)_{f-k_j}}{\prod_{f\in F_1} f!\prod_{f\in F_2} f!\prod_{i=1}^{\min\{k_1,k_2\}}(\alpha+i)_{k_1+k_2-2i+1}}\\ &\quad\times\prod_{f\in F_1}\prod_{g\in F_2}(\alpha+f+g+1).\end{aligned} \tag{5.6}$$

Proof. The proof of (5.6) follows by a careful computation using that $L_n^\alpha(0) = \frac{(1+\alpha)_n}{n!}$ and standard determinant techniques. Because of the value above of the Laguerre polynomials at 0, $\Omega_\mathcal{F}^\alpha(0)$ is clearly a polynomial in α; one can also see that the right hand side of (5.6) is a polynomial because each factor of the form $\alpha+s$ in the denominator cancels with one in the numerator. It is now easy to see that the right hand side of (5.6) only vanishes in some negative integers. □

Passing again to the limit, we can transform the second order difference operator (3.15) in a second order differential operator with respect to which the polynomials $L_n^{\alpha;F}$, $n\in\sigma_\mathcal{F}$, are eigenfunctions.

Theorem 5.2. *Given a real number $\alpha\neq-1,-2,\dots$ and a pair $\mathcal{F}$ of finite sets of positive integers, the polynomials $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_\mathcal{F}$, are common eigenfunctions of the second order differential operator*

$$D_F = x\partial^2 + h_1(x)\partial + h_0(x), \tag{5.7}$$

where $\partial = d/dx$ and

$$h_1(x) = \alpha+k+1-x-2x\frac{\Omega_\mathcal{F}'(x)}{\Omega_\mathcal{F}(x)}, \tag{5.8}$$

$$h_0(x) = -k_1-u_\mathcal{F}+(x-\alpha-k)\frac{\Omega_\mathcal{F}'(x)}{\Omega_\mathcal{F}(x)}+x\frac{\Omega_\mathcal{F}''(x)}{\Omega_\mathcal{F}(x)}. \tag{5.9}$$

More precisely $D_\mathcal{F}(L_n^\mathcal{F}) = -nL_n^\mathcal{F}(x)$.

Proof. We omit the proof because proceeds as that of Theorem 5.1 in [12] and it is a matter of calculation using carefully the basic limit (2.32) and its consequences

$$\lim_{a\to1^-}(1-a)^{\beta_\mathcal{F}}\Omega_\mathcal{F}^{a,c}(x_a) = (-1)^{\epsilon_\mathcal{F}}\Omega_\mathcal{F}^\alpha(x), \tag{5.10}$$

$$\lim_{a\to 1^-}(1-a)^{\beta_{\mathcal{F}}-1}(\Omega_{\mathcal{F}}^{a,c}(x_a+1)-\Omega_{\mathcal{F}}^{a,c}(x_a))=(-1)^{\epsilon_{\mathcal{F}}}(\Omega_{\mathcal{F}}^{\alpha})'(x), \tag{5.11}$$
$$\lim_{a\to 1^-}(1-a)^{\beta_{\mathcal{F}}-2}(\Omega_{\mathcal{F}}^{a,c}(x_a+1)-2\Omega_{\mathcal{F}}^{a,c}(x_a)+\Omega_{\mathcal{F}}^{a,c}(x_a-1))=(-1)^{\epsilon_{\mathcal{F}}}(\Omega_{F}^{\alpha})''(x).$$

where $c=\alpha+1$, $\beta_{\mathcal{F}}=u_{\mathcal{F}}+k_1(1-k_2)$, $x_a=x/(1-a)$ and $\epsilon_{\mathcal{F}}=\sum_{f\in F_1}f$. $\square$

To prove the completeness of the exceptional Laguerre polynomials in the associated L^2 space, we will need the following characterization of the linear space generated by $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_{\mathcal{F}}$.

Lemma 5.3. *Given a real number $\alpha\neq -1,-2,\dots$ and a pair $\mathcal{F}$ of finite sets of positive integers, consider the linear space $\mathbb{A}$ generated by the polynomials $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_{\mathcal{F}}$. Then $p\in\mathbb{A}$ if and only if*

$$(-2xp'+(x-\alpha-k)p)\Omega_{\mathcal{F}}'+xp\Omega_{\mathcal{F}}'' \tag{5.12}$$

is divisible by $\Omega_{\mathcal{F}}$.

Proof. Write $\mathbb{B}=\{p\in\mathbb{P}:D_{\mathcal{F}}(p)\in\mathbb{P}\}$. From the definition of the second order differential operator $D_{\mathcal{F}}$ (5.7), one easily sees that $p\in\mathbb{B}$ if and only if the polynomial (5.12) is divisible by $\Omega_{\mathcal{F}}$.

Since each polynomial $L_n^{\alpha;\mathcal{F}}$, $n\in\sigma_{\mathcal{F}}$, is an eigenfunction for $D_{\mathcal{F}}$, we get that $L_n^{\alpha;\mathcal{F}}\in\mathbb{B}$ and hence $\mathbb{A}\subset\mathbb{B}$.

Consider the set of nonnegative integers $S=\mathbb{N}\setminus\sigma_{\mathcal{F}}$. The definition of $\sigma_{\mathcal{F}}$ (2.12) shows that S is finite and $s=|S|=u_{\mathcal{F}}+k_1=\deg(\Omega_{\mathcal{F}})$. We can write $\mathbb{P}=\mathbb{A}\oplus H_1$, where $H_1=\langle x^j:j\in S\rangle$. Hence $\dim H_1=\deg(\Omega_{\mathcal{F}})$.

On the other hand, observe that the divisibility of (5.12) by $\Omega_{\mathcal{F}}$ imposes $\deg(\Omega_{\mathcal{F}})$ linearly independent homogeneous conditions on the coefficients of p. We can then construct linearly independent polynomials q_j, $j=1,\dots,\deg(\Omega_{\mathcal{F}})$, such that $\mathbb{P}=\mathbb{B}\oplus H_2$, where $H_2=\langle q_j: j=1,\dots,\deg(\Omega_{\mathcal{F}})\rangle$. Hence $\dim H_2=\deg(\Omega_{\mathcal{F}})$. Since $\mathbb{A}\subset\mathbb{B}$ and $\dim H_1=\dim H_2$, we get that actually $\mathbb{A}=\mathbb{B}$. $\square$

Again passing to the limit, from the factorization in Lemma 3.4 we can factorize the second order differential operator $D_{\mathcal{F}}$ as a product of two first order differential operators. This can be done by choosing one of the components of $\mathcal{F}=(F_1,F_2)$ and removing one element in the chosen component. An iteration shows that the system $(D_{\mathcal{F}},(L_n^{\alpha;\mathcal{F}})_{n\in\sigma_{\mathcal{F}}})$ can be constructed by applying a sequence of k Darboux transforms to the Laguerre system (see Definition 2.1). We display the details in the following lemma, where we remove one element of the component F_2, and hence we have to assume $F_2\neq\emptyset$. A similar result can be proved by removing one element of the component F_1.

Lemma 5.4. *Let $\mathcal{F}=(F_1,F_2)$ be a pair of finite sets of positive integers and assume $F_2\neq\emptyset$. We define the first order differential operators $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ as*

$$A_{\mathcal{F}}=-\frac{\Omega_{\mathcal{F}}(x)}{\Omega_{\mathcal{F}_{2,\{k_2\}}}(x)}\partial+\frac{\Omega_{\mathcal{F}}'(x)+\Omega_{\mathcal{F}}(x)}{\Omega_{\mathcal{F}_{2,\{k_2\}}}(x)}, \tag{5.13}$$

$$B_{\mathcal{F}}=\frac{-x\Omega_{\mathcal{F}_{2,\{k_2\}}}(x)}{\Omega_{\mathcal{F}}(x)}\partial+\frac{x\Omega_{\mathcal{F}_{2,\{k_2\}}}'(x)-(\alpha+k)\Omega_{\mathcal{F}_{2,\{k_2\}}}(x)}{\Omega_{\mathcal{F}}(x)}, \tag{5.14}$$

where k_2 is the number of elements of F_2 and the pair $\mathcal{F}_{2,\{k_2\}}$ is defined by (2.14). Then $L_n^{\alpha;\mathcal{F}}(x) = A_{\mathcal{F}}(L^{\alpha;\mathcal{F}_{2,\{k_2\}}}_{n-f_{k_2}^{2\rceil}+k_2-1})(x)$, $n \in \sigma_F$. *Moreover*

$$D_{\mathcal{F}_{2,\{k_2\}}} = B_{\mathcal{F}}A_{\mathcal{F}} + (\alpha + f_{k_2}^{2\rceil} - u_{\mathcal{F}_{2,\{k_2\}}} + 1)Id,$$

$$D_{\mathcal{F}} = A_{\mathcal{F}}B_{\mathcal{F}} + (\alpha + f_{k_2}^{2\rceil} - u_{\mathcal{F}} + 1)Id.$$

Proof. The lemma can be proved applying limits in Lemma 3.4. □

6. Exceptional Laguerre polynomials

In the previous section, given a real number $\alpha \neq -1, -2, \dots$, we have associated to each pair $\mathcal{F}$ of finite sets of positive integers the polynomials $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, which are always eigenfunctions of a second order differential operator with rational coefficients. We are interested in the cases when, in addition, those polynomials are orthogonal and complete with respect to a positive measure.

Definition 6.1. The polynomials $L_n^{\alpha;F}$, $n \in \sigma_{\mathcal{F}}$, defined by (5.1) are called exceptional Laguerre polynomials, if they are orthogonal and complete with respect to a positive measure.

The following lemma and theorem show that the admissibility of $\alpha + 1$ and $\mathcal{F}$ will be the key to construct exceptional Laguerre polynomials.

Lemma 6.2. *Given a real number $\alpha \neq -1, -2, \dots$ and a pair $\mathcal{F}$ of finite sets of positive integers, if $\alpha + 1$ and $\mathcal{F}$ are admissible then $\alpha + k > -1$ and $\Omega_{\mathcal{F}}^{\alpha}$ (5.3) does not vanish in* $[0, +\infty)$.

Proof. First of all, the part 1 of Lemma 2.6 gives that $\alpha + k > -1$.

Lemma 5.1 shows that for $\alpha \neq -1, -2, \dots$, $\Omega_{\mathcal{F}}^{\alpha}(0) \neq 0$. Hence it is enough to prove that $\Omega_{\mathcal{F}}^{\alpha} \neq 0$ for $x > 0$.

Write $c = \alpha + 1$. For $0 < a < 1$, consider the measure τ_a^c defined by

$$\tau_a^c = (1-a)^{c-k}\sum_{x=0}^{\infty}\frac{a^x\Gamma(x+c+k)(m_{u_{\mathcal{F}}}^{a,c;\mathcal{F}}(x))^2}{x!\Omega_{\mathcal{F}}^{a,c}(x)\Omega_{\mathcal{F}}^{a,c}(x+1)}\delta y_{a,x},$$

where

$$y_{a,x} = (1-a)x. \tag{6.1}$$

Since $\alpha + 1 = c$ and $\mathcal{F}$ are admissible, we have that the measure τ_a^c is positive (Theorem 4.4).

Consider the positive integer $s_{\mathcal{F}}$ and the pair $\mathcal{F}_{\Downarrow}$ defined by (2.16) and (2.15), respectively. We need the following limits

$$\lim_{a\to 1^-}(1-a)^{u_{\mathcal{F}}+k_1(1-k_2)}\Omega_{\mathcal{F}}^{a,c}(x/(1-a)) = \epsilon_0\Omega_{\mathcal{F}}^{\alpha}(x), \tag{6.2}$$

$$\lim_{a\to 1^-}(1-a)^{u_{\mathcal{F}}+k-(k_1+1)k_2}\Omega_{\mathcal{F}}^{a,c}(x/(1-a)+1) = \epsilon_0\Omega_{\mathcal{F}}^{\alpha}(x), \tag{6.3}$$

$$\lim_{a\to 1^-}(1-a)^{u_{\mathcal{F}}-(k_1+1)k_2}m_{u_{\mathcal{F}}}^{a,c;\mathcal{F}}(x/(1-a)) = \epsilon_1\Omega_{\mathcal{F}_{\Downarrow}}^{\alpha+s_{\mathcal{F}}}(x), \tag{6.4}$$

$$\lim_{a\to 1^-}\frac{(1-a)^{c+k-1}a^{x/(1-a)}\Gamma(x/(1-a)+c+k)}{\Gamma(x/(1-a)+1)} = x^{\alpha+k}e^{-x}, \tag{6.5}$$

uniformly in compact sets of $(0, +\infty)$, where the ϵ's are the signs

$$\epsilon_0 = (-1)^{\sum_{f\in F_1} f}, \qquad \epsilon_1 = (-1)^{\sum_{f\in F_1} f+\binom{s_{\mathcal{F}}}{2}+s_{\mathcal{F}}k_1}.$$

The first limit is (5.10). The second one is a consequence of (5.11). The third one is a consequence of (5.2) and (5.5). The fourth one is a consequence of the asymptotic behavior of $\Gamma(z+u)/\Gamma(z+v)$ when $z\to\infty$ (see [14], vol. I (4), p. 47).

We proceed in three steps.

First step. Assume that $\alpha+1$ and $\mathcal{F}$ are admissible and that there exists $x_0>0$ with $\Omega^{\alpha}_{\mathcal{F}}(x_0)=0$. We will now show that $\Omega^{\alpha+s_{\mathcal{F}}}_{\mathcal{F}_{\Downarrow}}(x_0)=0$.

Notice first that according to (5.5), the polynomials $\Omega^{\alpha+s_{\mathcal{F}}}_{\mathcal{F}_{\Downarrow}}$ and $L^{\alpha;\mathcal{F}}_{u_{\mathcal{F}}}$ coincide up to a sign.

We take a real number v with $x_0<v$ such that $\Omega^{\alpha}_{\mathcal{F}}(x)\neq 0$ for $x\in(x_0,v]$. For a real number u with $x_0<u<v$, write $I=[u,v]$. Then $\Omega^{\alpha}_{\mathcal{F}}$ does not vanish in I. Applying Hurwitz's Theorem to the limits (6.2) and (6.3) we can choose a countable set $X=\{a_n : n\in\mathbb{N}\}$ of numbers in $(0,1)$ with $\lim_n a_n=1$ such that $\Omega^{a,c}_{\mathcal{F}}(x/(1-a))\Omega^{a,c}_{\mathcal{F}}(x/(1-a)+1)\neq 0$, $x\in I$ and $a\in X$.

Hence, we can combine the limits (6.2)–(6.5) to get

$$\lim_{a\to 1; a\in X} h_a(x)=h(x), \quad \text{uniformly in } I, \tag{6.6}$$

where

$$h_a(x)=(1-a)^{c-k-1}\frac{a^{x/(1-a)}\Gamma(x/(1-a)+c+k)(m^{a,c;\mathcal{F}}_{u_{\mathcal{F}}}(x/(1-a)))^2}{\Gamma(x/(1-a)+1)\Omega^{a,c}_{\mathcal{F}}(x/(1-a))\Omega^{a,c}_{\mathcal{F}}(x/(1-a)+1)},$$
$$h(x)=\frac{x^{\alpha+k}e^{-x}(\Omega^{\alpha+s_{\mathcal{F}}}_{\mathcal{F}_{\Downarrow}})^2(x)}{(\Omega^{\alpha}_{\mathcal{F}})^2(x)}.$$

We now prove that

$$\lim_{a\to 1; a\in X}\tau^c_a(I)=\int_I h(x)dx. \tag{6.7}$$

To do that, write $I_a=\{x\in\mathbb{N} : u/(1-a)\le x\le v/(1-a)\}$. The numbers $y_{a,x}$, $x\in I_a$, form a partition of the interval I with $y_{a,x+1}-y_{a,x}=(1-a)$ (see (6.1)). Since the function h is continuous in the interval I, we get that

$$\int_I h(x)dx=\lim_{a\to 1; a\in X} S_a,$$

where S_a is the Cauchy sum

$$S_a=\sum_{x\in I_a} h(y_{a,x})(y_{a,x+1}-y_{a,x}).$$

On the other hand, since $x\in I_a$ if and only if $u\le y_{a,x}\le v$ (6.1), we get

$$\tau^c_a(I)=(1-a)^{c-k}\sum_{x\in I_a}\frac{a^x\Gamma(x+c+k)(m^{a,c;\mathcal{F}}_{u_{\mathcal{F}}}(x))^2}{x!\Omega^{a,c}_{\mathcal{F}}(x)\Omega^{a,c}_{\mathcal{F}}(x+1)}=(1-a)\sum_{x\in I_a}h_a(y_{a,x})$$
$$=\sum_{x\in I_a}h_a(y_{a,x})(y_{a,x+1}-y_{a,x}).$$

The limit (6.7) now follows from the uniform limit (6.6).

The identity (4.8) says that $\tau_a^c(\mathbb{R}) = a^{k_1-2k}\gamma_{\mathcal{F}}^c$, where the positive constant

$$\gamma_{\mathcal{F}}^c = \rho_{a,c}^{\mathcal{F}}(u_{\mathcal{F}}) = \Gamma(c)\prod_{f\in F_1}(-f)\prod_{f\in F_2}(c+f)$$

does not depend on a. This gives $\tau_a^c(I) \le a^{k_1-2k}\gamma_{\mathcal{F}}^c$. And so from the limit (6.7) we get

$$\int_I h(x)dx \le \gamma_{\mathcal{F}}^c.$$

That is

$$\int_u^v \frac{x^{\alpha+k}e^{-x}(\Omega_{\mathcal{F}_{\Downarrow}}^{\alpha+s_{\mathcal{F}}})^2(x)}{(\Omega_{\mathcal{F}}^{\alpha})^2(x)}dx \le \gamma_{\mathcal{F}}^c.$$

On the other hand, if $\Omega_{\mathcal{F}_{\Downarrow}}^{\alpha+s_{\mathcal{F}}}(x_0) \neq 0$, since $\Omega_{\mathcal{F}}^{\alpha}(x_0) = 0$ we get

$$\lim_{u\to x_0^+}\int_u^v \frac{x^{\alpha+k}e^{-x}(\Omega_{F_{\Downarrow}}^{\alpha+s_{\mathcal{F}}})^2(x)}{(\Omega_{F}^{\alpha})^2(x)}dx = +\infty.$$

Hence $\Omega_{\mathcal{F}_{\Downarrow}}^{\alpha+s_{\mathcal{F}}}(x_0) = 0$.

The proof of the theorem proceeds now by induction on $\max F_1$.

Second step. Assume $\alpha+1$ and $\mathcal{F}$ are admissible and $\max F_1 = -1$ (that is $F_1 = \emptyset$). We will now show that

$$\Omega_{\mathcal{F}}^{\alpha}(x) \neq 0, \quad x > 0. \tag{6.8}$$

Since $F_1 = \emptyset$, the part 3 of Lemma 2.6 implies that the assumption $\alpha+1$ and $\mathcal{F}$ are admissible is equivalent to the assumption $\alpha > -1$.

We prove this step by induction on k_2. For $k_2 = 1$, we have that F_2 is a singleton $F_2 = \{f\}$, and then $\Omega_{\mathcal{F}}^{\alpha}(x) = L_f^{\alpha}(-x)$. The usual properties of the zeros of Laguerre polynomials ($\alpha > -1$) imply (6.8).

Assume now that (6.8) holds for $k_2 \le s$ and $\alpha > -1$, and take a finite set of positive integers F_2, with $k_2 = s+1$ elements. We notice that according to the definition of s_{F_1} (2.7) for $F_1 = \emptyset$, we have $s_{F_1} = 1$. Hence we also have $s_{\mathcal{F}} = 1$ (see (2.16)). If there exists $x_0 > 0$ such that $\Omega_{\mathcal{F}}^{\alpha}(x_0) = 0$, using the first step, we get that also $\Omega_{\mathcal{F}_{\Downarrow}}^{\alpha+1}(x_0) = 0$. Since $F_1 = \emptyset$, we have $\mathcal{F} = \mathcal{F}_{\Downarrow}$ (see (2.15)), and hence, we can conclude that $\Omega_{\mathcal{F}}^{\alpha+j}(x_0) = 0$, $j = 0, 1, 2, \ldots$.

For a positive integer $m \ge \max F_2 + 1 \ge s+1$ consider the $(s+1)\times(m+1)$ matrix

$$M = \left(\begin{bmatrix} L_f^{\alpha}(-x_0) & L_f^{\alpha+1}(-x_0) & \cdots & L_f^{\alpha+m}(-x_0) \\ f\in F_2 & & & \end{bmatrix}\right).$$

Write c_i, $i = 1, \ldots, m+1$, for the columns of M (from left to right). We see that the minor of M formed by its first s rows and first s columns is equal to $\Omega_{\mathcal{F}_{2,\{s+1\}}}^{\alpha}(x_0)$, where the pair $\mathcal{F}_{2,\{s+1\}}$ is defined by (2.14). Since the set $F_2 \setminus \{f_{s+1}^{2]}\}$ has s elements and $\alpha > -1$, the induction hypothesis says that $\Omega_{\mathcal{F}_{2,\{s+1\}}}^{\alpha}(x_0) \neq 0$, and hence the columns c_i, $i = 1, \ldots, s$, of M are linearly independent. On the other hand, consider $j \ge 0$ and the $(s+1)\times(s+1)$ submatrix M_j of M formed by the consecutive columns c_{j+i}, $i = 1, \ldots, s+1$, of M. It is clear that $\det M_j = \Omega_{\mathcal{F}}^{\alpha+j}(x_0) = 0$.

That is, for $j \geq 0$ the consecutive columns c_{j+i}, $i = 1, \ldots, s+1$, of M are linearly dependent. Using Lemma 2.2, we conclude that $\operatorname{rank} M = s$. Write now

$$\tilde{M} = \left(\begin{bmatrix} L_f^\alpha(-x_0) & -(L_f^\alpha)'(-x_0) & \cdots & (-1)^m (L_f^\alpha)^{(m)}(-x_0) \\ f \in F_2 & & & \end{bmatrix} \right).$$

Using (2.30) and (2.31), we see that $\operatorname{rank} \tilde{M} = \operatorname{rank} M = s$. Then there exist numbers e_f, $f \in F_2$, not all zero such that the polynomial $p(x) = \sum_{f \in F_2} e_f L_f^\alpha(x)$ is non null and has a zero of multiplicity m in $-x_0$. But the polynomial p has degree at most $\max F_2$, and since $m \geq \max F_2 + 1 > \deg p$, this shows that $p = 0$, which is a contradiction. This proves the second step.

Assume now that $\alpha + 1$ and $\mathcal{F}$ are admissible and

$$\Omega_{\mathcal{F}}^\alpha(x) \neq 0, \quad x > 0, \tag{6.9}$$

holds for $\max F_1 \leq s$.

Third step. We now show that if $\max F_1 = s + 1$, then (6.9) continues to hold.

Consider the pair $\mathcal{F}_\Downarrow = \{(F_1)_\Downarrow, F_2\}$ defined by (2.15). Since $F_1 \neq \emptyset$, we have that $\max (F_1)_\Downarrow \leq s$. The part 4 of Lemma 2.6 says that if $\alpha + 1$ and $\mathcal{F}$ are admissible then $\alpha + 1 + s_{\mathcal{F}}$ and $\mathcal{F}_\Downarrow$ are admissible as well. The induction hypothesis (6.9) then says that $\Omega_{\mathcal{F}_\Downarrow}^{\alpha+s_{\mathcal{F}}}(x) \neq 0$ for $x > 0$. The first step then gives that also $\Omega_{\mathcal{F}}^\alpha(x) \neq 0$, for $x > 0$. $\square$

As we pointed out in the Introduction, we guess that the converse of the previous theorem is true. However, the condition $\Omega_{\mathcal{F}}^\alpha(x) \neq 0, x \geq 0$, is not enough to guarantee the admissibility of $\alpha + 1$ and $\mathcal{F}$. Indeed, consider $F_1 = \{1\}$, $F_2 = \emptyset$ and $\mathcal{F} = (F_1, F_2)$. The definition (2.17) straightforwardly gives that $\alpha + 1$ and $\mathcal{F}$ are admissible if and only if $-2 < \alpha < -1$. On the other hand, it is also easy to see that $\Omega_{\mathcal{F}}^\alpha(x) = \alpha + 1 - x$, Hence $\Omega_{\mathcal{F}}^\alpha \neq 0, x \geq 0$, as far as $\alpha < -1$. Hence for $\alpha < -2$, $\Omega_{\mathcal{F}}^\alpha \neq 0, x \geq 0$, but $\alpha + 1$ and $\mathcal{F}$ are not admissible.

Theorem 6.3. *Given a real number $\alpha \neq -1, -2, \ldots$ and a pair $\mathcal{F}$ of finite sets of positive integers, if $\alpha + 1$ and $\mathcal{F}$ are admissible then the polynomials $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are orthogonal with respect to the positive weight*

$$\omega_{\alpha;\mathcal{F}}(x) = \frac{x^{\alpha+k} e^{-x}}{(\Omega_{\mathcal{F}}^\alpha(x))^2}, \quad x > 0, \tag{6.10}$$

and their linear combinations are dense in $L^2(\omega_{\alpha;\mathcal{F}})$. Hence $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, are exceptional Laguerre polynomials.

Proof. First of all, notice that the positive weight (6.10) has finite moments of any order because since $\alpha + 1$ and $\mathcal{F}$ are admissible we have $\alpha + k > -1$ (part 1 of Lemma 2.6).

Write $\mathbb{A}$ for the linear space generated by the polynomials $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$. Using Lemma 2.6 of [12], it is easy to check that the second order differential operator $D_{\mathcal{F}}$ (5.7) is symmetric with respect to the pair $(\omega_{\alpha;\mathcal{F}}, \mathbb{A})$ (6.10). Since the polynomials $L_n^{\alpha;F}$, $n \in \sigma_{\mathcal{F}}$, are eigenfunctions of $D_{\mathcal{F}}$ with different eigenvalues Lemma 2.4 of [12] implies that they are orthogonal with respect to $\omega_{\alpha;F}$.

In order to prove the completeness of $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, we use the approach of [21], Th. 6.3 for the exceptional Hermite polynomials and proceed in two steps.

Step 1. For each $r > 0$ and $\alpha > -1$, the linear space $\{(1 + x^r)p : p \in \mathbb{P}\}$ is dense in $L^2(x^\alpha e^{-x})$.

Since $1 + x^r > 0, x > 0$, this is equivalent to the density of $\mathbb{P}$ in $L^2((1 + x^r)x^\alpha e^{-x})$. But this follows straightforwardly taking into account that $(1+x^r)x^\alpha e^{-x}dx$, $x > 0$, is a determinate measure (see the last comment in the Preliminaries of this paper).

Step 2. $\mathbb{A}$ is dense in $L^2(\omega_{\alpha;\mathcal{F}})$.

Take a function $f \in L^2(\omega_{\alpha;\mathcal{F}})$. Write $r = \deg(\Omega_{\mathcal{F}}^\alpha)$ and define the function $g(x) = (1 + x^r) f(x)/(\Omega_{\mathcal{F}}^\alpha(x))^2$. Since $\alpha + 1$ and $\mathcal{F}$ are admissible, we get from the previous lemma that $\alpha + k > -1$ and $\Omega_{\mathcal{F}}^\alpha(x) \neq 0, x \geq 0$. Hence $g \in L^2(x^{\alpha+k}e^{-x})$. Given $\epsilon > 0$ and using the first step, we get a polynomial p such that

$$\int |g(x) - (1 + x^r)p(x)|^2 x^{\alpha+k}e^{-x}dx < \epsilon. \tag{6.11}$$

Write $\gamma = \inf\{(1 + x^r)/\Omega_{\mathcal{F}}^\alpha(x), x \geq 0\}$. We then get

$$\begin{aligned}\int |g(x) - (1 + x^r)p(x)|^2 x^{\alpha+k}e^{-x}dx &= \int \left|\frac{1 + x^r}{\Omega_{\mathcal{F}}^\alpha(x)}\right|^2 |f(x) - (\Omega_{\mathcal{F}}^\alpha(x))^2 p(x)|^2 \omega_{\alpha;\mathcal{F}}dx \\ &\geq \gamma^2 \int |f(x) - (\Omega_{\mathcal{F}}^\alpha(x))^2 p(x)|\omega_{\alpha;\mathcal{F}}(x)dx.\end{aligned}$$

Using (6.11), we can conclude that the linear space $\{(\Omega_{\mathcal{F}}^\alpha(x))^2 p : p \in \mathbb{P}\}$ is dense in $L^2(\omega_{\alpha;\mathcal{F}})$.

Lemma 5.3 gives that $\{(\Omega_{\mathcal{F}}^\alpha(x))^2 p : p \in \mathbb{P}\} \subset \mathbb{A}$. This proves the second step and the theorem. □

Proceeding as in the first step of Lemma 6.2, one can find the norm of the polynomials $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, from the norm of the polynomials $m_n^{a,c;\mathcal{F}}$ (see (4.8)).

Corollary 6.4. *Given a real number $\alpha \neq -1, -2, \ldots$ and a pair $\mathcal{F}$ of finite sets of positive integers, assume that $\alpha + 1$ and $\mathcal{F}$ are admissible. Then for $n \in \sigma_{\mathcal{F}}$, we have*

$$\int_0^\infty (L_n^{\alpha;\mathcal{F}}(x))^2 \frac{x^{\alpha+k}e^{-x}}{(\Omega_{\mathcal{F}}^\alpha(x))^2}dx = \frac{\mathfrak{p}_{\mathcal{F}}(n - u_{\mathcal{F}})\Gamma(n - u_{\mathcal{F}} + \alpha + 1)}{(n - u_{\mathcal{F}})!},$$

where $\mathfrak{p}_{\mathcal{F}}(x)$ is the polynomial defined by $\mathfrak{p}_{\mathcal{F}}(x) = \prod_{f\in F_1}(x - f)\prod_{f\in F_2}(x + \alpha + f + 1)$.

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Appendix

When the determinants $\Omega_{\mathcal{F}}^{a,c}(n) \neq 0$ (3.6), $n \geq 0$ (or equivalently, $\Phi_n^{\mathcal{F}} \neq 0$ (3.10), $n \geq 0$), the following alternative construction of the polynomial $q_n^{a,c;\mathcal{F}}$ (3.4) has been given in [13].

For a pair $\mathcal{F} = (F_1, F_2)$ of finite sets of positive integers, consider the involuted sets $I(F_1) = G_1$ and $I(F_2) = G_2$, where the involution I is defined by (2.6) and the number $v_{\mathcal{F}}$ defined by (2.11). Write $m = m_1 + m_2$, where m_1, m_2 are the number of elements of G_1 and G_2, respectively.

Assume that $\Omega_{\mathcal{F}}^{a,c}(n) \neq 0, n \geq 0$, and write $\tilde{c} = c + M_{F_1} + M_{F_2} + 2$, where M_F denotes the maximum element of F. Using Theorem 1.1 of [13], we have

$$q_n^{a,c;\mathcal{F}}(x) = \alpha_n \begin{vmatrix} m_n^{a,\tilde{c}}(x-v_{\mathcal{F}}) & (\frac{a}{a-1})m_{n-1}^{a,\tilde{c}}(x-v_{\mathcal{F}}) & \cdots & (\frac{a}{a-1})^m m_{n-m}^{a,\tilde{c}}(x-v_{\mathcal{F}}) \\ \begin{bmatrix} m_g^{a,2-\tilde{c}}(-n-1) \\ g \in G_1 \end{bmatrix} & a m_g^{a,2-\tilde{c}}(-n) & \cdots & \begin{bmatrix} a^m m_g^{a,2-\tilde{c}}(-n+m-1) \\ \end{bmatrix} \\ \begin{bmatrix} m_g^{1/a,2-\tilde{c}}(-n-1) \\ g \in G_2 \end{bmatrix} & m_g^{1/a,2-\tilde{c}}(-n) & \cdots & \begin{bmatrix} m_g^{1/a,2-\tilde{c}}(-n+m-1) \\ \end{bmatrix} \end{vmatrix}, \tag{A.1}$$

where the positive integer $v_{\mathcal{F}}$ is defined by (2.11) and $\alpha_n, n \geq 0$, is certain normalization constant.

The dualities (2.26) and (3.5) then provide an alternative definition of the polynomial $m_n^{a,c;\mathcal{F}}$, $n \geq v_{\mathcal{F}}$ (3.1). Indeed, write r_j^c, $j \geq 0$, for the polynomial of degree m defined by $r_j^c(x) = (c+x-m)_{m-j}(x-j+1)_j$, and, as before, $\tilde{c} = c + M_{F_1} + M_{F_2} + 2$. After an easy calculation, we conclude that

$$m_n^{a,c;\mathcal{F}}(x) = \beta_n \begin{vmatrix} r_0^{\tilde{c}}(x)m_{n-v_{\mathcal{F}}}^{a,\tilde{c}}(x) & r_1^{\tilde{c}}(x)m_{n-v_{\mathcal{F}}}^{a,\tilde{c}}(x-1) & \cdots & r_m^{\tilde{c}}(x)m_{n-v_{\mathcal{F}}}^{a,\tilde{c}}(x-m) \\ \begin{bmatrix} m_g^{a,2-\tilde{c}}(-x-1) \\ g \in G_1 \end{bmatrix} & a m_g^{a,2-\tilde{c}}(-x) & \cdots & \begin{bmatrix} a^m m_g^{a,2-\tilde{c}}(-x+m-1) \\ \end{bmatrix} \\ \begin{bmatrix} m_g^{1/a,2-\tilde{c}}(-x-1) \\ g \in G_2 \end{bmatrix} & m_g^{1/a,2-\tilde{c}}(-x) & \cdots & \begin{bmatrix} m_g^{1/a,2-\tilde{c}}(-x+m-1) \\ \end{bmatrix} \end{vmatrix}, \tag{A.2}$$

where $\beta_n, n \geq 0$, is certain normalization constant.

When the sum of the cardinalities of the involuted sets $G_1 = I(F_1)$ and $G_2 = I(F_2)$ is less than the sum of the cardinalities of F_1 and F_2, the expression (A.2) will provide a more efficient way than (3.1) for an explicit computation of the polynomials $m_n^{a,c;\mathcal{F}}$, $n \geq v_{\mathcal{F}}$. For instance, take $F_1 = \{1, \ldots, k\}$, $F_2 = \{1, \ldots, k-2, k\}$. Since $I(F_1) = \{k\}$, $I(F_2) = \{1, k\}$, the determinant in (A.2) has order 4 while the determinant in (3.1) has order $2k$.

Assume now that $\alpha+1$ and $\mathcal{F}$ are admissible (2.17). Write $c = \alpha+1$. According to Lemma 4.2, this gives for all $0 < a < 1$ that $\Gamma(x+c+k)\Omega_{\mathcal{F}}^{a,c}(x)\Omega_{\mathcal{F}}^{a,c}(x+1) > 0$ for $x \in \mathbb{N}$, where $\Omega_{\mathcal{F}}^{a,c}$ is the polynomial (3.6) associated to the Meixner family. In particular $\Omega_{\mathcal{F}}^{a,c}(x) \neq 0$, for all nonnegative integers x. Write $w_n^{\alpha,j} = j!\binom{n+\alpha}{j}$ and $\tilde{\alpha} = \alpha + M_{F_1} + M_{F_2} + 2$. Hence, if instead of (3.3) we take limit in (A.2), we get the following alternative expression for the polynomials $L_n^{\alpha;F}$, $n \geq v_{\mathcal{F}}$,

$$L_n^{\alpha;\mathcal{F}}(x) = \gamma_n \begin{vmatrix} x^m L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}}(x) & w_{n-v_{\mathcal{F}}}^{\tilde{\alpha},1} x^{m-1} L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}-1}(x) & \cdots & w_{n-v_{\mathcal{F}}}^{\tilde{\alpha},m} L_{n-v_{\mathcal{F}}}^{\tilde{\alpha}-m}(x) \\ \begin{bmatrix} L_g^{-\tilde{\alpha}}(-x) \\ g \in G_1 \end{bmatrix} & L_g^{-\tilde{\alpha}+1}(-x) & \cdots & \begin{bmatrix} L_g^{-\tilde{\alpha}+m}(-x) \\ \end{bmatrix} \\ \begin{bmatrix} L_g^{-\tilde{\alpha}}(x) \\ g \in G_2 \end{bmatrix} & (L_g^{-\tilde{\alpha}})'(x) & \cdots & \begin{bmatrix} (L_g^{-\tilde{\alpha}})^{(m)}(x) \\ \end{bmatrix} \end{vmatrix}, \tag{A.3}$$

where γ_n is certain normalization constant.

Both determinantal constructions (A.2) and (A.3) for the polynomials $m_n^{a,c;\mathcal{F}}$ and $L_n^{\alpha;\mathcal{F}}$, $n \in \sigma_{\mathcal{F}}$, respectively, imply a couple of factorizations of the second order difference and differential operators $D_{\mathcal{F}}$ (see (3.15) and (5.7), respectively) in two first order difference and differential operators, respectively. These factorizations are different to the factorizations displayed in Lemmas 3.4 and 5.4, respectively. We do not include the details here but both factorizations can be worked out as Lemmas 3.7 and 5.3 of [12].

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