

On the two-dimensional Marcinkiewicz means with respect to Walsh–Kaczmarz system[☆]

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Abstract

In this paper we prove that the maximal operator of the Marcinkiewicz means of two-dimensional integrable functions with respect to the Walsh–Kaczmarz system is of weak type $(1, 1)$. Moreover, the Marcinkiewicz means $\mathcal{M}_n f$ converge to f almost everywhere, for any integrable function f .

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1. Introduction

In 1948 Šneider [19] introduced the Walsh–Kaczmarz system and showed that the inequality

$$\limsup_{n \rightarrow \infty} \frac{D_n^K(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [14] and Young [21] proved that the Walsh–Kaczmarz system is a convergence system. Skvorcov in 1981 [18] showed that the Fejér means converges uniformly to f for any continuous functions f . Gát [5,7] proved, for any integrable functions, that the Fejér means converges almost everywhere to the function.

In 1939 Marcinkiewicz [11] proved for two-dimensional trigonometric system that the Marcinkiewicz means of a function converge to the function itself almost everywhere for all

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$f \in L \log L([0, 2\pi]^2)$ (that is $\mathcal{M}_n f \rightarrow f$ a.e. ($n \rightarrow \infty$)). Zhizhiashvili [22] improved this result for $f \in L([0, 2\pi]^2)$. Dyachenko [4] proved this result for dimensions greater than 2. In 1996 Weisz proved [20] that the maximal operator of Marcinkiewicz means of double Walsh–Fourier series is bounded from Hardy–Lorentz space to Lorentz space for $p > 2/3$ and is weak type $(1, 1)$. In 2003 Goginava proved [10] that the Marcinkiewicz means of a function with respect to the d -dimensional Walsh–Paley system converge to the function almost everywhere. In 2004 Gát generalized [6] this result with respect to two-dimensional bounded Vilenkin systems. Motivated by the work of Gát [6] and Goginava [9,10], we prove that:

Theorem 1. For all $f \in L^1(G \times G)$

$$\mathcal{M}_n^k f \rightarrow f \quad \text{a.e.}$$

relation holds.

First, we give a brief introduction to the theory of dyadic analysis [17,1].

Denote by \mathbf{Z}_2 the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on \mathbf{Z}_2 is given in the way that the measure of a singleton is $1/2$. Let

$$G := \prod_{k=0}^{\infty} \mathbf{Z}_2.$$

G is called the Walsh group. The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbf{N}$) ($\mathbf{N} := \{0, 1, \dots\}$, $\mathbf{P} := \mathbf{N} \setminus \{0\}$).

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Consequently, G is a compact Abelian group. Dyadic intervals are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G$, $n \in \mathbf{P}$. They form a base for the neighborhoods of G . Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Denote the usual Lebesgue spaces on G by $L^p(G)$ (with the corresponding norm $\|\cdot\|_p$), \mathcal{A}_n the σ -algebra generated by the sets $I_n(x)$ ($x \in G$) and E_n the conditional expectation operator with respect to \mathcal{A}_n ($n \in \mathbf{N}$). The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \quad (i \in \mathbf{N}),$$

where only a finite number of n_i 's are different from zero. Let the order of n be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$.

Define the Walsh–Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}$$

and the Walsh–Kaczmarz functions by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

$\omega := (\omega_n : n \in \mathbf{N})$ is called the Walsh–Paley system and $\kappa := (\kappa_n : n \in \mathbf{N})$ is called the Walsh–Kaczmarz system. It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_k : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{N}$ and $\kappa_0 = \omega_0$.

Let the transformation $\tau_A : G \rightarrow G$ be defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots)$$

for $A \in \mathbf{N}$. By the definition of τ_A , we have

$$\kappa_n(x) = r_{|n|}(x) \omega_n(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G)$$

(it was given by Skvorcov, see [18]).

For a function f in $L^1(G)$ the Fourier coefficients, the partial sums of Fourier series, the Dirichlet kernels, the Fejér means and the Fejér kernels [2,8] are defined as follows:

$$\begin{aligned} \hat{f}^\alpha(n) &:= \int_G f \bar{\alpha}_n, \quad S_n^\alpha f := \sum_{k=0}^{n-1} \hat{f}^\alpha(k) \alpha_k, \\ D_n^\alpha &:= \sum_{k=0}^{n-1} \alpha_k, \quad \sigma_n^\alpha f := \frac{1}{n} \sum_{k=0}^n S_k^\alpha f, \quad K_n^\alpha := \frac{1}{n} \sum_{k=0}^n D_k^\alpha, \end{aligned}$$

where $\alpha_n = \omega_n$ or κ_n ($n \in \mathbf{P}$). $D_0^\alpha := 0$, $K_0^\alpha := 0$. The 2^n th Dirichlet kernels have a closed form (see e.g. [17])

$$D_{2^n}^\omega(x) = D_{2^n}^\kappa(x) = \begin{cases} 0 & \text{if } x \notin I_n, \\ 2^n & \text{if } x \in I_n. \end{cases}$$

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Walsh group be $G \times G$ and the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels, the Marcinkiewicz means and Marcinkiewicz kernels be defined as

$$\begin{aligned} \hat{f}^\alpha(n_1, n_2) &:= \int_{G \times G} f \alpha_{n_1} \alpha_{n_2} d\mu, \\ S_{n_1, n_2}^\alpha f(x^1, x^2) &:= \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} \hat{f}^\alpha(k, l) \alpha_k(x^1) \alpha_l(x^2), \\ D_{n_1, n_2}^\alpha(x^1, x^2) &:= D_{n_1}^\alpha(x^1) D_{n_2}^\alpha(x^2), \\ \mathcal{M}_n^\alpha f &:= \frac{1}{n} \sum_{k=0}^n S_{k,k}^\alpha f, \quad \mathcal{K}_n^\alpha := \frac{1}{n} \sum_{k=0}^n D_{k,k}^\alpha, \end{aligned}$$

where $\alpha_n =$ either ω_n or κ_n ($n \in \mathbf{P}$). It is well known that

$$\begin{aligned} S_{\underline{n}}^{\alpha} f(\underline{y}) &= \int_{G \times G} f(\underline{x} + \underline{y}) D_{\underline{n}}^{\alpha}(\underline{x}) d\mu(\underline{x}), \\ \mathcal{M}_{\underline{n}}^{\alpha} f(\underline{y}) &= \int_{G \times G} f(\underline{x} + \underline{y}) \mathcal{K}_{\underline{n}}^{\alpha}(\underline{x}) d\mu(\underline{x}), \\ (\underline{n} &= (n_1, n_2), \underline{y} \in G \times G, f \in L^1(G \times G)). \end{aligned}$$

Let $\mathcal{A}_{n,n}$ denote the σ -algebra generated by the sets $I_n(x) \times I_n(y)$ ($x, y \in G$) and $E_{n,n}$ the conditional expectation operator with respect to $\mathcal{A}_{n,n}$ ($n \in \mathbf{N}$). Define the maximal operator of the Marcinkiewicz means and the maximal function of a function $f \in L^1(G \times G)$ by

$$\mathcal{M}^{\kappa*} f := \sup_{n \in \mathbf{P}} |\mathcal{M}_n^{\kappa} f|, \quad f^* := \sup_{n \in \mathbf{N}} |E_{n,n} f|,$$

where

$$E_{n,n} f(y^1, y^2) = S_{2^n, 2^n} f(y^1, y^2) = 2^{2n} \int_{I_n(y^1) \times I_n(y^2)} f(x^1, x^2) d\mu(x^1, x^2),$$

We say that the operator $T : L^1 \rightarrow L^0$ is of type (p, p) , if $\|Tf\|_p \leq c_p \|f\|_p$ for some constant c_p for all $f \in L^p$ ($1 \leq p \leq \infty$); finally, T is of weak type $(1, 1)$, if there exists a $c > 0$ such that $\mu(\{y : Tf(y) > \lambda\}) \leq c \|f\|_1 / \lambda$ for all $\lambda > 0$ and $f \in L^1$.

The following lemma plays an important role during the proof of the main theorem (see e.g. [12,16]).

Lemma 1. *The maximal function f^* of a function f is of type (p, p) for all $1 < p \leq \infty$ and weak type $(1, 1)$.*

Let

$$\mathcal{K}_{a,b}^{\alpha} := \sum_{j=a}^{a+b-1} D_{j,j}^{\alpha}, \quad K_{a,b}^{\alpha} := \sum_{j=a}^{a+b-1} D_j^{\alpha} \quad (a, b \in \mathbf{N}, \alpha = \omega, \kappa)$$

and $n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i$ ($n, s \in \mathbf{N}$). By simple calculations we get

$$\begin{aligned} n\mathcal{K}_n^{\alpha} &= \sum_{s=0}^{|n|} n_s \mathcal{K}_{n^{(s+1)}, 2^s}^{\alpha} + D_{n,n}^{\alpha} =: n\tilde{\mathcal{K}}_n^{\alpha} + D_{n,n}^{\alpha}, \\ nK_n^{\alpha} &= \sum_{s=0}^{|n|} n_s K_{n^{(s+1)}, 2^s}^{\alpha} + D_n^{\alpha} =: n\tilde{K}_n^{\alpha} + D_n^{\alpha} \quad (\alpha = \omega, \kappa). \end{aligned}$$

We use the following lemmas of Gát (see [5]).

Lemma 2 (Gát [5]). *Let $A, t \in \mathbf{N}$, $A > t$. Suppose that $x \in I_t \setminus I_{t+1}$. Then*

$$K_{2^A}^{\omega}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_A, \\ 2^{t-1} & \text{if } x - x_t e_t \in I_A. \end{cases}$$

If $x \in I_A$, then $K_{2^A}^{\omega}(x) = \frac{2^A + 1}{2}$.

Lemma 3 (Gát [5]). Suppose that $s, t, n \in \mathbf{N}$ and $x \in I_t \setminus I_{t+1}$. If $s \leq t \leq |n|$, then $|K_{n^{(s+1)}, 2^s}^\omega(x)| \leq c2^{s+t}$, while if $t < s \leq |n|$, then

$$K_{n^{(s+1)}, 2^s}^\omega(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x) 2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases}$$

The proof of the main theorems will be based on the following decomposition of the Marcinkiewicz kernels (see [13]).

Lemma 4. For $k \in \mathbf{P}$ and $(x^1, x^2) \in G \times G$ holds

$$\begin{aligned} k\mathcal{K}_k^\kappa(x^1, x^2) &= 1 + \sum_{j=0}^{|k|-1} 2^j D_{2^j, 2^j}(x^1, x^2) + \sum_{j=0}^{|k|-1} 2^j D_{2^j}(x^1) r_j(x^2) K_{2^j}^\omega(\tau_j(x^2)) \\ &\quad + \sum_{j=0}^{|k|-1} 2^j D_{2^j}(x^2) r_j(x^1) K_{2^j}^\omega(\tau_j(x^1)) \\ &\quad + \sum_{j=0}^{|k|-1} 2^j r_j(x^1 + x^2) \mathcal{K}_{2^j}^\omega(\tau_j(x^1), \tau_j(x^2)) \\ &\quad + (k - 2^{|k|})(D_{2^{|k|}, 2^{|k|}}(x^1, x^2) + D_{2^{|k|}}(x^1) r_{|k|}(x^2) K_{k-2^{|k|}}^\omega(\tau_{|k|}(x^2)) \\ &\quad + D_{2^{|k|}}(x^2) r_{|k|}(x^1) K_{k-2^{|k|}}^\omega(\tau_{|k|}(x^1)) \\ &\quad + r_{|k|}(x^1 + x^2) \mathcal{K}_{k-2^{|k|}}^\omega(\tau_{|k|}(x^1), \tau_{|k|}(x^2))). \end{aligned}$$

To prove the main theorem, as the decomposition lemma shows, we need to know the exact values of the Marcinkiewicz kernels (see [5,7]). (Gát gave [6] estimations for the kernels only, but in the case of Walsh–Kaczmarz system this is not enough.) Thus, we need the following lemma and corollaries (see [13]).

Lemma 5. Suppose that $s, t, n \in \mathbf{N}$, $(x^1, x^2) \in I_{|n|+1} \times (I_t \setminus I_{t+1})$. If $s \leq t \leq |n|$, then

$$\left| \mathcal{K}_{n^{(s+1)}, 2^s}^\omega(x^1, x^2) \right| \leq c2^{s+t} (n^{(s+1)} + 2^s).$$

If $t < s \leq |n|$ then we have

$$\begin{aligned} &\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(x^1, x^2) \\ &= \begin{cases} 0 & \text{if } \exists l, \quad t < t+l < s, \quad x^2 - x_t^2 e_t - e_{t+l} \notin I_s, \quad x_{t+l}^2 \neq 0, \\ \omega_{n^{(s+1)}}(x^2) 2^{2t+s+l-2} & \text{if } \exists l, \quad t < t+l < s, \quad x^2 - x_t^2 e_t - e_{t+l} \in I_s, \quad x_{t+l}^2 \neq 0, \\ \omega_{n^{(s+1)}}(x^2) 2^{t-2} n(s, t) & \text{if } x^2 - x_t^2 e_t \in I_s, \end{cases} \end{aligned}$$

where $n(s, t) = n^{(s+1)} 2^{s+1} - 2^t (2^s - 2^{t-1} + \frac{1}{2}) - 2^s (2^s - 2)$.

Corollary 1. Let $A, t, l \in \mathbb{N}$, $(x^1, x^2) \in I_A \times (I_t \setminus I_{t+1})$ and $t < t + l < A$. Then

$$\begin{aligned} & \mathcal{K}_{2^A}^\omega(x^1, x^2) \\ &= \begin{cases} 0 & \text{if } \exists l, \quad t < t + l < A, \quad x^2 - x_t^2 e_t - e_{t+l} \notin I_A, \quad x_{t+l}^2 \neq 0, \\ 2^{2t+l-2} & \text{if } \exists l, \quad t < t + l < A, \quad x^2 - x_t^2 e_t - e_{t+l} \in I_A, \quad x_{t+l}^2 \neq 0, \\ 2^{t-2} n(A, t) & \text{if } x^2 - x_t^2 e_t \in I_A, \end{cases} \end{aligned}$$

where $n(A, t) = -\frac{2^t}{2^A} (2^A - 2^{t-1} + \frac{1}{2}) - (2^A - 2)$.

If $(x^1, x^2) \in I_A \times I_A$, then

$$\mathcal{K}_{2^A}^\omega(x^1, x^2) = \frac{(2^A + 1)(2^{A+1} + 1)}{6}.$$

Corollary 2. Let $A, t^1, t^2 \in \mathbb{N}$, $t^1 \leq t^2 < A$, $(x^1, x^2) \in (I_{t^1} \setminus I_{t^1+1}) \times (I_{t^2} \setminus I_{t^2+1})$. Then

$$\begin{aligned} & \mathcal{K}_{2^A}^\omega(x^1, x^2) \\ &= \begin{cases} 0 & \text{if } \exists i \in B_1, \quad x_i^1 \neq x_i^2, \\ 0 & \text{if } \forall i \in B_1, \quad x_i^1 = x_i^2, \quad \exists l \in B_2, \quad x^1 - e_{t^1} - e_l \notin I_{t^2+1}, \quad x_l^1 = 1, \\ 2^{t^1+l-2} & \text{if } \forall i \in B_1, \quad x_i^1 = x_i^2, \quad \exists l \in B_2, \quad x^1 - e_{t^1} - e_l \in I_{t^2+1}, \quad x_l^1 = 1, \\ 2^{2t^1-1} & \text{if } x^1 - e_{t^1} \in I_{t^2+1} \quad (\forall i \in B_1, \quad x_i^1 = x_i^2), \end{cases} \end{aligned}$$

where $B_1 = \{t^2 + 1, \dots, A - 1\}$, $B_2 = \{t^1 + 1, \dots, t^2\}$.

2. The proof of the Marcinkiewicz theorem

Define the operators SR , SL , SM , SRR , SRL and SMR by the sums of shift operators as follows:

$$SRf(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} |f|^*(y^1, y^2 + e_t), \quad SLf(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} |f|^*(y^1 + e_t, y^2),$$

$$SMf(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} |f|^*(y^1 + e_t, y^2 + e_t),$$

$$SRRf(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} \sum_{T=0}^{\infty} 2^{-T} |f|^*(y^1, y^2 + e_t + e_T) = \sum_{T=0}^{\infty} 2^{-T} SRf(y^1, y^2 + e_T),$$

$$SRLf(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} \sum_{T=0}^{\infty} 2^{-T} |f|^*(y^1 + e_t, y^2 + e_T) = \sum_{T=0}^{\infty} 2^{-T} SLf(y^1, y^2 + e_T),$$

$$\begin{aligned} SMRf(y^1, y^2) &:= \sum_{t=0}^{\infty} \sum_{T=0}^{\infty} 2^{-t-T} |f|^*(y^1 + e_t, y^2 + e_t + e_T) \\ &= \sum_{t=0}^{\infty} 2^{-t} SRf(y^1 + e_t, y^2 + e_t) \end{aligned}$$

$((y^1, y^2) \in G \times G, f \in L^1(G \times G))$.

Lemma 6. *The operators SR , SL , SM , SRR , SRL and SMR are of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.*

Proof. It is easy to see that SR is of type (∞, ∞) . Now, we show that the operator SR is of weak type $(1, 1)$.

$$\begin{aligned} \mu(SRf > c\lambda) &\leq \mu\left(\sum_{t=0}^{\infty} 2^{-t} |f|^*(\cdot, \cdot + e_t) > c\lambda\right) \\ &\leq \mu\left(\bigcup_{t=0}^{\infty} \{|f|^*(\cdot, \cdot + e_t) > 2^{t/2} c\lambda\}\right) \leq \sum_{t=0}^{\infty} \mu(|f|^*(\cdot, \cdot + e_t) > 2^{t/2} c\lambda) \\ &\leq c \sum_{t=0}^{\infty} 2^{-t/2} \frac{\|f(\cdot, \cdot + e_t)\|_1}{\lambda} \leq c \frac{\|f\|_1}{\lambda}. \end{aligned}$$

Using this method repeatedly for the operators SL , SM , SRR , SRL and SMR we have that the operators SL , SM , SRR , SRL and SMR are of weak type $(1, 1)$. (Sometimes we calculate with $2^{3t/4}$, $2^{-3t/4}$ ($2^{2t/3}$, $2^{-2t/3}$) instead of $2^{t/2}$, $2^{-t/2}$.) \square

Remark 1. Define the modified shift operator of SRR by

$$\tilde{SRR}f(y^1, y^2) := \sum_{t=0}^{\infty} 2^{-t} \sum_{T=t}^{\infty} 2^{-T} |f|^*(y^1, y^2 + e_t + e_T).$$

Following the proof of lemma, we have that any modified operator, defined in a similar way as above, is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Lemma 7. *Set*

$$Lf(y^1, y^2) := \sup_{A \in \mathbb{N}} \left| \int_{G \times G} f(x^1 + y^1, x^2 + y^2) r_A(x^1 + x^2) \mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) d\mu(x^1, x^2) \right|$$

((y^1, y^2) $\in G \times G$, $f \in L^1(G \times G)$). Then the operator L is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Proof. To discuss $Lf(y^1, y^2)$, introduce the following notation:

$$\begin{aligned} frk(A, x^1, x^2, y^1, y^2) &:= f(x^1 + y^1, x^2 + y^2) r_A(x^1 + x^2) \mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \text{ and} \\ fr(A, x^1, x^2, y^1, y^2) &:= f(x^1 + y^1, x^2 + y^2) r_A(x^1 + x^2). \end{aligned}$$

$$\begin{aligned} Lf(y^1, y^2) &\leq \sup_{A \in \mathbb{N}} \left| \int_{I_A \times I_A} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sup_{A \in \mathbb{N}} \left| \int_{I_A \times \bar{I}_A} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sup_{A \in \mathbb{N}} \left| \int_{\bar{I}_A \times I_A} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sup_{A \in \mathbb{N}} \left| \int_{\bar{I}_A \times \bar{I}_A} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &=: L_1 f(y^1, y^2) + L_2 f(y^1, y^2) + L_3 f(y^1, y^2) + L_4 f(y^1, y^2). \end{aligned}$$

By Corollary 1 we have

$$\begin{aligned} L_1 f(y^1, y^2) &= \sup_{A \in \mathbb{N}} \frac{(2^A + 1)(2^{A+1} + 1)}{6} \left| \int_{I_A \times I_A} f r(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq c|f|^*(y^1, y^2) \end{aligned}$$

by Lemma 1 the operator L_1 is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$, then we need to discuss the other three operators. Next, we decompose the set \bar{I}_A

$$\bar{I}_A = \bigcup_{t=0}^{A-1} (I_t \setminus I_{t+1}),$$

the set $J_t := I_t \setminus I_{t+1}$ can be represented as the disjoint union

$$I_t \setminus I_{t+1} = \bigcup_{T=t+1}^{\infty} I_t^T \cup \{e_t\},$$

where

$$I_t^T := \{x \in G : x_t = x_T = 1 \text{ and } x_i = 0 \text{ for } i < T, i \neq t\}$$

for any integer $T > t$.

$$\begin{aligned} L_2 f(y^1, y^2) &= \sup_{A \in \mathbb{N}} \left| \sum_{t=0}^{A-1} \int_{I_A \times J_t} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq \sum_{t=0}^{\infty} \sup_{A > t} \left| \int_{I_A \times J_t} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq \sum_{t=0}^{\infty} \sup_{A > t} \sum_{T=t+1}^{A-1} \left| \int_{I_A \times I_t^T} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sum_{t=0}^{\infty} \sup_{A > t} \sum_{T=A}^{\infty} \left| \int_{I_A \times I_t^T} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &=: \sum_{2,1} + \sum_{2,2}. \end{aligned}$$

If $T \geq A$, then $x_t^2 - x_t^1 e_t \in I_A$ and $\tau_A(x^2) - (\tau_A(x^2))_{A-t-1} e_{A-t-1} \in I_A$. By Corollary 1 we get $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{A-t-3} n(A, t)$ where $n(A, t) = -2^{-t-1}(-2^{A-t-2} + 2^A + \frac{1}{2}) - (2^A - 2)$. Thus,

$$\begin{aligned} \sum_{2,2} &= \sum_{t=0}^{\infty} \sup_{A > t} \sum_{T=A}^{\infty} \left| 2^{A-t-3} n(A, t) \int_{I_A \times I_t^T} f r(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} 2^{-t} \sup_{A > t} \left| 2^{2A} \int_{I_A \times I_A(e_t)} f r(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} 2^{-t} |f|^*(y^1, y^2 + e_t) = cSRf(y^1, y^2). \end{aligned}$$

If $t < T < A$ and $x^2 \in I_t^T$, then we have two cases $x^2 - x_t^2 e_t - x_T^2 e_T \in I_A$ or $x^2 - x_t^2 e_t - x_T^2 e_T \notin I_A$. In the second case $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 0$. Thus, we have to discuss the first case only. Set $x^2 - x_t^2 e_t - x_T^2 e_T \in I_A$, then $\tau_A(x^2) - (\tau_A(x^2))_{A-t-1} e_{A-t-1} - (\tau_A(x^2))_{A-T-1} e_{A-T-1} \in I_A$ and $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2A-t-T-4}$ (see Corollary 1). Therefore, we decompose the set I_t^T as the following:

$$I_t^T = I_A(e_t + e_T) \cup \left(I_t^T \setminus I_A(e_t + e_T) \right).$$

$$\begin{aligned} \sum_{2,1} &\leq c \sum_{t=0}^{\infty} \sup_{A>t} \sum_{T=t+1}^{A-1} 2^{2A-t-T} \left| \int_{I_A \times I_A(e_t+e_T)} fr(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} 2^{-t} \sum_{T=0}^{\infty} 2^{-T} |f|^*(y^1, y^2 + e_t + e_T) = cSRRf(y^1, y^2). \end{aligned}$$

By Lemma 6 the operator L_2 is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. In the same way we can investigate the operator L_3 .

At last, we will investigate $L_4 f(y^1, y^2)$.

$$\begin{aligned} L_4 f(y^1, y^2) &\leq \sup_{A \in \mathbb{N}} \left| \int_{\bar{I}_A \times \bar{I}_A} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq \sup_{A \in \mathbb{N}} \sum_{t^1=0}^{A-1} \sum_{t^2=0}^{A-1} \left| \int_{J_{t^1} \times J_{t^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq \sum_{t^1=0}^{\infty} \sum_{t^2=0}^{\infty} \sup_{\substack{A>t^1 \\ A>t^2}} \left| \int_{J_{t^1} \times J_{t^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq \sum_{t^1=0}^{\infty} \sum_{t^2=t^1}^{\infty} \sup_{A>t^2} \left| \int_{J_{t^1} \times J_{t^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} \left| \int_{J_{t^1} \times J_{t^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &=: i_1 + i_2. \end{aligned}$$

We will discuss i_2 (i_1 can be discussed in a similar way) by the help of Corollary 2.

$$\begin{aligned} i_2 &\leq \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^2+1}^{A-1} \left| \int_{I_{t^1}^{T^1} \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=A}^{\infty} \left| \int_{I_{t^1}^{T^1} \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \end{aligned}$$

$$\begin{aligned}
& + \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} \sum_{T^1=A}^{\infty} \sum_{T^2=t^2+1}^{A-1} \left| \int_{I_{t^1}^{T^1} \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
& + \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} \sum_{T^1=A}^{\infty} \sum_{T^2=A}^{\infty} \left| \int_{I_{t^1}^{T^1} \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
& =: \sum_L^1 + \sum_L^2 + \sum_L^3 + \sum_L^4.
\end{aligned}$$

First of all, we discuss \sum_L^4 . Set $T^1, T^2 \geq A, t^2 \leq t^1$ and $(x^1, x^2) \in I_{t^1}^{T^1} \times I_{t^2}^{T^2}$, then $x^1 - e_{t^1} \in I_A, x^2 - e_{t^2} \in I_A$, and $\tau_A(x^1) - e_{A-t^1-1} \in I_A, \tau_A(x^2) - e_{A-t^2-1} \in I_A$. The facts that $\tau_A(x^1) \in I_{A-t^1-2} \setminus I_{A-t^1-1}$ and $\tau_A(x^2) \in I_{A-t^2-2} \setminus I_{A-t^2-1}$ give that $(\tau_A(x^1))_i = (\tau_A(x^2))_i = 0$ for all $i = A-t^2, \dots, A-1$ and $\tau_A(x^1) - (\tau_A(x^1))_{A-t^1-1} e_{A-t^1-1} \in I_{A-t^2-1}$ that is $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2A-2t^1-3}$.

$$\begin{aligned}
\sum_L^4 & \leq c \sum_{t^2=0}^{\infty} \sum_{t^1=t^2}^{\infty} \sup_{A>t^1} 2^{2A-2t^1} \left| \int_{I_A(e_{t^1}) \times I_A(e_{t^2})} fr(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
& \leq c \sum_{t^2=0}^{\infty} \sum_{t^1=0}^{\infty} 2^{-t^1-t^2} |f|^*(y^1 + e_{t^1}, y^2 + e_{t^2}) = cSRLf(y^1, y^2).
\end{aligned}$$

Now, we investigate \sum_L^3 . (\sum_L^2 can be investigated in a similar way.) Next, set $A > t^1 \geq t^2, T^1 \geq A, t^2 < T^2 < A, x^1 \in I_{t^1}^{T^1}$ and $x^2 \in I_{t^2}^{T^2}$. In this case $x^1 - x_{t^1}^1 e_{t^1} \in I_A$ and $\tau_A(x^1) - \tau_A(x^1)_{A-t^1-1} e_{A-t^1-1} \in I_A, \tau_A(x^2) \in I_j \setminus I_{j+1}$ for some $j = 0, \dots, A-T^2-2$, if $A-t^1-1 \leq j$ holds, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 0$. Thus, we have to discuss $A-t^1-1 > j$. Set $A-t^1-1 > j$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$ only in that case $t^1 = t^2$. Consequently, set $A > t^1 = t^2$.

If $j = A-T^2-2$, then $\tau_A(x^2) - \tau_A(x^2)_{A-T^2-1} e_{A-T^2-1} \in I_{A-t^1-1}$ and $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2A-2T^2-3}$ (only in the case $\tau_A(x^2)_{A-t^1} = \dots = \tau_A(x^2)_{A-1} = 0$).

For some $j \in \{0, \dots, A-T^2-3\}$ we have $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$ only in that case $\tau_A(x^2) - \tau_A(x^2)_j e_j - \tau_A(x^2)_{A-T^2-1} e_{A-T^2-1} \in I_{A-t^1-1}$, and $\tau_A(x^2)_{A-t^1} = \dots = \tau_A(x^2)_{A-1} = 0$, in this case we have $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{j+A-T^2-3}$.

$$\begin{aligned}
\sum_L^3 & = \sum_{t^1=0}^{\infty} \sup_{A>t^1} \sum_{T^1=A}^{\infty} \sum_{T^2=t^1+1}^{A-1} \left| \int_{I_{t^1}^{T^1} \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
& = \sum_{t^1=0}^{\infty} \sup_{A>t^1} \sum_{T^2=t^1+1}^{A-1} \left| \int_{I_A(e_{t^1}) \times I_{t^2}^{T^2}} frk(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
& \leq c \sum_{t^1=0}^{\infty} \sup_{A>t^1} \sum_{T^2=t^1+1}^{A-1} 2^{2A-2T^2} \int_{I_A(e_{t^1}) \times I_A(e_{t^1}+e_{T^2})} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2)
\end{aligned}$$

$$\begin{aligned}
& + c \sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^2=t^1+1}^{A-1} \sum_{j=0}^{A-T^2-3} 2^{j+A-T^2} \\
& \times \int_{I_A(e_{t^1}) \times I_A(e_{t^1}+e_{T^2}+e_{A-j-1})} |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\
& \leq c \sum_{t^1=0}^{\infty} \sum_{T^2=0}^{\infty} 2^{-t^1-T^2} |f|^*(y^1+e_{t^1}, y^2+e_{t^1}+e_{T^2}) \\
& + c \sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^2=t^1+1}^{A-1} \sum_{j=0}^{A-T^2-3} 2^{j-A-T^2} |f|^*(y^1+e_{t^1}, y^2+e_{t^1}+e_{T^2}+e_{A-j-1}) \\
& \leq cSMRf(y^1, y^2) + cTf(y^1, y^2).
\end{aligned}$$

To discuss the operator T set $l := A - j - 1$.

$$\begin{aligned}
Tf(y^1, y^2) & \leq \sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^2=t^1}^A \sum_{l=T^2+2}^{A-1} 2^{l-T^2} |f|^*(y^1+e_{t^1}, y^2+e_{t^1}+e_{T^2}+e_l) \\
& \leq \sum_{t^1=0}^{\infty} \sum_{T^2=t^1+1}^{\infty} 2^{-T^2} \sum_{l=T^2+2}^{\infty} 2^{-l} |f|^*(y^1+e_{t^1}, y^2+e_{t^1}+e_{T^2}+e_l).
\end{aligned}$$

That is, the operator T is bounded by a modified shift operator, Remark 1 give that the operator T is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Finally, we will discuss \sum_L^1 by the help of Corollary 2. Set $A > t^1 \geq t^2$, $A > T^1 > t^1$, $A > T^2 > t^2$ and $(x^1, x^2) \in I_{t^1}^{T^1} \times I_{t^2}^{T^2}$. $\tau_A(x^1) \in I_{j^1} \setminus I_{j^1+1}$ for some $j^1 \in \{0, 1, \dots, A - T^1 - 2\}$ and $\tau_A(x^2) \in I_{j^2} \setminus I_{j^2+1}$ for some $j^2 \in \{0, 1, \dots, A - T^2 - 2\}$. Set $j^1 \leq j^2$ ($j^1 > j^2$ can be discussed similarly), then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$ only in that case $t^1 = t^2$, $T^1 \geq T^2$ and $(\tau_A(x^1))_i = (\tau_A(x^2))_i$ for all $i \in \{j^2 + 1, \dots, A - T^1 - 1\}$.

If $j^2 < A - T^2 - 2$, then $T^1 = T^2$ is needed for $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$. We have three subcases:

(ai) $\tau_A(x^1) - e_{j^1} \in I_{j^2+1}$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2j^1-1}$,

(aii) there exists an index $l \in \{j^1+1, \dots, j^2\}$ for which $(\tau_A(x^1))_l = 1$ and $\tau_A(x^1) - e_{j^1} - e_l \in I_{j^2+1}$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{j^1+l-2}$,

(aiii) there exists an index $l \in \{j^1+1, \dots, j^2\}$ for which $(\tau_A(x^1))_l = 1$ and $\tau_A(x^1) - e_{j^1} - e_l \notin I_{j^2+1}$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 0$.

If $j^2 = A - T^2 - 2$, then $T^1 \geq T^2$ and we have four subcases while $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$:

(bi) if $j^1 = A - T^1 - 2$ ($T^1 \geq T^2$), then $\tau_A(x^1) - e_{j^1} \in I_{j^2+1}$ and $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2j^1-1} = 2^{2A-2T^1-5}$,

(bii) if $j^1 < A - T^1 - 2 < j^2 = A - T^2 - 2$ ($T^1 > T^2$), then $\tau_A(x^1) - e_{j^1} - e_{T^1} \in I_{j^2+1}$ and $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{j^1+T^1-2}$,

(biii) if $j^1 < A - T^1 - 2 = j^2 = A - T^2 - 2$ ($T^1 = T^2$) and $\tau_A(x^1) - e_{j^1} \in I_{j^2+1}$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{2j^1-2}$,

(biv) if $j^1 < A - T^1 - 2 = j^2 = A - T^2 - 2$ ($T^1 = T^2$) and there exists an index $l \in \{j^1 + 1, \dots, j^2\}$ for which $(\tau_A(x^1))_l = 1$ and $\tau_A(x^1) - e_{j^1} - e_l \in I_{j^2+1}$, then $\mathcal{K}_{2A}^\omega(\tau_A(x^1), \tau_A(x^2)) = 2^{j^1+l-2}$.

These mean that subcases (ai) and (biii) (see $R_1 f$), (aii) and (biv) (see $R_2 f$) can be discussed in the same time, but (bi) (see $R_3 f$) and (bii) (see $R_4 f$) cannot.

$$\begin{aligned}
\sum_L^1 &\leq c \sum_{t^1=0}^\infty \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{j^1=0}^{A-T^1-2} \sum_{j^2=j^1}^{A-T^1-2} \sum_{\substack{i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{2j^1-2A} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_{A-j^1-1} + x, y^2 + e_{t^1} + e_{T^1} + e_{A-j^2-1} + x) \\
&+ c \sum_{t^1=0}^\infty \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{j^1=0}^{A-T^1-2} \sum_{j^2=j^1}^{A-T^1-2} \sum_{l=j^1+1}^{j^2} \sum_{\substack{i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{j^1+l-2A} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_{A-j^1-1} + e_{A-l-1} + x, y^2 + e_{t^1} + e_{T^1} + e_{A-j^2-1} + x) \\
&+ c \sum_{t^1=0}^\infty \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1} \sum_{j^2=A-T^1-1}^{A-T^2-1} \sum_{\substack{i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{-2T^1} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + x, y^2 + e_{t^1} + e_{T^2} + e_{A-j^2-1} + x) \\
&+ c \sum_{t^1=0}^\infty \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1-1} \sum_{j^1=0}^{A-T^1-3} \sum_{j^2=j^1+1}^{A-T^2-2} \sum_{\substack{i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{j^1+T^1-2A} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_{A-j^1-1} + x, y^2 + e_{t^1} + e_{T^2} + e_{A-j^2-1} + x) \\
&=: cR_1 f(y^1, y^2) + cR_2 f(y^1, y^2) + cR_3 f(y^1, y^2) + cR_4 f(y^1, y^2).
\end{aligned}$$

Now, we investigate $R_1 f(y^1, y^2)$. Set $l := A - j^1 - 1$ and $k := A - j^2 - 1$

$$\begin{aligned}
R_1 f(y^1, y^2) &\leq c \sum_{t^1=0}^\infty \sum_{T^1=t^1+1}^\infty \sum_{l=T^1+1}^\infty \sum_{k=T^1+1}^l \sum_{\substack{i \in \{T^1+1, \dots, k-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{-2l} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_l + x, y^2 + e_{t^1} + e_{T^1} + e_k + x).
\end{aligned}$$

That is, $R_1 f$ is bounded by a modified shift operator, by Remark 1 the operator R_1 is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. The investigation of the operator R_2 goes by the same way as above.

Discuss the operator R_3 (and set $l := A - j^2 - 1$).

$$\begin{aligned}
 R_3 f(y^1, y^2) &= \sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1} \sum_{j^2=A-T^1-1}^{A-T^2-1} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{-2T^1} \\
 &\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + x, y^2 + e_{t^1} + e_{T^2} + e_{A-j^2-1} + x) \\
 &\leq \sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1} \sum_{l=T^2}^{T^1} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{-2T^1} \\
 &\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + x, y^2 + e_{t^1} + e_{T^2} + e_l + x) \\
 &\leq \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1} \sum_{l=T^2}^{T^1} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{-2T^1} \\
 &\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + x, y^2 + e_{t^1} + e_{T^2} + e_l + x).
 \end{aligned}$$

Using the notation $|f|^*(y^1, y^2, x, t^1, T^1, T^2, l) := |f|^*(y^1 + e_{t^1} + e_{T^1} + x, y^2 + e_{t^1} + e_{T^2} + e_l + x)$, we have that

$$\begin{aligned}
 \mu(R_3 f > c\lambda) &\leq \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1} \sum_{l=T^2}^{T^1} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 \mu(|f|^*(., ., x, t^1, T^1, T^2, l) > c\lambda 2^{T^1}) \\
 &\leq c \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1} \sum_{l=T^2}^{T^1} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{-T^1} \frac{\|f(., ., x, t^1, T^1, T^2, l)\|_1}{\lambda} \\
 &\leq c \frac{\|f\|_1}{\lambda} \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} (T^1 - t^1) 2^{-T^1} \leq c \frac{\|f\|_1}{\lambda}.
 \end{aligned}$$

For more information see Remark 1.

At last, we have to discuss the operator R_4 . Set $l := A - j^2 - 1$ and $k := A - j^1 - 1$.

$$\begin{aligned}
 &\sum_{t^1=0}^{\infty} \sup_{A > t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1-1} \sum_{j^1=0}^{A-T^1-3} \sum_{j^2=j^1+1}^{A-T^2-2} \sum_{\substack{x_i=0 \\ i \in \{T^1+1, \dots, A-j^2-2\} \\ x_i=0 \text{ otherwise}}}^1 2^{j^1+T^1-2A} \\
 &\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_{A-j^1-1} + x, y^2 + e_{t^1} + e_{T^2} + e_{A-j^2-1} + x)
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{t^1=0}^{\infty} \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1-1} \sum_{j^1=0}^{A-T^1-3} \sum_{l=T^2}^{A-j^1-2} \sum_{\substack{i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{j^1+T^1-2A} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_{A-j^1-1} + x, y^2 + e_{t^1} + e_{T^2} + e_l + x) \\
&\leq c \sum_{t^1=0}^{\infty} \sup_{A>t^1} \sum_{T^1=t^1+1}^{A-1} \sum_{T^2=t^1+1}^{T^1-1} \sum_{k=T^1+2}^{A-1} \sum_{l=T^2}^{k-1} \sum_{\substack{i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{T^1-A-k} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_k + x, y^2 + e_{t^1} + e_{T^2} + e_l + x) \\
&\leq c \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1-1} \sum_{k=T^1+2}^{\infty} \sum_{l=T^2}^{k-1} \sum_{\substack{i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{T^1-2k} \\
&\quad \times |f|^*(y^1 + e_{t^1} + e_{T^1} + e_k + x, y^2 + e_{t^1} + e_{T^2} + e_l + x).
\end{aligned}$$

Using the notation $|f|_{x, t^1, T^1}^{T^2, k, l*}(y^1, y^2) := |f|^*(y^1 + e_{t^1} + e_{T^1} + e_k + x, y^2 + e_{t^1} + e_{T^2} + e_l + x)$, we have that

$$\begin{aligned}
&\mu(R_4 f > c\lambda) \\
&\leq \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1-1} \sum_{k=T^1+2}^{\infty} \sum_{l=T^2}^{k-1} \sum_{\substack{i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 \mu(|f|_{x, t^1, T^1}^{T^2, k, l*} > c\lambda 2^{\frac{2}{3}(2k-T^1)}) \\
&\leq c \frac{\|f\|_1}{\lambda} \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} \sum_{T^2=t^1+1}^{T^1-1} \sum_{k=T^1+2}^{\infty} \sum_{l=T^2}^{k-1} \sum_{\substack{i \in \{T^1+1, \dots, l-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{\frac{2}{3}(T^1-2k)} \\
&\leq c \frac{\|f\|_1}{\lambda} \sum_{t^1=0}^{\infty} \sum_{T^1=t^1+1}^{\infty} (T^1 - t^1) 2^{-\frac{2}{3}T^1} \leq c \frac{\|f\|_1}{\lambda}.
\end{aligned}$$

This and Remark 1 complete the proof of Lemma 7. \square

Lemma 8. Set

$$\begin{aligned}
&Mf(y^1, y^2) \\
&:= \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} \left| \int_{G \times G} f(x^1 + y^1, x^2 + y^2) r_A(x^1 + x^2) \mathcal{K}_n^\omega(\tau_A(x^1), \tau_A(x^2)) d\mu(x^1, x^2) \right|
\end{aligned}$$

$((y^1, y^2) \in G \times G, f \in L^1(G \times G))$. Then the operator M is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Proof. To discuss $Mf(y^1, y^2)$, introduce the following notation:

$$\begin{aligned} fr\tilde{k}(A, x^1, x^2, y^1, y^2) &:= f(x^1 + y^1, x^2 + y^2)r_A(x^1 + x^2)\tilde{\mathcal{K}}_n^\omega(\tau_A(x^1), \tau_A(x^2)), \\ frk(A, s, x^1, x^2, y^1, y^2) &:= f(x^1 + y^1, x^2 + y^2)r_A(x^1 + x^2)\mathcal{K}_{n(s+1), 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)), \\ frd(A, n, x^1, x^2, y^1, y^2) &:= f(x^1 + y^1, x^2 + y^2)r_A(x^1 + x^2)D_{n,n}^\omega(\tau_A(x^1), \tau_A(x^2)), \end{aligned}$$

$$M_1 f(y^1, y^2) := \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \frac{1}{n} \left| \int_{G \times G} frd(A, n, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right|$$

and

$$M_2 f(y^1, y^2) := \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \left| \int_{G \times G} fr\tilde{k}(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right|.$$

To show that M_1 is of weak type $(1, 1)$, we will prove that M_1 is of type $(1, 1)$. First, fix an $q, A \in \mathbb{N}$ and set $|n| = q \leq A$. By the theorem of Fubini we have

$$\begin{aligned} & \int_{G \times G} \sup_{\substack{n \in \mathbb{N} \\ |n|=q}} \left| \int_{G \times G} frd(A, n, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ & \leq \|f\|_1 \int_{G \times G} \sup_{\substack{n \in \mathbb{N} \\ |n|=q}} |D_{n,n}^\omega(\tau_A(z^1), \tau_A(z^2))| d\mu(z^1, z^2). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_G \sup_{\substack{n \in \mathbb{N} \\ |n|=q}} |D_n^\omega(\tau_A(z))| d\mu(z) & \leq \sum_{j=0}^{q-1} \int_{\{z | z_{A-1} = \dots = z_{A-j} = 0, z_{A-j-1} = 1\}} 2^{j+1} dz \\ & + \int_{\{z | z_{A-1} = \dots = z_{A-q} = 0\}} 2^{q+1} dz \leq \sum_{j=0}^{q-1} 2 + 2 \leq cq. \end{aligned}$$

These inequalities imply that

$$\begin{aligned} \|M_1 f\|_1 & \leq \sum_{A=0}^{\infty} \frac{1}{2^A} \sum_{q=0}^A \int_{G \times G} \sup_{\substack{n \in \mathbb{N} \\ |n|=q}} \left| \int_{G \times G} frd(A, n, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ & \quad \times d\mu(y^1, y^2) \\ & \leq c \|f\|_1 \sum_{A=0}^{\infty} \frac{1}{2^A} \sum_{q=0}^A q^2 \leq c \|f\|_1 \end{aligned}$$

and

$$\|M_1 f\|_{\infty} \leq c \|f\|_{\infty} \sum_{A=0}^{\infty} \frac{1}{2^A} \sum_{q=0}^A q^2 \leq c \|f\|_{\infty}.$$

Discuss the operator M_2 .

$$\begin{aligned}
 M_2 f(y^1, y^2) &\leq \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \left| \int_{I_A \times I_A} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &+ \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \frac{1}{n} \left| \sum_{s=0}^{|n|} \int_{I_A \times \bar{I}_A} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &+ \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \frac{1}{n} \left| \sum_{s=0}^{|n|} \int_{\bar{I}_A \times I_A} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &+ \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} \frac{n}{2^A} \frac{1}{n} \left| \sum_{s=0}^{|n|} \int_{\bar{I}_A \times \bar{I}_A} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &=: M_2^1 f(y^1, y^2) + M_2^2 f(y^1, y^2) + M_2^3 f(y^1, y^2) + M_2^4 f(y^1, y^2).
 \end{aligned}$$

By the definition of \tilde{K}_n ($|n| < A$) we have for $x \in I_A$ that $\tau_A(x) \in I_A$ and $\tilde{K}_n = \frac{1}{n} \sum_{k=0}^{n-1} D_{n,n}^\omega \leq \frac{1}{n} \sum_{k=0}^{n-1} k^2 \leq c 2^{2A}$.

$$M_2^1 f(y^1, y^2) \leq c \sup_{\substack{n, A \in \mathbb{N} \\ |n| < A}} 2^{2A} \int_{I_A \times I_A} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \leq c |f|^*(y^1, y^2).$$

This and Lemma 1 give that the operator M_2^1 is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. Next, we decompose the set \bar{I}_A and the set $J_t := I_t \setminus I_{t+1}$

$$\begin{aligned}
 \bar{I}_A &= \bigcup_{t=0}^{A-1} (I_t \setminus I_{t+1}), \quad I_t \setminus I_{t+1} = \bigcup_{T=t+1}^{\infty} I_t^T \cup \{e_t\}. \\
 M_2^2 f(y^1, y^2) &\leq \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{s=0}^{|n|} \sum_{T=t+1}^{\infty} \left| \int_{I_A \times I_t^T} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &\leq \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{s=0}^{|n|} \sum_{T=t+1}^{A-1} \left| \int_{I_A \times I_t^T} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &+ \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{s=0}^{|n|} \sum_{T=A}^{\infty} \left| \int_{I_A \times I_t^T} f r k(A, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\
 &:= \sum_M^1 + \sum_M^2.
 \end{aligned}$$

If $T \geq A$, then $x_t^2 - x_t^2 e_t \in I_A$ and $\tau_A(x^2) - (\tau_A(x^2))_{A-1-t} e_{A-1-t} \in I_A$. By Lemma 5, if $A - t - 1 < s$, then $|\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2))| = 2^{A-t-3} n(s, t) \leq c 2^{A-t+2s}$ where $n(s, t) = n^{(s+1)} 2^{s+1} - 2^{A-t-3} (-2^{A-t-3} + 2^s + \frac{1}{2}) - 2^s (2^s - 2)$, if $A - t - 1 \geq s$, then $|\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2))| \leq c 2^{s+A-t} n$. Thus,

$$\begin{aligned} \sum_M^2 &\leq c \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{s=0}^{|n|} 2^{s+A-t} n \int_{I_A \times I_A(e_t)} |f(x^1 + y^2, x^2 + y^2)| d\mu(x^1, x^2) \\ &\leq c S R f(y^1, y^2). \end{aligned}$$

If $t < T < A$ and $x^2 \in I_t^T$, then $\tau_A(x^2) \in I_j \setminus I_{j+1}$ for some $j \in \{0, 1, \dots, A - T - 2\}$. We have the following four cases:

(i) If $j < A - T - 1 < A - t - 1 \leq s$, then $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) \neq 0$ only in the case $\tau_A(x^2) - \tau_A(x^2)_{A-T-1} e_{A-T-1} - \tau_A(x^2)_{A-t-1} e_{A-t-1} \in I_s$. In this case $\tau_A(x^2) - \tau_A(x^2)_{A-T-1} e_{A-T-1} - \tau_A(x^2)_{A-t-1} e_{A-t-1} \in I_A$ and $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) = \omega_{n^{(s+1)}}(\tau_A(x^2)) 2^{2A-T-t+s-4}$ (by Lemma 5).

(ii) Set $j < A - T - 1 < s < A - t - 1$. If $\tau_A(x^2) - (\tau_A(x^2))_{A-T-1} e_{A-T-1} \in I_s$, then $\tau_A(x^2) - (\tau_A(x^2))_{A-T-1} e_{A-T-1} - (\tau_A(x^2))_{A-t-1} e_{A-t-1} \in I_A$, and $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) = \omega_{n^{(s+1)}}(\tau_A(x^2)) 2^{A-T-3} [n^{(s+1)} 2^{s+1} - 2^{A-T-1} (2^s - 2^{A-T-2} + \frac{1}{2}) - 2^s (2^s - 2)]$. If $\tau_A(x^2) - (\tau_A(x^2))_{A-T-1} e_{A-T-1} - (\tau_A(x^2))_j e_j \in I_s$, then $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) = \omega_{n^{(s+1)}}(\tau_A(x^2)) 2^{A-T+j+s-3}$ (by Lemma 5).

(iii) Set $j < s \leq A - T - 1 < A - t - 1$. If $\tau_A(x^2) - (\tau_A(x^2))_j e_j \in I_s$, $(\tau_A(x^2) - (\tau_A(x^2))_j e_j) \in I_A(e_{A-T-1} + e_{A-t-1})$, then $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) = \omega_{n^{(s+1)}}(\tau_A(x^2)) 2^{j-2} [n^{(s+1)} 2^{s+1} - 2^j (2^s - 2^{j-1} + \frac{1}{2}) - 2^s (2^s - 2)]$.

If $\tau_A(x^2) - (\tau_A(x^2))_j e_j - (\tau_A(x^2))_l e_l \in I_s$ for some $j < l < s$, then $\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2)) = \omega_{n^{(s+1)}}(\tau_A(x^2)) 2^{j+l+s-2}$ (by Lemma 5).

(iv) If $s \leq j$, then $|\mathcal{K}_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^1), \tau_A(x^2))| \leq c 2^{s+j} (n^{(s+1)} + 2^s)$ (by Lemma 5).

$$\begin{aligned} \sum_M^1 &\leq c \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{T=t+1}^{A-1} \sum_{s=0}^{A-1} \left| \int_{I_A \times I_t^T} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{T=t+1}^{A-1} \sum_{s=A-T-1}^{A-1} \left| \int_{I_A \times I_t^T} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + c \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{T=t+1}^{A-1} \sum_{s=A-T}^{A-T-2} \left| \int_{I_A \times I_t^T} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + c \sum_{t=0}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ t, |n| < A}} \frac{1}{2^A} \sum_{T=t+1}^{A-1} \sum_{s=0}^{A-T-1} \left| \int_{I_A \times I_t^T} f r k(A, s, x^1, x^2, y^1, y^2) d\mu(x^1, x^2) \right| \\ &\quad + c \sum_M^{1,1} + c \sum_M^{1,2} + c \sum_M^{1,3}. \end{aligned}$$

Set $l := A - s$.

$$\begin{aligned} \sum_M^{1,1} &\leq c \sum_{t=0}^{\infty} \sup_{\substack{A \in \mathbb{N} \\ t < A}} \sum_{T=t+1}^{A-1} \sum_{s=A-t-1}^{A-1} 2^{A-T-t+s} \int_{I_A \times I_A(e_T+e_t)} |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\ &\leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} 2^{-T-t} \sup_{\substack{A \in \mathbb{N} \\ T < A}} \sum_{l=1}^{t+1} 2^{-l} |f|^*(y^1, y^2 + e_T + e_t) \leq c SRRf(y^1, y^2). \end{aligned}$$

Set $l := A - j - 1$.

$$\begin{aligned} \sum_M^{1,2} &\leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ |n|, T < A}} \sum_{s=A-T}^{A-t-2} 2^{s-T} n^{(s+1)} \int_{I_A \times I_A(e_T+e_t)} |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\ &\quad + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ |n|, T < A}} \sum_{s=A-T}^{A-t-2} \sum_{j=0}^{A-T-2} 2^{s+j-T} \int_{I_A \times I_A(e_T+e_t+e_{A-j-1})} \\ &\quad \times |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\ &\leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} 2^{-T-t} |f|^*(y^1, y^2 + e_T + e_t) \\ &\quad + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} 2^{-T-t} \sup_{\substack{A \in \mathbb{N} \\ T < A}} \sum_{l=T+1}^{A-1} 2^{2A-l} \int_{I_A \times I_A(e_T+e_t+e_l)} \\ &\quad \times |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\ &\leq c SRRf(y^1, y^2) + c \sum_{t=0}^{\infty} 2^{-t} SRRf(y^1, y^2 + e_t). \end{aligned}$$

Thus, by Lemma 6 the operator $\sum_M^{1,2}$ is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. Now, we investigate the operator $\sum_M^{1,3}$.

$$\begin{aligned} \sum_M^{1,3} &\leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ |n|, T < A}} \sum_{j=0}^{A-T-2} \sum_{s=j+1}^{A-T-1} 2^{s+j} \int_{I_A \times I_A(e_T+e_t+e_{A-j-1})} \\ &\quad \times |f(x^1+y^1, x^2+y^2)| d\mu(x^1, x^2) \\ &\quad + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ |n|, T < A}} \sum_{j=0}^{A-T-2} \sum_{s=j+1}^{A-T-1} \sum_{l=j+1}^{s-1} 2^{s+j+l-A} \end{aligned}$$

$$\begin{aligned}
& \times \int_{I_A \times I_A(e_T + e_t + e_{A-j-1} + e_{A-l-1})} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \\
& + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{n, A \in \mathbb{N} \\ |n|, T < A}} \sum_{j=0}^{A-T-2} \sum_{s=0}^j \sum_{\substack{i \in \{T+1, \dots, A-j-1\} \\ x_i=0 \text{ otherwise}}}^1 2^{s+j} \\
& \times \int_{I_A \times I_A(e_T + e_t + x)} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2).
\end{aligned}$$

Set $k := A - j - 1$ and $p := A - l - 1$.

$$\begin{aligned}
\sum_M^{1,3} & \leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{A \in \mathbb{N} \\ T < A}} \sum_{k=T+1}^{A-1} 2^{2A-T-k} \int_{I_A \times I_A(e_T + e_t + e_k)} \\
& \times |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \\
& + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{A \in \mathbb{N} \\ T < A}} \sum_{k=T+1}^{A-1} \sum_{s=A-k}^{A-T-1} \sum_{p=A-s}^{k-1} 2^{A+s-k-p} \int_{I_A \times I_A(e_T + e_t + e_k + e_p)} \\
& \times |f(x^1 + y^1, x^2 + y^2)| + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sup_{\substack{A \in \mathbb{N} \\ T < A}} \sum_{k=T+1}^{A-1} \\
& \times \sum_{\substack{i \in \{T+1, \dots, k\} \\ x_i=0 \text{ otherwise}}}^1 2^{2A-2k} \int_{I_A \times I_A(e_T + e_t + x)} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \\
& \leq c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sum_{k=T+1}^{\infty} 2^{-T-k} |f|^*(y^1, y^2 + e_T + e_t + e_k) \\
& + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sum_{k=T+1}^{\infty} \sum_{s=-k}^{-T-1} \sum_{p=-s}^{k-1} 2^{s-k-p} |f|^*(y^1, y^2 + e_T + e_t + e_k + e_p) \\
& + c \sum_{t=0}^{\infty} \sum_{T=t+1}^{\infty} \sum_{k=T+1}^{\infty} 2^{-2k} \sum_{\substack{i \in \{T+1, \dots, k\} \\ x_i=0 \text{ otherwise}}}^1 |f|^*(y^1, y^2 + e_T + e_t + x) \\
& := MT_1 f(y^1, y^2) + MT_2 f(y^1, y^2) + MT_3 f(y^1, y^2).
\end{aligned}$$

The operators MT_1 , MT_2 and MT_3 are modified shift operators, by Remark 1 these operators are of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

This means that the operator M_2^2 is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. In the same way we can investigate the operator M_2^3 .

At last, we investigate $M_2^4 f(y^1, y^2)$. We can write

$$\begin{aligned} \mathcal{K}_{n^{(s+1)}, 2^s}^\omega(x^1, x^2) &= \sum_{k=0}^{2^s-1} D_{n^{(s)}+k}^\omega(x^1) D_{n^{(s)}+k}^\omega(x^2) \\ &= \sum_{k=0}^{2^s-1} [D_{n^{(s)}}^\omega(x^1) + n_s r_s(x^1) D_k^\omega(x^1)] [D_{n^{(s)}}^\omega(x^2) + n_s r_s(x^2) D_k^\omega(x^2)] \\ &= 2^s D_{n^{(s)}}^\omega(x^1) D_{n^{(s)}}^\omega(x^2) + D_{n^{(s)}}^\omega(x^1) n_s r_s(x^2) \sum_{k=0}^{2^s-1} D_k^\omega(x^2) \\ &\quad + D_{n^{(s)}}^\omega(x^2) n_s r_s(x^1) \sum_{k=0}^{2^s-1} D_k^\omega(x^1) + n_s r_s(x^1) r_s(x^2) \sum_{k=0}^{2^s-1} D_k^\omega(x^1) D_k^\omega(x^2). \end{aligned}$$

This allows us to define the operators T_1 , T_2 , T_3 and T_4 by

$$\begin{aligned} T_1 f(y^1, y^2) &:= \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{s-A} \int_{\bar{I}_A(y^1) \times \bar{I}_A(y^2)} f(x^1, x^2) r_A(y^1 - x^1 + y^2 - x^2) \right. \\ &\quad \left. \times D_{n^{(s)}}^\omega(\tau_A(y^1 - x^1)) D_{n^{(s)}}^\omega(\tau_A(y^2 - x^2)) d\mu(x^1, x^2) \right|, \\ T_2 f(y^1, y^2) &:= \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{-A} \int_{\bar{I}_A(y^1) \times \bar{I}_A(y^2)} f(x^1, x^2) r_A(y^1 - x^1 + y^2 - x^2) \right. \\ &\quad \left. \times D_{n^{(s)}}^\omega(\tau_A(y^1 - x^1)) n_s r_s(\tau_A(y^2 - x^2)) \sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(y^2 - x^2)) d\mu(x^1, x^2) \right|, \\ T_4 f(y^1, y^2) &:= \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{-A} \int_{\bar{I}_A(y^1) \times \bar{I}_A(y^2)} f(x^1, x^2) r_A(y^1 - x^1 + y^2 - x^2) n_s r_s(\tau_A(y^1 - x^1)) \right. \\ &\quad \left. \times r_s(\tau_A(y^2 - x^2)) \sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(y^1 - x^1)) D_k^\omega(\tau_A(y^2 - x^2)) d\mu(x^1, x^2) \right|. \end{aligned}$$

The operator T_3 can be defined and investigated in a similar way as the operator T_2 . Now, we discuss the operator T_1 .

If $s = A$, then $D_{n^{(s)}}^\omega(\tau_A(y^1 - x^1)) = D_{2^A}^\omega(\tau_A(y^1 - x^1)) = 0$ for any $x^1 \in \bar{I}_A(y^1)$ (since $\tau_A(y^1 - x^1) \in \bar{I}_A$).

Set $s < A$, $x^1 \in J_j(y^1) := \{x | x_{A-1} = y_{A-1}, \dots, x_{A-j} = y_{A-j}, x_{A-j-1} \neq y_{A-j-1}\}$ and $x^2 \in J_k(y^2)$, then $\tau_A(y^1 - x^1) \in I_j \setminus I_{j+1}$ and $\tau_A(y^2 - x^2) \in I_k \setminus I_{k+1}$.

$$|D_{n^{(s)}}^\omega(x)| \leq D_{2^A}^\omega(x) + D_{2^{A-1}}^\omega(x) + \dots + D_{2^s}^\omega(x)$$

gives that $D_{n(s)}^\omega(\tau_A(y^1 - x^1)) = 0$ for $s > j$ and $D_{n(s)}^\omega(\tau_A(y^2 - x^2)) = 0$ for $s > k$. This implies that $D_{n(s)}^\omega(\tau_A(y^1 - x^1))D_{n(s)}^\omega(\tau_A(y^2 - x^2)) \neq 0$ only in the case when $j, k \geq s$. Set $j, k \geq s$, then $|D_{n(s)}^\omega(\tau_A(y^1 - x^1))D_{n(s)}^\omega(\tau_A(y^2 - x^2))| \leq 2^{j+k+2}$. For a fix A define the operator T_1^A by

$$\begin{aligned} T_1^A f(y^1, y^2) &:= \sup_{\substack{n \in \mathbf{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{s-A} \int_{\bar{I}_A(y^1) \times \bar{I}_A(y^2)} f(x^1, x^2) r_A(y^1 - x^1 + y^2 - x^2) \right. \\ &\quad \left. \times D_{n(s)}^\omega(\tau_A(y^1 - x^1)) D_{n(s)}^\omega(\tau_A(y^2 - x^2)) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{s=0}^A 2^{s-A} \sum_{j,k=s}^A \int_{J_j(y^1) \times J_k(y^2)} |f(x^1, x^2)| 2^{j+k} d\mu(x^1, x^2). \end{aligned}$$

($T_1 f \leq \sup_{A \in \mathbf{N}} T_1^A f$.) The operator T_1^A is of type $(1, 1)$ and of type (∞, ∞) . By the theorem of Fubini

$$\|T_1^A f\|_p \leq c \|f\|_p \sum_{s=0}^A 2^{s-A} (A-s)^2 \leq c \|f\|_p$$

for $p = 1$ or ∞ . Moreover, the operator T_1 is of type (∞, ∞) . By the Marcinkiewicz interpolation theorem the operator T_1^A is of type (p, p) for any $1 \leq p \leq \infty$ (uniformly in A). Moreover, $T_1^A f = T_1^A(E_{A+1, A+1} f) = T_1^A(E_{A+1, A+1} f - E_{A, A} f)$. These give that the operator T_1 is of type $(2, 2)$.

$$\begin{aligned} \|\sup_{A \in \mathbf{N}} |T_1^A f|\|_2^2 &\leq \sum_{A \in \mathbf{N}} \|T_1^A f\|_2^2 \leq \sum_{A \in \mathbf{N}} \|T_1^A(E_{A+1, A+1} f - E_{A, A} f)\|_2^2 \\ &\leq c \sum_{A \in \mathbf{N}} \|E_{A+1, A+1} f - E_{A, A} f\|_2^2 \leq c \|f\|_2^2. \end{aligned}$$

Define the stopping time ν in the following way:

$$\nu(x^1, x^2) := \inf(k \in \mathbf{N} : E_{k, k}(|f|)(x^1, x^2) > \lambda) \quad (\inf \emptyset = \infty).$$

It is known that $\mu(\nu < \infty) \leq \|f\|_1 / \lambda$. Denote the characteristic function of set $B \subset G$ by 1_B . This and

$$\begin{aligned} 1_G &= 1_{\{\nu < \infty\}} + 1_{\{\nu = \infty\}} \\ &= 1_{\{\nu > A+1\}} + 1_{\{\nu < A+1\}} + 1_{\{\nu = A+1\}} \end{aligned}$$

give

$$\begin{aligned} \mu(T_1 f > \lambda) &\leq \mu(1_{\{\nu < \infty\}} T_1 f > \lambda/2) \\ &\quad + \mu(1_{\{\nu = \infty\}} \sup_{A \in \mathbf{N}} T_1^A (1_{\{\nu > A+1\}} (E_{A+1, A+1} f - E_{A, A} f)) > \lambda/6) \\ &\quad + \mu(1_{\{\nu = \infty\}} \sup_{A \in \mathbf{N}} T_1^A (1_{\{\nu < A+1\}} (E_{A+1, A+1} f - E_{A, A} f)) > \lambda/6) \\ &\quad + \mu(1_{\{\nu = \infty\}} \sup_{A \in \mathbf{N}} T_1^A (1_{\{\nu = A+1\}} (E_{A+1, A+1} f - E_{A, A} f)) > \lambda/6) \\ &=: J^1 + J^2 + J^3 + J^4. \end{aligned}$$

We already have $J^1 \leq c \|f\|_1 / \lambda$. $T_1^A(1_{\{v < A+1\}}(E_{A+1,A+1}f - E_{A,A}f)) = 0$ on the set $\{v = \infty\}$ implies $\sup_{A \in \mathbb{N}} T_1^A(1_{\{v < A+1\}}(E_{A+1,A+1}f - E_{A,A}f)) = 0$ on the set $\{v = \infty\}$. Thus, $J^3 = 0$. We give an upper bound for J^4 , by using the $(1, 1)$ typeness of the operators T_1^A (uniformly in A).

$$\begin{aligned} J^4 &\leq \frac{c}{\lambda} \left\| \sup_{A \in \mathbb{N}} T_1^A(1_{\{v=A+1\}}(E_{A+1,A+1}f - E_{A,A}f)) \right\|_1 \\ &\leq \frac{c}{\lambda} \sum_{A=0}^{\infty} \|T_1^A(1_{\{v=A+1\}}(E_{A+1,A+1}f - E_{A,A}f))\|_1 \\ &\leq \frac{c}{\lambda} \sum_{A=0}^{\infty} \|1_{\{v=A+1\}}(E_{A+1,A+1}f - E_{A,A}f)\|_1 \\ &\leq \frac{c}{\lambda} \sum_{A=0}^{\infty} \|f 1_{\{v=A+1\}}\|_1 \leq c \frac{\|f\|_1}{\lambda}. \end{aligned}$$

At last, we give an upper bound for J^2 . The lemma of Burkholder [3,15] gives that

$$\sum_{A=0}^{\infty} \|1_{\{v>A+1\}}(E_{A+1,A+1}f - E_{A,A}f)\|_2^2 \leq c \|f\|_1 \lambda.$$

By this and the $(2, 2)$ typeness of the operators T_1^A (uniformly in A) we have

$$\begin{aligned} J^2 &\leq \frac{c}{\lambda^2} \sum_{A=0}^{\infty} \|T_1^A(1_{\{v>A+1\}}(E_{A+1,A+1}f - E_{A,A}f))\|_2^2 \\ &\leq \frac{c}{\lambda^2} \sum_{A=0}^{\infty} \|1_{\{v>A+1\}}(E_{A+1,A+1}f - E_{A,A}f)\|_2^2 \leq c \frac{\|f\|_1}{\lambda}. \end{aligned}$$

This gives that the operator T_1 is of weak type $(1, 1)$. By the Marcinkiewicz interpolation theorem the operator T_1 is of type (p, p) for all $1 < p \leq \infty$.

To investigate the operator T_2 set $x^1 \in J_j(y^1)$ and $x^2 \in J_k(y^2)$, then $\tau_A(y^1 - x^1) \in I_j \setminus I_{j+1}$ (and $D_{n(s)}^\omega(\tau_A(y^1 - x^1)) = 0$ if $s > j$, $|D_{n(s)}^\omega(\tau_A(y^1 - x^1))| \leq 2^{j+1}$ if $j \geq s$) and $\tau_A(y^2 - x^2) \in I_k \setminus I_{k+1}$. If $k \geq s$, then $\sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(y^2 - x^2)) = c2^s$. If $k < s$, then by the lemma of Gát

$$\sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(y^2 - x^2)) = (2^s K_{2^s}^\omega - D_{2^s}^\omega)(\tau_A(y^2 - x^2)) = \begin{cases} 2^{k-1} & \text{if } x^2 \in J_k^s(y^2), \\ 0 & \text{otherwise,} \end{cases}$$

where $J_k^s(y) := \{x \in J_k : (\tau_A(x))_l = (\tau_A(y))_l \text{ for } l = k+2, \dots, s\}$. For a fix A define the operator T_2^A by

$$\begin{aligned} T_2^A f(y^1, y^2) &:= \sup_{\substack{n \in \mathbb{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{-A} \int_{\bar{I}_A(y^1) \times \bar{I}_A(y^2)} f(x^1, x^2) r_A(y^1 - x^1 + y^2 - x^2) \right. \\ &\quad \left. \times D_{n(s)}^\omega(\tau_A(y^1 - x^1)) n_s r_s(\tau_A(y^2 - x^2)) \sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(y^2 - x^2)) d\mu(x^1, x^2) \right| \end{aligned}$$

$$\begin{aligned} &\leq c \sum_{s=0}^A 2^{-A} \sum_{j,k=s}^A \int_{J_j(y^1) \times J_k(y^2)} |f(x^1, x^2)| 2^{j+2s} d\mu(x^1, x^2) \\ &\quad + c \sum_{s=0}^A 2^{s-A} \sum_{j=s}^A \sum_{k=0}^{s-1} \int_{J_j(y^1) \times J_k^s(y^2)} |f(x^1, x^2)| 2^{j+k} d\mu(x^1, x^2). \end{aligned}$$

The operator T_2^A is of type $(1, 1)$ and of type (∞, ∞) (uniformly in A). Moreover, the operator T_2 is of type (∞, ∞) . By the theorem of Fubini

$$\|T_2^A f\|_p \leq c \|f\|_p \sum_{s=0}^A 2^{s-A} (A-s) \leq c \|f\|_p$$

for $p = 1$ or ∞ . By the Marcinkiewicz interpolation theorem the operator T_2^A is of type (p, p) for any $1 \leq p \leq \infty$ (uniformly in A). Moreover, $T_2^A f = T_2^A(E_{A+1, A+1}f) = T_2^A(E_{A+1, A+1}f - E_{A, A}f)$. These give that the operator T_2 is of type $(2, 2)$. Following the method on T^1 we could prove that the operator T_2 is of weak type $(1, 1)$.

For a fix A define the operator T_4^A by

$$\begin{aligned} T_4^A f(y^1, y^2) := & \sup_{\substack{n \in \mathbb{N} \\ |n| \leq A}} \left| \sum_{s=0}^A 2^{-A} \int_{\bar{I}_A \times \bar{I}_A} f(x^1 + y^1, x^2 + y^2) r_A(x^1 + x^2) n_s r_s(\tau_A(x^1)) \right. \\ & \left. \times r_s(\tau_A(x^2)) \sum_{k=0}^{2^s-1} D_k^\omega(\tau_A(x^1)) D_k^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right|. \end{aligned}$$

By the theorem of Fubini

$$\begin{aligned} \|T_4^A f\|_1 &\leq c \|f\|_1 \sum_{s=0}^A 2^{-A} \|(\mathcal{K}_{2^s}^\omega - D_{2^s, 2^s})^\omega(\tau_A(\cdot), \tau_A(\cdot))\|_1 \\ &\leq c \|f\|_1 \sum_{s=0}^A 2^{s-A} \leq c \|f\|_1 \end{aligned}$$

(uniformly in A) and

$$\begin{aligned} \|T_4^A f\|_\infty &\leq c \|f\|_\infty \sum_{s=0}^A 2^{-A} \|(\mathcal{K}_{2^s}^\omega - D_{2^s, 2^s})^\omega(\tau_A(\cdot), \tau_A(\cdot))\|_1 \\ &\leq c \|f\|_\infty \sum_{s=0}^A 2^{s-A} \leq c \|f\|_\infty \end{aligned}$$

(uniformly in A), thus the operator T_4 is of type (∞, ∞) . We have $T_4^A f = T_4^A(E_{A+1, A+1}f) = T_4^A(E_{A+1, A+1}f - E_{A, A}f)$. Following the method on the operator T_1 we have that the operator T_4 is of weak type $(1, 1)$. This completes the proof of Lemma 8. \square

Lemma 9. Set

$$Nf(y^1, y^2) := \sup_{A \in \mathbb{N}} \left| \int_{G \times G} f(x^1 + y^1, x^2 + y^2) D_{2^A}(x^1) r_A(x^2) K_{2^A}^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right|$$

$((y^1, y^2) \in G \times G, f \in L^1(G \times G))$. Then the operator N is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Proof. By the Gát's Lemma 2 we have

$$\begin{aligned} Nf(y^1, y^2) &\leq \sup_{A \in \mathbb{N}} \left| \int_{I_A \times I_A} \frac{2^A(2^A + 1)}{2} f(x^1 + y^1, x^2 + y^2) r_A(x^2) d\mu(x^1, x^2) \right| \\ &\quad + \sup_{A \in \mathbb{N}} \left| \int_{I_A \times \bar{I}_A} 2^A f(x^1 + y^1, x^2 + y^2) r_A(x^2) K_{2^A}^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right| \\ &=: c|f|^*(y^1, y^2) + \tilde{N}f(y^1, y^2). \end{aligned}$$

Next, we investigate the operator \tilde{N} . Set $t < A$ and $x \in I_t^T$ for some $T > t$.

If $T < A$, then $x_t^2 = (\tau_A(x^2))_{A-t-1} = 1$, $x_T^2 = (\tau_A(x^2))_{A-T-1} = 1$ and $K_{2^A}^\omega(\tau_A(x^2)) = 0$.

If $T \geq A$, then $x_t^2 - x_t^2 e_t \in I_A$, $\tau_A(x^2) - (\tau_A(x^2))_{A-t-1} e_{A-t-1} \in I_A$ and $K_{2^A}^\omega(\tau_A(x^2)) = 2^{A-t-2}$. Therefore

$$\begin{aligned} \tilde{N}f(y^1, y^2) &\leq \sum_{t=0}^{\infty} \sup_{A>t} \left| \int_{I_A \times (I_t \setminus I_{t+1})} 2^A f(x^1 + y^1, x^2 + y^2) r_A(x^2) K_{2^A}^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right| \\ &\leq \sum_{t=0}^{\infty} \sup_{A>t} \sum_{T=A}^{\infty} \left| \int_{I_A \times I_t^T} 2^{2A-t-2} f(x^1 + y^1, x^2 + y^2) r_A(x^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} 2^{-t} \sup_{A>t} \left| \int_{I_A \times I_A(e_t)} 2^{2A} f(x^1 + y^1, x^2 + y^2) r_A(x^2) d\mu(x^1, x^2) \right| \\ &\leq c \sum_{t=0}^{\infty} 2^{-t} |f|^*(y^1, y^2 + e_t) = cSRf(y^1, y^2). \end{aligned}$$

This and Lemma 6 complete the proof of Lemma 9. \square

Lemma 10. Set

$$Of(y^1, y^2) := \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \left| \int_{G \times G} f(x^1 + y^1, x^2 + y^2) D_{2^A}(x^1) r_A(x^2) K_n^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right|$$

$((y^1, y^2) \in G \times G, f \in L^1(G \times G))$. Then the operator O is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Proof. Since $|\frac{1}{n} D_n^\omega| \leq 1$, we have to consider the modified kernel \tilde{K}_n^ω and

$$\tilde{O}f(y^1, y^2) := \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} 2^A \left| \int_{I_A \times G} f(x^1 + y^1, x^2 + y^2) r_A(x^2) \tilde{K}_n^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right|,$$

where $(y^1, y^2) \in G \times G$. Set $A, t \in \mathbb{N}$ and $A > t \geq 1$, define the set J_t^A by $J_t^A := \{x \in G : x_{A-1} = \dots = x_{A-t} = 0, x_{A-t-1} = 1\}$ and $J_0^A := \{x \in G : x_{A-1} = 1\}$ for $A \geq 1$. Then for

every $1 \leq A \in \mathbb{N}$ we can decompose G as the disjoint union $G = I_A \cup \bigcup_{t=0}^{A-1} J_t^A$. If $x^2 \in I_A$, then $\tilde{K}_n^\omega(\tau_A(x^2)) \leq cn \leq c2^A$. This means that

$$\begin{aligned} & \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} 2^A \left| \int_{I_A \times I_A} f(x^1 + y^1, x^2 + y^2) r_A(x^2) \tilde{K}_n^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right| \\ & \leq c \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} 2^{2A} \int_{I_A \times I_A} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \leq c|f|^*(y^1, y^2). \end{aligned}$$

By Lemma 3 of Gát we have

$$\begin{aligned} & \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \frac{n}{2^A} 2^A \left| \int_{I_A \times (G \setminus I_A)} f(x^1 + y^1, x^2 + y^2) r_A(x^2) \tilde{K}_n^\omega(\tau_A(x^2)) d\mu(x^1, x^2) \right| \\ & \leq \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \int_{I_A \times (G \setminus I_A)} |f(x^1 + y^1, x^2 + y^2)| \sum_{s=0}^{|n|} |K_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^2))| d\mu(x^1, x^2) \\ & \leq \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \sum_{t=0}^{A-1} \sum_{s=0}^{|n|} \int_{I_A \times J_t^A} |f(x^1 + y^1, x^2 + y^2)| \cdot |K_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^2))| d\mu(x^1, x^2) \\ & \leq c \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \sum_{t=0}^{A-1} \sum_{s=0}^t \int_{I_A \times J_t^A} |f(x^1 + y^1, x^2 + y^2)| 2^{s+t} d\mu(x^1, x^2) \\ & \quad + \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \sum_{t=0}^{A-1} \sum_{s=t+1}^A \int_{I_A \times J_t^A} |f(x^1 + y^1, x^2 + y^2)| \cdot |K_{n^{(s+1)}, 2^s}^\omega(\tau_A(x^2))| d\mu(x^1, x^2) \\ & =: c\tilde{O}_1 f(y^1, y^2) + \tilde{O}_2 f(y^1, y^2). \end{aligned}$$

First, we will investigate the operator \tilde{O}_1 . Set $l := A - t$, then we have

$$\begin{aligned} \tilde{O}_1 f(y^1, y^2) & \leq \sup_{\substack{n, A \in \mathbb{N} \\ |n| \leq A}} \sum_{t=0}^{A-1} 2^{2t} \int_{I_A \times J_t^A} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \\ & \leq \sum_{l=1}^{\infty} 2^{-l} \sup_{A \in \mathbb{N}, A > l} 2^{2A-l} \int_{I_A \times J_{A-l}^A} |f(x^1 + y^1, x^2 + y^2)| d\mu(x^1, x^2) \\ & \leq \sum_{l=1}^{\infty} 2^{-l} T_l |f(y^1, y^2)|, \end{aligned}$$

where $T_l f$ is defined by

$$T_l f(y^1, y^2) := \sup_{A \geq l} 2^{2A-l} \left| \int_{I_A \times \{x \in G : x_l = x_{l+1} = \dots = x_{A-1} = 0\}} f(x^1 + y^1, x^2 + y^2) d\mu(x^1, x^2) \right|.$$

The operator $T_l f$ is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$ (uniformly in l). To prove this set

$$g(z^1, z^2) := 2^{-l} \sum_{\substack{a_i \in \{0,1\} \\ i \in \{0,1,\dots,l-1\}}} f(z^1, a_0 e_0 + \dots + a_{l-1} e_{l-1} + z^2)$$

and follow the method of Gát in his paper [5]. These give that

$$E_{n,n} g(y^1, y^2) = 2^{2n-l} \int_{I_n \times \{x \in G : x_l = x_{l+1} = \dots = x_{n-1} = 0\}} f(x^1 + y^1, x^2 + y^2) d\mu(x^1, x^2),$$

$$T_l f(y^1, y^2) = \sup_{n \geq l} |E_{n,n} g(y^1, y^2)| \leq g^*(y^1, y^2),$$

and

$$\mu(T_l f > \lambda) \leq \mu(g^* > \lambda) \leq c \|g\|_1 / \lambda \leq c \|f\|_1 / \lambda$$

$$\|T_l f\|_p \leq \|g^*\|_p \leq c_p \|g\|_p \leq c_p \|f\|_p$$

for all $1 < p \leq \infty$.

At last, we have to discuss the operator \tilde{O}_2 . For $l, t \in \mathbb{N}$, $l < t$ define the operator $T_{l,t}$ by

$$T_{l,t} f(y^1, y^2) = \sup_{A \geq t} |2^{2A-l} \int_{I_A \times \{x \in G : x_l = \dots = x_{t-1} = 0, x_t = 1, x_{t+1} = \dots = x_{A-1} = 0\}} \\ \times f(x^1 + y^1, x^2 + y^2) d\mu(x^1, x^2)|.$$

$T_{l,t} f(y^1, y^2) \leq T_l f(y^1, y^2 + e_t)$ and $\|f(\cdot, \cdot)\|_p = \|f(\cdot, \cdot + e_t)\|_p$ ($1 < p \leq \infty$) give that the operator $T_{l,t}$ is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$ (uniformly in l, t). Following Gát's method we have that

$$\tilde{O}_2 f(y^1, y^2) \leq c \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} 2^{-m} T_{l,m-1} |f|(y^1, y^2),$$

this implies that the operator \tilde{O}_2 is weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. This completes the proof of this lemma. \square

Theorem 2. *The operator $\mathcal{M}^{*\kappa}$ is of weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.*

Proof. By the decomposition Lemmas 4 and 1, 7–10, we have $\mathcal{M}^{*\kappa}$ is of weak type $(1, 1)$.

$$\mu(\mathcal{M}^{*\kappa} f(\cdot) > c\lambda) \leq \mu\left(\sup_{n \in \mathbb{P}} \left| \int_{G \times G} f(\underline{x} + \cdot) \frac{1}{n} d\mu(\underline{x}) \right| > c\lambda\right) \\ + \mu\left(\sum_{j=0}^{|n|-1} \frac{2^j}{n} |f^*| > c\lambda\right) + 2\mu\left(\sum_{j=0}^{|n|-1} \frac{2^j}{n} Nf > c\lambda\right)$$

$$\begin{aligned}
& + \mu \left(\sum_{j=0}^{|n|-1} \frac{2^j}{n} Lf > c\lambda \right) + \mu \left(\sup_{n \in \mathbf{P}} \left(1 - \frac{2^{|n|}}{n} \right) |f^*| > c\lambda \right) \\
& + 2\mu \left(\sup_{n \in \mathbf{P}} \left(1 - \frac{2^{|n|}}{n} \right) Of > c\lambda \right) + \mu \left(\sup_{n \in \mathbf{P}} \left(1 - \frac{2^{|n|}}{n} \right) Mf > c\lambda \right) \\
& \leq c \frac{\|f\|_1}{\lambda}.
\end{aligned}$$

Similarly, we have $\mathcal{M}^{*\kappa}$ is of type (p, p) for all $1 < p \leq \infty$. \square

Proof of Theorem 1. The maximal operator $\mathcal{M}^{*\kappa}$ is of weak type $(1, 1)$, the set of Walsh–Kaczmarz polynomials is dense in $L^1(G \times G)$, and the fact that $\lim_{n \rightarrow \infty} \mathcal{M}_n^\kappa P \rightarrow P$ everywhere, by the well-known density argument give that $\mathcal{M}_n^\kappa f \rightarrow f$ a.e. for all integrable two-variable function. \square

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