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Exceptional Charlier and Hermite orthogonal polynomials☆

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Abstract

Using Casorati determinants of Charlier polynomials $(c_n^a)_n$, we construct for each finite set F of positive integers a sequence of polynomials c_n^F, $n \in \sigma_F$, which are eigenfunctions of a second order difference operator, where σ_F is certain infinite set of nonnegative integers, $\sigma_F \subsetneq \mathbb{N}$. For suitable finite sets F (we call them admissible sets), we prove that the polynomials c_n^F, $n \in \sigma_F$, are actually exceptional Charlier polynomials; that is, in addition, they are orthogonal and complete with respect to a positive measure. By passing to the limit, we transform the Casorati determinant of Charlier polynomials into a Wronskian determinant of Hermite polynomials. For admissible sets, these Wronskian determinants turn out to be exceptional Hermite polynomials.

Keywords: Orthogonal polynomials; Exceptional orthogonal polynomial; Difference operators; Differential operators; Charlier polynomials; Hermite polynomials

1. Introduction

Exceptional orthogonal polynomials p_n, $n \in X \subsetneq \mathbb{N}$, are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order

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differential operator. They extend the classical families of Hermite, Laguerre and Jacobi. The last few years have seen a great deal of activity in the area of exceptional orthogonal polynomials (see, for instance, [9,17,18] (where the adjective exceptional for this topic was introduced), [19,20,22,28,30,31,35,36,38] and the references therein).

The most apparent difference between classical orthogonal polynomials and exceptional orthogonal polynomials is that the exceptional families have gaps in their degrees, in the sense that not all degrees are present in the sequence of polynomials (as it happens with the classical families) although they form a complete orthonormal set of the underlying L^2 space defined by the orthogonalizing positive measure. This means in particular that they are not covered by the hypotheses of Bochner's classification theorem [4]. Exceptional orthogonal polynomials have been applied to shape-invariant potentials [35], supersymmetric transformations [19], discrete quantum mechanics [31], mass-dependent potentials [28], and quasi-exact solvability [38].

In the same way, exceptional discrete orthogonal polynomials are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order difference operator, extending the discrete classical families of Charlier, Meixner, Krawtchouk and Hahn. As far as the author knows the only known example of what can be called exceptional Charlier polynomials appeared in [39]. If orthogonal discrete polynomials on nonuniform lattices and orthogonal q-polynomials are considered, then one should add [31–34] where exceptional Wilson, Racah, Askey–Wilson and q-Racah polynomials are considered.

The purpose of this paper (and the forthcoming ones) is to introduce a systematic way of constructing exceptional discrete orthogonal polynomials using the concept of dual families of polynomials (see [27]). One can then also construct examples of exceptional orthogonal polynomials by taking limits in some of the parameters in the same way as one goes from classical discrete polynomials to classical polynomials in the Askey tableau.

Definition 1.1. Given two sets of nonnegative integers $U, V \subset \mathbb{N}$, we say that the two sequences of polynomials $(p_u)_{u\in U}$, $(q_v)_{v\in V}$ are dual if there exist a couple of sequences of numbers $(\xi_u)_{u\in U}$, $(\zeta_v)_{v\in V}$ such that

$$\xi_u p_u(v) = \zeta_v q_v(u), \quad u \in U, v \in V. \tag{1.1}$$

Duality has shown to be a fruitful concept regarding discrete orthogonal polynomials, and its utility will be again manifest in the exceptional discrete polynomials world. Indeed, it turns out that duality interchanges exceptional discrete orthogonal polynomials with the so-called Krall discrete orthogonal polynomials. A Krall discrete orthogonal family is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$, p_n of degree n, orthogonal with respect to a positive measure which, in addition, are also eigenfunctions of a higher order difference operator. A huge amount of families of Krall discrete orthogonal polynomials have been recently introduced by the author by means of certain Christoffel transform of the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see [10,11,14]). A Christoffel transform is a transformation which consists in multiplying a measure μ by a polynomial r. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E.B. Christoffel (see [7] and also [37]) studied it for the particular case $r(x) = x$.

In this paper we will concentrate on exceptional Charlier and Hermite polynomials (Meixner, Krawtchouk, Hahn, Laguerre and Jacobi families will be considered in the forthcoming papers).

The content of this paper is as follows. In Section 2, we include some preliminary results about symmetric operators, Christoffel transforms and finite sets of positive integers.

In Section 3, using Casorati determinants of Charlier polynomials we associate to each finite set F of positive integers a sequence of polynomials which are eigenfunctions of a second order difference operator. Indeed, write the finite set F of positive integers as $F = \{f_1, \dots, f_k\}$, $f_i < f_{i+1}$ (k is then the number of elements of F and f_k the maximum element of F). We define the nonnegative integer u_F by $u_F = \sum_{f\in F} f - \binom{k+1}{2}$ and the infinite set of nonnegative integers σ_F by

$$\sigma_F = \{u_F, u_F+1, u_F+2, \dots\} \setminus \{u_F+f, f\in F\}.$$

Given $a \in \mathbb{R}\setminus\{0\}$, we then associate to F the sequence of polynomials $c_n^{a;F}$, $n\in\sigma_F$, defined by

$$c_n^{a;F}(x) = \begin{vmatrix} c^a_{n-u_F}(x) & c^a_{n-u_F}(x+1) & \cdots & c^a_{n-u_F}(x+k) \\ c^a_{f_1}(x) & c^a_{f_1}(x+1) & \cdots & c^a_{f_1}(x+k) \\ \vdots & \vdots & \ddots & \vdots \\ c^a_{f_k}(x) & c^a_{f_k}(x+1) & \cdots & c^a_{f_k}(x+k) \end{vmatrix}, \tag{1.2}$$

where $(c_n^a)_n$ are the Charlier polynomials (see (2.24)) orthogonal with respect to the discrete measure

$$\rho_a = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x.$$

Consider now the measure

$$\rho_a^F = (x-f_1)\cdots(x-f_k)\rho_a. \tag{1.3}$$

Orthogonal polynomials with respect to ρ_a^F are eigenfunctions of higher order difference operators (see [10,14]). It turns out that the sequence of polynomials $c_n^{a;F}$, $n \in \sigma_F$, and the sequence of orthogonal polynomials $(q_n^F)_n$ with respect to the measure ρ_a^F are dual sequences (see Lemma 3.2). As a consequence we get that the polynomials $c_n^{a;F}$, $n\in\sigma_F$, are always eigenfunctions of a second order difference operator D_F (whose coefficients are rational functions); see Theorem 3.3. Charlier-type orthogonal polynomials considered in [39] corresponds with the case $F=\{1,2\}$ (duality is also used in [39]).

In Section 4, we study the most interesting case: it appears when the measure ρ_a^F (1.3) is positive. This gives rise to the concept of admissible sets of positive integers. Split up the set F, $F=\bigcup_{i=1}^K Y_i$, in such a way that $Y_i\cap Y_j=\emptyset$, $i\neq j$, the elements of each Y_i are consecutive integers and $1+\max Y_i < \min Y_{i+1}$, $i=1,\dots,K-1$; we then say that F is admissible if each Y_i, $i=1,\dots,K$, has an even number of elements. It is straightforward to see that F is admissible if and only if $\prod_{f\in F}(x-f)\geq 0$, $x\in\mathbb{N}$, or in other words, (if $a>0$) the measure ρ_a^F (1.3) is positive. This concept of admissibility has appeared several times in the literature. Relevant to this paper, because of the relationship with exceptional polynomials are [25,1] where the concept appears in connection with the zeros of certain Wronskian determinants associated with eigenfunctions of second order differential operators of the form $-d^2/dx^2+U$. Admissibility was also considered in [24,39].

We prove (Theorems 4.4 and 4.5) that if F is an admissible set and $a>0$, then the polynomials $c_n^{a;F}$, $n\in\sigma_F$, are orthogonal and complete with respect to the positive measure

$$\omega_{a;F} = \sum_{x=0}^{\infty} \frac{a^x}{x!\,\Omega_F^a(x)\Omega_F^a(x+1)}\delta_x,$$

where Ω_F^a is the polynomial defined by

$$\Omega_F^a(x) = \begin{vmatrix} c_{f_1}^a(x) & c_{f_1}^a(x+1) & \cdots & c_{f_1}^a(x+k-1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{f_k}^a(x) & c_{f_k}^a(x+1) & \cdots & c_{f_k}^a(x+k-1) \end{vmatrix}. \tag{1.4}$$

In particular we characterize admissible sets F as those for which the Casorati determinant $\Omega_F^a(x)$ has constant sign for $x \in \mathbb{N}$ (Lemma 4.3).

Casorati determinants like (1.4) for Charlier and other discrete orthogonal polynomials were considered by Karlin and Szegő in [24] (see also [23]). In particular, Karlin and Szegő proved that when F is admissible then $\Omega_F^a(x)$ $(a > 0)$ has a constant sign for $x \in \mathbb{N}$.

Although it is out of the scope of this paper, we point out here that the duality transforms the higher order difference operator with respect to which the polynomials $(q_n^F)_n$ are eigenfunctions in a higher order recurrence relation for the polynomials $c_n^{a;F}$. This higher order recurrence relation has the form

$$h(x)c_n^{a;F}(x) = \sum_{j=-u_F-k-1}^{u_F+k+1} u_{n,j}^{a;F} c_{n+j}^{a;F}(x)$$

where h is a polynomial in x of degree $u_F + k + 1$ satisfying $h(x) - h(x-1) = \Omega_F^a(x)$, and $u_{n,j}^{a;F}$, $j = -u_F - k - 1, \ldots, u_F + k + 1$, are rational functions in n depending on a and F but not on x.

In Sections 5 and 6, we construct exceptional Hermite polynomials by taking limit (in a suitable way) in the exceptional Charlier polynomials when $a \to +\infty$. We then get (see Theorem 5.1) that for each finite set F of positive integers, the polynomials

$$H_n^F(x) = \begin{vmatrix} H_{n-u_F}(x) & H'_{n-u_F}(x) & \cdots & H^{(k)}_{n-u_F}(x) \\ H_{f_1}(x) & H'_{f_1}(x) & \cdots & H^{(k)}_{f_1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H^{(k)}_{f_k}(x) \end{vmatrix}, \tag{1.5}$$

$n \in \sigma_F$, are eigenfunctions of a second order differential operator.

When F is admissible, the Wronskian determinant Ω_F defined by

$$\Omega_F(x) = \begin{vmatrix} H_{f_1}(x) & H'_{f_1}(x) & \cdots & H^{(k-1)}_{f_1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H^{(k-1)}_{f_k}(x) \end{vmatrix} \tag{1.6}$$

does not vanish in $\mathbb{R}$. For admissible sets F, we then prove that the polynomials H_n^F, $n \in \sigma_F$, are orthogonal with respect to the positive weight

$$\omega_F(x) = \frac{e^{-x^2}}{\Omega_F^2(x)}, \quad x \in \mathbb{R}.$$

Moreover, they form a complete orthogonal system in $L^2(\omega_F)$ (see Theorem 6.4). The exceptional Hermite family introduced in [15] corresponds with $F = \{1, 2, \ldots, 2k\}$ (for the case $k = 1$ see also [9,5]). Simultaneously with this paper, exceptional Hermite polynomials as the Wronskian determinant of Hermite polynomials have been introduced and studied (using a different approach) in [22].

We guess that the non vanishing property of the Wronskian determinant (1.6) in $\mathbb{R}$ is actually true for orthogonal polynomials with respect to a positive measure. Moreover, we conjecture that this property characterizes admissible sets:

Conjecture. Let $F = \{f_1, \dots, f_k\}$ and μ be a finite set of positive integers and a positive measure with finite moments and infinitely many points in its support, respectively. Consider the monic sequence $(p_n)_n$ of orthogonal polynomials with respect to μ and write Ω_F^μ for the Wronskian determinant defined by $\Omega_F^\mu(x) = |p_{f_i}^{(j-1)}(x)|_{i,j=1}^k$. Then the following conditions are equivalent.

1. F is admissible.
2. For all positive measures μ as above the Wronskian determinant $\Omega_F^\mu(x)$ does not vanish in $\mathbb{R}$.

Wronskian determinants like (1.6) for orthogonal polynomials were considered by Karlin and Szegő in [24] for the particular case of finite sets F formed by consecutive positive integers. In particular, Karlin and Szegő proved the implication (1) $\Rightarrow$, (2) when F is formed by an even number of consecutive positive integers.

When F is an admissible set and $a > 0$, exceptional Charlier and Hermite polynomials $c_n^{a;F}$ and H_n^F, $n \in \sigma_F$, can be constructed in an alternative way. Indeed, consider the involution I in the set of all finite sets of positive integers defined by

$$I(F) = \{1, 2, \dots, f_k\} \setminus \{f_k - f, f \in F\}.$$

The set $I(F)$ will be denoted by G: $G = I(F)$. We also write $G = \{g_1, \dots, g_m\}$ with $g_i < g_{i+1}$ so that m is the number of elements of G and g_m the maximum element of G. We also need the nonnegative integer v_F defined by

$$v_F = \sum_{f\in F} f + f_k - \frac{(k-1)(k+2)}{2}.$$

For $n \geq v_F$, we then have

$$c_n^{a;F}(x) = \beta_n \begin{vmatrix} c^a_{n-v_F}(x) & \frac{x}{a}c^a_{n-v_F}(x-1) & \cdots & \frac{(x-m+1)_m}{a^m}c^a_{n-v_F}(x-m+1) \\ c^{-a}_{g_1}(-x-1) & c^{-a}_{g_1}(-x) & \cdots & c^{-a}_{g_1}(-x+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ c^{-a}_{g_m}(-x-1) & c^{-a}_{g_m}(-x) & \cdots & c^{-a}_{g_m}(-x+m-1) \end{vmatrix}, \quad (1.7)$$

$$H_n^F(x) = \gamma_n \begin{vmatrix} H_{n-v_F}(x) & -H_{n-v_F-1}(x) & \cdots & (-1)^m H_{n-v_F-m}(x) \\ H_{g_1}(-ix) & H'_{g_1}(-ix) & \cdots & H^{(m)}_{g_1}(-ix) \\ \vdots & \vdots & \ddots & \vdots \\ H_{g_m}(-ix) & H'_{g_m}(-ix) & \cdots & H^{(m)}_{g_m}(-ix) \end{vmatrix}, \quad (1.8)$$

where β_n and γ_n, $n \geq v_F$, are certain normalization constants (see (3.37) and (5.5)).

We have however computational evidence that shows that both identities (1.7) and (1.8) are true for every finite set F of positive integers.

Both determinantal definitions (1.2) and (1.7) of the polynomials $c_n^{a;F}$, $n \in \sigma_F$, automatically imply a couple of factorizations of its associated second order difference operator D_F in two first order difference operators. Using these factorizations, we prove that the sequence $c_n^{a;F}$, $n \in \sigma_F$, and the operator D_F can be constructed in two different ways using Darboux transforms (see Definition 3.5). If we consider (1.2) the Darboux transform uses the sequence $c_n^{a,F_{\{k\}}}$, $n \in \sigma_{F_{\{k\}}}$, where $F_{\{k\}} = \{f_1, \ldots, f_{k-1}\}$. On the other hand, if we consider (1.7) the Darboux transform uses the sequence $c_n^{a;F_\Downarrow}$, $n \in \sigma_{F_\Downarrow}$, where

$$F_\Downarrow = \begin{cases} \emptyset, & \text{if } F = \{1, 2, \ldots, k\}, \\ \{f_{s_F} - s_F, \ldots, f_k - s_F\}, & \text{if } F \neq \{1, 2, \ldots, k\}, \end{cases}$$

and for $F \neq \{1, 2, \ldots, k\}$, we write $s_F = \min\{s \geq 1 : s < f_s\}$. The second factorization seems to be more interesting because the operator $F \to F_\Downarrow$ preserves the admissibility of the set F. The same happens with the determinantal definitions of the exceptional Hermite polynomials H_n^F (1.5) and (1.8). This fact agrees with the Gómez-Ullate–Kamran–Milson conjecture and its corresponding discrete version (see [21]): exceptional and exceptional discrete orthogonal polynomials can be obtained by applying a sequence of Darboux transforms to a classical or classical discrete orthogonal family, respectively.

We finish this Introduction by pointing out that there is a very nice invariant property of the polynomial Ω_F^a (1.4) underlying the fact that the polynomials $c_n^{a;F}$, $n \in \sigma_F$, admit both determinantal definitions (1.2) and (1.7) (see [12,13,8]): except for a sign, Ω_F^a remains invariant if we change F to $G = I(F)$, x to $-x$ and a to $-a$; that is

$$\Omega_F^a(x) = (-1)^{k+u_F} \Omega_G^{-a}(-x).$$

This invariant property gives rise to the corresponding one for the Wronskian determinant (1.6) (see (5.8)).

2. Preliminaries

Let μ be a Borel measure (positive or not) on the real line. The nth moment of μ is defined by $\int_{\mathbb{R}} t^n d\mu(t)$. When μ has finite moments for any $n \in \mathbb{N}$, we can associate it a bilinear form defined in the linear space of polynomials by

$$\langle p, q \rangle = \int pq d\mu. \tag{2.1}$$

Given an infinite set X of nonnegative integers, we say that the polynomials p_n, $n \in X$, p_n of degree n, are orthogonal with respect to μ if they are orthogonal with respect to the bilinear form defined by μ; that is, if they satisfy

$$\int p_n p_m d\mu = 0, \quad n \neq m, \quad n, m \in X.$$

When $X = \mathbb{N}$ and the degree of p_n is n, $n \geq 0$, we get the usual definition of orthogonal polynomials with respect to a measure. When $X = \mathbb{N}$, orthogonal polynomials with respect to a measure are unique up to multiplication by non null constant. Let us remark that this property is not true when $X \neq \mathbb{N}$. Positive measures μ with finite moments of any order and infinitely many points in its support has always a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$, p_n of degree n (it is enough to apply the Gram–Schmidt orthogonalizing process to $1, x, x^2, \ldots$); in this case the orthogonal polynomials have positive norm: $\langle p_n, p_n \rangle > 0$. Moreover, given a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$ with respect to a measure μ (positive or not) the bilinear form (2.1) can be represented by a positive measure if and only if $\langle p_n, p_n \rangle > 0$, $n \geq 0$.

When $X=\mathbb{N}$, Favard's Theorem establishes that a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials, p_n of degree n, is orthogonal (with non null norm) with respect to a measure if and only if it satisfies a three term recurrence relation of the form ($p_{-1}=0$)

$$xp_n(x)=a_np_{n+1}(x)+b_np_n(x)+c_np_{n-1}(x),\quad n\geq 0,$$

where $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are sequences of real numbers with $a_{n-1}c_n\neq 0$, $n\geq 1$. If, in addition, $a_{n-1}c_n>0$, $n\geq 1$, then the polynomials $(p_n)_{n\in\mathbb{N}}$ are orthogonal with respect to a positive measure with infinitely many points in its support, and conversely. Again, Favard's Theorem is not true for a sequence of orthogonal polynomials $(p_n)_{n\in X}$ when $X\neq\mathbb{N}$.

We will also need the following three lemmas. The first one is Sylvester's determinant identity (for the proof and a more general formulation of Sylvester's identity see [16], p. 32).

Lemma 2.1. *For a square matrix $M=(m_{i,j})_{i,j=1}^k$, and for each $1\leq i,j\leq k$, denote by M_i^j the square matrix that results from M by deleting the ith row and the jth column. Similarly, for $1\leq i,j,p,q\leq k$ denote by $M_{i,j}^{p,q}$ the square matrix that results from M by deleting the ith and jth rows and the pth and qth columns. Sylvester's determinant identity establishes that for i_0,i_1,j_0,j_1 with $1\leq i_0<i_1\leq k$ and $1\leq j_0<j_1\leq k$,*

$$\det(M)\det(M_{i_0,i_1}^{j_0,j_1})=\det(M_{i_0}^{j_0})\det(M_{i_1}^{j_1})-\det(M_{i_0}^{j_1})\det(M_{i_1}^{j_0}).$$

The second and third lemmas establish some (more or less straightforward) technical properties about second order difference operators.

Lemma 2.2. *Write A, B and D for the following two first order and a second order difference operators*

$$A=a_0\mathfrak{s}_0+a_1\mathfrak{s}_1,\qquad B=b_{-1}\mathfrak{s}_{-1}+b_0\mathfrak{s}_0,\qquad D=f_{-1}\mathfrak{s}_{-1}+f_0\mathfrak{s}_0+f_1\mathfrak{s}_1,$$

where $\mathfrak{s}_l$ denotes the shift operator $\mathfrak{s}_l(p)(x)=p(x+l)$. Then $D=BA$ if and only if

$$b_{-1}(x)=\frac{f_{-1}(x)}{a_0(x-1)},\quad b_0(x)=\frac{f_1(x)}{a_1(x)},\quad f_0(x)=\frac{f_{-1}(x)a_1(x-1)}{a_0(x-1)}+\frac{f_1(x)a_0(x)}{a_1(x)}.$$

On the other hand, $D=AB$ if and only if

$$a_0(x)=\frac{f_{-1}(x)}{b_{-1}(x)},\quad a_1(x)=\frac{f_1(x)}{b_0(x+1)},\quad f_0(x)=\frac{f_{-1}(x)b_0(x)}{b_{-1}(x)}+\frac{f_1(x)b_{-1}(x+1)}{b_0(x+1)}.$$

Lemma 2.3. *Let D and $\tilde{D}$ be two second order difference operators with rational coefficients. Assume that there exist polynomials p_1, p_2 and p_3 with degrees d_1, d_2 and d_3, respectively, such that $D(p_i)=\tilde{D}(p_i)$, $i=1,2,3$. If $d_i>0$ and $d_i\neq d_j$, $i\neq j$, then $D=\tilde{D}$.*

Proof. Since $d_i\neq d_j$, $i\neq j$, we deduce from Lemma 3.4 of [14] that the polynomial

$$P(x)=\begin{vmatrix}p_1(x+1) & p_1(x) & p_1(x-1)\\ p_2(x+1) & p_2(x) & p_2(x-1)\\ p_3(x+1) & p_3(x) & p_3(x-1)\end{vmatrix}$$

has degree $d=d_1+d_2+d_3-3>0$. Hence $P(x)\neq 0$ for $x\notin X_P$, where X_P is formed by at most d complex numbers. Given any three rational functions g_1,g_2,g_3, write Y for the set formed by their poles. Then, for each $x\notin X_P\cup Y$, the linear system of equations $D(p_i)(x)=g_i(x)$ defines uniquely the value at x of the coefficients of the second order difference operator D. Since $D(p_i)=\tilde{D}(p_i)$, $i=1,2,3$, we can conclude that $D=\tilde{D}$ since their coefficients are equal. $\square$

Given a finite set of numbers $F = \{f_1, \ldots, f_k\}$ we denote by V_F the Vandermonde determinant defined by

$$V_F = \prod_{1=i<j=k} (f_j - f_i). \tag{2.2}$$

2.1. Symmetric operators

Consider a measure μ with finite moments of any order (so that we can integrate polynomials with respect to μ). Let $\mathbb{A}$ be a linear subspace of the linear space of polynomials $\mathbb{P}$. We say that a linear operator $T : \mathbb{A} \to \mathbb{P}$ is symmetric with respect to the pair $(\mu, \mathbb{A})$ if $\langle T(p), q\rangle_\mu = \langle p, T(q)\rangle_\mu$ for all polynomials $p, q \in \mathbb{A}$, where the bilinear form $\langle\cdot,\cdot\rangle_\mu$ is defined by (2.1). The following lemma is then straightforward.

Lemma 2.4. *Let T be a symmetric operator with respect to the pair $(\mu, \mathbb{A})$. Assume we have polynomials $r_n \in \mathbb{A}$, $n \in X \subset \mathbb{N}$, which are eigenfunctions for the operator T with different eigenvalues, that is, $T(r_n) = \lambda_n r_n$, $n \in X$, and $\lambda_n \neq \lambda_m$, $n \neq m$. Then the polynomials r_n, $n \in X$, are orthogonal with respect to μ.*

When μ is a discrete measure, the symmetry of a finite order difference operator $D = \sum_{l=-r}^{r} h_l \mathfrak{s}_l$ with respect to a pair $(\mu, \mathbb{A})$ can be guaranteed by a finite set of difference equations together with certain boundary conditions. The proof follows as that of Theorem 3.2 in [10] and it is omitted.

Lemma 2.5. *Let μ be a discrete measure supported on a countable set X, $X \subset \mathbb{R}$. Consider a finite order difference operator $T : \mathbb{A} \to \mathbb{P}$ of the form $T = \sum_{l=-r}^{r} h_l \mathfrak{s}_l$, where $\mathbb{A}$ is a linear subspace of the linear space of polynomials $\mathbb{P}$. Assume that the measure μ and the coefficients h_l, $l = -r, \ldots, r$, of T satisfy the difference equations*

$$h_l(x-l)\mu(x-l) = h_{-l}(x)\mu(x), \quad \text{for } x \in (l+X)\cap X \text{ and } l = 1, \ldots, r, \tag{2.3}$$

and the boundary conditions

$$h_l(x-l) = 0, \quad \text{for } x \in (l+X)\setminus X \text{ and } l = 1, \ldots, r, \tag{2.4}$$

$$h_{-l}(x) = 0, \quad \text{for } x \in X\setminus(l+X) \text{ and } l = 1, \ldots, r. \tag{2.5}$$

Then T is symmetric with respect to the pair $(\mu, \mathbb{A})$. (Let us remind that for a set of numbers A and a number b, we denote by $b + A$ the set $b + A = \{b + a : a \in A\}$.)

On the other hand, when μ has a smooth density with respect to the Lebesgue measure (i.e. $d\mu = f(x)dx$), the symmetry of a second order differential operator with respect to a pair $(\mu, \mathbb{A})$ can be guaranteed by the usual Pearson equation. The proof follows by performing an integration by parts and it is omitted.

Lemma 2.6. *Let μ be a measure having a positive $\mathcal{C}^2(I)$ density f with respect to the Lebesgue measure in an interval $I \subset \mathbb{R}$. Consider a second order differential operator $T : \mathbb{A} \to \mathbb{P}$ of the form $T = a_2(x)\partial^2 + a_1(x)\partial + a_0(x)$, where $\mathbb{A}$ is a linear subspace of the linear space of polynomials $\mathbb{P}$, $\partial = d/dx$ and a_2 and a_1 are $\mathcal{C}^1(I)$ functions. Assume that f and the coefficients a_2, a_1 of T satisfy the Pearson equation*

$$(a_2(x)f(x))' = a_1(x)f(x), \quad x \in I,$$

and the boundary conditions that the limit of the functions $x^n a_2(x) f(x)$ *and* $x^n (a_2(x) f(x))'$ *vanish at the endpoints of* I *for* $n \geq 0$*. Then* T *is symmetric with respect to the pair* $(\mu, \mathbb{A})$*.*

2.2. *Christoffel transform*

Let μ be a measure (positive or not) and assume that μ has a sequence of orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$, p_n with degree n and $\langle p_n, p_n\rangle \neq 0$ (as we mentioned above, that always happens if μ is positive, with finite moments and infinitely many points in its support). Favard's theorem implies that the sequence of polynomials $(p_n)_n$ satisfies the three term recurrence relation ($p_{-1} = 0$)

$$x p_n(x) = a_n^P p_{n+1}(x) + b_n^P p_n(x) + c_n^P p_{n-1}(x). \tag{2.6}$$

Given a finite set F of real numbers, $F = \{f_1, \ldots, f_k\}$, $f_i < f_{i+1}$, we write Φ_n, $n \geq 0$, for the $k \times k$ determinant

$$\Phi_n = |p_{n+j-1}(f_i)|_{i,j=1,\ldots,k}. \tag{2.7}$$

Notice that Φ_n, $n \geq 0$, depends on both, the finite set F and the measure μ. In order to stress this dependence, we sometimes write in this section $\Phi_n^{\mu,F}$ for Φ_n.

Along this section we assume that the set $\Theta_\mu^F = \{n \in \mathbb{N} : \Phi_n^{\mu,F} = 0\}$ is finite. We denote $\theta_\mu^F = \max \Theta_\mu^F$. If $\Theta_\mu^F = \emptyset$ we take $\theta_\mu^F = -1$.

The Christoffel transform of μ associated to the annihilator polynomial $\mathfrak{p}_F$ of F,

$$\mathfrak{p}_F(x) = (x - f_1)\cdots(x - f_k),$$

is the measure defined by $\mu_F = \mathfrak{p}_F \mu$.

Orthogonal polynomials with respect to μ_F can be constructed by means of the formula

$$q_n(x) = \frac{1}{\mathfrak{p}_F(x)} \det \begin{pmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+k}(x) \\ p_n(f_1) & p_{n+1}(f_1) & \cdots & p_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(f_k) & p_{n+1}(f_k) & \cdots & p_{n+k}(f_k) \end{pmatrix}. \tag{2.8}$$

Notice that the degree of q_n is equal to n if and only if $n \notin \Theta_\mu^F$. In that case the leading coefficient λ_n^Q of q_n is equal to $(-1)^k \lambda_{n+k}^P \Phi_n$, where λ_n^P denotes the leading coefficient of p_n.

The next lemma follows easily using [37], Th. 2.5.

Lemma 2.7. *The measure* μ_F *has a sequence* $(q_n)_{n=0}^\infty$*,* q_n *of degree* n*, of orthogonal polynomials if and only if* $\Theta_\mu^F = \emptyset$*. In that case, an orthogonal polynomial of degree* n *with respect to* μ_F *is given by* (2.8) *and also* $\langle q_n, q_n\rangle_{\mu_F} \neq 0$*,* $n \geq 0$*. If* $\Theta_\mu \neq \emptyset$*, the polynomial* q_n (2.8) *has still degree* n *for* $n \notin \Theta_\mu^F$*, and satisfies* $\langle q_n, r\rangle_{\mu_F} = 0$ *for all polynomials* r *with degree less than* n *and* $\langle q_n, q_n\rangle_{\mu_F} \neq 0$*.*

The three term recurrence relation for the polynomials $(q_n)_n$ can be derived from the corresponding recurrence relation for the polynomials $(p_n)_n$ (2.6). In addition to the determinant Φ_n (2.7), $n \geq 0$, we also consider the $k \times k$ determinant

$$\Psi_n = \begin{vmatrix} p_n(f_1) & p_{n+1}(f_1) & \cdots & p_{n+k-2}(f_1) & p_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_n(f_k) & p_{n+1}(f_k) & \cdots & p_{n+k-2}(f_k) & p_{n+k}(f_k) \end{vmatrix}. \tag{2.9}$$

Lemma 2.8. *For $n > \theta_\mu^F + 1$, the polynomials q_n (2.8) satisfy the three term recurrence relation*

$$xq_n(x) = a_n^Q q_{n+1}(x) + b_n^Q q_n(x) + c_n^Q q_{n-1}(x), \tag{2.10}$$

where

$$a_n^Q = a_n^P \frac{\lambda_{n+1}^P \lambda_{n+k}^P}{\lambda_n^P \lambda_{n+k+1}^P} \frac{\Phi_n}{\Phi_{n+1}},$$
$$b_n^Q = b_{n+k}^P + \frac{\lambda_{n+k}^P}{\lambda_{n+k+1}^P} \frac{\Psi_{n+1}}{\Phi_{n+1}} - \frac{\lambda_{n+k-1}^P}{\lambda_{n+k}^P} \frac{\Psi_n}{\Phi_n},$$
$$c_n^Q = c_n^P \frac{\Phi_{n+1}}{\Phi_n}.$$

Moreover,

$$\langle q_n, q_n \rangle_{\mu_F} = (-1)^k \frac{\lambda_{n+k}^P}{\lambda_n^P} \Phi_n \Phi_{n+1} \langle p_n, p_n \rangle_\mu, \quad n > \theta_\mu^F + 1. \tag{2.11}$$

If $\Theta_\mu^F = \emptyset$, then (2.10) and (2.11) hold for $n \geq 0$, with initial condition $q_{-1} = 0$.

Proof. We can assume that p_n are monic (that is, $\lambda_n^P = 1$). Write $\hat{q}_n(x) = q_n(x)/\lambda_n^Q$, $n > \theta_\mu^F + 1$. It is then enough to prove that

$$x\hat{q}_n(x) = a_n \hat{q}_{n+1}(x) + b_n \hat{q}_n(x) + c_n \hat{q}_{n-1}(x)$$

with

$$a_n = 1, \tag{2.12}$$
$$b_n = b_{n+k}^P + \frac{\Psi_{n+1}}{\Phi_{n+1}} - \frac{\Psi_n}{\Phi_n}, \tag{2.13}$$
$$c_n = c_n^P \frac{\Phi_{n-1} \Phi_{n+1}}{\Phi_n^2}, \tag{2.14}$$
$$\langle q_n, q_n \rangle_{\mu_F} = \frac{\langle p_n, p_n \rangle_\mu \Phi_{n+1}}{\Phi_n}. \tag{2.15}$$

We write $u_n(x) = x^n$ for $n \leq \theta_\mu^F + 1$, and $u_n(x) = \hat{q}_n(x)$ for $n > \theta_\mu^F + 1$. Then the polynomials u_n, $n \geq 0$, form a basis of $\mathbb{P}$. From the previous lemma, we also have for $n > \theta_\mu^F$ that $\langle \hat{q}_n, u_j \rangle_{\mu_F} = 0$, $j = 0, \ldots, n-1$. Taking this into account, it is easy to deduce that the polynomials $\hat{q}_n$, $n > \theta_\mu^F + 1$, satisfy a three term recurrence relation

$$x\hat{q}_n(x) = a_n \hat{q}_{n+1}(x) + b_n \hat{q}_n(x) + c_n \hat{q}_{n-1}(x).$$

Since they are monic, we straightforwardly have $a_n = 1$, that is, (2.12). We compute c_n as

$$c_n = \frac{\langle x\hat{q}_n, \hat{q}_{n-1} \rangle_{\mu_F}}{\langle \hat{q}_{n-1}, \hat{q}_{n-1} \rangle_{\mu_F}} = \frac{\langle x\hat{q}_n, p_{n-1} \rangle_{\mu_F}}{\langle \hat{q}_{n-1}, \hat{q}_{n-1} \rangle_{\mu_F}}.$$

Using (2.8), we get

$$\begin{aligned}
\langle x\hat{q}_n, p_{n-1}\rangle_{\mu_F} &= \frac{(-1)^k}{\Phi_n}\begin{vmatrix} \langle xp_n, p_{n-1}\rangle_\mu & \langle xp_{n+1}, p_{n-1}\rangle_\mu & \cdots & \langle xp_{n+k}, p_{n-1}\rangle_\mu \\ p_n(f_1) & p_{n+1}(f_1) & \cdots & p_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(f_k) & p_{n+1}(f_k) & \cdots & p_{n+k}(f_k) \end{vmatrix} \\
&= \frac{(-1)^k}{\Phi_n}\begin{vmatrix} c_n^P\langle p_{n-1}, p_{n-1}\rangle_\mu & 0 & \cdots & 0 \\ p_n(f_1) & p_{n+1}(f_1) & \cdots & p_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(f_k) & p_{n+1}(f_k) & \cdots & p_{n+k}(f_k) \end{vmatrix} \\
&= (-1)^k \frac{c_n^P\langle p_{n-1}, p_{n-1}\rangle_\mu \Phi_{n+1}}{\Phi_n}.
\end{aligned}$$

In a similar way, one finds that

$$\langle \hat{q}_{n-1}, \hat{q}_{n-1}\rangle_{\mu_F} = (-1)^k \frac{\langle p_{n-1}, p_{n-1}\rangle_\mu \Phi_n}{\Phi_{n-1}}.$$

(2.14) and (2.15) can now be easily deduced. (2.13) can be proved analogously. □

For $\Theta_\mu^F = \emptyset$, the previous lemma has already appeared in the literature (see for instance [39]).

2.3. *Finite set of positive integers*

From now on, F will denote a finite set of positive integers. We will write $F = \{f_1, \ldots, f_k\}$, with $f_i < f_{i+1}$. Hence k is the number of elements of F and f_k is the maximum element of F.

We associate to F the nonnegative integers u_F and v_F and the infinite set of nonnegative integers σ_F defined by

$$u_F = \sum_{f\in F} f - \binom{k+1}{2}, \tag{2.16}$$

$$v_F = \sum_{f\in F} f + f_k - \frac{(k-1)(k+2)}{2}, \tag{2.17}$$

$$\sigma_F = \{u_F, u_F+1, u_F+2, \ldots\} \setminus \{u_F + f, f \in F\}. \tag{2.18}$$

The infinite set σ_F will be the set of indices for the exceptional Charlier or Hermite polynomials associated to F.

Notice that $v_F = u_F + f_k + 1$; hence $\{v_F, v_F+1, v_F+2, \ldots\} \subset \sigma_F$. Notice also that u_F is an increasing function with respect to the inclusion order, that is, if $F \subset \tilde{F}$ then $u_F \le u_{\tilde{F}}$.

Consider the set Υ formed by all finite sets of positive integers:

$$\Upsilon = \{F : F \text{ is a finite set of positive integers}\}.$$

We consider the involution I in Υ defined by

$$I(F) = \{1, 2, \ldots, f_k\} \setminus \{f_k - f, f \in F\}. \tag{2.19}$$

The definition of I implies that $I^2 = Id$.

For the involution I, the bigger the holes in F (with respect to the set $\{1, 2, \dots, f_k\}$), the bigger the involuted set $I(F)$. Here it is a couple of examples

$$I(\{1, 2, 3, \dots, k\}) = \{k\}, \qquad I(\{1, k\}) = \{1, 2, \dots, k-2, k\}.$$

The set $I(F)$ will be denoted by G: $G = I(F)$. We also write $G = \{g_1, \dots, g_m\}$ with $g_i < g_{i+1}$ so that m is the number of elements of G and g_m the maximum element of G. Notice that

$$f_k = g_m, \quad m = f_k - k + 1.$$

We also define the number s_F by

$$s_F = \begin{cases} 1, & \text{if } F = \emptyset, \\ k+1, & \text{if } F = \{1, 2, \dots, k\}, \\ \min\{s \geq 1 : s < f_s\}, & \text{if } F \neq \{1, 2, \dots, k\}. \end{cases} \tag{2.20}$$

For $1 \leq i \leq k$, we denote by $F_{\{i\}}$ and $F_{\Downarrow}$ the finite sets of positive integers defined by

$$F_{\{i\}} = F \setminus \{f_i\}, \tag{2.21}$$

$$F_{\Downarrow} = \begin{cases} \emptyset, & \text{if } F = \{1, 2, \dots, k\}, \\ \{f_{s_F} - s_F, \dots, f_k - s_F\}, & \text{if } F \neq \{1, 2, \dots, k\}. \end{cases} \tag{2.22}$$

The following relation is straightforward from (2.19), (2.21) and (2.22):

$$F_{\Downarrow} = I\left(G_{\{m\}}\right) \tag{2.23}$$

(where as indicated above $G = I(F)$ and m is the number of elements of G).

2.4. Charlier and Hermite polynomials

We include here basic definitions and facts about Charlier and Hermite polynomials, which we will need in the following sections.

For $a \neq 0$, we write $(c_n^a)_n$ for the sequence of Charlier polynomials (the next formulas can be found in [6], pp. 170–1; see also [26], pp., 247–9 or [29], ch. 2) defined by

$$c_n^a(x) = \frac{1}{n!} \sum_{j=0}^{n} (-a)^{n-j} \binom{n}{j} \binom{x}{j} j!. \tag{2.24}$$

The Charlier polynomials are orthogonal with respect to the measure

$$\rho_a = \sum_{x=0}^{\infty} \frac{a^x}{x!} \delta_x, \quad a \neq 0, \tag{2.25}$$

which is positive only when $a > 0$ and then

$$\langle c_n^a, c_n^a \rangle = \frac{a^n}{n!} e^a. \tag{2.26}$$

The three-term recurrence formula for $(c_n^a)_n$ is $(c_{-1}^a = 0)$

$$x c_n^a = (n+1) c_{n+1}^a + (n+a) c_n^a + a c_{n-1}^a, \quad n \geq 0. \tag{2.27}$$

They are eigenfunctions of the following second-order difference operator

$$D_a = -x \mathfrak{S}_{-1} + (x+a) \mathfrak{S}_0 - a \mathfrak{S}_1, \qquad D_a(c_n^a) = n c_n^a, \quad n \geq 0, \tag{2.28}$$

where $\mathfrak{S}_j(f) = f(x+j)$. They also satisfy

$$\Delta(c_n^a) = c_{n-1}^a, \qquad \frac{d}{da}(c_n^a) = -c_{n-1}^a, \tag{2.29}$$

and the duality

$$(-1)^m a^m n! c_n^a(m) = (-1)^n a^n m! c_m^a(n), \quad n, m \geq 0. \tag{2.30}$$

We write $(H_n)_n$ for the sequence of Hermite polynomials (the next formulas can be found in [6], Ch. V; see also [26], pp. 250–3) defined by

$$H_n(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2x)^{n-2j}}{j!(n-2j)!}. \tag{2.31}$$

The Hermite polynomials are orthogonal with respect to the weight function e^{-x^2}, $x \in \mathbb{R}$. They are eigenfunctions of the following second-order differential operator

$$D = \partial^2 - 2x\partial, \qquad D(H_n) = -2nH_n, \quad n \geq 0, \tag{2.32}$$

where $\partial = d/dx$.

They also satisfy $H_n'(x) = 2nH_{n-1}(x)$.

One can obtain Hermite polynomials from Charlier polynomials using the limit

$$\lim_{a\to\infty} \left(\frac{2}{a}\right)^{n/2} c_n^a(\sqrt{2a}x + a) = \frac{1}{n!} H_n(x) \tag{2.33}$$

see [26], p. 249 (take into account that if we write $(C_n^a)_n$ for the polynomials defined by (9.14.1) in [26], p. 247, then $c_n^a = (-a)^n C_n^a/n!$). The previous limit is uniform in compact sets of $\mathbb{C}$.

3. Constructing polynomials which are eigenfunctions of second order difference operators

As in Section 2.3, F will denote a finite set of positive integers. We will write $F = \{f_1, \ldots, f_k\}$, with $f_i < f_{i+1}$. Hence k is the number of elements of F and f_k is the maximum element of F.

We associate to each finite set F of positive integers the polynomials $c_n^{a;F}$, $n \in \sigma_F$, displayed in the following definition. It turns out that these polynomials are always eigenfunctions of a second order difference operator with rational coefficients. We call them exceptional Charlier polynomials when, in addition, they are orthogonal and complete with respect to a positive measure (this will happen as long as the finite set F is admissible; see Definition 4.2 in the next section).

Definition 3.1. For a given real number $a \neq 0$ and a finite set F of positive integers, we define the polynomials $c_n^{a;F}$, $n \in \sigma_F$, as

$$c_n^{a;F}(x) = \begin{vmatrix} c_{n-u_F}^a(x) & c_{n-u_F}^a(x+1) & \cdots & c_{n-u_F}^a(x+k) \\ c_{f_1}^a(x) & c_{f_1}^a(x+1) & \cdots & c_{f_1}^a(x+k) \\ \vdots & \vdots & \ddots & \vdots \\ c_{f_k}^a(x) & c_{f_k}^a(x+1) & \cdots & c_{f_k}^a(x+k) \end{vmatrix}, \tag{3.1}$$

where the number u_F and the infinite set of nonnegative integers σ_F are defined by (2.16) and (2.18), respectively.

To simplify the notation, we will sometimes write $c_n^F = c_n^{a;F}$.

Using Lemma 3.4 of [14], we deduce that c_n^F, $n \in \sigma_F$, is a polynomial of degree n with leading coefficient equal to

$$\frac{\prod_{i=1}^{k}(f_i - n + u_F)}{(n-u_F)!\prod_{f\in F} f!}V_F, \tag{3.2}$$

where V_F is the Vandermonde determinant (2.2). With the convention that $c_n^a = 0$ for $n < 0$, the determinant (3.1) defines a polynomial for any $n \geq 0$, but for $n \notin \sigma_F$ we have $c_n^F = 0$.

Combining columns in (3.1) and taking into account the first formula in (2.29), we have the alternative definition

$$c_n^F(x) = \begin{vmatrix} c^a_{n-u_F}(x) & c^a_{n-u_F-1}(x) & \cdots & c^a_{n-u_F-k}(x) \\ c^a_{f_1}(x) & c^a_{f_1-1}(x) & \cdots & c^a_{f_1-k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c^a_{f_k}(x) & c^a_{f_k-1}(x) & \cdots & c^a_{f_k-k}(x) \end{vmatrix}. \tag{3.3}$$

The polynomials c_n^F, $n \in \sigma_F$, are strongly related by duality with the polynomials q_n^F, $n \geq 0$, defined by

$$q_n^F(x) = \frac{\begin{vmatrix} c^a_n(x-u_F) & c^a_{n+1}(x-u_F) & \cdots & c^a_{n+k}(x-u_F) \\ c^a_n(f_1) & c^a_{n+1}(f_1) & \cdots & c^a_{n+k}(f_1) \\ \vdots & \vdots & \ddots & \vdots \\ c^a_n(f_k) & c^a_{n+1}(f_k) & \cdots & c^a_{n+k}(f_k) \end{vmatrix}}{\prod_{f\in F}(x-f-u_F)}. \tag{3.4}$$

Lemma 3.2. *If u is a nonnegative integer and $v \in \sigma_F$, then*

$$q_u^F(v) = \xi_u \zeta_v c_v^F(u), \tag{3.5}$$

where

$$\xi_u = \frac{(-a)^{(k+1)u}}{\prod_{i=0}^{k}(u+i)!}, \qquad \zeta_v = \frac{(-a)^{-v}(v-u_F)!\prod_{f\in F} f!}{\prod_{f\in F}(v-f-u_F)}.$$

Proof. It is a straightforward consequence of the duality (2.30) for the Charlier polynomials. $\square$

We now prove that the polynomials c_n^F, $n \in \sigma_F$, are eigenfunctions of a second order difference operator with rational coefficients. To establish the result in full, we need some more notations. We denote by $\Omega_F^a(x)$ and $\Lambda_F^a(x)$ the polynomials

$$\Omega_F^a(x) = |c^a_{f_i}(x+j-1)|_{i,j=1}^k, \tag{3.6}$$

$$\Lambda_F^a(x) = \begin{vmatrix} c_{f_1}^a(x) & c_{f_1}^a(x+1) & \cdots & c_{f_1}^a(x+k-2) & c_{f_1}^a(x+k) \\ \vdots & \vdots & \ddots & \vdots & \\ c_{f_k}^a(x) & c_{f_k}^a(x+1) & \cdots & c_{f_k}^a(x+k-2) & c_{f_k}^a(x+k) \end{vmatrix}. \tag{3.7}$$

To simplify the notation, we will sometimes write $\Omega^F = \Omega_F^a$ and $\Lambda^F = \Lambda_F^a$.

Using Lemma 3.4 of [14] and the definition of u_F (2.16), we deduce that the degree of both Ω_F and Λ_F is $u_F + k$. From (3.1) and (3.6), we have

$$\Omega_F(x) = (-1)^{k-1} c_{f_k+u_{F_{\{k\}}}}^{F_{\{k\}}}(x), \tag{3.8}$$

where the finite set of positive integers $F_{\{k\}}$ is defined by (2.21).

As for c_n^F (see (3.3)), we have for Ω_F the following alternative definition

$$\Omega_F(x) = |c_{f_i-j+1}^a(x)|_{i,j=1}^k. \tag{3.9}$$

From here and (3.3), it is easy to deduce that

$$c_{u_F}^F(x) = \Omega_{F_\Downarrow}(x), \tag{3.10}$$

where the finite set of positive integers $F_\Downarrow$ is defined by (2.22).

A simple calculation using the third formula in (2.29) shows that

$$\Lambda_F^a(x) = k\Omega_F^a(x) - \frac{d}{da}\Omega_F^a(x). \tag{3.11}$$

We also need the determinants Φ_n^F and Ψ_n^F, $n \geq 0$, defined by

$$\Phi_n^F = |c_{n+j-1}^a(f_i)|_{i,j=1}^k, \tag{3.12}$$

$$\Psi_n^F = \begin{vmatrix} c_n^a(f_1) & c_{n+1}^a(f_1) & \cdots & c_{n+k-2}^a(f_1) & c_{n+k}^a(f_1) \\ \vdots & \vdots & \ddots & \vdots & \\ c_n^a(f_k) & c_{n+1}^a(f_k) & \cdots & c_{n+k-2}^a(f_k) & c_{n+k}^a(f_k) \end{vmatrix}. \tag{3.13}$$

Using the duality (2.30), we have

$$\Omega_F(n) = \frac{\prod_{i=0}^{k-1}(n+i)!}{(-a)^{k(n-1)-u_F}\prod_{f\in F} f!}\Phi_n^F, \tag{3.14}$$

$$\Lambda_F(n) = \frac{(n+k)!\prod_{i=0}^{k-2}(n+i)!}{(-a)^{k(n-1)-u_F+1}\prod_{f\in F} f!}\Psi_n^F. \tag{3.15}$$

According to Lemma 2.7, as long as $\Phi_n^F \neq 0$, $n \geq 0$, the polynomials q_n^F, $n \geq 0$, are orthogonal with respect to the measure

$$\rho_a^F = \sum_{x=u_F}^{\infty} \prod_{f\in F}(x-f-u_F)\frac{a^{x-u_F}}{(x-u_F)!}\delta_x. \tag{3.16}$$

Notice that the measure ρ_a^F is supported in the infinite set of nonnegative integers σ_F (2.18).

Theorem 3.3. *Let F be a finite set of positive integers. Then the polynomials c_n^F, $n \in \sigma_F$, (3.1) are common eigenfunctions of the second order difference operator*

$$D_F = h_{-1}(x)\mathfrak{S}_{-1} + h_0(x)\mathfrak{S}_0 + h_1(x)\mathfrak{S}_1, \tag{3.17}$$

where

$$h_{-1}(x) = -x\frac{\Omega_F(x+1)}{\Omega_F(x)}, \tag{3.18}$$

$$h_0(x) = x + k + a + u_F - a\frac{\Lambda_F(x+1)}{\Omega_F(x+1)} + a\frac{\Lambda_F(x)}{\Omega_F(x)}, \tag{3.19}$$

$$h_1(x) = -a\frac{\Omega_F(x)}{\Omega_F(x+1)}. \tag{3.20}$$

Moreover $D_F(c_n^F) = nc_n^F$, $n \in \sigma_F$.

Proof. Consider the set Θ_a^F of nonnegative integers defined by $\Theta_a^F = \{n \in \mathbb{N} : \Phi_n^F = 0\}$. Using (3.14), we get $\Theta_a^F = \{x \in \mathbb{N} : \Omega_F(x) = 0\}$. Since Ω_F is a polynomial in x, we conclude that Θ_a^F is finite. Define then $\theta_a^F = \max \Theta_a^F$, with the convention that if $\Theta_a^F = \emptyset$ then $\theta_a^F = -1$.

Write $p_n(x) = c_n^a(x - u_F)$ and $q_n(x) = q_n^F(x)$ (see (3.4)). With the notation of Section 2.2, we have

$$\lambda_n^P = \frac{1}{n!}, \qquad \lambda_n^Q = \frac{(-1)^k \Phi_n^F}{(n+k)!}.$$

Using the three term recurrence relations (2.27) for the Charlier polynomials and (2.10) for q_n, $n > \theta_a^F + 1$, we conclude after an easy calculation that for $u > \theta_a^F + 1$ and $v \in \mathbb{R}$

$$vq_u^F(v) = a_u^Q q_{u+1}^F(v) + b_u^Q q_u^F(v) + c_u^Q q_{u-1}^F(v), \tag{3.21}$$

where

$$a_n^Q = (n+k+1)\frac{\Phi_n^F}{\Phi_{n+1}^F}, \tag{3.22}$$

$$b_n^Q = (n+k+a+u_F) + (n+k+1)\frac{\Psi_{n+1}^F}{\Phi_{n+1}^F} - (n+k)\frac{\Psi_n^F}{\Phi_n^F}, \tag{3.23}$$

$$c_n^Q = a\frac{\Phi_{n+1}^F}{\Phi_n^F}. \tag{3.24}$$

Assume now that $v \in \sigma_F$. Then, using the dualities (3.5), (3.14) and (3.15), we get from (3.21) after straightforward calculations

$$uc_v^F(u) = h_1(u)c_v^F(u+1) + h_0(u)c_v^F(u) + h_{-1}(u)c_v^F(u-1), \tag{3.25}$$

for all nonnegative integers $u > \theta_a^F + 1$, where h_1, h_0 and h_{-1} are given by (3.18)–(3.20), respectively. Since c_v^F, $v \in \sigma_F$, are polynomials and h_1, h_0 and h_{-1} are rational functions, we have that (3.25) holds also for all complex numbers u. In other words, the polynomials c_n^F, $n \in \sigma_F$, are eigenfunctions of the second order difference operator D_F (3.17). □

The determinant which defines Ω_F^a (3.6) enjoys a very nice invariant property with respect to the involution I defined by (2.19). Indeed, for a finite set $F = \{f_1, \ldots, f_k\}$ of positive integers, consider the involuted set $I(F) = G = \{g_1, \dots, g_m\}$ with $g_i < g_{i+1}$. We also need the associated functions $\tilde{\Omega}_F^a$ and $\tilde{\Lambda}_F^a$ defined by

$$\tilde{\Omega}_F^a(x) = |c_{g_i}^{-a}(-x + j - 1)|_{i,j=1}^m, \tag{3.26}$$

$$\tilde{\Lambda}_F^a(x) = \begin{vmatrix} c_{g_1}^{-a}(-x) & c_{g_1}^{-a}(-x+1) & \cdots & c_{g_1}^{-a}(-x+m-2) & c_{g_1}^{-a}(-x+m) \\ \vdots & \vdots & \ddots & \vdots & \\ c_{g_m}^{-a}(-x) & c_{g_m}^{-a}(-x+1) & \cdots & c_{g_m}^{-a}(-x+m-2) & c_{g_m}^{-a}(-x+m) \end{vmatrix}. \tag{3.27}$$

Using the definition of the involution I, we have that both $\tilde{\Omega}_F^a$ and $\tilde{\Lambda}_F^a$ are polynomials of degree $u_F + k$ (this last can be deduced using Lemma 3.4 of [14]).

The invariant property mentioned above for Ω_F^a (3.6) is the following: except for a sign, Ω_F^a remains invariant if we change F to $G = I(F)$, x to $-x$ and a to $-a$. In other words, except for a sign, Ω_F^a and $\tilde{\Omega}_F^a$ are equal:

$$\Omega_F^a(x) = (-1)^{k+u_F}\tilde{\Omega}_F^a(x). \tag{3.28}$$

For finite sets F formed by consecutive positive integers this invariance was conjecture in [12] and proved in [13]. The proof for all finite set of positive integers will be included in [8].

According to this invariant property, we can rewrite as follows the second order difference operator D_F for which the polynomials c_n^F, $n \in \sigma_F$, are common eigenfunctions.

Theorem 3.4. *Let F be a finite set of positive integers. Then the coefficients h_{-1}, h_0, h_1 of the operator D_F (3.17) can be rewritten in the form*

$$h_{-1}(x) = -x\frac{\tilde{\Omega}_F(x+1)}{\tilde{\Omega}_F(x)}, \tag{3.29}$$

$$h_0(x) = x + m + a + u_G + a\frac{\tilde{\Lambda}_F(x+1)}{\tilde{\Omega}_F(x+1)} - a\frac{\tilde{\Lambda}_F(x)}{\tilde{\Omega}_F(x)}, \tag{3.30}$$

$$h_1(x) = -a\frac{\tilde{\Omega}_F(x)}{\tilde{\Omega}_F(x+1)}. \tag{3.31}$$

Proof. Using (3.28) and (3.11), we straightforwardly get (3.29), (3.30) and (3.31) from (3.18)–(3.20). □

We next show that the polynomials c_n^F, $n \in \sigma_F$, (3.1) and the corresponding difference operator D_F (3.17) can be constructed by applying a sequence of at most k Darboux transforms to the Charlier system (where k is the number of elements of F).

Definition 3.5. Given a system $(T, (\phi_n)_n)$ formed by a second order difference operator T and a sequence $(\phi_n)_n$ of eigenfunctions for T, $T(\phi_n) = \pi_n\phi_n$, by a Darboux transform of the system $(T, (\phi_n)_n)$ we mean the following. For a real number λ, we factorize $T - \lambda Id$ as the product of two first order difference operators $T = BA + \lambda Id$ (Id denotes the identity operator). We then produce a new system consisting in the operator $\hat{T}$, obtained by reversing the order of the factors, $\hat{T} = AB + \lambda Id$, and the sequence of eigenfunctions $\hat{\phi}_n = A(\phi_n)$: $\hat{T}(\hat{\phi}_n) = \pi_n\hat{\phi}_n$. We say that the system $(\hat{T}, (\hat{\phi}_n)_n)$ has been obtained by applying a Darboux transformation with parameter λ to the system $(T, (\phi_n)_n)$.

Lemma 3.6. *Let $F = \{f_1, \ldots, f_k\}$ be a finite set of positive integers and write $F_{\{k\}} = \{f_1, \ldots, f_{k-1}\}$ (see (2.21)). We define the first order difference operators A_F and B_F as*

$$A_F = \frac{\Omega_F(x+1)}{\Omega_{F_{\{k\}}}(x+1)}\mathfrak{s}_0 - \frac{\Omega_F(x)}{\Omega_{F_{\{k\}}}(x+1)}\mathfrak{s}_1, \tag{3.32}$$

$$B_F = -x\frac{\Omega_{F_{\{k\}}}(x+1)}{\Omega_F(x)}\mathfrak{s}_{-1} + a\frac{\Omega_{F_{\{k\}}}(x)}{\Omega_F(x)}\mathfrak{s}_0. \tag{3.33}$$

Then $c_n^F(x) = A_F(c^{F_{\{k\}}}_{n-f_k+k})(x)$, $n \in \sigma_F$. Moreover

$$D_{F_{\{k\}}} = B_F A_F + (f_k + u_{F_{\{k\}}})Id,$$
$$D_F = A_F B_F + (f_k + u_F)Id.$$

In other words, the system $(D_F, (c_n^F)_{n\in\sigma_F})$ can be obtained by applying a Darboux transform to the system $(D_{F_{\{k\}}}, (c_n^{F_{\{k\}}})_{n\in\sigma_{F_{\{k\}}}})$.

Proof. First of all, we point out that $\sigma_F = f_k - k + \sigma_{F_{\{k\}}}$ (that is an easy consequence of (2.16) and (2.18)). In particular $u_F = u_{F_{\{k\}}} + f_k - k$.

If we apply Sylvester's identity with $i_0 = j_0 = 1$, $i_1 = j_1 = k$ (see Lemma 2.1) to the determinant (3.1), we get

$$\begin{aligned} c_n^F(x) &= \frac{\Omega_F(x+1)}{\Omega_{F_{\{k\}}}(x+1)}c^{F_{\{k\}}}_{n-f_k+k}(x) - \frac{\Omega_F(x)}{\Omega_{F_{\{k\}}}(x+1)}c^{F_{\{k\}}}_{n-f_k+k}(x+1) \\ &= A_F(c^{F_{\{k\}}}_{n-f_k+k})(x). \end{aligned}$$

Write now $D_{F_{\{k\}}} = h_{-1}^{F_{\{k\}}}\mathfrak{s}_{-1} + h_0^{F_{\{k\}}}\mathfrak{s}_0 + h_1^{F_{\{k\}}}\mathfrak{s}_1$. Using Lemma 2.2, the factorization $D_{F_{\{k\}}} = B_F A_F - (f_k + u_{F_{\{k\}}})Id$ will follow if we prove

$$h_0^{F_{\{k\}}}(x) - (f_k + u_{F_{\{k\}}}) = -h_{-1}^{F_{\{k\}}}(x)\frac{\Omega_F(x-1)}{\Omega_F(x)} - h_1^{F_{\{k\}}}(x)\frac{\Omega_F(x+1)}{\Omega_F(x)}.$$

This can be rewritten as

$$D_{F_{\{k\}}}(\Omega_F) = (f_k + u_{F_{\{k\}}})\Omega_F. \tag{3.34}$$

But this is a consequence of the identity $\Omega_F(x) = (-1)^{k-1}c^{F_{\{k\}}}_{f_k+u_{F_{\{k\}}}}(x)$ (3.8).

We finally prove the factorization $D_F = A_F B_F - f_k Id$. Since $D_F(c_n^F) = nc_n^F$, $n \in \sigma_F$, using Lemma 2.3, it will be enough to prove that $A_F B_F(c_n^F) = (n - f_k - u_F)c_n^F$, $n \in \sigma_F$:

$$\begin{aligned} A_F B_F(c_n^F) &= A_F B_F A_F(c^{F_{\{k\}}}_{n-f_k+k}) = A_F[D^{F_{\{k\}}} - (f_k + u_{F_{\{k\}}})Id](c^{F_{\{k\}}}_{n-f_k+k}) \\ &= A_F[(n - f_k - u_F)(c^{F_{\{k\}}}_{n-f_k+k})] = (n - f_k - u_F)c_n^F. \quad \square \end{aligned}$$

Analogous factorization can be obtained by using any of the sets $F_{\{i\}}$, $1 \le i < k$ (see (2.21)) instead of $F_{\{k\}}$.

When the determinants $\Omega_F(n) \neq 0$ (3.6), $n \ge 0$ (or equivalently, $\Phi_n^F \neq 0$ (3.12), $n \ge 0$), the following alternative construction of the polynomial q_n^F (3.4) has been given in [14]. For a finite set $F = \{f_1, \ldots, f_k\}$ of positive integers, consider the involuted set $I(F) = G = \{g_1, \ldots, g_m\}$

with $g_i < g_{i+1}$, where the involution I is defined by (2.19). Assuming that $\Omega_F(n) \neq 0$, $n \geq 0$, using the invariance (3.28) and Theorem 1.1 of [14], we have

$$q_n^F(x) = \alpha_n \begin{vmatrix} c_n^a(x - v_F) & -c_{n-1}^a(x - v_F) & \cdots & (-1)^m c_{n-m}^a(x - v_F) \\ c_{g_1}^{-a}(-n-1) & c_{g_1}^{-a}(-n) & \cdots & c_{g_1}^{-a}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{g_m}^{-a}(-n-1) & c_{g_m}^{-a}(-n) & \cdots & c_{g_m}^{-a}(-n+m-1) \end{vmatrix}, \tag{3.35}$$

where α_n, $n \geq 0$, is the normalization constant

$$\alpha_n = (-1)^{k(n+1)} \frac{a^{k(n-1)-u_F} \prod_{f \in F} f!}{\prod_{i=1}^{k} (n+i)!}.$$

The duality (3.5) then provides an alternative definition of the polynomial c_n^F, $n \geq v_F$. Indeed, after an easy calculation, we conclude that

$$c_n^F(x) = \beta_n \begin{vmatrix} c_{n-v_F}^a(x) & \frac{x}{a} c_{n-v_F}^a(x-1) & \cdots & \frac{(x-m+1)_m}{a^m} c_{n-v_F}^a(x-m+1) \\ c_{g_1}^{-a}(-x-1) & c_{g_1}^{-a}(-x) & \cdots & c_{g_1}^{-a}(-x+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{g_m}^{-a}(-x-1) & c_{g_m}^{-a}(-x) & \cdots & c_{g_m}^{-a}(-x+m-1) \end{vmatrix}, \tag{3.36}$$

where β_n, $n \geq 0$, is the normalization constant

$$\beta_n = (-1)^{m+k+u_F} \frac{a^m (n - v_F)! V_F \prod_{g \in G} g! \prod_{i=1}^{k} (f_i - n + u_F)}{(n - u_F)! V_G \prod_{f \in F} f!}. \tag{3.37}$$

When the cardinality of the involuted set $G = I(F)$ is less than the cardinality of F, (3.36) will provide a more efficient way than (3.1) for an explicit computation of the polynomials c_n^F, $n \geq v_F$. For instance, take $F = \{1, \ldots, k\}$. Since $I(F) = \{k\}$, the determinant in (3.36) has order 2 while the determinant in (3.1) has order $k + 1$.

Applying Sylvester's identity to the determinant (3.36), we get an alternative way to construct the system (D_F, c_n^F) by applying a sequence of at most m Darboux transforms to the Charlier system.

Lemma 3.7. *Given a real number $a \neq 0$ and a finite set F of positive integers for which $\Omega_F^a(n) \neq 0$, $n \geq 0$, define the first order difference operators C_F and E_F as*

$$C_F = -\frac{x \tilde{\Omega}_F(x+1)}{a \tilde{\Omega}_{F_\Downarrow}(x)} \mathfrak{s}_{-1} + \frac{\tilde{\Omega}_F(x)}{\tilde{\Omega}_{F_\Downarrow}(x)} \mathfrak{s}_0, \tag{3.38}$$

$$E_F = a \frac{\tilde{\Omega}_{F_\Downarrow}(x+1)}{\tilde{\Omega}_F(x+1)} \mathfrak{s}_0 - a \frac{\tilde{\Omega}_{F_\Downarrow}(x)}{\tilde{\Omega}_F(x+1)} \mathfrak{s}_1, \tag{3.39}$$

where $F_{\Downarrow}$ is the finite set of positive integers defined by (2.22). Then $D_{F_{\Downarrow}} = E_F C_F + (u_F - k - 1)Id$ and $D_F = C_F E_F + u_F Id$. Moreover

$$C_F(c_{n-k-1}^{F_{\Downarrow}}) = (-1)^{u_F + u_{F_{\Downarrow}} + 1} \frac{n - u_F}{a} c_n^F(x), \quad n \geq v_F, \tag{3.40}$$

where $n_{F_{\Downarrow}}$ is the number of elements of $F_{\Downarrow}$.

Proof. Write $D_{F_{\Downarrow}} = h_{-1}^{F_{\Downarrow}} \mathfrak{S}_{-1} + h_0^{F_{\Downarrow}} \mathfrak{S}_0 + h_1^{F_{\Downarrow}} \mathfrak{S}_1$. Using Lemma 2.2, the factorization $D_{F_{\Downarrow}} = E_F C_F - (f_k + u_{F_0})Id$ will follow if we prove

$$h_0^{F_{\Downarrow}}(x) - (u_F - k - 1) = -\frac{a}{x} h_{-1}^{F_{\Downarrow}}(x) \frac{\tilde{\Omega}_F(x)}{\tilde{\Omega}_F(x+1)} - \frac{x+1}{a} h_1^{F_{\Downarrow}}(x) \frac{\tilde{\Omega}_F(x+2)}{\tilde{\Omega}_F(x+1)}. \tag{3.41}$$

If we set $a \to -a$, $x \to -x - 1$ and use the invariant property of Ω (3.28), this can be rewritten as

$$D_{G_{\{m\}}}(\Omega_G) = (g_m + u_{G_{\{m\}}})\Omega_G,$$

where $G_{\{m\}}$ is the finite set of positive integers defined by (2.21). (3.41) then follows by taking into account that $\Omega_G(x) = (-1)^{m-1} c_{g_m + u_{G_{\{m\}}}}^{G_{\{m\}}}(x)$ (3.8).

For $n \geq v_F$, the identity (3.40) follows by applying Sylvester's identity to the determinant (3.36) and using (2.23).

The factorization $D_F = C_F E_F + u_F Id$ can be proved as in Lemma 3.6. □

We have computational evidences which show that (3.40) also holds for $n \in \sigma_F$, $n < v_F$.

4. Exceptional Charlier polynomials

In the previous section, we have associated to each finite set F of positive integers the polynomials c_n^F, $n \in \sigma_F$, which are always eigenfunctions of a second order difference operator with rational coefficients. We are interested in the cases when, in addition, those polynomials are orthogonal and complete with respect to a positive measure.

Definition 4.1. The polynomials $c_n^{a;F}$, $n \in \sigma_F$, defined by (3.1) are called exceptional Charlier polynomials, if they are orthogonal and complete with respect to a positive measure.

We next introduce the key concept for finite sets F such that the polynomials c_n^F, $n \in \sigma_F$, are exceptional Charlier polynomials.

Definition 4.2. Let F be a finite set of positive integers. Split up the set F, $F = \bigcup_{i=1}^K Y_i$, in such a way that $Y_i \cap Y_j = \emptyset$, $i \neq j$, the elements of each Y_i are consecutive integers and $1 + \max(Y_i) < \min Y_{i+1}$, $i = 1, \ldots, K-1$. We say that F is admissible if each Y_i, $i = 1, \ldots, K$, has an even number of elements.

Admissible sets F can be characterized in terms of the positivity of the measure ρ_a^F (3.16) and the sign of the Casorati polynomial Ω_F in $\mathbb{N}$.

Lemma 4.3. *Given a positive real number a and a finite set F of positive integers, the following conditions are equivalent.*

1. *The measure ρ_a^F (3.16) is positive.*

2. *The finite set F is admissible.*
3. $\Omega_F^a(n)\Omega_F^a(n+1) > 0$ *for all nonnegative integers n, where the polynomial Ω_F^a is defined by* (3.6).

Proof. It is clear that the definition of an admissible set F is equivalent to $\prod_{f\in F}(x-f) \geq 0$, for all $x \in \mathbb{N}$. The equivalence between (1) and (2) is then an easy consequence of the definition of the measure ρ_a^F.

We now prove the equivalence between (1) and (3).

(1) $\Rightarrow$ (3). Since the measure ρ_a^F is positive, the polynomials $(q_n^F)_n$ (3.4) are orthogonal with respect to the measure ρ_a^F and have a positive L^2-norm. According to (2.11) in Lemma 2.8, we have

$$\langle q_n^F, q_n^F\rangle = (-1)^k \frac{n!}{(n+k)!}\langle c_n^a, c_n^a\rangle \Phi_n^F \Phi_{n+1}^F. \tag{4.1}$$

We deduce then that $(-1)^k \Phi_n^F \Phi_{n+1}^F > 0$ for all n. Using the duality (3.14), we conclude that $\Omega_F(n)\Omega_F(n+1) > 0$ for all nonnegative integers n.

(3) $\Rightarrow$ (1). Using Lemma 2.7, the duality (3.14) and proceeding as before, we conclude that the polynomials $(q_n^F)_n$ are orthogonal with respect to ρ_a^F and have a positive L^2-norm. This implies that there exists a positive measure μ with respect to which the polynomials $(q_n^F)_n$ are orthogonal. Taking into account that the Fourier transform of ρ_a^F is an entire function, using moment problem standard techniques (see, for instance, [2]), it is not difficult to prove that μ has to be equal to ρ_a^F. Hence the measure ρ_a^F is positive. □

In the two following theorems we prove that for admissible sets F the polynomials c_n^F, $n \in \sigma_F$, are orthogonal and complete with respect to a positive measure.

Theorem 4.4. *Given a real number $a \neq 0$ and a finite set F of positive integers, assume that $\Omega_F^a(n) \neq 0$ for all nonnegative integers n. Then the polynomials $c_n^{a;F}$, $n \in \sigma_F$, are orthogonal with respect to the (possibly signed) discrete measure*

$$\omega_{a;F} = \sum_{x=0}^{\infty} \frac{a^x}{x!\,\Omega_F^a(x)\Omega_F^a(x+1)}\delta_x. \tag{4.2}$$

Moreover, for $a < 0$ the measure $\omega_{a;F}$ is never positive, and for $a > 0$ the measure $\omega_{a;F}$ is positive if and only if F is admissible.

Proof. Write $\mathbb{A}$ for the linear space generated by the polynomials c_n^F, $n \in \sigma_F$. Using Lemma 2.5, the definition of the measure $\omega_{a;F}$ and the expressions for the difference coefficients of the operator D_F (see Theorem 3.3), it is straightforward to check that D_F is symmetric with respect to the pair $(\omega_{a;F}, \mathbb{A})$. Since the polynomials c_n^F, $n \in \sigma_F$, are eigenfunctions of D_F with different eigenvalues, Lemma 2.4 implies that they are orthogonal with respect to $\omega_{a;F}$.

If $a < 0$ and the measure $\omega_{a;F}$ is positive, we conclude that $\Omega_F(2n+1)\Omega_F(2n+2) < 0$ for all positive integers n. But this would imply that Ω_F has at least a zero in each interval $(2n+1, 2n+2)$, which is impossible since Ω_F is a polynomial.

If $a > 0$, according to Lemma 4.3, F is admissible if and only if $\Omega_F(x)\Omega_F(x+1) > 0$ for all nonnegative integers x. □

Theorem 4.5. *Let a and F be a positive real number and an admissible finite set of positive integers, respectively. Then the linear combinations of the polynomials $c_n^{a;F}$, $n \in \sigma_F$, are dense* *in $L^2(\omega_{a;F})$, where $\omega_{a;F}$ is the positive measure (4.2). Hence $c_n^{a;F}$, $n \in \sigma_F$, are exceptional Charlier polynomials.*

Proof. Using Lemma 4.3 and taking into account that F is admissible, it follows that the measure ρ_a^F (3.16) is positive. We remark that this positive measure is also determinate (that is, there is not other measure with the same moments as those of ρ_a^F). As we pointed out above, this can be proved using moment problem standard techniques (taking into account, for instance, that the Fourier transform of ρ_a^F is an entire function). Since for determinate measures the polynomials are dense in the associated L^2 space, we deduce that the sequence $(q_n^F/\|q_n^F\|_2)_n$ (where q_n^F is the polynomial defined by (3.4)) is an orthonormal basis in $L^2(\rho_a^F)$.

For $s \in \sigma_F$, consider the function $h_s(x) = \begin{cases} 1/\rho_a^F(s), & x = s \\ 0, & x \neq s, \end{cases}$ where by $\rho_a^F(s)$ we denote the mass of the discrete measure ρ_a^F at the point s. Since the support of ρ_a^F is σ_F, we get that $h_s \in L^2(\rho_a^F)$. Its Fourier coefficients with respect to the orthonormal basis $(q_n^F/\|q_n^F\|_2)_n$ are $q_n^F(s)/\|q_n^F\|_2$, $n \geq 0$. Hence

$$\sum_{n=0}^{\infty} \frac{q_n^F(s)q_n^F(r)}{\|q_n^F\|_2^2} = \langle h_s, h_r \rangle_{\rho_a^F} = \frac{1}{\rho_a^F(s)} \delta_{s,r}. \tag{4.3}$$

This is the dual orthogonality associated to the orthogonality

$$\sum_{u \in \sigma_F} q_n^F(u) q_m^F(u) \rho_a^F(u) = \langle q_n^F, q_n^F \rangle \delta_{n,m}$$

of the polynomials q_n^F, $n \geq 0$, with respect to the positive measure ρ_a^F (see, for instance, [3], Appendix III, or [26], Th. 3.8).

Using (4.1), (2.26) and the duality (3.14), we get

$$\frac{1}{\|q_n^F\|_2^2} = \omega_{a;F}(n) x_n, \tag{4.4}$$

where x_n is the positive number given by

$$x_n = \frac{a^k}{e^a} \left(\frac{\prod_{i=0}^{k} (n+i)!}{a^{(k+1)n-u_F} \prod_{f \in F} f!} \right)^2. \tag{4.5}$$

Using now the duality (3.5), we can rewrite (4.3) for $n = m$ as

$$\langle c_n^{a;F}, c_n^{a;F} \rangle_{\omega_a^F} = \frac{a^{n-u_F-k} e^a \prod_{f \in F} (n - f - u_F)}{(n - u_F)!}. \tag{4.6}$$

Consider now a function f in $L^2(\omega_{a;F})$ and write $g(n) = (-1)^n f(n)/x_n^{1/2}$, where x_n is the positive number given by (4.5). Using (4.4), we get

$$\sum_{n=0}^{\infty} \frac{|g(n)|^2}{\langle q_n^F, q_n^F \rangle_{\rho_a^F}} = \sum_{n=0}^{\infty} \omega_{a;F}(n) |f(n)|^2 = \|f\|_2^2 < \infty.$$

Define now

$$v_r = \sum_{n=0}^{\infty} \frac{g(n) q_n^F(r)}{\langle q_n^F, q_n^F \rangle_{\rho_a^F}}.$$

Using Theorem III.2.1 of [3], we get

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \frac{|g(n)|^2}{\langle q_n^F, q_n^F \rangle_{\rho_a^F}} = \sum_{r \in \sigma_F} |v_r|^2 \rho_a^F(r). \tag{4.7}$$

On the other hand, using the duality (3.5) and (4.4)–(4.6), we have

$$v_r = \frac{(-1)^r}{(\rho_a^F(r))^{1/2}} \sum_{n=0}^{\infty} f(n) \frac{c_r^{a;F}(n)}{\|c_r^{a;F}\|_2} \omega_{a;F}(n).$$

This is saying that $(-1)^r (\rho_a^F(r))^{1/2} v_r$, $r \in \sigma_{\mathrm{U}}$, are the Fourier coefficients of f with respect to the orthonormal system $(c_n^{a;F}/\|c_n^{a;F}\|_2)_n$. Hence, the identity (4.7) is Parseval's identity for the function f. From where we deduce that the orthonormal system $(c_n^{a;F}/\|c_n^{a;F}\|_2)_n$ is complete in $L^2(\omega_{a;F})$. □

5. Constructing polynomials which are eigenfunctions of second order differential operators

One can construct exceptional Hermite polynomials by taking limit in the exceptional Charlier polynomials. We use the basic limit (2.33).

Given a finite set of positive integers F, using the expression (3.3) for the polynomials $c_n^{a;F}$, $n \in \sigma_F$, setting $x \to \sqrt{2a}x + a$ and taking limit as $a \to +\infty$, we get (up to normalization constants) the polynomials, $n \in \sigma_F$,

$$H_n^F(x) = \begin{vmatrix} H_{n-u_F}(x) & H'_{n-u_F}(x) & \cdots & H^{(k)}_{n-u_F}(x) \\ H_{f_1}(x) & H'_{f_1}(x) & \cdots & H^{(k)}_{f_1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H^{(k)}_{f_k}(x) \end{vmatrix}. \tag{5.1}$$

More precisely

$$\lim_{a \to +\infty} \left(\frac{2}{a}\right)^{n/2} c_n^F(\sqrt{2a}x + a) = \frac{1}{(n-u_F)!\nu_F} H_n^F(x) \tag{5.2}$$

uniformly in compact sets, where

$$\nu_F = 2^{\binom{k+1}{2}} \prod_{f \in F} f!. \tag{5.3}$$

Notice that H_n^F is a polynomial of degree n with leading coefficient equal to

$$2^{n+\binom{k+1}{2}} V_F \prod_{f \in F} (f - n + u_F),$$

where V_F is the Vandermonde determinant defined by (2.2).

Assume now that F is admissible (4.2). According to Lemma 4.3, this gives for all $a > 0$ that $\Omega_F^a(x)\Omega_F^a(x+1) > 0$ for $x \in \mathbb{N}$, where Ω_F^a is the polynomial (3.6) associated to the Charlier family. In particular $\Omega_F^a(x) \neq 0$, for all nonnegative integers x. Hence, if instead of (3.3) we use (3.36), we get the following alternative expression for the polynomials H_n^F, $n \geq v_F$, (i denotes the imaginary unit $i = \sqrt{-1}$)

$$H_n^F(x) = \gamma_n \begin{vmatrix} H_{n-v_F}(x) & -iH_{n-v_F+1}(x) & \cdots & (-i)^m H_{n-v_F+m}(x) \\ H_{g_1}(-ix) & H'_{g_1}(-ix) & \cdots & H_{g_1}^{(m)}(-ix) \\ \vdots & \vdots & \ddots & \vdots \\ H_{g_m}(-ix) & H'_{g_m}(-ix) & \cdots & H_{g_m}^{(m)}(-ix) \end{vmatrix}, \tag{5.4}$$

where γ_n is the normalization constant

$$\gamma_n = i^{u_G} 2^{\binom{k+1}{2}-\binom{m}{2}} \frac{V_F}{V_G} \prod_{f \in F} (f - n + u_F), \tag{5.5}$$

and as in the previous sections G denotes the involuted set $G = I(F)$ (see (2.19)).

We introduce the associated polynomials

$$\Omega_F(x) = \begin{vmatrix} H_{f_1}(x) & H'_{f_1}(x) & \cdots & H_{f_1}^{(k-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H_{f_k}^{(k-1)}(x) \end{vmatrix}, \tag{5.6}$$

$$\tilde{\Omega}_F(x) = i^{u_G+m} \begin{vmatrix} H_{g_1}(-ix) & H'_{g_1}(-ix) & \cdots & H_{g_1}^{(m-1)}(-ix) \\ \vdots & \vdots & \ddots & \vdots \\ H_{g_m}(-ix) & H'_{g_m}(-ix) & \cdots & H_{g_m}^{(m-1)}(-ix) \end{vmatrix}. \tag{5.7}$$

Since $u_G + m = u_F + k$, we have that both Ω_F and $\tilde{\Omega}_F$ are polynomials of degree $u_F + k$.

The invariant property (3.28) gives

$$\Omega_F(x) = 2^{\binom{k}{2}-\binom{m}{2}} \frac{V_F}{V_G} \tilde{\Omega}_F(x). \tag{5.8}$$

We also straightforwardly have

$$H_{u_F}^F(x) = \frac{2^{k-s_F+1} v_F}{v_{F_\Downarrow}} \Omega_{F_\Downarrow}(x), \tag{5.9}$$

where the numbers v_F and s_F are defined by (5.3) and (2.20), respectively, and the finite set of integers $F_\Downarrow$ is defined by (2.22).

Proceeding in a similar way, we can transform the second order difference operator (3.17) into a second order differential operator with respect to which the polynomials H_n^F, $n \in \sigma_F$, are eigenfunctions:

Theorem 5.1. *Let F be a finite set of positive integers. Then the polynomials H_n^F, $n \in \sigma_F$, are common eigenfunctions of the second order differential operator*

$$D_F = -\partial^2 + h_1(x)\partial + h_0(x), \tag{5.10}$$

where $\partial = d/dx$ *and*

$$h_1(x) = 2\left(x + \frac{\Omega'_F(x)}{\Omega_F(x)}\right), \tag{5.11}$$

$$h_0(x) = 2\left(k + u_F - x\frac{\Omega'_F(x)}{\Omega_F(x)}\right) - \frac{\Omega''_F(x)}{\Omega_F(x)}. \tag{5.12}$$

More precisely $D_F(H_n^F) = 2nH_n^F(x)$.

Proof. The proof is a matter of calculation using carefully the basic limit (2.33), hence we only sketch it.

We assume that k is even (the case for k odd being similar).

Using that $c_n^a(x+k) = \sum_{j=0}^k \binom{k}{j} c_{n-j}^a(x)$, the basic limit (2.33) and the alternative definition (3.9) for Ω_F^a, we can get the limits

$$\lim_{a\to\infty}\left(\frac{2}{a}\right)^{(u_F+k)/2} \Omega_F^a(x_a) = \frac{2^k\Omega_F(x)}{\nu_F}, \tag{5.13}$$

$$\lim_{a\to\infty}\left(\frac{2}{a}\right)^{(u_F+k-1)/2} (\Omega_F^a(x_a+1) - \Omega_F^a(x_a)) = \frac{2^{k-1}\Omega'_F(x)}{\nu_F}, \tag{5.14}$$

$$\lim_{a\to\infty}\left(\frac{2}{a}\right)^{(u_F+k-2)/2} (\Omega_F^a(x_a+1) - 2\Omega_F^a(x_a) + \Omega_F^a(x_a-1)) = \frac{2^{k-2}\Omega''_F(x)}{\nu_F},$$

where ν_F is defined by (5.3) and $x_a = \sqrt{2a}x + a$.

Taking into account that $c_n^{a;F}(x) = \Omega_{F_n}^a(x)$, where $F_n = \{f_1, \dots, f_k, n - u_F\}$, we can get similar limits for the polynomials $c_n^{a;F}(x)$, $n \in \sigma_F$.

We next write the spectral equation $D_F^a(c_n^{a;F}) = nc_n^{a;F}$ (where we write D_F^a for the second order difference operator (3.17)) in the form

$$\begin{aligned} &h_{-1}^a(x)\left[c_n^F(x+1) - 2c_n^F(x) + c_n^F(x-1)\right] + (h_1^a(x) - h_{-1}^a(x))\left[c_n^F(x+1) - c_n^F(x)\right] \\ &\quad + (h_0^a(x) + h_1^a(x) + h_{-1}^a(x))c_n^F(x) = nc_n^F(x), \end{aligned}$$

where h_{-1}^a, h_0^a and h_1^a are given by (3.18)–(3.20), respectively. It is then enough to set $x \to x_a$ and take carefully limit as $a \to \infty$ using (3.18), (3.19), (3.20) and the previous limits. □

We can factorize the second order differential operator D_F as a product of two first order differential operators. As a consequence the system $(D_F, (H_n^F)_{n\in\sigma_F})$ can be constructed by applying a sequence of k Darboux transforms to the Hermite system.

Lemma 5.2. *Let* $F = \{f_1, \dots, f_k\}$ *be a finite set of positive integers and write* $F_{\{k\}} = \{f_1, \dots, f_{k-1}\}$. *We define the first order differential operators* A_F *and* B_F *as*

$$A_F = -\frac{\Omega_F(x)}{\Omega_{F_{\{k\}}}(x)}\partial + \frac{\Omega'_F(x)}{\Omega_{F_{\{k\}}}(x)}, \tag{5.15}$$

$$B_F = \frac{\Omega_{F_{\{k\}}}(x)}{\Omega_F(x)}\partial - \frac{2x\Omega_{F_k} + \Omega'_{F_{\{k\}}}(x)}{\Omega_F(x)}. \tag{5.16}$$

Then $H_n^F(x) = A_F(H_{n-f_k+k}^{F_{\{k\}}})(x)$, $n \in \sigma_F$. *Moreover*

$$D_{F_{\{k\}}} = B_F A_F + 2(f_k + u_{F_{\{k\}}})Id,$$
$$D_F = A_F B_F + 2(f_k + u_F)Id.$$

Proof. The lemma can be proved applying limits in Lemma 3.6, or by applying Silvester's identity (for rows $(1, k)$ and columns $(k-1, k)$) in the definition (5.1) of the polynomials H_n^F, $n \in \sigma_F$. $\square$

When F is admissible, using the alternative expression (5.4) for the polynomials H_n^F, $n \in \sigma_F$, we get other factorization for the differential operator D_F.

Lemma 5.3. *Let F be an admissible finite set of positive integers and write $F_{\Downarrow}$ for the finite set of positive integers defined by (2.22). We define the first order differential operators C_F and E_F as*

$$C_F = \frac{\tilde{\Omega}_F(x)}{\tilde{\Omega}_{F_{\Downarrow}}(x)}\partial - \frac{\tilde{\Omega}'_F(x) + 2x\tilde{\Omega}_F(x)}{\tilde{\Omega}_{F_{\Downarrow}}(x)}, \tag{5.17}$$

$$E_F = -\frac{\tilde{\Omega}_{F_{\Downarrow}}(x)}{\tilde{\Omega}_F(x)}\partial + \frac{\tilde{\Omega}'_{F_{\Downarrow}}(x)}{\tilde{\Omega}_F(x)}. \tag{5.18}$$

Then $D_{F_{\Downarrow}} = E_F C_F + 2(u_F - k - 1)Id$ *and* $D_F = C_F E_F + 2u_F Id$. *Moreover*

$$C_F(H_{n-k-1}^{F_{\Downarrow}}) = \frac{-2^{m+\binom{k-s_F+2}{2}-\binom{k+1}{2}-1}\prod_{j=1}^{m-1}(g_m - g_j)}{\prod_{j=1}^{s_F-1}(j-1)!(j-n+u_F)\prod_{f\in F; f>s_F}(f-j)} H_n^F(x), \quad n \geq v_F, \tag{5.19}$$

where $G = I(F) = \{g_1, \dots, g_m\}$ *and* s_F *is defined by (2.20).*

6. Exceptional Hermite polynomials

In the previous section, we have associated to each finite set F of positive integers the polynomials H_n^F, $n \in \sigma_F$, which are always eigenfunctions of a second order differential operator with rational coefficients. We are interested in the cases when, in addition, those polynomials are orthogonal and complete with respect to a positive measure.

Definition 6.1. The polynomials H_n^F, $n \in \sigma_F$, defined by (5.1) are called exceptional Hermite polynomials, if they are orthogonal and complete with respect to a positive measure.

As it was mentioned in the Introduction, simultaneously with this paper, exceptional Hermite polynomials as Wronskian determinant of Hermite polynomials have been introduced and studied (using a different approach) in [22]. In that paper, exceptional Hermite polynomials are defined for a given non-decreasing finite sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_l)$, and are denoted by $H_j^{(\lambda)}(x)$; the degree of $H_j^{(\lambda)}(x)$ is $2\sum_{j=1}^{l}\lambda_j - 2l + j$. The relationship between the exceptional Hermite polynomials introduced in [22] and the ones in this paper is the following: given a non-decreasing finite sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_l)$, we form a finite set of positive integers F as follows: $F = \{f_1, f_2, \dots, f_{2l-1}, f_{2l}\}$, where $f_{2j-1} = \lambda_j + 2j - 2$ and $f_{2j} = \lambda_j + 2j - 1$, $j = 1, \dots, l$; then $H_j^{(\lambda)}(x) = H^F_{2\sum_{j=1}^{l} \lambda_j - 2l + j}(x)$.

The following lemma and theorem show that again the admissibility of F will be the key to construct exceptional Hermite polynomials.

Lemma 6.2. *Let F be a finite set of positive integers. Then F is admissible if and only if the Wronskian determinant Ω_F (5.6) does not vanish in $\mathbb{R}$.*

Proof. Consider a second order differential operator T of the form $T = -d^2/dx^2 + U$, and write ϕ_n, $n \geq 0$, for a sequence of eigenfunctions for T, where U and ϕ_n satisfy suitable boundary conditions. For a finite set of positive integers $F = \{f_1, \dots, f_k\}$, consider the Wronskian determinant $\Omega_F^T(x) = |\phi_{f_l}^{(j-1)}(x)|_{l,j=1}^{k}$. For operators defined in a half-line, Krein proved [25] that F is admissible if and only if Ω_F^T does not vanish in the real line. A similar result was proved by Adler [1] for operators defined in a bounded interval. Adler's result can easily be extended to the whole real line (in fact, he considered in [1] the case of Wronskian determinant of Hermite polynomials). The lemma is then an easy consequence of this result for the functions $H_n(x)e^{-x^2}$.

Anyway, for the sake of completeness, we prove by passing to the limit from Lemma 4.3 the implication $\Rightarrow$ in the lemma (which it is what we need in the following theorem).

For $a > 0$, consider the positive measure τ_a defined by

$$\tau_a = \frac{a^k}{e^a} \sum_{x=0}^{\infty} \frac{a^x (c_{u_F}^{a;F}(x))^2}{x! \Omega_F^a(x) \Omega_F^a(x+1)} \delta y_{a,x},$$

where

$$y_{a,x} = (x - a)/\sqrt{2a}. \tag{6.1}$$

We also need the following limits

$$\lim_{a \to +\infty} \frac{\Omega_F^a(\sqrt{2a}x + a)}{a^{(k+u_F)/2}} = \frac{2^{(k-u_F)/2} \Omega_F(x)}{\nu_F}, \tag{6.2}$$

$$\lim_{a \to +\infty} \frac{\Omega_F^a(\sqrt{2a}x + a + 1)}{a^{(k+u_F)/2}} = \frac{2^{(k-u_F)/2} \Omega_F(x)}{\nu_F}, \tag{6.3}$$

$$\lim_{a \to +\infty} \frac{c_{u_F}^{a;F}(\sqrt{2a}x + a)}{a^{u_F/2}} = \frac{2^{k-u_F/2-s_F+1} \Omega_{F_\Downarrow}(x)}{\nu_{F_\Downarrow}}, \tag{6.4}$$

$$\lim_{a \to +\infty} \frac{\sqrt{2a} a^{\sqrt{2a}x+a}}{e^a \Gamma(\sqrt{2a}x + a + 1)} = e^{-x^2}/\sqrt{\pi}, \tag{6.5}$$

uniformly in compact sets. The first limit is (5.13). The second one is a consequence of (5.14). The third one is a consequence of (5.2) and (5.9). The fourth one is a consequence of Stirling's formula.

We proceed by complete induction on $s = \max F$. Since F is admissible, the first case to be considered is $s = 2$ which it corresponds with $F = \{1, 2\}$. Then $\Omega_F(x) = 8x^2 + 4$ which it clearly satisfies $\Omega_F(x) \neq 0$, $x \in \mathbb{R}$.

Assume that $\Omega_F(x) \neq 0$, $x \in \mathbb{R}$, if $\max F \leq s$ and take an admissible set F with $\max F = s + 1$. The definition of $F_\Downarrow$ (2.22) then says that $\max F_\Downarrow \leq s$. The induction hypothesis then implies that $\Omega_{F_\Downarrow}(x) \neq 0$, $x \in \mathbb{R}$. We now proceed by *reductio ad absurdum*. Hence, we assume that the polynomial Ω_F vanishes in $\mathbb{R}$. Write $x_0 = \max\{x \in \mathbb{R} : \Omega_F(x) = 0\}$. Take real numbers u, v with $x_0 < u < v$ and write $I = [u, v]$. Since $\Omega_F(x) \neq 0$, $x \in I$, applying Hurwitz's Theorem to the limits (6.2) and (6.3) we can choose a countable set $X = \{a_n : n \in \mathbb{N}\}$ of positive numbers with $\lim_n a_n = +\infty$ such that $\Omega_F^a(\sqrt{2a}x + a)\Omega_F^a(\sqrt{2a}x + a + 1) \neq 0$, $x \in I$ and $a \in X$.

Hence, we can combine the limits (6.2)–(6.5) to get

$$\lim_{a\to+\infty; a\in X} h_a(x) = d_3 h(x), \quad \text{uniformly in } I, \tag{6.6}$$

where

$$h_a(x) = \frac{a^k \sqrt{2a}\, a^{\sqrt{2a}x+a} (c_{u_F}^{a;F}(\sqrt{2a}x + a))^2}{e^a \Gamma(\sqrt{2a}x + a + 1)\Omega_F^a(\sqrt{2a}x + a)\Omega_F^a(\sqrt{2a}x + a + 1)},$$

$$h(x) = \frac{e^{-x^2}\Omega_{F_\Downarrow}^2(x)}{\Omega_F^2(x)},$$

and $d_3 = 2^{k-2s_F+2}\nu_F^2/(\sqrt{\pi}\nu_{F_\Downarrow}^2)$. We now prove that

$$\lim_{a\to+\infty; a\in X} \tau_a(I) = d_3 \int_I h(x)dx. \tag{6.7}$$

To do that, write $I_a = \{x \in \mathbb{N} : a + u\sqrt{2a} \le x \le a + v\sqrt{2a}\}$. The numbers $y_{a,x}$, $x \in I_a$, form a partition of the interval I with $y_{a,x+1} - y_{a,x} = 1/\sqrt{2a}$ (see (6.1)). Since the function h is continuous in the interval I, we get that

$$\int_I h(x)dx = \lim_{a\to+\infty; a\in X} S_a,$$

where S_a is the Cauchy sum

$$S_a = \sum_{x\in I_a} h(y_{a,x})(y_{a,x+1} - y_{a,x}).$$

On the other hand, since $x \in I_a$ if and only if $u \le y_{a,x} \le v$ (6.1), we get

$$\begin{aligned}\tau_a(I) &= \frac{a^k}{e^a}\sum_{x\in I_a}\frac{a^x (c_{u_F}^F(x))^2}{x!\Omega_F^a(x)\Omega_F^a(x+1)} = \frac{1}{\sqrt{2a}}\sum_{x\in I_a} h_a(y_{a,x}) \\ &= \sum_{x\in I_a} h_a(y_{a,x})(y_{a,x+1} - y_{a,x}).\end{aligned}$$

The limit (6.7) now follows from the uniform limit (6.6).

The identity (4.6) for $n = u_F$ says that $\tau_a(\mathbb{R}) = d_F$, where the positive constant $d_F = \prod_{f\in F} f$ does not depend on a. This gives $\tau_a(I) \le d_F$. And so from the limit (6.7) we get

$$\int_I h(x)dx \le \frac{d_F}{d_3}.$$

That is

$$\int_u^v \frac{e^{-x^2}\Omega_{F_\Downarrow}^2(x)}{\Omega_F^2(x)}dx \le \frac{d_F}{d_3}.$$

On the other hand, since $\Omega_F(x_0) = 0$ and $\Omega_{F_\Downarrow}(x) \neq 0$, $x \in \mathbb{R}$, we get

$$\lim_{u \to x_0^+} \int_u^v \frac{e^{-x^2} \Omega^2_{F_\Downarrow}(x)}{\Omega^2_F(x)} dx = \infty,$$

which is a contradiction. $\square$

Corollary 6.3. *Given an admissible finite set F of positive integers, we have for $n \in \sigma_F$,*

$$\langle H_n^F, H_n^F \rangle_{\omega_F} = \sqrt{\pi} 2^{n-u_F+k} (n - u_F)! \prod_{f \in F} (n - f - u_F). \tag{6.8}$$

Proof. The proof is similar to that of the previous theorem (using (4.6)) and it is omitted. $\square$

Theorem 6.4. *Let F be an admissible finite set of positive integers. Then the polynomials H_n^F, $n \in \sigma_F$, are orthogonal with respect to the positive weight*

$$\omega_F(x) = \frac{e^{-x^2}}{\Omega^2_F(x)}, \quad x \in \mathbb{R}, \tag{6.9}$$

and their linear combinations are dense in $L^2(\omega_F)$. Hence H_n^F, $n \in \sigma_F$, are exceptional Hermite polynomials.

Proof. Write $\mathbb{A}_F$ for the linear space generated by the polynomials H_n^F, $n \in \sigma_F$. Using Lemma 2.6, it is easy to check that the second order differential operator D_F (5.10) is symmetric with respect to the pair $(\omega_F, \mathbb{A}_F)$ (6.9). Since the polynomials H_n^F, $n \in \sigma_F$, are eigenfunctions of D_F with different eigenvalues Lemma 2.4 implies that they are orthogonal with respect to ω_F.

The completeness of H_n^F, $n \in \sigma_F$, in $L^2(\omega_F)$ can be proved in a similar way to that of Proposition 5.8 in [22] and it is omitted. $\square$

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