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Hardy–Littlewood–Pólya relation in the best dominated approximation in symmetric spaces

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Abstract

We investigate a correspondence between strict K-monotonicity, K-order continuity and the best dominated approximation problems with respect to the Hardy–Littlewood–Pólya relation $\prec$. Namely, we study, in terms of an *LKM* point and a *UKM* point, a necessary condition for uniqueness of the best dominated approximation under the relation $\prec$ in a symmetric space E. Next, we characterize a relation between a point of K-order continuity and an existence of a best dominated approximant with respect to $\prec$. Finally, we present a compete criteria, written in a notion of K-order continuity, under which a closed and K-bounded above subset of a symmetric space E is proximinal.

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1. Introduction

The natural motivation for exploring of the geometrical structure of Banach spaces for many decades was an application to the best approximation. Recently, many authors have been investigated intensively the relations between local structure of monotonicity and rotundity properties of

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Banach spaces and the best approximation (see [7,9,12,13,16]). One of the most essential results was published by W. Kurc in [16], who has established a connection between the best dominated approximation problems and order continuity as well as strict and uniformly monotone properties. It is worth mentioning that in view of the previous results in the paper [8], local approach of strict and uniform monotone properties and order continuity in Banach lattice has been researched with application to the best dominated approximation. The intension of this paper is to find a complete characterization of existence and uniqueness of the best approximant from the cone of all nonnegative and decreasing functions that oscillate around the given function x in a symmetric space E. This induces the crucial question of replacing the partial order with respect to the relation $\leq$ in the best dominated approximation problems by the Hardy–Littlewood–Pólya relation $\prec$. It is worth mentioning that recently in papers [4,10,5] there has been researched local and global monotone properties in the sense of the Hardy–Littlewood–Pólya relation $\prec$ in symmetric spaces (so-called K-monotone properties). In the spirit of the previous investigations, it is natural to expect a correspondence between K-monotone properties as well as K-order continuity and the best dominated approximation in the sense of the Hardy–Littlewood–Pólya relation.

In Section 2, we recall the needed terminology.

Section 3 is devoted to connections between the best dominated approximation problems under the Hardy–Littlewood–Pólya relation $\prec$ and K-order continuity and also K-monotone properties.

First, we establish a useful property for a vanishing decreasing rearrangement at infinity and its maximal function for any element in $L^1 + L^\infty$. Next, we discuss the sufficient condition, in terms of the best dominated approximation with respect to the Hardy–Littlewood–Pólya relation $\prec$, for an *LKM* point and a *UKM* point in symmetric space E. In our investigation of the best dominated approximation under the relation $\prec$ and strict K-monotonicity, we restrict ourself to closed subsets of a symmetric space E which are bounded above or below with respect to $\prec$. In this problem, we also answer the key question whether a uniqueness of the best approximation for bounded subsets in E in the sense of $\prec$ yields strict K-monotonicity of E. We present an example of Lorentz spaces $\Gamma_{p,w}$ strictly K-monotone in which uniqueness of the best dominated approximation in the sense of $\prec$ is not fulfilled. Next, we characterize under which conditions any closed subset of a symmetric space E, additionally bounded above or below in the sense of $\prec$, is proximinal. Finally, we present complete criteria for K-order continuity in symmetric spaces expressed in terms of the best dominated approximation with respect to the relation $\prec$.

2. Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of reals and positive integers, respectively. Let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the closed unit ball) of a Banach space $(X, \|\cdot\|_X)$. Denote by L^0 the set of all (equivalence classes of) extended real valued Lebesgue measurable functions on $I = [0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions:

(1) If $x \in L^0$, $y \in E$ and $|x| \leq |y|$ a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
(2) There exists a strictly positive $x \in E$.

By E^+ we denote the positive cone of E, i.e. $E^+ = \{x \in E : x \geq 0\}$. We use the notation $A^c = I \setminus A$ for any measurable set A. By μ denote the Lebesgue measure on I. A point $x \in E$ is said to be a *point of order continuity* if for any sequence $(x_n) \subset E^+$ such that $x_n \leq |x|$ and $x_n \to 0$ a.e. we have $\|x_n\|_E \to 0$. A Köthe space E is called *order continuous* (shortly

$E \in (OC)$) if every element x of E is a point of order continuity (see [17]). Unless we say otherwise, we assume in the whole paper that E has the *Fatou property*, i.e. if $(x_n) \subset E^+$, $\sup_{n\in\mathbb{N}} \|x_n\|_E < \infty$ and $x_n \uparrow x \in L^0$, then $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$. For any function $x \in L^0$ we define its *distribution function* by

$$d_x(\lambda) = \mu\{s \in I : |x(s)| > \lambda\}, \quad \lambda \geq 0,$$

and its *decreasing rearrangement* by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t \geq 0.$$

A function $x \in L^0$ is said to be **regular* if

$$m\left(\left\{t \in \operatorname{supp}(x) : |x(t)| < x^*(\alpha)\right\}\right) = 0.$$

The above notion is given under the following convention $x^*(\infty) = \lim_{t\to\infty} x^*(t)$ if $\alpha = \infty$ and $x^*(\infty) = 0$ if $\alpha = 1$. Given $x \in L^0$ we denote the *maximal function* of x^* by

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

It is well known that $x^* \leq x^{**}$, x^{**} is decreasing and subadditive, i.e.

$$(x+y)^{**} \leq x^{**} + y^{**}$$

for any $x, y \in L^0$. Two functions $x, y \in L^0$ are said to be *equimeasurable* (shortly $x \sim y$) if $d_x = d_y$. The notion $\prec$ so-called *Hardy–Littlewood–Pólya relation* is given for any x, y in $L^1 + L^\infty$ by

$$x \prec y \Leftrightarrow x^{**}(t) \leq y^{**}(t) \quad \text{for all } t > 0.$$

A Banach function space $(E, \|\cdot\|_E)$ is called *rearrangement invariant* (r.i. for short) or *symmetric* if whenever $x \in L^0$ and $y \in E$ with $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. For more properties of d_x, x^* and x^{**} see [1,15]. Let $0 < p < \infty$ and $w \in L^0$ be a nonnegative weight function, the Lorentz space $\Gamma_{p,w}$ is a subspace of L^0 such that

$$\|x\|_{\Gamma_{p,w}} := \left(\int_0^\alpha x^{**p}(t)w(t)dt\right)^{1/p} < \infty.$$

Additionally, we assume that w is from class D_p, i.e.

$$W(s) := \int_0^s w(t)dt < \infty \quad \text{and} \quad W_p(s) := s^p\int_s^\alpha t^{-p}w(t)dt < \infty$$

for all $0 < s \leq 1$ if $\alpha = 1$ and for all $0 < s < \infty$ otherwise. These two conditions guarantee that the Lorentz space $\Gamma_{p,w}$ is nontrivial. It is well known that $\left(\Gamma_{p,w}, \|\cdot\|_{\Gamma_{p,w}}\right)$ is a symmetric space with the Fatou property. It is easy to see that for $\alpha = 1$ by the Lebesgue Dominated Convergence Theorem, $\Gamma_{p,w}$ is order continuous. It was proved in [14] that in the case when $\alpha = \infty$ the space $\Gamma_{p,w}$ has order continuous norm if and only if $\int_0^\infty w(t)\,dt = \infty$. The spaces $\Gamma_{p,w}$ were introduced by A.P. Calderón in [3] and it is an interpolation space between L^1 and L^∞ yielded by the Lions–Peetre K-method [2,15]. For more details about the properties of $\Gamma_{p,w}$ the reader is referred to [14,7,8].

A symmetric space E is called K-*monotone* (shortly $E \in (KM)$) if for any $x \in L^1 + L^\infty$ and $y \in E$ such that $x \prec y$, we have $x \in E$ and $\|x\|_E \leq \|y\|_E$. Recall that a symmetric space is K-monotone if and only if E is exact interpolation space between L^1 and L^∞. Moreover, it is well known that a symmetric spaces with Fatou property or with an order continuous norm is K-monotone (see [15]). A point $x \in E$ is said to be a *point of* K*-order continuity* of E if and only if for any $(x_n) \subset E$ such that $x_n \prec x$ and $x_n^* \to 0$ a.e. we have $\|x_n\|_E \to 0$. A symmetric space E is called K*-order continuous* (shortly $E \in (KOC)$) if every element x of E is a point of K-order continuity. A point $x \in E$ is a *point of upper* K*-monotonicity* (*lower* K*-monotonicity*) shortly a *UKM point* (an *LKM point*) of E if and only if for any $y \in E$, $x^* \neq y^*$ with $x \prec y$ (with $y \prec x$), we have $\|x\|_E < \|y\|_E$ ($\|y\|_E < \|x\|_E$), respectively. Clearly, a symmetric space E is *strictly* K*-monotone* (shortly $E \in (SKM)$) if every point of E is a *UKM* point or equivalently if every point of E is an *LKM* point [5]. For more details about K-monotonicity the reader is referred to [4,10,11].

Let $(E, \|\cdot\|_E)$ be a symmetric space and let $Y \subset X$ be a nonempty subset. For $x \in X$ define

$$P_Y(x) = \{y \in Y : \|x - y\| = dist(x, Y)\}.$$

Any element $y \in P_Y(x)$ is called a best approximant in Y to x. A nonempty set $Y \subset X$ is called *proximinal* or *set of existence* if $P_Y(x) \neq \emptyset$ for any $x \in X$. A nonempty set Y is said to be a *Chebyshev set* if it is proximinal and $P_Y(x)$ is a singleton for any $x \in E$.

3. Best K-dominated approximation problems

First, we present auxiliary result which will be useful in our investigation. We omit the proof of the following lemma since it follows instantly by L'Hospital's rule.

Lemma 3.1. *Let $x \in L^1 + L^\infty$ and $x^*(\infty) = 0$, then $x^{**}(\infty) = 0$.*

Now, concerning a local approach, we characterize strict K-monotonicity of symmetric spaces E in terms of the best dominated approximation in the sense of the Hardy–Littlewood–Pólya relation $\prec$. We start our investigation of a *UKM* point and an *LKM* point examining uniqueness of the best approximation for a closed set which is also bounded below in the sense of $\prec$.

Theorem 3.2. *Let E be a symmetric space and $x \in E^+$. If for any closed subset K of the space E such that $x \prec K$, the set $P_K(x)$ is at most singleton, then x is a UKM point and an LKM point.*

Proof. For a contrary we assume that x is no *UKM* point, i.e. there exists $y \in E$ such that $x^* \neq y^*$, $x \prec y$ and $\|x\|_E = \|y\|_E$. Define

$$K = \left\{x + u^* : u \in E, x \prec u \prec y\right\}.$$

Observe that for every $z = x + u^* \in K$ we have $x \prec z$ and $\|x - z\|_E = \|u\|_E$. Since $x \prec u \prec y$, by assumption that E is a symmetric space it follows that $\|x\|_E \leq \|u\|_E \leq \|y\|_E$ (see [1,15]). Therefore, $\|x\|_E = \|x - z\|_E = \inf_{w \in K} \|x - w\|_E$. Moreover, since $x^* \neq y^*$ we obtain $x + x^* \neq x + y^*$ and so K is no singleton. Consequently, $P_K(x) = K$ which contradicts with assumption that $P_K(x)$ is a singleton. Now, we show that K is closed. Assume $(x_n) \subset K$, $x_0 \in E$ and x_n converges to x_0 in norm of E. Then, there exist $(u_n) \subset E$ such that $x_n = x + u_n^*$ and $x \prec u_n \prec y$ for any $n \in \mathbb{N}$. Define $u = x_0 - x$. Observe that

$$\left\|u_n^* - u\right\|_E = \left\|u_n^* - (x_0 - x)\right\|_E = \|x_n - x_0\|_E \to 0 \tag{1}$$

whence, by symmetry of E, passing to subsequence if necessary we obtain that u_n^* converges to u a.e. By symmetry of E and by Lemma 3.2 [15] for any $n \in \mathbb{N}$ and $t > 0$ we have

$$\left(u_n^* - u^*\right)^{**}(t) \left\|\chi_{(0,t)}\right\|_E \leq \left\|u_n^* - u^*\right\|_E \leq \left\|u_n^* - u\right\|_E .$$

Consequently, by condition (1) and by the triangle inequality for the maximal function it follows that u_n^{**} converges to u^{**} for all $t > 0$ and also $u^* = u$ a.e. Thus, by assumption $x \prec u_n \prec y$ for any $n \in \mathbb{N}$ we get $x \prec u \prec y$. So $x_0 \in K$.

Now, suppose that x is not an *LKM* point, then there exists $y \in E$ such that $x^* \neq y^*$ and $y \prec x$ and $\|x\|_E = \|y\|_E$. Denoting

$$K = \left\{x + u^* : u \in E, y \prec u \prec x\right\}$$

and proceeding analogously as above we finish the proof. □

Now, we continue our research of a *UKM* point and an *LKM* point in the notion of uniqueness of the best approximation for a set bounded above with respect to $\prec$.

Theorem 3.3. *Let E be a symmetric space and $x \in E^+$ be *regular. If for any closed subset K of the space E such that $K \prec x$, the set $P_K(x)$ is at most singleton, then x is an LKM point. Additionally, if $y^*(\infty) = 0$ for all $y \in E$, then x is a UKM point.*

Proof. Let x be *regular. By Lemma 2.2 [8] there exists a measure preserving transformation $\sigma : \mathrm{supp}(x) \to \mathrm{supp}(x^*)$ such that $x^* \circ \sigma = x$ a.e. on $\mathrm{supp}(x)$. Clearly, if $\mu(\mathrm{supp}(x)) < \infty$ then we may construct the measure preserving transformation $\sigma : I \to I$ with $x^* \circ \sigma = x$ a.e. on I (see [1,18]). Assume that x is not an *LKM* point. Then, there exists $y \in E$ such that $x^* \neq y^*$, $y \prec x$ and $\|x\|_E = \|y\|_E$. Define

$$K = \left\{\frac{x}{2} - \frac{u^* \circ \sigma}{2} : u \in E, y \prec u \prec x\right\}.$$

Notice that $K \neq \emptyset$ is no singleton. Let $z \in K$, there exists $u \in E$ such that $z = (x - u^* \circ \sigma)/2$ and $y \prec u \prec x$. Then, we have

$$z = \frac{x - u^* \circ \sigma}{2} \prec \frac{x^* + u^*}{2} \quad \text{and} \quad y \prec \frac{x^* + u^*}{2} \prec x.$$

In consequence, $K \prec x$ and also by symmetry of the space E and by assumption that $\|x\|_E = \|y\|_E$ we get

$$\|x - z\|_E = \frac{1}{2}\left\|x + u^* \circ \sigma\right\|_E = \frac{1}{2}\left\|x^* \circ \sigma + u^* \circ \sigma\right\|_E = \frac{1}{2}\left\|x^* + u^*\right\|_E = \|x\|_E.$$

Therefore,

$$\inf_{z \in K} \|x - z\|_E = \|x\|_E$$

which implies that the set $P_K(x) = K$ is no singleton. Now, we prove that K is closed. Let $(z_n) \subset K$ and $z \in E$ such that z_n converges to z in norm of E. Then, there is a sequence $(u_n) \subset E$ such that $z_n = (x - u_n^* \circ \sigma)/2$ and $y \prec u_n \prec x$ for every $n \in \mathbb{N}$. Define $u = x - 2z$. Then,

$$\frac{1}{2}\left\|u - u_n^* \circ \sigma\right\|_E = \frac{1}{2}\left\|x - u_n^* \circ \sigma - 2z\right\|_E = \|z_n - z\|_E \to 0. \tag{2}$$

Thus, by symmetry of E, passing to subsequence and relabelling if necessary we may assume that $u_n^* \circ \sigma$ converges to u a.e. By symmetry of E and by Lemma 3.2 [15] for any $n \in \mathbb{N}$ and $t > 0$ we have

$$\left(u_n^* - u^*\right)^{**}(t)\left\|\chi_{(0,t)}\right\|_E \le \left\|u_n^* - u^*\right\|_E \le \left\|u_n^* \circ \sigma - u\right\|_E$$

and also

$$\begin{aligned}\left\|u^*\circ\sigma-u\right\|_E &\leq \left\|u_n^*\circ\sigma-u\right\|_E+\left\|u^*\circ\sigma-u_n^*\circ\sigma\right\|_E\\ &=\left\|u_n^*\circ\sigma-u\right\|_E+\left\|u^*-u_n^*\right\|_E\\ &\leq 2\left\|u_n^*\circ\sigma-u\right\|_E.\end{aligned}$$

Hence, by condition (2) we obtain u_n^{**} converges to u^{**} for all $t>0$ and $u^*\circ\sigma=u$ a.e. Therefore, $z=(x-u^*\circ\sigma)/2$ and by assumption that $y\prec u_n\prec x$ for any $n\in\mathbb{N}$ we get $y\prec u\prec x$ and so $z\in K$. Now, we assume that x is no *UKM* point. Then, there exists $y\in E$ with $y^*\neq x^*$, $x\prec y$ and $\|x\|_E=\|y\|_E$. We continue the proof in cases.

Case 1. Assume $y^*\geq x^*$. Denote

$$K=\left\{-u:u\in E^+,u\leq y^*\circ\sigma-x,u\prec x\right\}.$$

Clearly, $K\neq\emptyset$ is no singleton and $K\prec x$. Let $z=-u\in K$. Then, $u\in E^+$, $u\leq y^*\circ\sigma-x$ and $u\prec x$. So, by symmetry of E and by assumption that $\|x\|_E=\|y\|_E$ we have $\|x-z\|_E=\|x+u\|_E=\|x\|_E$ for any $-u\in K$. Hence, $P_K(x)=K$. We claim that K is closed. Suppose that $(u_n)\subset E^+$ with $u_n\leq y^*\circ\sigma-x$, $u_n^*\prec x^*$ for every $n\in\mathbb{N}$ and also u_n converges to $u\in E$ in norm of E. Now, by symmetry of E, passing to subsequence and relabelling if necessary we obtain $u_n\to u$ a.e. and $u\leq y^*\circ\sigma-x$ a.e. and also $u\in E^+$. Moreover, by Lemma 3.2 [15] for any $n\in\mathbb{N}$ and $t>0$ we get

$$\left(u_n^*-u^*\right)^{**}(t)\left\|\chi_{(0,t)}\right\|_E\leq\left\|u_n^*-u^*\right\|_E\leq\|u_n-u\|_E.$$

Hence, since $u_n\prec x$, by assumption $\|u_n-u\|_E\to 0$ and by the triangle inequality of the maximal function we conclude that $u\prec x$ and so $u\in K$.

Case 2. Suppose $y^*\ngeq x^*$. Since x^* and y^* are decreasing and right-continuous, by assumption $x\prec y$, there are $0<\gamma<\beta$ such that $x^*\leq y^*$ on $[0,\gamma)$ and $y^*<x^*$ on (γ,β) and also

$$\int_0^\gamma(y^*-x^*)\geq\int_\gamma^\beta(x^*-y^*). \tag{3}$$

Moreover, we may find an interval $[a,b]\subset[0,\gamma]$ and $\epsilon\in(0,x^*(a))$ such that

$$x^*\leq x^*+2\epsilon\chi_{[a,b]}\leq y^*\quad\text{on }[0,\gamma]. \tag{4}$$

Since $x^*>y^*$ on (γ,β) there exist $\epsilon_1\in(0,\epsilon)$ and $\delta\in(0,\min\{a/2,b-a\})$ such that

$$\int_{\beta-\delta}^\beta x^*\geq\int_{\beta-\delta}^\beta y^*+2\delta\epsilon_1.$$

Hence, by condition (3) we get for any $t\in[\gamma,\beta-\delta]$,

$$\begin{aligned}\int_0^\gamma(y^*-x^*)&\geq\int_\gamma^\beta(x^*-y^*)=\int_\gamma^{\beta-\delta}(x^*-y^*)+\int_{\beta-\delta}^\beta(x^*-y^*)\\ &\geq\int_\gamma^{\beta-\delta}(x^*-y^*)+2\delta\epsilon_1\geq\int_\gamma^t(x^*-y^*)+2\delta\epsilon_1.\end{aligned}$$

Therefore, by the triangle inequality for the maximal function we obtain

$$\int_0^t y^*\geq\int_0^t x^*+2\delta\epsilon_1\geq\int_0^t\left(x^*+\epsilon_1\chi_{[a,a+\delta]}\right)^*+\delta\epsilon_1 \tag{5}$$

for any $t \in [\gamma, \beta - \delta]$. Furthermore, by condition (4) we have $x^* + \epsilon_1 \chi_{[a,a+\delta]} \leq y^*$ on $[0, \gamma]$. Then, for any $t \in [0, \gamma]$ we have

$$\int_0^t y^* \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^* . \tag{6}$$

Now, consider that x^* is not constant on (γ, β). Then, there is $c, d \in (\gamma, \beta)$ such that $x^*(c) > x^*(d)$. Hence, by right-continuity and monotonicity of x^* we may find $0 < \delta < \min\{b - a, d - c, \beta - d\}$ and $\epsilon_1 \in (0, \epsilon]$ such that $x^*(d) < x^*(c + \delta) - \epsilon_1$. Thus, we have

$$x^*(t) = \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[c,c+\delta]}\right)^* (t)$$

for all $t \in (\beta - \delta, \infty)$ and also

$$(x^* + \epsilon_1 \chi_{[a,a+\delta]})^*(t) = \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[c,c+\delta]}\right)^* (t)$$

for any $t \in [0, \gamma]$. In consequence, since $x \prec y$, by condition (6) it follows that

$$\int_0^t y^* \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[c,c+\delta]}\right)^* \tag{7}$$

for any $t \in [0, \gamma) \cup (\beta - \delta, \infty)$. Now, according to condition (5) and by the triangle inequality for the maximal function, we conclude

$$\int_0^t y^* \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^* + \delta\epsilon_1 \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[c,c+\delta]}\right)^* \tag{8}$$

for all $t \in [\gamma, \beta - \delta]$. If x^* is constant on (γ, β), then denoting $\lambda = \sup\{\omega > 0 : x^*(w) = x^*(\gamma)\}$ we may assume without loss of generality that $x^*(\lambda) < x^*(\lambda^-)$. Indeed, because otherwise by assumption $x \prec y$ and by right-continuity of x^* we are able to find $\beta > \lambda$ such that $y^* < x^*$ and x^* is not constant on (γ, β) and then we proceed as above. Now, assuming $\beta = \lambda$ it is easy to see that there exist $0 < \delta < \{b - a, \beta - \gamma\}$ and $0 < \epsilon_1 < \min\{\epsilon, x^*(\beta - \delta) - y^*(\beta - \delta), x^*(\beta^-) - x^*(\beta)\}$ such that for all $t > 0$ we get

$$\left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^* (t) - \epsilon_1 \chi_{[\beta-\delta,\beta)}(t) = \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[\beta-\delta,\beta)}\right)^* (t).$$

Hence, x^* is constant and y^* is decreasing on (γ, β), by condition (3) it follows that for all $t \in (\beta - \delta, \infty)$,

$$\begin{aligned}
\int_0^\gamma \left(y^* - \left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^*\right) &= \int_0^\gamma (y^* - x^*) - \int_0^\gamma \epsilon_1 \chi_{[a,a+\delta]} \\
&\geq \int_\gamma^\beta (x^* - y^*) - \int_\gamma^\beta \epsilon_1 \chi_{[\beta-\delta,\beta)} \\
&= \int_\gamma^{\beta-\delta} (x^* - y^*) + \int_{\beta-\delta}^\beta (x^* - \epsilon_1 - y^*) \\
&= \int_\gamma^{\beta-\delta} (x^* - y^*) + \int_{\beta-\delta}^\beta \left((x^* - \epsilon_1 \chi_{[\beta-\delta,\beta)})^* - y^*\right) \\
&\geq \int_\gamma^{\beta-\delta} (x^* - y^*) + \int_{\beta-\delta}^t ((x^* - \epsilon_1 \chi_{[\beta-\delta,\beta)})^* - y^*).
\end{aligned}$$

Therefore, for any $t \in (\beta - \delta, \infty)$ we have

$$\int_0^t y^* \geq \int_0^\gamma \left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^* + \int_\gamma^{\beta-\delta} x^* + \int_{\beta-\delta}^t \left(x^* - \epsilon_1 \chi_{[\beta-\delta,\beta)}\right)^*$$
$$= \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[\beta-\delta,\beta)}\right)^*. \tag{9}$$

Moreover, by conditions (5) and (6) for any $t \in [0, \beta - \delta]$ we obtain

$$\int_0^t y^* \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]}\right)^* = \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[\beta-\delta,\beta)}\right)^*. \tag{10}$$

Now, we define a function z by

$$z = x^* + \epsilon_1 \chi_{[a,a+\delta]} - \begin{cases} \epsilon_1 \chi_{[c,c+\delta)}, & \text{if } x^* \text{ is not constant on } (\gamma, \beta), \\ \epsilon_1 \chi_{[\beta-\delta,\beta)}, & \text{if } x^* \text{ is constant on } (\gamma, \beta). \end{cases}$$

Furthermore, since $[a, a + \delta] \subset [0, \gamma]$ and $(c, c + \delta) \subset (\gamma, \beta)$ and by the Hardy–Littlewood inequality (see [1]) for any $t > 0$ we have

$$\int_0^t z^* \geq \int_0^t \left(x^* + \epsilon_1 \chi_{[a,a+\delta]} - \epsilon_1 \chi_{[\beta-\delta,\beta)}\right) \geq \int_0^t x^*.$$

Hence, by conditions (7)–(10), and by symmetry of E, in view of assumption $\|x\|_E = \|y\|_E$, we get

$$x \prec z \prec y \quad \text{and} \quad \|x\|_E = \|z\|_E\,. \tag{11}$$

Denote

$$u = \epsilon_1 \chi_{[a,a+\delta]} - \begin{cases} \epsilon_1 \chi_{[c,c+\delta)}, & \text{if } x^* \text{ is not constant on } (\gamma, \beta), \\ \epsilon_1 \chi_{[\beta-\delta,\beta)}, & \text{if } x^* \text{ is constant on } (\gamma, \beta), \end{cases}$$

and

$$K = \{-\lambda u \circ \sigma : \lambda \in [0, 1]\}\,.$$

Notice that $|u \circ \sigma| \leq \epsilon_1 \leq x^*(a)$, $\delta < a/2$ and so $\lambda u \prec x$ for any $\lambda \in [0, 1]$, which means that $K \prec x$. Moreover, by condition (11) and by definition of u and z replacing ϵ_1 by $\lambda\epsilon_1 > 0$ where $\lambda \in [0, 1]$ we can easily observe

$$x \prec x^* + \lambda u \prec y$$

for any $\lambda \in [0, 1]$. Then, assuming $v \in K$, there is $\lambda \in [0, 1]$ such that $v = -\lambda u \circ \sigma$ and

$$\|x - v\|_E = \|x + \lambda u \circ \sigma\|_E = \left\|x^* \circ \sigma + \lambda u \circ \sigma\right\|_E = \left\|x^* + \lambda u\right\|_E = \|x\|_E\,.$$

Hence $P_K(x) = K \neq \emptyset$ is no singleton. Obviously, K is closed. $\square$

The immediate consequence of Theorems 3.2 and 3.3 are the following results.

Corollary 3.4. *Let E be a symmetric space. If for any closed subset K of the space E and $x \in E$ with $x \prec K$, the set $P_K(|x|)$ is at most singleton, then E is strictly K-monotone.*

Corollary 3.5. *Let E be a symmetric space such that $x^*(\infty) = 0$ for any $x \in E$. If for any closed subset K of the space E and $x \in E$ with $K \prec x$ the set $P_K(|x|)$ is at most singleton, then E is strictly K-monotone.*

The following example shows that the conditions in Corollaries 3.4 and 3.5 are not necessary for E to be strictly K-monotone.

Example 3.6. Let us consider the Lorentz space $\Gamma_{p,w}$ with a weight $w > 0$, $\int_0^\infty w < \infty$ and $p > 0$. Then, by Theorem 2.10 [10] it follows that $\Gamma_{p,w}$ is strictly K-monotone. Denote $A = \bigcup_{n=0}^{\infty}[2n, 2n+1)$. Define

$$x = 2\chi_I + \frac{1}{1+t}\chi_A, \qquad u = 1\chi_I + \frac{1}{1+t}\chi_A, \qquad v = \left(1 + \frac{1}{1+t}\right)\chi_I$$

and denote an order interval $\mathcal{K} = [u, v]$. Then, by Proposition 2.1 [6] we obtain that $\Gamma_{p,w}$ contains an order-isometric copy of l^∞, whence $x \in \Gamma_{p,w}$ and $\mathcal{K} \subset \Gamma_{p,w}$. Moreover, it is easy to see that $\mathcal{K} \prec x$ and $(x - z)^* = 1$ for any $z \in \mathcal{K}$. Consequently,

$$\inf_{z\in\mathcal{K}} \|x - z\|_{\Gamma_{p,w}} = \|x - u\|_{\Gamma_{p,w}} = \|x - v\|_{\Gamma_{p,w}}$$

which implies that the set $P_{\mathcal{K}}(x) = \mathcal{K}$ is no singleton. Now, we define

$$x = \left(1 + \frac{1}{1+t}\right)\chi_I, \qquad u = 2\chi_I \quad v = u + \frac{1}{1+t}\chi_A, \quad \text{and} \quad \mathcal{K} = [u, v].$$

Clearly, $x \in \Gamma_{p,w}$, $\mathcal{K} \subset \Gamma_{p,w}$, $x \prec \mathcal{K}$ and $(x-z)^* = 1$ for any $z \in \mathcal{K}$, whence the set $P_{\mathcal{K}}(x) = \mathcal{K}$ is no singleton.

Now, we investigate a relation between K-order continuity and proximinality of the best dominated approximation with respect to $\prec$.

Theorem 3.7. *Let E be a symmetric space and let $\mathcal{A} \subset E$ be a closed subset such that for any $a \in \mathcal{A}$ we have $a^* \in \mathcal{A}$. If $x \in E$ is a point of K-order continuity with $x^*(\infty) = 0$ and $\mathcal{A} \prec x$, then the set $P_{\mathcal{A}}(x^*) \neq \emptyset$.*

Proof. Let $\mathcal{A} \prec x$ and $(u_n) \subset \mathcal{A}$ be a minimizing sequence from $\mathcal{A}$, i.e.

$$d = \inf_{a\in\mathcal{A}} \left\|x^* - a\right\|_E = \lim_{n\to\infty} \left\|x^* - u_n\right\|_E . \tag{12}$$

Hence, by Lemma 3.2 [15] we may assume without loss of generality that $u_n = u_n^*$ for any $n \in \mathbb{N}$. Since $u_n \prec x$ for every $n \in \mathbb{N}$, by symmetry of E it follows that

$$u_n^*(t) \left\|\chi_{(0,t)}\right\|_E \le u_n^{**}(t) \left\|\chi_{(0,t)}\right\|_E \le x^{**}(t) \left\|\chi_{(0,t)}\right\|_E \le \|x\|_E$$

for any $t > 0$ and $n \in \mathbb{N}$. Thus, by Helly's Selection Theorem [19] passing to subsequence and relabelling if necessary there exists $u = u^*$ such that u_n^* converges to u^* a.e. on I. Therefore, since $u_n \prec x$ for each $n \in \mathbb{N}$ and by Fatou's lemma [18] we get

$$\int_0^t u^* \le \liminf_{n\to\infty} \int_0^t u_n^* \le \int_0^t x^*$$

for any $t > 0$ which proves $u \prec x$. Thus, by assumption that $x^*(\infty) = 0$ and by Lemma 3.1, we have $u^*(\infty) = 0$. Define $v_n = \bigwedge_{k=n}^{\infty} u_k^*$ for any $n \in \mathbb{N}$. Then, $v_n = v_n^* \le u_n^*$ for every $n \in \mathbb{N}$ and $v_n^* \uparrow u^*$ a.e. on I. Therefore, for any $n \in \mathbb{N}$, $0 \le u^* - v_n^* \le u^*$ and $u^* - v_n^* \downarrow 0$ a.e. on I. Hence, applying the fact $u^*(\infty) = 0$ and property 2.12 [15] we obtain

$$(u^* - v_n^*)^*(t) \to 0 \tag{13}$$

for all $t \in I$. Moreover, by assumption that x is a point of K-order continuity, since $u^* - v_n^* \prec x$ and by condition (13) we get

$$\left\|u^* - v_n^*\right\|_E \to 0. \tag{14}$$

Now, we claim that $\left\|u_n^* - v_n^*\right\|_E \to 0$. Let $\epsilon > 0$. Since $u^*(\infty) = 0$ and $u_n^* \to u^*$ a.e. on I, by monotonicity of decreasing rearrangement passing to subsequence if necessary we may find $t_\epsilon \in I$ such that $u_n^*(t) < \epsilon$ for any $t \geq t_\epsilon$ and $n \in \mathbb{N}$. Thus, since $u_n^* - v_n^* \to 0$ locally in measure and $0 \leq u_n^* - v_n^* \leq u_n^*$ for any $n \in \mathbb{N}$ it follows that $u_n^* - v_n^* \to 0$ in measure on I. Therefore, by Property 2.11 in [15] we get $(u_n^* - v_n^*)^*(t) \to 0$ for all $t \in I$, whence by assumption that x is a point of K-order continuity and by the inequality $u_n^* - v_n^* \prec x$ for any $n \in \mathbb{N}$ we have

$$\left\|u_n^* - v_n^*\right\|_E \to 0.$$

Hence, by condition (14) and by the triangle inequality of the norm in E we obtain

$$\left\|u_n^* - u^*\right\|_E \to 0$$

which implies that $u^* \in \mathcal{A}$, in view of the assumption $\mathcal{A}$ is closed. In consequence, by condition (12) we finish the proof. □

Theorem 3.8. *Let E be a symmetric space and $x \in E^+$ be *regular. If for any closed subset $\mathcal{A}$ of E such that $\mathcal{A} \prec x$ we have the set $P_{\mathcal{A}}(x) \neq \emptyset$, then x is a point of K-order continuity.*

Proof. Let us suppose for a contrary that there exists $(x_n) \subset E$ such that $x_n^* \to 0$ a.e. on I and $x_n \prec x$ for any $n \in \mathbb{N}$ and also $\delta = \inf_{n\in\mathbb{N}} \|x_n\|_E > 0$. Since x is *regular, by Lemma 2.2 [8] there exists a measure preserving transformation $\sigma : \operatorname{supp}(x) \to \operatorname{supp}(x^*)$ such that $x^* \circ \sigma = x$ a.e. on $\operatorname{supp}(x)$. It is obvious that if $\mu(\operatorname{supp}(x)) < \infty$ then we may construct the measure preserving transformation $\sigma : I \to I$ with $x^* \circ \sigma = x$ a.e. on I (see [1,18]). Notice that

$$x_n^* = x_n^* \wedge x^* + (x_n^* - x^*)^+$$

for a.e. on I and for any $n \in \mathbb{N}$. Hence, by triangle inequality for any $n \in \mathbb{N}$ we get

$$\delta = \inf_{n\in\mathbb{N}} \|x_n\|_E \leq \left\|x_n^* \wedge x^*\right\|_E + \left\|(x_n^* - x^*)^+\right\|_E .$$

Therefore, we can easily see that

$$\left\|x_n^* \wedge x^*\right\|_E \geq \frac{\delta}{2} \quad \text{or} \quad \left\|(x_n^* - x^*)^+\right\|_E \geq \frac{\delta}{2}$$

for any $n \in \mathbb{N}$. Now, we consider two cases.

Case 1. Assume that there exist a subsequence $(n_k) \subset \mathbb{N}$ and $\delta_1 \in (0, \delta/2]$ such that

$$\inf_{k\in\mathbb{N}} \left\|x_{n_k}^* \wedge x^*\right\|_E = \delta_1.$$

Now, passing to subsequence and relabelling if necessary we obtain $\left\|x_n^* \wedge x^*\right\|_E \geq \delta_1$ for all $n \in \mathbb{N}$. Define for every $n \in \mathbb{N}$,

$$y_n = \frac{1}{2}\left(1 + \frac{1}{n}\right) \bigvee_{k=n}^{\infty} x_k^* \wedge x^*.$$

Clearly, $y_{n+1} \leq y_n \leq x^*$ a.e. on I and for any $n \in \mathbb{N}$. Moreover,

$$\|y_n\|_E > \|y_{n+1}\|_E > \inf_{k\in\mathbb{N}} \|y_k\|_E \geq \delta_1/2 \tag{15}$$

for any $n \in \mathbb{N}$. Denote $\mathcal{A} = \{x - y_n \circ \sigma : n \in \mathbb{N}\}$. By definition of y_n we observe $0 \leq x - y_n \circ \sigma \leq x$ for any $n \in \mathbb{N}$, and consequently $\mathcal{A} \prec x$. Furthermore, by condition (15) we have

$$\|x - (x - y_n \circ \sigma)\|_E = \|y_n\|_E > \inf_{k \in \mathbb{N}} \|y_k\|_E = \inf_{a \in \mathcal{A}} \|x - a\|_E$$

for any $n \in \mathbb{N}$. Hence, the set $P_{\mathcal{A}}(x) = \emptyset$. Now, we claim that $\mathcal{A}$ is closed. Indeed, taking a subsequence $(n_k) \subset \mathbb{N}$ and $y \in E$ such that $\left\|y_{n_k} - y\right\|_E \to 0$, by symmetry of E passing to subsequence and relabelling if necessary we conclude y_n converges to y a.e. on I. Since $x_n^* \to 0$ a.e. on I and by definition of y_n this yields that $y_n \to 0$ a.e. on I, which implies $y = 0$ a.e. on I. By assumption that $\left\|x_n^* \wedge x^*\right\|_E \geq \delta_1$ for any $n \in \mathbb{N}$ we have

$$\frac{\delta_1}{2} \leq \frac{1}{2}\left(1 + \frac{1}{n}\right)\left\|x_n^* \wedge x^*\right\|_E \leq \|y_n\|_E$$

for all $n \in \mathbb{N}$, which contradicts with the fact $\|y_n\|_E \to 0$ and finishes case 1.

Case 2. Assume $\left\|x_n^* \wedge x^*\right\|_E \to 0$ and there exist $(n_k) \subset \mathbb{N}$ and $\delta_1 \in (0, \delta/2]$ such that

$$\inf_{k \in \mathbb{N}} \left\|(x_{n_k}^* - x^*)^+\right\|_E = \delta_1.$$

Since $(x_{n_k}^* - x^*)^+ \geq 0$ for all $k \in \mathbb{N}$, passing to subsequence and relabelling if necessary we may suppose without loss of generality that

$$\inf_{n \in \mathbb{N}} \left\|x^* + \frac{1}{2}(x_n^* - x^*)^+\right\|_E = \delta_2 \geq \frac{\delta_1}{2} \quad \text{and} \quad \left\|x_n^* \wedge x^*\right\|_E \to 0.$$

Now, by definition of infimum, passing to subsequence and relabelling again if necessary we may assume that for any $n \in \mathbb{N}$,

$$\left\|x^* + \frac{1}{2}(x_n^* - x^*)^+\right\|_E \geq \left\|x^* + \frac{1}{2}(x_{n+1}^* - x^*)^+\right\|_E \geq \delta_2. \tag{16}$$

Denote

$$y_n = \left(1 - \frac{1}{n}\right)\frac{x^*}{2} - \frac{1}{4}\left(1 + \frac{1}{n}\right)(x_n^* - x^*)^+$$

for every $n \in \mathbb{N}$ and also $\mathcal{A} = \{y_n \circ \sigma : n \in \mathbb{N}\}$. It is easy to see that $(x_n^* - x^*)^+ \leq x_n^* \prec x$ for any $n \in \mathbb{N}$, whence

$$|y_n| \leq \left(1 - \frac{1}{n}\right)\frac{x^*}{2} + \frac{1}{4}\left(1 + \frac{1}{n}\right)x_n^* \leq \frac{1}{2}x_n^* + \frac{1}{2}x^* \prec x$$

for each $n \in \mathbb{N}$, which implies that $\mathcal{A} \prec x$. Moreover, by symmetry of E and by condition (16) we have

$$\begin{aligned}
\|x - y_n \circ \sigma\|_E = \left\|(x^* - y_n) \circ \sigma\right\|_E &= \frac{1}{2}\left(1 + \frac{1}{n}\right)\left\|x^* + \frac{1}{2}(x_n^* - x^*)^+\right\|_E \\
&> \frac{1}{2}\left(1 + \frac{1}{n+1}\right)\left\|x^* + \frac{1}{2}(x_{n+1}^* - x^*)^+\right\|_E \\
&= \|x - y_{n+1} \circ \sigma\|_E \\
&> \frac{\delta_2}{2} = \inf_{a \in \mathcal{A}} \|x - a\|_E
\end{aligned}$$

for all $n \in \mathbb{N}$. In consequence, $P_{\mathcal{A}}(x) = \emptyset$. Now, we show that $\mathcal{A}$ is closed. Suppose for a contrary that it is not true, then there are $(n_k) \subset \mathbb{N}$ and $y \in E \setminus \mathcal{A}$ such that $\left\|y_{n_k} - y\right\|_E \to 0$. Thus, by symmetry of E passing to subsequence and relabelling if necessary we get y_n converges to y a.e. on I. Moreover, by assumption $x_n^* \to 0$ a.e. on I and by construction of y_n it follows that $y_n \to x^*/2$ a.e. on I and so $y = x^*/2$ a.e. on I. Hence, we have

$$\|y_n - y\|_E = \left\|\frac{x^*}{2n} + \frac{1}{4}\left(1 + \frac{1}{n}\right)(x_n^* - x^*)^+\right\|_E \to 0. \tag{17}$$

Furthermore, in view of the assumption that $\left\|(x_n^* - x^*)^+\right\|_E \geq \delta_1$ for every $n \in \mathbb{N}$ and by the triangle inequality we obtain

$$\frac{\delta_1}{4} \leq \frac{1}{4}\left(1 + \frac{1}{n}\right)\left\|(x_n^* - x^*)^+\right\|_E \leq \left\|\frac{x^*}{2n} + \frac{1}{4}\left(1 + \frac{1}{n}\right)(x_n^* - x^*)^+\right\|_E + \frac{\|x^*\|_E}{2n}$$

for any $n \in \mathbb{N}$. Thus, by condition (17) we obtain a contradiction and complete the proof. □

Immediately, by the previous theorems we obtain the following corollaries.

Corollary 3.9. *Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$ and let $\mathcal{A} \subset E$ be a closed subset such that for any $a \in \mathcal{A}$ we have $a^* \in \mathcal{A}$. If E is K-order continuous and $\mathcal{A} \prec x$ with $x \in E$, then the set $P_{\mathcal{A}}(x^*)$ is proximinal.*

Corollary 3.10. *Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$. If for any closed subset $\mathcal{A}$ of E and for any $x \in E$ with $\mathcal{A} \prec x$ the set $P_{\mathcal{A}}(x)$ is proximinal, then E is K-order continuous.*

A point $a \in E$ is called a K-upper bound of a subset $\mathcal{A} \subset E$ if for any $a' \in \mathcal{A}$ we have $a' \prec a$. If there exists a K-upper bound of a subset $\mathcal{A} \subset E$, then the set $\mathcal{A}$ is said to be K-bounded above. Now, we discuss proximinality of the best dominated approximation under the additional notion K-bounded above.

Theorem 3.11. *Let E be a symmetric space, $x \in E$ and let $\mathcal{A} \subset E$ be a closed K-bounded above subset such that $x \prec \mathcal{A}$, $a^* \in \mathcal{A}$ for any $a \in \mathcal{A}$. If a K-upper bound of $\mathcal{A}$ is a point of K-order continuity with finite distribution, then the set $P_{\mathcal{A}}(x^*) \neq \emptyset$.*

Proof. Let $x \in E$ and $\mathcal{A} \subset E$ be a closed K-bounded above subset with $x \prec \mathcal{A}$ and let $(u_n) \subset \mathcal{A}$ be a minimizing sequence, i.e.

$$\inf_{a \in \mathcal{A}} \left\|x^* - a\right\|_E = \lim_{n \to \infty} \left\|x^* - u_n\right\|_E. \tag{18}$$

Let $b \in E$ be a K-upper bound of $\mathcal{A}$. Thus, $u_n \prec b$ for all $n \in \mathbb{N}$ and so by symmetry of E we get $\|u_n\|_E \leq \|b\|_E$ for every $n \in \mathbb{N}$. Hence, proceeding analogously as in the proof of Theorem 3.7 we may assume that $u_n = u_n^*$ converges to u^* a.e. on I.

Define $v_n = \bigwedge_{k=n}^{\infty} u_k^*$ for any $n \in \mathbb{N}$. Then, $v_n = v_n^*$ for all $n \in \mathbb{N}$ and

$$0 \leq u^* - v_n^* \downarrow 0 \quad \text{and} \quad 0 \leq u_n^* - v_n^* \to 0 \tag{19}$$

a.e. on I. Hence, since $0 \leq u_n^* - v_n^* \leq u_n^* \prec b$ for any $n \in \mathbb{N}$ and $b^*(\infty) = 0$ and by Lemma 3.1 it is easy to see that $u_n^* - v_n^* \to 0$ in measure. Thus, by property 2.11 in [15] we obtain

$$(u_n^* - v_n^*)^*(t) \to 0,$$

for any $t > 0$. So, by assumption that b is a point of K-order continuity we conclude

$$\left\|u_n^* - v_n^*\right\|_E \to 0. \tag{20}$$

Now, according to condition (19), by Lebesgue Monotone Convergence Theorem (see [18]) we obtain $v_n^{**}(t) \to u^{**}(t)$ and in view of the following inequality (see [1]).

$$(u_n^* - v_n^*)^{**}(t) \left\|\chi_{(0,t)}\right\|_E \le \left\|u_n^* - v_n^*\right\|_E$$

for all $t > 0$ and by condition (20) we get $u_n^{**}(t) \to u^{**}(t)$ for all $t > 0$. Hence, since $x \prec u_n \prec b$ for each $n \in \mathbb{N}$ we conclude $x \prec u \prec b$. Therefore, by assumption $b^*(\infty) = 0$ applying Lemma 3.1 and by condition (19) we can easily prove that $u_n^* \to u^*$ in measure. Consequently, according to property 2.11 in [15] we obtain $(u_n^* - u^*)^*(t) \to 0$ for all $t > 0$. Then, since $u_n^* - u^* \prec 2b$ and by assumption that b is a point of K-order continuity it follows that

$$\left\|u_n^* - u^*\right\|_E \to 0.$$

Finally, by assumption that $\mathcal{A}$ is closed and by condition (18) it follows that $P_{\mathcal{A}}(x^*) \neq \emptyset$. □

Theorem 3.12. *Let E be a symmetric space and $x \in E$. If for any closed K-bounded above subset $\mathcal{A}$ of E such that $x \prec \mathcal{A}$ and $a^* \in \mathcal{A}$ for any $a \in \mathcal{A}$ we have the set $P_{\mathcal{A}}(x^*) \neq \emptyset$, then x is a point of K-order continuity.*

We present the proof of the above theorem for the sake of completeness and convenience of the reader although it is similar in some parts to the proof of Proposition 3.3 in [16].

Proof. Let $(x_n) \subset E$ and $x \in E$ be such that $x_n^* \to 0$ a.e., $x_n \prec x$ for any $n \in \mathbb{N}$ and $\delta = \inf_{n\in\mathbb{N}} \|x_n\|_E > 0$. By definition of δ passing to subsequence and relabelling if necessary we may assume that $\|x_n\|_E \ge \|x_{n+1}\|_E \ge \delta$ for any $n \in \mathbb{N}$. Denote $y_n = \frac{1}{2}(1 + \frac{1}{n})x_n^*$ for any $n \in \mathbb{N}$. Clearly, $y_n = y_n^* \to 0$ a.e. and also $\|y_n\|_E > \|y_{n+1}\| > \inf_{n\in\mathbb{N}} \|y_n\|_E = \delta/2$ for any $n \in \mathbb{N}$. Define $\mathcal{A} = \{x^* + y_n : n \in \mathbb{N}\}$. Notice that $x^* \le a = a^*$ for any $a \in \mathcal{A}$. Hence, $x \prec \mathcal{A}$ and for any $n \in \mathbb{N}$,

$$\frac{\delta}{2} = \inf_{a\in\mathcal{A}} \left\|x^* - a\right\|_E < \|y_n\|_E = \left\|x^* - (x^* + y_n)\right\|_E . \tag{21}$$

Therefore, the set $P_{\mathcal{A}}(x^*) = \emptyset$. We claim that $\mathcal{A}$ is closed. Suppose to a contrary that there exists $(a_n) \subset \mathcal{A}$ and $a \notin \mathcal{A}$ such that $\|a_n - a\|_E \to 0$. So, by symmetry of E passing to subsequence and relabelling if necessary we obtain $a_n \to a$ a.e. Thus, since $x^* + y_n \to x^*$ a.e. we get $a = x^*$ a.e. Consequently, by condition (21) we conclude a contradiction which proves our claim. Finally, since $x_n \prec x$ for all $n \in \mathbb{N}$ it follows that $\mathcal{A} \prec 2x$, so the set $\mathcal{A}$ is K-bounded above. □

The immediate consequence of Theorems 3.11 and 3.12 is the following characterization of K-order continuity in symmetric spaces.

Theorem 3.13. *Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$. Then, the following conditions are equivalent.*

(i) *E is K-order continuous.*
(ii) *For any $x \in E$ and $\mathcal{A} \subset E$ a closed K-bounded above subset such that $x \prec \mathcal{A}$, $a^* \in \mathcal{A}$ for any $a \in \mathcal{A}$, we have $P_{\mathcal{A}}(x^*) \neq \emptyset$.*

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References

[1] C. Bennett, R. Sharpley, Interpolation of Operators, in: Pure and Applied Mathematics Series, vol. 129, Academic Press Inc., 1988.
[2] Yu.A. Brudnyi, N.Ya. Kruglyak, Interpolation Functors and Intepolation Spaces, North-Holland, Amsterdam, 1991.
[3] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190.
[4] V.I. Chilin, P.G. Dodds, A.A. Sedaev, F.A. Sukochev, Characterizations of Kadec-Klee properties in symmetric spaces of measurable functions, Trans. Amer. Math. Soc. 348 (12) (1996) 4895–4918.
[5] M. Ciesielski, On geometric structure of symmetric spaces, J. Math. Anal. Appl. 430 (1) (2015) 98–125.
[6] M. Ciesielski, A. Kamińska, P. Kolwicz, R. Płuciennik, Monotonicity and rotundity of Lorentz spaces $\Gamma_{p,w}$, Nonlinear Anal. 75 (2012) 2713–2723.
[7] M. Ciesielski, A. Kamińska, R. Płuciennik, Gâteaux derivatives and their applications to approximation in Lorentz spaces $\Gamma_{p,w}$, Math. Nachr. 282 (9) (2009) 1242–1264.
[8] M. Ciesielski, P. Kolwicz, A. Panfil, Local monotonicity structure of symmetric spaces with applications, J. Math. Anal. Appl. 409 (2014) 649–662.
[9] M. Ciesielski, P. Kolwicz, R. Płuciennik, Local approach to Kadec-Klee properties in symmetric function spaces, J. Math. Anal. Appl. 426 (2015) 700–726.
[10] M. Ciesielski, P. Kolwicz, R. Płuciennik, A note on strict K-monotonicity of some symmetric function spaces, Commun. Math. 53 (2) (2013) 311–322.
[11] H. Hudzik, A. Kamińska, M. Mastyło, On geometric properties of Orlicz-Lorentz spaces, Canad. Math. Bull. 40 (3) (1997) 316–329.
[12] H. Hudzik, W. Kurc, Monotonicity properties of Musielak-Orlicz spaces and dominated best approximant in Banach lattices, J. Approx. Theory 95 (3) (1998) 353–368.
[13] A. Kamińska, H.J. Lee, G. Lewicki, Extreme and smooth points in Lorentz and Marcinkiewicz spaces with applications to contractive projections, Rocky Mountain J. Math. 39 (5) (2009) 1533–1572.
[14] A. Kamińska, L. Maligranda, On Lorentz spaces $\Gamma_{p,w}$, Israel J. Math. 140 (2004) 285–318.
[15] S.G. Krein, Yu.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, in: Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence, RI, 1982, Translated from the Russian by J. Szűcs.
[16] W. Kurc, Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation, J. Approx. Theory 69 (2) (1992) 173–187.
[17] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces. II. Function Spaces, Springer-Verlag, Berlin-New York, 1979.
[18] H.L. Royden, Real Analysis, third ed., Macmillan Publishing Company, New York, 1988.
[19] Š. Schwabik, M. Tvrdý, O. Vejvoda, Differential and Integral Equations. Boundary Value Problems and Adjoints, D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London, 1979.