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Multivariate Markov-type and Nikolskii-type inequalities for polynomials associated with downward closed multi-index sets

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Abstract

We present novel Markov-type and Nikolskii-type inequalities for multivariate polynomials associated with arbitrary downward closed multi-index sets in any dimension. Moreover, we show how the constant of these inequalities changes, when the polynomial is expanded in series of tensorized Legendre or Chebyshev or Gegenbauer or Jacobi orthogonal polynomials indexed by a downward closed multi-index set. The proofs of these inequalities rely on a general result concerning the summation of tensorized polynomials over arbitrary downward closed multi-index sets.

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1. Introduction

Polynomials and polynomial inequalities are ubiquitous in mathematics. Nowadays several monographs address polynomials, orthogonal polynomials and their properties, *e.g.* [21,19,2,11]. Many related topics and computational issues are covered as well, with countless applications in physics and applied mathematics. The univariate analysis is far more developed than its multivariate counterpart, see *e.g.* the monograph [7] specifically targeted to the multivariate case. In this paper we deal with multivariate polynomial inequalities of Markov type and Nikolskii type. The results proposed can find possible applications, among others, in the fields of polynomial approximation techniques for aleatory functions [6,18,14], for parametric and stochastic partial differential equations [17,4,14], spectral methods [3] and high-order finite element methods [20].

In recent years, the importance of Nikolskii-type inequalities between L^∞ and L^2_ρ has arisen in the analysis of the stability and accuracy properties of polynomial approximation based on discrete least squares with random evaluations [6,18,17,4,14]. The constant of the $L^\infty - L^2_\rho$ inverse inequality plays a role in [6,18], which concern the analysis of discrete least squares in the univariate case. The multivariate case is more challenging, since there are more degrees of freedom to enrich the multivariate polynomial space. The multivariate polynomial space can be characterized by means of multi-indices. In the present paper we focus on polynomial spaces associated with downward closed multi-index sets, also known as lower sets, see *e.g.* [8]. Multivariate interpolation on polynomial spaces of this type has been analyzed in [8] and references therein. In the multivariate case, Nikolskii-type $L^\infty - L^2_\rho$ and $L^4_\rho - L^2_\rho$ inequalities for tensorized Legendre polynomials have been derived in the specific case of tensor product, total degree and hyperbolic cross polynomial spaces in [14, Appendix B], and more general Nikolskii-type $L^\infty - L^2_\rho$ inequalities for tensorized Legendre and Chebyshev polynomials of the first kind have been derived in [4,5] in polynomial spaces associated with arbitrary downward closed multi-index sets.

In several contributions over the past decades, the analyses of Markov-type and Nikolskii-type inequalities, in univariate and multivariate tensor product and total degree polynomial spaces, have been developed for general domains and general weights, see *e.g.* [2,10,9] and the references therein.

In the present work we prove a general result in Theorem 1 concerning the summation of tensorized polynomials over arbitrary downward closed multi-index sets in any dimension. Afterwards, using Theorem 1, we derive Markov-type and Nikolskii-type inequalities over multivariate polynomial spaces associated with arbitrary downward closed multi-index sets in any dimension. Moreover, we show how the constant of these inequalities changes when the polynomial is expanded in series of tensorized Legendre or Chebyshev or Gegenbauer or Jacobi orthogonal polynomials indexed by a downward closed multi-index set.

In [22,13,12] multivariate Markov-type inequalities have been proposed in the case of polynomial spaces of total degree type. Recent results on Markov-type inequalities for the mixed derivatives have been proposed in [1], showing a relation between the L^∞ norm of the gradient of a polynomial and its mixed derivatives.

In the present paper we propose novel Markov-type inequalities for the mixed derivatives of any general multivariate polynomial associated with an arbitrary downward closed multi-index set in any dimension, and refine the proposed result depending on the series of orthogonal polynomials used in the expansion.

The outline of the paper is the following. In Section 2 we introduce the settings of polynomial approximation and the notation. In Section 3 we prove Theorem 1 concerning the summation of tensorized polynomials over arbitrary downward closed multi-index sets in any dimension. In

Section 4 we review the most common families of orthogonal polynomials and their orthonormalization. In Section 5 we present novel multivariate polynomial inequalities. We begin to recall some one-dimensional Markov inequalities in Section 5.1. Then in Section 5.2 we prove multivariate Markov-type inequalities for the mixed derivatives and in Section 5.3 we prove multivariate Nikolskii-type $L^\infty - L^2_\rho$ inequalities.

2. Multivariate polynomial spaces

Let d be a positive integer, $D_q := [-1, 1] \subset \mathbb{R}$ be a compact interval and $\rho_q : D_q \to \mathbb{R}_0^+$ be a univariate weight function for all $q = 1, \dots, d$. Define the compact set $D := \prod_{q=1}^d D_q = [-1, 1]^d \subset \mathbb{R}^d$ as the d-dimensional hypercube in the d-dimensional euclidean space. Consider the d-dimensional weight function $\rho := \prod_{q=1}^d \rho_q : D \to \mathbb{R}_0^+$, the L^2_ρ weighted inner product

$$\langle f_1, f_2 \rangle_{L^2_\rho(D)} := \int_D f_1(y) f_2(y) \rho(y) dy, \quad \forall f_1, f_2 \in L^2_\rho(D), \tag{1}$$

and its associated L^2_ρ norm $\| \cdot \|_{L^2_\rho(D)} := \langle \cdot, \cdot \rangle^{1/2}_{L^2_\rho(D)}$. Moreover, we denote by $\langle f_1, f_2 \rangle_{L^2(D)}$ the standard L^2 inner product with its associated L^2 norm $\| \cdot \|_{L^2(D)} := \langle \cdot, \cdot \rangle^{1/2}_{L^2(D)}$, and by $\| \cdot \|_{L^\infty(D)}$ the standard L^∞ norm. On any compact set D, for any continuous real-valued function $f \in \mathcal{C}(D)$ it holds $\|f\|_{L^\infty(D)} = \max_{y \in D} |f(y)|$. We denote the integral of ρ by

$$W_\rho := \int_D \rho(y) dy. \tag{2}$$

For any $n, k \in \mathbb{N}_0$ we denote by δ_{nk} the Kronecker delta, equal to one if the indices are equal, and equal to zero otherwise. For any $q = 1, \dots, d$, denote by $\{\varphi^q_n\}_{n \geq 0}$ the family of univariate polynomials orthonormal w.r.t. (1) with the weight ρ_q, *i.e.* $\langle \varphi_n \varphi_k \rangle_{L^2_{\rho_q}(D_q)} = \delta_{nk}$. Denote by $\Lambda \subset \mathbb{N}_0^d$ a finite multi-index set, and by $\#\Lambda$ its cardinality. For any $\boldsymbol{\nu} \in \Lambda$ define the tensorized (multivariate) polynomials $\psi_{\boldsymbol{\nu}}$, orthonormal w.r.t. (1), as

$$\psi_{\boldsymbol{\nu}}(y) = \prod_{q=1}^d \varphi_{\nu_q}(y_q), \quad y \in D. \tag{3}$$

The space of polynomials $\mathbb{P}_\Lambda = \mathbb{P}_\Lambda(D)$ associated with the multi-index set Λ is defined as follows:

$$\mathbb{P}_\Lambda := \mathrm{span}\{\psi_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \Lambda\}.$$

It holds that $\dim(\mathbb{P}_\Lambda) = \#\Lambda$. Denoting by w a nonnegative integer, common isotropic polynomial spaces $\mathbb{P}_{\Lambda_{\mathrm{w}}}$ are

$$\text{Tensor Product (TP):} \quad \Lambda_{\mathrm{w}} = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^d : \|\boldsymbol{\nu}\|_{\ell_\infty(\mathbb{N}_0^d)} \leq \mathrm{w} \right\},$$

$$\text{Total Degree (TD):} \quad \Lambda_{\mathrm{w}} = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^d : \|\boldsymbol{\nu}\|_{\ell_1(\mathbb{N}_0^d)} \leq \mathrm{w} \right\},$$

$$\text{Hyperbolic Cross (HC):} \quad \Lambda_{\mathrm{w}} = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^d : \prod_{q=1}^d (\nu_q + 1) \leq \mathrm{w} + 1 \right\}.$$

Anisotropic variants of these spaces can be defined by replacing $\mathrm{w} \in \mathbb{N}_0$ with a multi-index. For example, the anisotropic tensor product space, with maximum degrees in each direction $\boldsymbol{w} = (w_1, \ldots, w_d) \in \mathbb{N}_0^d$, is defined as

$$\text{anisotropic Tensor Product (aTP): } \Lambda_{\boldsymbol{w}} = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^d : \nu_q \le w_q,\ \forall q = 1, \ldots, d \right\}. \tag{4}$$

In the present paper we confine to multi-index sets Λ featuring the following property, see also [8].

Definition (*Downward Closedness of the Multi-Index Set* Λ)**.** The finite multi-index set $\Lambda \subset \mathbb{N}_0^d$ is downward closed (or it is a lower set) if

$$(\boldsymbol{\nu} \in \Lambda \text{ and } \boldsymbol{\mu} \le \boldsymbol{\nu}) \Rightarrow \boldsymbol{\mu} \in \Lambda,$$

where $\boldsymbol{\mu} \le \boldsymbol{\nu}$ means that $\mu_q \le \nu_q$ for all $q = 1, \ldots, d$.

Hence, also the multi-index set $\Lambda = \{\boldsymbol{\nu} : \nu_q = 0 \text{ for all } q = 1, \ldots, d\}$ containing only the null multi-index is by definition downward closed.

3. Summations of tensorized polynomials over a downward closed multi-index set

Given $\eta \in \mathbb{N}_0$ and $\eta + 1$ real nonnegative coefficients $\alpha_0, \ldots, \alpha_\eta$, we define the univariate polynomial $p \in \mathbb{P}_\eta(\mathbb{N}_0)$ of degree η as

$$p : \mathbb{N}_0 \to \mathbb{R} : n \mapsto p(n) := \sum_{l=0}^{\eta} \alpha_l n^l, \tag{5}$$

with the convention that $0^0 = 1$ to avoid the splitting of the summation. In any dimension d and given an arbitrary downward closed multi-index set Λ, we define the quantity $K_p(\Lambda)$ as

$$K_p(\Lambda) := \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} p(\boldsymbol{\nu}_q) = \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} \left(\alpha_0 + \alpha_1 \boldsymbol{\nu}_q + \cdots + \alpha_\eta \boldsymbol{\nu}_q^\eta \right), \tag{6}$$

which depends only on Λ when p is fixed. This quantity has shown considerable importance in the analyses of the stability and convergence properties of polynomial approximation based on discrete least squares with random evaluations [6,18,4,17,14,15], or with evaluations in low-discrepancy point sets [16]. In the particular case where $\eta = 1$, *i.e.* $p(n) = \alpha_0 + \alpha_1 n$, $K_p(\Lambda)$ has been analyzed in [6] in the univariate case, in [4,5] with tensorized Legendre polynomials and in [4] with tensorized Chebyshev polynomials of the first kind.

We introduce the following condition concerning the coefficients of the polynomial p.

Definition (*Binomial Condition*)**.** The polynomial p defined in (5) satisfies the binomial condition if its coefficients $\alpha_0, \ldots, \alpha_\eta$ satisfy

$$\alpha_l \le \binom{\eta + 1}{l}, \quad \text{for any } l = 0, \ldots, \eta. \tag{7}$$

Throughout the paper, given any $\eta \in \mathbb{N}_0$, we denote by

$$\widetilde{p} : \mathbb{N}_0 \to \mathbb{R} : n \mapsto \widetilde{p}(n) := \sum_{l=0}^{\eta} \binom{\eta+1}{l} n^l \tag{8}$$

the unique polynomial of degree η whose coefficients sharply satisfy (with equalities) the binomial condition (7).

The multinomial coefficient is defined as $\binom{\eta}{k_0,\dots,k_r} := \frac{\eta!}{k_0!\cdots k_r!}$, for any $\eta, r, k_0, \dots, k_r \in \mathbb{N}_0$ such that $k_0 + \cdots + k_r = \eta$.

Lemma 1. *For any $M, \eta \in \mathbb{N}_0$ and any choice of $M+1$ real nonnegative numbers $\lambda_0, \dots, \lambda_M$ it holds*

$$\sum_{r=0}^{M} \sum_{\substack{k_0+\cdots+k_r=\eta+1 \\ k_r>0}} \binom{\eta+1}{k_0,\dots,k_r} \prod_{z=0}^{r} \lambda_z^{k_z} = \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_0,\dots,k_M \in \mathbb{N}_0}} \binom{\eta+1}{k_0,\dots,k_M} \prod_{r=0}^{M} \lambda_r^{k_r}. \tag{9}$$

Proof. We expand the outermost summation with r ranging from 0 to M, then we manipulate the rightmost term, merge the rightmost and rightmost but one terms, and proceed backward till when only one term remains:

$$\begin{aligned}
&\sum_{r=0}^{M} \sum_{\substack{k_0+\cdots+k_r=\eta+1 \\ k_r>0}} \binom{\eta+1}{k_0,\dots,k_r} \prod_{z=0}^{r} \lambda_z^{k_z} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0,\dots,k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \cdots + \sum_{\substack{k_0+k_1=\eta+1 \\ k_1>0}} \binom{\eta+1}{k_0,k_1} \prod_{r=0}^{1} \lambda_r^{k_r} \\
&\quad + \sum_{\substack{k_0=\eta+1 \\ k_0>0}} \binom{\eta+1}{k_0} \prod_{r=0}^{0} \lambda_r^{k_r} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0,\dots,k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \cdots + \sum_{\substack{k_0+k_1=\eta+1 \\ k_1>0}} \binom{\eta+1}{k_0,k_1} \prod_{r=0}^{1} \lambda_r^{k_r} \\
&\quad + \sum_{\substack{k_0+k_1=\eta+1 \\ k_1=0}} \binom{\eta+1}{k_0,k_1} \prod_{r=0}^{1} \lambda_r^{k_r} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0,\dots,k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \cdots + \sum_{\substack{k_0+k_1+k_2=\eta+1 \\ k_2>0}} \binom{\eta+1}{k_0,k_1,k_2} \prod_{r=0}^{2} \lambda_r^{k_r} \\
&\quad + \sum_{k_0+k_1=\eta+1} \binom{\eta+1}{k_0,k_1} \prod_{r=0}^{1} \lambda_r^{k_r} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0,\dots,k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \cdots + \sum_{\substack{k_0+k_1+k_2=\eta+1 \\ k_2>0}} \binom{\eta+1}{k_0,k_1,k_2} \prod_{r=0}^{2} \lambda_r^{k_r}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{k_0+k_1+k_2=\eta+1 \\ k_2=0}} \binom{\eta+1}{k_0, k_1, k_2} \prod_{r=0}^{2} \lambda_r^{k_r} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0, \ldots, k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \cdots + \sum_{k_0+k_1+k_2=\eta+1} \binom{\eta+1}{k_0, k_1, k_2} \prod_{r=0}^{2} \lambda_r^{k_r} \\
&\;\vdots \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M>0}} \binom{\eta+1}{k_0, \ldots, k_M} \prod_{r=0}^{M} \lambda_r^{k_r} + \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_M=0}} \binom{\eta+1}{k_0, \ldots, k_M} \prod_{r=0}^{M} \lambda_r^{k_r} \\
&= \sum_{\substack{k_0+\cdots+k_M=\eta+1 \\ k_0,\ldots,k_M\in\mathbb{N}_0}} \binom{\eta+1}{k_0, \ldots, k_M} \prod_{r=0}^{M} \lambda_r^{k_r}. \quad \square
\end{aligned}
$$

Theorem 1. *In any dimension d, for any downward closed multi-index set Λ and for any $\eta \in \mathbb{N}_0$, if the coefficients $\alpha_0, \ldots, \alpha_\eta$ of the polynomial p satisfy the binomial condition (7) then the quantity $K_p(\Lambda)$ defined in (6) satisfies*

$$
K_p(\Lambda) \le (\#\Lambda)^{\eta+1}. \tag{10}
$$

Proof. We prove (10) by induction. The relation (10) trivially holds when Λ contains only the null multi-index *i.e.* $\Lambda = \{\boldsymbol{\nu} : \nu_q = 0 \text{ for all } q = 1, \ldots, d\}$, since in this case $K_p(\Lambda) = 1$ and $\#\Lambda = 1$.

Now we have to prove the induction step, *i.e.* we suppose that (10) holds for any arbitrarily given downward closed multi-index set $\hat{\Lambda}$ with $\#\hat{\Lambda} \ge 1$, and we prove that (10) still holds for any $\Lambda = \hat{\Lambda} \cup \boldsymbol{\mu}$, with $\boldsymbol{\mu} \notin \hat{\Lambda}$ and such that Λ remains downward closed.

The directions can be arbitrarily reordered, so without loss of generality we suppose that $\boldsymbol{\nu}_1 \neq 0$ for some $\boldsymbol{\nu} \in \Lambda$, and we denote by $J := \max_{\boldsymbol{\nu}\in\Lambda} \boldsymbol{\nu}_1$ the maximal value of the first component of the multi-indices $\nu \in \Lambda$.

For any $r \in \mathbb{N}_0$, we denote by $\Lambda_r := \{\hat{\boldsymbol{\nu}} \in \mathbb{N}_0^{d-1} : (r, \hat{\boldsymbol{\nu}}) \in \Lambda\}$ the "sections" of the set Λ w.r.t. the current first component according to the lexicographical ordering. Moreover, for any $r = 1, \ldots, J$ it holds

$$
\Lambda_J \subseteq \cdots \subseteq \Lambda_r \subseteq \Lambda_{r-1} \subseteq \cdots \subseteq \Lambda_0, \tag{11}
$$

and each one of these sets is also finite and downward closed. For any $r > J$ it holds $\Lambda_r = \emptyset$. Moreover, $\#\Lambda_0 \le \#\hat{\Lambda} = \#\Lambda - 1$ and therefore the inclusions (11) imply that the induction hypothesis holds for all the sets $\Lambda_0, \ldots, \Lambda_J$ as well.

In addition, for any $r = 1, \ldots, J$ we have

$$
r\#\Lambda_r \le \sum_{z=0}^{r-1} \#\Lambda_z. \tag{12}
$$

Finally we prove the induction step when the coefficients $\alpha_0, \ldots, \alpha_\eta$ satisfy the binomial condition (7):

$$
\begin{aligned}
K_p(\Lambda) &= \sum_{\boldsymbol{\nu}\in\Lambda}\prod_{q=1}^{d}\left(\alpha_0+\alpha_1\boldsymbol{\nu}_q+\cdots+\alpha_\eta\boldsymbol{\nu}_q^\eta\right)\\
&= \sum_{r=0}^{J}\left(\alpha_0+\alpha_1 r+\cdots+\alpha_\eta r^\eta\right)K_p(\Lambda_r)\\
&= \alpha_0 K_p(\Lambda_0)+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\alpha_l r^l K_p(\Lambda_r)\quad\left[\text{induction hypotheses on }\Lambda_0,\ldots,\Lambda_J\right]\\
&\le \alpha_0(\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\alpha_l r^l(\#\Lambda_r)^{\eta+1}\\
&= \alpha_0(\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\alpha_l (r\#\Lambda_r)^l(\#\Lambda_r)^{\eta+1-l}\quad\left[\text{using (12)}\right]\\
&\le \alpha_0(\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\alpha_l\left(\sum_{z=0}^{r-1}\#\Lambda_z\right)^l(\#\Lambda_r)^{\eta+1-l}\\
&\quad\left[\text{using the binomial condition (7)}\right]\\
&\le (\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\binom{\eta+1}{l}\left(\sum_{z=0}^{r-1}\#\Lambda_z\right)^l(\#\Lambda_r)^{\eta+1-l}\\
&\quad\left[\text{using the multinomial theorem}\right]\\
&= (\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\binom{\eta+1}{l}\sum_{\substack{k_0+\cdots+k_{r-1}=l\\ k_0,\ldots,k_{r-1}\in\mathbb{N}_0}}\binom{l}{k_0,\ldots,k_{r-1}}\\
&\quad\times\prod_{z=0}^{r-1}(\#\Lambda_z)^{k_z}(\#\Lambda_r)^{\eta+1-l}\\
&= (\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{l=0}^{\eta}\sum_{\substack{k_0+\cdots+k_{r-1}=l\\ k_0,\ldots,k_{r-1}\in\mathbb{N}_0}}\binom{\eta+1}{k_0,\ldots,k_{r-1},\eta+1-l}\\
&\quad\times\prod_{z=0}^{r-1}(\#\Lambda_z)^{k_z}(\#\Lambda_r)^{\eta+1-l}\\
&= (\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{k_r=1}^{\eta+1}\sum_{k_0+\cdots+k_{r-1}=\eta+1-k_r}\binom{\eta+1}{k_0,\ldots,k_r}\prod_{z=0}^{r-1}(\#\Lambda_z)^{k_z}(\#\Lambda_r)^{k_r}\\
&= (\#\Lambda_0)^{\eta+1}+\sum_{r=1}^{J}\sum_{\substack{k_0+\cdots+k_r=\eta+1\\ k_r>0}}\binom{\eta+1}{k_0,\ldots,k_r}\prod_{z=0}^{r}(\#\Lambda_z)^{k_z}
\end{aligned}
$$

$$= \sum_{r=0}^{J} \sum_{\substack{k_0+\cdots+k_r=\eta+1 \\ k_r>0}} \binom{\eta+1}{k_0,\ldots,k_r} \prod_{z=0}^{r} (\#\Lambda_z)^{k_z} \quad \text{[using (9)]}$$

$$= \sum_{\substack{k_0+\cdots+k_J=\eta+1 \\ k_0,\ldots,k_J\in\mathbb{N}_0}} \binom{\eta+1}{k_0,\ldots,k_J} \prod_{r=0}^{J} (\#\Lambda_r)^{k_r} \quad \text{[using the multinomial theorem]}$$

$$= \left(\sum_{r=0}^{J} \#\Lambda_r\right)^{\eta+1}$$

$$= (\#\Lambda)^{\eta+1},$$

and the proof of the induction step is completed. □

Theorem 1 can be further generalized to allow any positive rational number (or even positive real number) in the exponents of the polynomial p, *e.g.* $p(n) = \sqrt{n}$, but the proof in this case requires the use of the generalized multinomial theorem and generalized multinomial coefficients.

In the next remark we state an optimality property of the combinatorial estimate (10) in the class of downward closed multi-index sets.

Remark 1. For any given $\eta \in \mathbb{N}_0$, consider the polynomial $\widetilde{p} = \widetilde{p}(n)$ defined in (8), with its coefficients sharply satisfying (with equalities) the binomial condition (7). In any dimension d, let Λ be any multi-index set of anisotropic tensor product type (4) with degrees $\boldsymbol{w} = (w_1, \ldots, w_d)$: its sections according to the first direction satisfy

$$\Lambda_0 = \Lambda_1 = \cdots = \Lambda_J \quad \text{and} \quad \#\Lambda_0 = \#\Lambda_1 = \cdots = \#\Lambda_J = \prod_{q=2}^{d} (w_q + 1);$$

hence, repeating the proof of the induction step in Theorem 1 with all the inequalities replaced by equalities, one can prove that

$$K_{\widetilde{p}}(\Lambda) = (\#\Lambda)^{\eta+1}.$$

Therefore the thesis of Theorem 1 with the polynomial $\widetilde{p}$ is optimal in the class of downward closed multi-index sets, in the sense that the equality in (10) is always attained for at least one multi-index set in the class.

We introduce a finite constant $\hat{C} \in \mathbb{R}_0^+$ defined as

$$\hat{C} := \max_{l=0,\ldots,\eta} \frac{\alpha_l}{\binom{\eta+1}{l}} \geq 0. \tag{13}$$

When $\hat{C} > 1$ the constant $\hat{C}$ quantifies how much the coefficients $\alpha_0, \ldots, \alpha_\eta$ violate the binomial condition (7). When $\hat{C} < 1$ it quantifies how much the coefficients $\alpha_0, \ldots, \alpha_\eta$ are far from the violation of the binomial condition (7). When $\hat{C} = 1$ at least one of the coefficients equals the corresponding binomial coefficient.

Lemma 2. *In any dimension d and for any downward closed multi-index set Λ, let p be the polynomial defined in (5) with arbitrary coefficients $\alpha_0, \ldots, \alpha_\eta \in \mathbb{R}_0^+$, and let $\hat{C}$ be their* *associated constant defined in (13). Let $\widetilde{p}$ be the polynomial defined in (8), and let $K_p(\Lambda)$ and $K_{\widetilde{p}}(\Lambda)$ be the quantities (6) associated with p and $\widetilde{p}$, respectively. It holds that*

$$K_p(\Lambda) \leq \hat{C}^d K_{\widetilde{p}}(\Lambda) \leq \hat{C}^d (\#\Lambda)^{\eta+1}. \tag{14}$$

Proof. To prove the left inequality in (14): from (13), $\alpha_l \leq \hat{C} \binom{\eta+1}{l}$ for any $l = 0, \dots, \eta$, and by linearity it follows that

$$\begin{aligned} K_p(\Lambda) &= \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} \left(\alpha_0 + \alpha_1 \boldsymbol{\nu}_q + \cdots + \alpha_\eta \boldsymbol{\nu}_q^\eta \right) \\ &\leq \hat{C}^d \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} \left(\binom{\eta+1}{0} + \binom{\eta+1}{1} \boldsymbol{\nu}_q + \cdots + \binom{\eta+1}{\eta} \boldsymbol{\nu}_q^\eta \right) = \hat{C}^d K_{\widetilde{p}}(\Lambda). \end{aligned} \tag{15}$$

To prove the right inequality in (14): in the rightmost expression of (15) the coefficients of $\widetilde{p}$ by definition satisfy the binomial condition (7), and we can apply Theorem 1 to bound $K_{\widetilde{p}}(\Lambda)$. $\square$

Remark 2. When $\hat{C} > 1$ and with the additional requirement that $\alpha_0 \leq 1$, by using similar techniques to those used in the proof of [4, Lemma 2], inequality (14) can be rewritten as

$$K_p(\Lambda) \leq (\#\Lambda)^{\eta+1+\theta},$$

where θ is a positive monotonic increasing function depending on $\hat{C}$, *i.e.* $\theta = \theta(\hat{C})$. Therefore, when $\hat{C} > 1$ and $\alpha_0 \leq 1$, one can get rid of the multiplicative constant $\hat{C}^d$ in the rightmost expression of (14) but at the price of a worse exponent.

4. Orthogonal polynomials

In this section we recall several common families of univariate orthogonal polynomials defined over the interval $[-1, 1] \subset \mathbb{R}$, some of their properties, their three-term recurrence relation, and derive their orthonormalization in the L^2_ρ norm. We denote by Γ the usual gamma function defined as $\Gamma(\alpha) := \int_0^{+\infty} y^{\alpha-1} e^{-y}\, dy$ with $\mathrm{Re}(\alpha) > 0$, and then extended by analytic continuation. We recall that $\Gamma(\alpha+1) = \alpha!$ when $\alpha \in \mathbb{N}_0$.

Univariate Jacobi polynomials. These polynomials are orthogonal w.r.t. the inner product (1) with the univariate weight

$$\rho_J(y) := (1-y)^\alpha (1+y)^\beta, \quad y \in [-1, 1], \tag{16}$$

and any real numbers $\alpha,\ \beta > -1$:

$$\begin{aligned} &\int_{-1}^{+1} \widetilde{J}_n^{\alpha,\beta}(y)\, \widetilde{J}_k^{\alpha,\beta}(y)\, (1-y)^\alpha (1+y)^\beta\, dy \\ &\quad = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1)} \delta_{nk}, \end{aligned} \tag{17}$$

see [21, Eq. (4.3.3)]. When $n = 0$, the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ has to be replaced by $\Gamma(\alpha+\beta+2)$. These polynomials satisfy the following three-term recurrence relation

[21, Eq. (4.5.1)]:

$$
\begin{aligned}
&\widetilde{J}_0^{\alpha,\beta}(y) \equiv 1, \qquad \widetilde{J}_1^{\alpha,\beta}(y) = (\alpha+1) + (\alpha+\beta+2)(y-1)/2,\\
&\widetilde{J}_n^{\alpha,\beta}(y) \;=\; \frac{(2n+\alpha+\beta-1)\left((2n+\alpha+\beta)(2n+\alpha+\beta-2)y+\alpha^2-\beta^2\right)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)}\widetilde{J}_{n-1}^{\alpha,\beta}(y)\\
&\qquad - \frac{2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)}\widetilde{J}_{n-2}^{\alpha,\beta}(y), \quad n = 2, 3, \ldots .
\end{aligned}
$$

From (17), the L^2_ρ-orthonormal Jacobi polynomials are defined as

$$
J_n^{\alpha,\beta}(y) := \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}\,\widetilde{J}_n^{\alpha,\beta}(y), \quad n \in \mathbb{N}_0.
$$

Denote $\gamma_m := \min(\alpha, \beta)$ and $\gamma_M := \max(\alpha, \beta)$. Thanks to [21, Theorem 7.32.1], in the case $\gamma_M \geq -1/2$ it holds that

$$
\begin{aligned}
\| J_n^{\alpha,\beta} \|_{L^\infty(-1,1)} &= \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}\binom{n+\gamma_M}{n}\\
&= \sqrt{\frac{(2n+\gamma_m+\gamma_M+1)\Gamma(n+\gamma_m+\gamma_M+1)\Gamma(n+\gamma_M+1)}{2^{\gamma_m+\gamma_M+1}\Gamma(n+\gamma_m+1)\Gamma(n+1)}}\,\frac{1}{\Gamma(\gamma_M+1)},\\
&n \in \mathbb{N}_0.
\end{aligned} \tag{18}
$$

We will not consider the case $\gamma_M < -1/2$ where the behavior of $\| J_n^{\alpha,\beta} \|_{L^\infty(-1,1)}$ is different, see [21, Theorem 7.32.1].

Univariate Gegenbauer polynomials (or ultraspherical polynomials). These polynomials belong to the family of Jacobi polynomials, and $\widetilde{S}_n^\alpha = \widetilde{J}_n^{\alpha,\alpha}$ for any real $\alpha > -1$ and any $n \in \mathbb{N}_0$. They are orthogonal w.r.t. the inner product (1) with the univariate weight

$$
\rho_S(y) := (1-y^2)^\alpha, \quad y \in [-1, 1], \tag{19}
$$

and any real $\alpha > -1$:

$$
\int_{-1}^{+1} \widetilde{S}_n^\alpha(y)\, \widetilde{S}_k^\alpha(y)\, (1-y^2)^\alpha\, dy = \frac{2^{2\alpha+1}(\Gamma(n+\alpha+1))^2}{(2n+2\alpha+1)\Gamma(n+1)\Gamma(n+2\alpha+1)}\delta_{nk}. \tag{20}
$$

When $n = 0$, the product $(2n+2\alpha+1)\Gamma(n+2\alpha+1)$ has to be replaced by $\Gamma(2\alpha+2)$. These polynomials satisfy the following three-term recurrence relation:

$$
\begin{aligned}
&\widetilde{S}_0^\alpha(y) \equiv 1, \qquad \widetilde{S}_1^\alpha(y) = (\alpha+1)y,\\
&\widetilde{S}_n^\alpha(y) \;=\; \frac{(2n+2\alpha-1)(n+\alpha)(n+\alpha-1)y}{n(n+2\alpha)(n+\alpha-1)}\widetilde{S}_{n-1}^\alpha(y) - \frac{(n+\alpha-1)^2(n+\alpha)}{n(n+2\alpha)(n+\alpha-1)}\widetilde{S}_{n-2}^\alpha(y),\\
&\quad n = 2, 3, \ldots .
\end{aligned}
$$

From (20), the L^2_ρ-orthonormal Gegenbauer polynomials are defined as

$$
S_n^\alpha(y) := \sqrt{\frac{(2n+2\alpha+1)\Gamma(n+1)\Gamma(n+2\alpha+1)}{2^{2\alpha+1}(\Gamma(n+\alpha+1))^2}}\,\widetilde{S}_n^\alpha(y), \quad n \in \mathbb{N}_0.
$$

Choosing $\beta = \alpha$ in (18), in the case $\alpha \geq -1/2$ we obtain

$$\|S_n^\alpha\|_{L^\infty(-1,1)} = \sqrt{\frac{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}{2^{2\alpha+1}\Gamma(n+1)}}\frac{1}{\Gamma(\alpha+1)}, \quad n \in \mathbb{N}_0. \tag{21}$$

Univariate Legendre polynomials. These polynomials belong to the family of Jacobi and Gegenbauer polynomials, and $\widetilde{L}_n = \widetilde{S}_n^0 = \widetilde{J}_n^{0,0}$ for any $n \in \mathbb{N}_0$. They are orthogonal w.r.t. the inner product (1) with the univariate weight

$$\rho_L(y) := \mathbb{I}_{[-1,1]}(y), \quad y \in [-1,1], \tag{22}$$

i.e.

$$\int_{-1}^{+1} \widetilde{L}_n(y)\, \widetilde{L}_k(y)\, dy = \frac{2}{2n+1}\delta_{nk}. \tag{23}$$

Notice that, when using the weight (22), the weighted L^2_ρ norm associated with the weighted inner product (1) reduces to the standard L^2 norm. These polynomials satisfy the following three-term recurrence relation:

$$\widetilde{L}_0(y) \equiv 1, \qquad \widetilde{L}_1(y) = y, \qquad \widetilde{L}_{n+1}(y) = \frac{2n+1}{n+1} y \widetilde{L}_n(y) - \frac{n}{n+1}\widetilde{L}_{n-1}(y), \quad n \in \mathbb{N}.$$

From (23), the L^2_ρ-orthonormal Legendre polynomials are defined as

$$L_n(y) := \sqrt{\frac{2n+1}{2}}\widetilde{L}_n(y), \quad n \in \mathbb{N}_0,$$

and, choosing $\alpha = \beta = 0$ in (18), it holds that

$$\|L_n\|_{L^\infty(-1,1)} = \sqrt{\frac{2n+1}{2}}, \quad n \in \mathbb{N}_0. \tag{24}$$

Univariate Chebyshev polynomials of the first kind. These polynomials belong to the family of Jacobi and Gegenbauer polynomials, and $\widetilde{T}_n = \widetilde{S}_n^{-1/2}$ for any $n \in \mathbb{N}_0$. They are orthogonal w.r.t. the inner product (1) with the univariate weight

$$\rho_T(y) := (1-y^2)^{-1/2}, \quad y \in [-1,1], \tag{25}$$

i.e.

$$\int_{-1}^{+1} \widetilde{T}_n(y)\, \widetilde{T}_k(y)\, (1-y^2)^{-1/2}\, dy = \begin{cases} \pi, & \text{if } n = k = 0, \\ \pi/2, & \text{if } n = k \geq 1, \\ 0, & \text{if } n \neq k. \end{cases} \tag{26}$$

They satisfy the following three-term recurrence relation:

$$\widetilde{T}_0(y) \equiv 1, \qquad \widetilde{T}_1(y) = y, \qquad \widetilde{T}_{n+1}(y) = 2y\widetilde{T}_n(y) - \widetilde{T}_{n-1}(y), \quad n \in \mathbb{N}.$$

From (26), the L^2_ρ-orthonormal Chebyshev polynomials of the first kind are defined as

$$T_0(y) := \sqrt{\frac{1}{\pi}}\widetilde{T}_0(y), \quad \text{and} \quad T_n(y) := \sqrt{\frac{2}{\pi}}\widetilde{T}_n(y), \quad n \in \mathbb{N}.$$

Choosing $\alpha = \beta = -1/2$ in (18) and exploiting classical properties of the gamma function, their norms equal

$$\|T_0\|_{L^\infty(-1,1)} = \sqrt{\frac{1}{\pi}}, \quad \text{and} \quad \|T_n\|_{L^\infty(-1,1)} = \sqrt{\frac{2}{\pi}}, \quad n \in \mathbb{N}. \tag{27}$$

The univariate families of L^2_ρ-orthonormal polynomials $\{J_n^{\alpha,\beta}\}_{n\geq 0}$, $\{S_n^\alpha\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$, $\{T_n\}_{n\geq 0}$, corresponding to Jacobi, Gegenbauer, Legendre and Chebyshev polynomials, are used to build the corresponding tensorized (multivariate) families of L^2_ρ-orthonormal polynomials. For each one of the four families of univariate L^2_ρ-orthonormal polynomials, we define the d-dimensional orthonormalization weights using the univariate weights (16), (19), (22) and (25):

$$\rho_J^d(y) := \prod_{q=1}^d \rho_J(y_q), \quad \text{(tensorized Jacobi polynomials)}, \tag{28}$$

$$\rho_S^d(y) := \prod_{q=1}^d \rho_S(y_q), \quad \text{(tensorized Gegenbauer polynomials)}, \tag{29}$$

$$\rho_L^d(y) := \prod_{q=1}^d \rho_L(y_q), \quad \text{(tensorized Legendre polynomials)}, \tag{30}$$

$$\rho_T^d(y) := \prod_{q=1}^d \rho_T(y_q), \quad \text{(tensorized Chebyshev polynomials of the first kind)}. \tag{31}$$

Given any arbitrary d-dimensional downward closed multi-index set Λ, we denote by $\{J_{\boldsymbol{\nu}}^{\alpha,\beta}\}_{\boldsymbol{\nu}\in\Lambda}$, $\{S_{\boldsymbol{\nu}}^\alpha\}_{\boldsymbol{\nu}\in\Lambda}$, $\{L_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\Lambda}$ and $\{T_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\Lambda}$ the tensorized families of Jacobi, Gegenbauer, Legendre and Chebyshev polynomials over the d-dimensional hypercube $[-1,1]^d \subset \mathbb{R}^d$, with each multivariate polynomial being built by tensorization as in (3) using the univariate families $\{J_n^{\alpha,\beta}\}_{n\geq 0}$, $\{S_n^\alpha\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$. Each one of these tensorized families is L^2_ρ-orthonormal w.r.t. the corresponding tensorized orthonormalization weight defined in (28)–(31).

In any dimension $d \geq 1$ and for any real numbers $\alpha, \beta > -1$, we introduce the integral of the weight (28) as

$$\begin{aligned} W(\alpha,\beta,d) &:= \int_D \rho_J^d(y)dy = \int_D \prod_{q=1}^d (1-y_q)^\alpha (1+y_q)^\beta dy \\ &= \left(\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \right)^d, \end{aligned} \tag{32}$$

where its evaluation is given by taking $n = k = 0$ in (17), in each one of the d directions. In any dimension $d \geq 1$ and for any real number $\alpha > -1$, choosing $\beta = \alpha$ in (32) yields the integral of the weight (29), *i.e.*

$$W(\alpha,\alpha,d) = \left(\frac{2^{2\alpha+1}(\Gamma(\alpha+1))^2}{\Gamma(2\alpha+2)} \right)^d. \tag{33}$$

The integral of the weight (30) equals

$$W(0,0,d) = 2^d, \tag{34}$$

and the integral of the weight (31) equals

$$W(-1/2,-1/2,d) = \pi^d. \tag{35}$$

Remark 3. Throughout the paper $\{J_n^{\alpha,\beta}\}_{n\geq 0}$, $\{S_n^\alpha\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ will always denote the families of univariate L^2_ρ-orthonormal polynomials over the interval $[-1,1]$, with their L^∞ norms satisfying (18), (21), (24) and (27), respectively. Analogously, $\{J_{\boldsymbol{\nu}}^{\alpha,\beta}\}_{\boldsymbol{\nu}\in\Lambda}$, $\{S_{\boldsymbol{\nu}}^\alpha\}_{\boldsymbol{\nu}\in\Lambda}$, $\{L_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\Lambda}$ and $\{T_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\Lambda}$ will always denote the tensorized families of L^2_ρ-orthonormal polynomials on $[-1,1]^d$ associated with the multi-index set Λ.

5. Multivariate polynomial inequalities

In this section we prove several Markov-type and Nikolskii-type inequalities for multivariate polynomials indexed by downward closed multi-index sets in any dimension. Throughout this section D will always denote the d-dimensional hypercube $D = [-1,1]^d \subset \mathbb{R}^d$.

5.1. Markov one-dimensional inequalities

Lemma 3 (*Markov One-Dimensional Inequality in L^2*)**.** *Given an interval $[a,b]\subset\mathbb{R}$, for any polynomial $u\in\mathbb{P}_{\mathrm{w}}(a,b)$ with maximum degree* w *it holds that*

$$\|u'\|_{L^2(a,b)} \leq \frac{2\sqrt{3}}{b-a}\mathrm{w}^2\|u\|_{L^2(a,b)}. \tag{36}$$

Proof. See [20]. □

Lemma 4 (*Derivative of Univariate Legendre Polynomials*)**.** *Given the interval $[-1,1]\subset\mathbb{R}$, for any L^2_ρ-orthonormal Legendre polynomial $L_n\in\mathbb{P}_n(-1,1)$ with degree $n\in\mathbb{N}_0$ it holds that*

$$\|L_n'\|_{L^2_\rho(-1,1)} = \sqrt{n\left(n+\frac{1}{2}\right)(n+1)}. \tag{37}$$

Proof. Thanks to the following identity [21, Eq. (4.7.29)] it holds that

$$\frac{d}{dy}\left(\widetilde{L}_{n+1}(y)-\widetilde{L}_{n-1}(y)\right) = (2n+1)\widetilde{L}_n(y), \quad \forall y\in D,\ \forall n\geq 1,$$

and, by recurrence and using (23), we obtain for any $n\geq 1$ that

$$\|\widetilde{L}'_{2n}\|^2_{L^2_\rho} = \sum_{r=1}^{n}(2(2r-1)+1)^2\|\widetilde{L}_{2r-1}\|^2_{L^2_\rho} = 2n(2n+1),$$

$$\|\widetilde{L}'_{2n+1}\|^2_{L^2_\rho} = \sum_{r=0}^{n}(2(2r)+1)^2\|\widetilde{L}_{2r}\|^2_{L^2_\rho} = 2(n+1)(2n+1).$$

Hence the L^2_ρ-orthonormal polynomials $(L_n)_{n\geq 2}$ satisfy (37). By direct calculation, L_0 and L_1 satisfy (37) as well. □

Lemma 5 (*Derivative of Univariate Chebyshev Polynomials*)**.** *Given the interval* $[-1,1]\subset\mathbb{R}$, *for any* L^2_ρ*-orthonormal Chebyshev polynomial of the first kind* $T_n\in\mathbb{P}_n(-1,1)$ *with degree* $n\in\mathbb{N}_0$ *it holds that*

$$\|T_n'\|_{L^2_\rho(-1,1)}=\sqrt{2}n^{3/2}. \tag{38}$$

Proof. Consider the following identity (see [19, p. 5])

$$\widetilde{T}_n=\frac{\widetilde{T}'_{n+1}}{2n+2}-\frac{\widetilde{T}'_{n-1}}{2n-2},\quad \forall n\geq 2.$$

When $n=1$ we have $\widetilde{T}_1=\widetilde{T}'_2/4$. By recurrence we obtain that

$$\widetilde{T}'_{n+1}=2(n+1)\left(\sum_{r=1}^{n/2}\widetilde{T}_{2r}+\widetilde{T}'_1/2\right),\quad \text{if } n \text{ is even},$$

$$\widetilde{T}'_{n+1}=2(n+1)\sum_{r=0}^{\lfloor n/2\rfloor}\widetilde{T}_{2r+1},\quad \text{if } n \text{ is odd}.$$

Therefore, since $\widetilde{T}'_1=\widetilde{T}_0$, we obtain

$$\|\widetilde{T}'_{n+1}\|^2_{L^2_\rho}=4(n+1)^2\left(\sum_{r=1}^{n/2}\|\widetilde{T}_{2r}\|^2_{L^2_\rho}+\|\widetilde{T}'_1\|^2_{L^2_\rho}/4\right)=\pi(n+1)^3,\quad n \text{ even},$$

and

$$\|\widetilde{T}'_{n+1}\|^2_{L^2_\rho}=4(n+1)^2\left(\sum_{r=0}^{\lfloor n/2\rfloor}\|\widetilde{T}_{2r+1}\|^2_{L^2_\rho}\right)=\pi(n+1)^3,\quad n \text{ odd}.$$

Thus the L^2_ρ-orthonormal polynomials $(T_n)_{n\geq 1}$ satisfy (38). By direct calculation T_0 satisfies (38) as well. □

Lemma 6 (*Derivative of Univariate Gegenbauer Polynomials*)**.** *Given the interval* $[-1,1]\subset\mathbb{R}$ *and any* $\alpha\in\mathbb{N}$, *for any* L^2_ρ*-orthonormal Gegenbauer polynomial* $S^\alpha_n\in\mathbb{P}_n(-1,1)$ *with degree* $n\in\mathbb{N}_0$ *it holds that*

$$\|(S^\alpha_n)'\|^2_{L^2_\rho(-1,1)}\leq\zeta^e(\alpha)\,(n+\alpha+1/2)\,(n^2+n(2\alpha+1)),\quad \textit{if } n \textit{ is even}, \tag{39}$$

$$\|(S^\alpha_1)'\|^2_{L^2_\rho(-1,1)}=\frac{(3+2\alpha)(2\alpha+1)!}{2^{2\alpha+1}(\alpha!)^2}=\frac{(3+2\alpha)(\alpha+1)}{2^{2\alpha+1}}\prod_{k=1}^{\alpha}\left(\frac{\alpha+1}{k}+1\right), \tag{40}$$

$$\|(S^\alpha_n)'\|^2_{L^2_\rho(-1,1)}\ \leq\ \zeta^o(\alpha)(n+\alpha+1/2)\left(n^2+n\,(2\alpha+1)-\frac{2\alpha(\alpha+1)}{2\alpha+1}\right),\quad \textit{if } n \textit{ is odd and } n\geq 3, \tag{41}$$

with $\zeta^e:\mathbb{N}\to\mathbb{R}^+$ *and* $\zeta^o:\mathbb{N}\to\mathbb{R}^+$ *being defined as*

$$\zeta^e(\alpha):=\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{(k+2)(\alpha+k+1)}\right)<1,$$

$$\zeta^o(\alpha) := \prod_{k=1}^{\alpha}\left(1 - \frac{3\alpha}{(k+3)(\alpha+k)}\right) < 1.$$

Proof. From the following identity [21, Eq. (4.7.29)] it holds

$$\frac{d}{dy}\left(\widetilde{S}^{\alpha}_{n+1}(y) - \widetilde{S}^{\alpha}_{n-1}(y)\right) = (2n+2\alpha+1)\widetilde{S}^{\alpha}_{n}(y), \quad \forall y \in D,\ \forall n \geq 1, \tag{42}$$

and, by recurrence and using (20), we obtain for any $n \geq 1$ that

$$\|(\widetilde{S}^{\alpha}_{2n})'\|^2_{L^2_\rho} = \sum_{r=1}^{n}(2(2r-1)+2\alpha+1)^2\|\widetilde{S}^{\alpha}_{2r-1}\|^2_{L^2_\rho},$$

$$\|(\widetilde{S}^{\alpha}_{2n+1})'\|^2_{L^2_\rho} = \sum_{r=1}^{n}(2(2r)+2\alpha+1)^2\|\widetilde{S}^{\alpha}_{2r}\|^2_{L^2_\rho} + (\alpha+1)^2\|\widetilde{S}^{\alpha}_{0}\|^2_{L^2_\rho}.$$

Hence the L^2_ρ-orthonormal Gegenbauer polynomials for any $n \geq 1$ and any $\alpha \in \mathbb{N}$ satisfy

$$\begin{aligned}
\|(S^{\alpha}_{2n})'\|^2_{L^2_\rho} &= \frac{(2(2n)+2\alpha+1)(2n)!(2n+2\alpha)!}{((2n+\alpha)!)^2}\sum_{r=1}^{n}(4r+2\alpha-1)\\
&\quad\times\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{2r-1+k+\alpha}\right)\\
&\leq \frac{(4n+2\alpha+1)(2n)!(2n+2\alpha)!}{((2n+\alpha)!)^2}\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{1+k+\alpha}\right)\sum_{r=1}^{n}(4r+2\alpha-1)\\
&= \frac{(4n+2\alpha+1)n(2n+2\alpha+1)(2n)!(2n+2\alpha)!}{((2n+\alpha)!)^2}\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{1+k+\alpha}\right)\\
&= (4n+2\alpha+1)n(2n+2\alpha+1)\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{1+k+\alpha}\right)\prod_{k=1}^{\alpha}\left(1+\frac{\alpha}{2n+k}\right)\\
&\leq (4n+2\alpha+1)n(2n+2\alpha+1)\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{1+k+\alpha}\right)\prod_{k=1}^{\alpha}\left(1+\frac{\alpha}{2+k}\right)\\
&= (4n+2\alpha+1)n(2n+2\alpha+1)\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{(k+2)(\alpha+k+1)}\right),\\
\|(S^{\alpha}_{2n+1})'\|^2_{L^2_\rho} &= \frac{(2(2n+1)+2\alpha+1)(2n+1)!(2n+1+2\alpha)!}{((2n+1+\alpha)!)^2}\\
&\quad\times\left(\sum_{r=1}^{n}(4r+2\alpha+1)\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{2r+k+\alpha}\right)+\frac{(\alpha+1)^2(\alpha!)^2}{(2\alpha+1)(2\alpha)!}\right)\\
&\leq \frac{(4n+2\alpha+3)(2n+1)!(2n+1+2\alpha)!}{((2n+1+\alpha)!)^2}\\
&\quad\times\prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{k+\alpha}\right)\left(\sum_{r=1}^{n}(4r+2\alpha+1)+\frac{(\alpha+1)^2}{(2\alpha+1)}\right)\\
&= \frac{(4n+2\alpha+3)(2n+1)!(2n+1+2\alpha)!}{((2n+1+\alpha)!)^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{k+\alpha}\right)\left(n(2n+2\alpha+3)+\frac{(\alpha+1)^2}{(2\alpha+1)}\right)\\
&= (4n+2\alpha+3)\left(2n^2+n(2\alpha+3)+\frac{(\alpha+1)^2}{(2\alpha+1)}\right)\\
&\times \prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{k+\alpha}\right)\prod_{k=1}^{\alpha}\left(1+\frac{\alpha}{2n+1+k}\right)\\
&\le (4n+2\alpha+3)\left(2n^2+n(2\alpha+3)+\frac{(\alpha+1)^2}{(2\alpha+1)}\right)\\
&\times \prod_{k=1}^{\alpha}\left(1-\frac{\alpha}{k+\alpha}\right)\prod_{k=1}^{\alpha}\left(1+\frac{\alpha}{3+k}\right)\\
&= (4n+2\alpha+3)\left(2n^2+n(2\alpha+3)+\frac{(\alpha+1)^2}{(2\alpha+1)}\right)\\
&\times \prod_{k=1}^{\alpha}\left(1-\frac{3\alpha}{(k+3)(\alpha+k)}\right).
\end{aligned}
$$

Hence the L^2_ρ-orthonormal polynomials $(S_n^\alpha)_{n\ge 2}$ satisfy (39)–(41) depending on the parity of n, since

$$
\begin{aligned}
&\|(S_n^\alpha)'\|^2_{L^2_\rho} \le \zeta^e(\alpha)(n+\alpha+1/2)n(n+2\alpha+1), \quad \text{when } n \text{ is even and } n\ge 2,\\
&\|(S_n^\alpha)'\|^2_{L^2_\rho} \le \zeta^o(\alpha)(n+\alpha+1/2)\left(n^2+n\,(2\alpha+1)-\frac{2\alpha(\alpha+1)}{2\alpha+1}\right),\\
&\quad \text{when } n \text{ is odd and } n\ge 3.
\end{aligned}
$$

By direct calculation, it holds that S_0^α satisfies (39) and S_1^α satisfies (40). □

5.2. Multivariate Markov-type inequalities

The result in Theorem 1 can be used to derive inequalities of Markov-type, for the mixed derivative of a multivariate polynomial $u\in\mathbb{P}_\Lambda(D)$ associated with an arbitrary downward closed multi-index set Λ in any dimension d.

Theorem 2. *For any d-variate polynomial $u\in\mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$
\left\|\frac{\partial^d}{\partial y_1\cdots\partial y_d}u\right\|_{L^2(D)} \le \left(\frac{3}{5}\right)^{d/2}(\#\Lambda)^{5/2}\,\|u\|_{L^2(D)}.
$$

Proof. We expand the polynomial $u\in\mathbb{P}_\Lambda(D)$ over any polynomial orthonormal basis $\{\psi_{\boldsymbol\nu}\}_{\boldsymbol\nu\in\Lambda}$ of $\mathbb{P}_\Lambda(D)$ of the form (3):

$$
u=\sum_{\boldsymbol\nu\in\Lambda}\beta_{\boldsymbol\nu}\psi_{\boldsymbol\nu}.
$$

Then, using the Cauchy–Schwarz inequality in $\mathbb{R}^{\#\Lambda}$ and (36) in each direction with $[a,b]=[-1,1]$, we proceed as follows:

$$
\begin{aligned}
\left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right\|^2_{L^2(D)} &= \int_D \left(\frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right)^2 \, dy \\
&= \int_{D_1} \cdots \int_{D_d} \left(\frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right)^2 \, dy_1 \cdots dy_d \\
&= \int_{D_1} \cdots \int_{D_d} \left(\frac{\partial^d}{\partial y_1 \cdots \partial y_d} \sum_{\boldsymbol{\nu} \in \Lambda} \beta_{\boldsymbol{\nu}} \psi_{\boldsymbol{\nu}} \right)^2 \, dy_1 \cdots dy_d \\
&= \int_{D_1} \cdots \int_{D_d} \left(\sum_{\boldsymbol{\nu} \in \Lambda} \beta_{\boldsymbol{\nu}} \frac{\partial^d}{\partial y_1 \cdots \partial y_d} \psi_{\boldsymbol{\nu}} \right)^2 \, dy_1 \cdots dy_d \\
&\leq \left(\sum_{\boldsymbol{\nu} \in \Lambda} \beta^2_{\boldsymbol{\nu}} \right) \int_{D_1} \cdots \int_{D_d} \sum_{\boldsymbol{\nu} \in \Lambda} \left(\frac{\partial^d}{\partial y_1 \cdots \partial y_d} \psi_{\boldsymbol{\nu}} \right)^2 \, dy_1 \cdots dy_d \\
&= \|\beta\|^2_{\ell_2} \sum_{\boldsymbol{\nu} \in \Lambda} \int_{D_1} \cdots \int_{D_d} \left(\frac{\partial^d}{\partial y_1 \cdots \partial y_d} \psi_{\boldsymbol{\nu}} \right)^2 \, dy_1 \cdots dy_d \\
&= \|u\|^2_{L^2(D)} \sum_{\boldsymbol{\nu} \in \Lambda} \left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} \psi_{\boldsymbol{\nu}} \right\|^2_{L^2(D)} \\
&= \|u\|^2_{L^2(D)} \sum_{\boldsymbol{\nu} \in \Lambda} \left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} \prod_{q=1}^{d} \varphi_{\nu_q}(y_q) \right\|^2_{L^2(D)} \\
&\leq \|u\|^2_{L^2(D)} \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} 3\nu_q^4 \\
&= \|u\|^2_{L^2(D)} \left(\frac{3}{5} \right)^d \sum_{\boldsymbol{\nu} \in \Lambda} \prod_{q=1}^{d} 5\nu_q^4 \\
&\leq \|u\|^2_{L^2(D)} (\#\Lambda)^5 \left(\frac{3}{5} \right)^d .
\end{aligned}
$$

In the last step we have applied Theorem 1 with $\eta = 4$ and $p(n) = 5n^4$. Notice that we are in the case where the binomial condition (7) is satisfied, since $\binom{5}{4} = 5$. $\square$

Theorem 3. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$
\left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right\|_{L^2_\rho(D)} \leq 2^{-d} (\#\Lambda)^2 \|u\|_{L^2_\rho(D)},
$$

with $\rho = \rho_L^d$ being defined in (30) as the weight associated with tensorized Legendre L^2_ρ-orthonormal polynomials.

Proof. Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Legendre polynomials $(L_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$. Following the lines of the proof of Theorem 2, but using (37) in each direction, we obtain

$$
\begin{aligned}
\left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right\|^2_{L^2(D)} &= \|u\|^2_{L^2(D)} \sum_{\boldsymbol{\nu}\in\Lambda} \left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} \prod_{q=1}^{d} L_{\nu_q}(y_q) \right\|^2_{L^2(D)} \\
&\leq \|u\|^2_{L^2(D)} \sum_{\boldsymbol{\nu}\in\Lambda} \prod_{q=1}^{d} \nu_q \left(\nu_q + \frac{1}{2} \right) \left(\nu_q + 1 \right) \\
&= \|u\|^2_{L^2(D)} 4^{-d} \sum_{\boldsymbol{\nu}\in\Lambda} \prod_{q=1}^{d} \left(4\nu_q^3 + 6\nu_q^2 + 2\nu_q \right) \\
&\leq \|u\|^2_{L^2(D)} 4^{-d} \sum_{\boldsymbol{\nu}\in\Lambda} \prod_{q=1}^{d} \widetilde{p}(\nu_q) \\
&\leq \|u\|^2_{L^2(D)} 4^{-d} (\#\Lambda)^4.
\end{aligned}
$$

In the last but one step we have used the polynomial $\widetilde{p} = \widetilde{p}(n)$ defined in (8) with $\eta = 3$, and then in the last step we have applied Theorem 1. □

Remark 4. The standard L^2 norm coincides with the weighted L^2_ρ norm when ρ is the weight (30) associated with tensorized Legendre L^2_ρ-orthonormal polynomials. Choosing $d = 1$ in the thesis of Theorem 3 yields a better constant than Lemma 3. Expanding the polynomial $u \in \mathbb{P}_\Lambda(D)$ in Legendre series is advantageous also when $d > 1$, since the inequality constant $(3/5)^{d/2}(\#\Lambda)^{5/2}$ in the thesis of Theorem 2 improves to $2^{-d}(\#\Lambda)^2$ in the thesis of Theorem 3.

Theorem 4. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$
\left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right\|_{L^2_\rho(D)} \leq 2^{-d/2} (\#\Lambda)^2 \|u\|_{L^2_\rho(D)},
$$

with $\rho = \rho_T^d$ being defined in (31) as the weight associated with tensorized Chebyshev of the first kind L^2_ρ-orthonormal polynomials.

Proof. Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Chebyshev polynomials of the first kind $(T_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$. It suffices to follow the lines of the proof of Theorem 3, but using (38) in each direction, take out the constant 2^{-d} from the summation, and then apply Theorem 1 with $\eta = 3$ and the polynomial $\widetilde{p} = \widetilde{p}(n)$ defined in (8). □

Theorem 5. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed and any $\alpha \in \mathbb{N}$, it holds that*

$$
\left\| \frac{\partial^d}{\partial y_1 \cdots \partial y_d} u \right\|_{L^2_\rho(D)} \leq (C_S(\alpha))^{d/2} (\#\Lambda)^2 \|u\|_{L^2_\rho(D)}, \tag{43}
$$

with $\rho = \rho_S^d$ being defined in (29) as the weight associated with tensorized Gegenbauer L^2_ρ-orthonormal polynomials, and with $C_S : \mathbb{N} \to \mathbb{R}^+$ being the function

$$
C_S(\alpha) := \max \left\{ \frac{\zeta^e(\alpha)\,(\alpha + 1/2)^2}{2}, \frac{(3 + 2\alpha)(2\alpha + 1)!}{8\,(\alpha!)^2\, 2^{2\alpha}} \right\} > 1. \tag{44}
$$

Proof. Consider the bounds of $\|(S_n^\alpha)'\|^2_{L^2_\rho}$ on the right-hand side of (39) and (41): they are polynomials of third degree in the variable n with coefficients depending on $\alpha \in \mathbb{N}$. We name these polynomials $p^e = p^e(n)$ if n is even, and $p^o = p^o(n)$ if n is odd and $n \geq 3$. The bound on the right-hand side of (40), when $n = 1$, can be associated with a polynomial $p^1 = p^1(n)$ of degree one. With the same notation, we extend the polynomials p^e, p^o and p^1 over any $n \in \mathbb{N}_0$. Since $\zeta^e(\alpha) \geq \zeta^o(\alpha)$ for any $\alpha \in \mathbb{N}$, it holds true that $p^e(n) \geq p^o(n)$ for any $n \in \mathbb{N}_0$. Using the polynomial $\widetilde{p} = \widetilde{p}(n)$ defined in (8) with $\eta = 3$, we seek a function $C(\alpha) : \mathbb{N} \to \mathbb{R}^+$ such that

$$\|(S_n^\alpha)'\|^2_{L^2_\rho} \leq C(\alpha)\widetilde{p}(n), \quad \forall n \in \mathbb{N}_0,\ \forall \alpha \in \mathbb{N}. \tag{45}$$

To this aim, we compute the constant (13) for the polynomial $p^e = p^e(n)$ with $\eta = 3$, *i.e.*

$$\hat{C}_S^e(\alpha) := \max\left\{ \frac{\zeta^e(\alpha)}{\binom{\eta+1}{3}}, \frac{\zeta^e(\alpha)(3\alpha + 3/2)}{\binom{\eta+1}{2}}, \frac{2\zeta^e(\alpha)(\alpha + 1/2)^2}{\binom{\eta+1}{1}}, 0 \right\},$$

and for the polynomial $p^1 = p^1(n)$ with $\eta = 3$ (albeit a linear function), *i.e.*

$$\hat{C}_S^1(\alpha) := \max\left\{ 0, 0, \frac{(3 + 2\alpha)(2\alpha + 1)!}{\binom{\eta+1}{1}\, 2^{2\alpha+1}(\alpha!)^2}, 0 \right\}.$$

The function $C_S(\alpha)$ defined in (44) satisfies $C_S(\alpha) = \max\left\{\hat{C}_S^e(\alpha), \hat{C}_S^1(\alpha)\right\}$ for any $\alpha \in \mathbb{N}$. Therefore, (45) holds true with $C(\alpha) = C_S(\alpha)$.

To prove (43), we follow the lines of the proof of Theorem 3. Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Gegenbauer polynomials $(S_{\boldsymbol{\nu}}^\alpha)_{\boldsymbol{\nu}\in\Lambda}$. Then we use (45) with $C(\alpha) = C_S(\alpha)$ to bound the derivatives in each direction, take out the constant $(C_S(\alpha))^d$ from the summation, apply Theorem 1 with $\eta = 3$ and finally obtain (43). □

Remark 5. The estimates proven in Lemma 6 can be extended to any real $\alpha > -1$, making use of the properties of the gamma function. The same extension can be achieved in Theorem 5, because the shape parameter α does not enter in the exponent η when applying Theorem 1.

5.3. Multivariate Nikolskii-type inequalities

Multivariate Nikolskii-type inequalities between L^∞ and L^2_ρ have been proven in [4, Lemmas 1 and 2] for Legendre and Chebyshev polynomials of the first kind. To keep the present article self contained we recall these results in the following, and afterwards we generalize them to the case of Gegenbauer and Jacobi polynomials using Theorem 1. The result in Theorem 6 can be proven with the same proof as in [4, Lemma 1], and taking into account the different orthonormalization of the Legendre polynomials, see also Remark 6. The result in Theorem 7 in the case of Chebyshev polynomials is only stated, since a specific treatment is needed to get the optimal exponent, see [4, Lemma 2] for the proof. In Theorem 8, with Gegenbauer polynomials, we confine to values of the parameter α such that $2\alpha + 1 \in \mathbb{N}$, and this allows us to include the Chebyshev polynomials of the second kind given by $\alpha = \beta = 1/2$. In Theorem 9, with Jacobi polynomials, we confine to integer values of the parameters $\alpha, \beta \in \mathbb{N}_0$. The analysis in the general case with $\alpha, \beta \in \mathbb{R}_0^+$ requires an extension of Theorem 1 to include (positive) real exponents η. The case of Legendre polynomials $\alpha = \beta = 0$ is included as a particular case in both theses of Theorems 8 and 9.

Theorem 6. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$\|u\|^2_{L^\infty(D)} \le (\#\Lambda)^2 W_\rho^{-1} \|u\|^2_{L^2_\rho(D)},$$

with $\rho = \rho_L^d$ being defined in (30) as the weight associated with tensorized Legendre L^2_ρ-orthonormal polynomials, and with $W_\rho = 2^d$ being its integral defined in (2).

Proof. From (24) we have that the univariate Legendre L^2_ρ-orthonormal polynomials satisfy

$$\|L_n\|^2_{L^\infty(-1,1)} = \frac{(2n+1)}{2} = \frac{\widetilde{p}(n)}{W(0,0,1)}. \tag{46}$$

In the rightmost expression of (46) we have used the polynomial $\widetilde{p}(n) = 2n+1$ defined in (8) with $\eta = 1$, divided by the constant $W(0,0,d)$ defined in (34) with $d = 1$.

Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Legendre polynomials $(L_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$:

$$u(y) = \sum_{\boldsymbol{\nu}\in\Lambda} \beta_{\boldsymbol{\nu}} \prod_{q=1}^{d} L_{\boldsymbol{\nu}_q}(y_q).$$

Therefore, using in sequence the Cauchy–Schwarz inequality in $\mathbb{R}^{\#\Lambda}$, (46) and Theorem 1, we have

$$\begin{aligned}
\|u(y)\|_{L^\infty(D)} &= \left\| \sum_{\boldsymbol{\nu}\in\Lambda} \beta_{\boldsymbol{\nu}} \prod_{q=1}^{d} L_{\boldsymbol{\nu}_q}(y_q) \right\|_{L^\infty(D)} \\
&\le \sqrt{\sum_{\boldsymbol{\nu}\in\Lambda} |\beta_{\boldsymbol{\nu}}|^2} \sqrt{\sum_{\boldsymbol{\nu}\in\Lambda} \left(\prod_{q=1}^{d} \left\| L_{\boldsymbol{\nu}_q}(y_q) \right\|_{L^\infty(-1,1)} \right)^2} \\
&\le \|u\|_{L^2_\rho(D)} \sqrt{W(0,0,d)^{-1}(\#\Lambda)^2},
\end{aligned}$$

and we obtain the thesis with $W_\rho = W(0,0,d)$. □

Theorem 7. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$\|u\|^2_{L^\infty(D)} \le (\#\Lambda)^{\frac{\ln 3}{\ln 2}} W_\rho^{-1} \|u\|^2_{L^2_\rho(D)},$$

with $\rho = \rho_T^d$ being defined in (31) as the weight associated with tensorized Chebyshev of the first kind L^2_ρ-orthonormal polynomials, and with $W_\rho = \pi^d$ being its integral defined in (2).

Proof. The result follows from the same proof as in [4, Lemma 2], and taking into account the different orthonormalization of the Chebyshev polynomials of the first kind. □

Theorem 8. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed it holds that*

$$\|u\|^2_{L^\infty(D)} \le (\#\Lambda)^{2\alpha+2} W_\rho^{-1} \|u\|^2_{L^2_\rho(D)}, \quad \textit{for any } \alpha : 2\alpha + 1 \in \mathbb{N}, \tag{47}$$

with $\rho = \rho_S^d$ being defined in (29) as the weight associated with tensorized Gegenbauer L^2_ρ-orthonormal polynomials, and with W_ρ being its integral defined in (2).

Proof. From (21) we have that the univariate Gegenbauer L^2_ρ-orthonormal polynomials with $2\alpha - 1 \in \mathbb{N}_0$ satisfy

$$\begin{aligned}
\|S_n^\alpha\|^2_{L^\infty(-1,1)} &= \frac{(2n+2\alpha+1)(n+2\alpha)!}{2^{2\alpha+1}(\Gamma(\alpha+1))^2 n!} \\
&= \frac{(2\alpha+1)}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}\left(\frac{2n}{2\alpha+1}+1\right)(2\alpha)!\prod_{k=1}^{2\alpha}\left(\frac{n}{k}+1\right) \\
&\le \frac{(2\alpha+1)\,(2\alpha)!}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(n+1)^{2\alpha+1} \\
&= \frac{(2\alpha+1)\,(2\alpha)!}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}\sum_{l=0}^{2\alpha+1}\binom{2\alpha+1}{l} n^l. \\
&= W(\alpha,\alpha,1)^{-1} p(n).
\end{aligned} \tag{48}$$

In the last but one step we have used the binomial theorem, with the restrictions on α ensuring that the exponent $2\alpha+1$ is a nonnegative integer. In the last step we have introduced the polynomial $p(n) := \sum_{l=0}^{2\alpha+1}\binom{2\alpha+1}{l} n^l$ divided by the constant $W(\alpha,\alpha,d)$ defined in (33) with $d=1$. The polynomial p has maximum degree $\eta = 2\alpha+1$, and its coefficients satisfy the binomial condition (7) since $\binom{2\alpha+1}{l} \le \binom{2\alpha+2}{l}$ for any $l = 0,\ldots,\eta$.

Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Gegenbauer polynomials $(S_{\boldsymbol{\nu}}^\alpha)_{\boldsymbol{\nu}\in\Lambda}$:

$$u(y) = \sum_{\boldsymbol{\nu}\in\Lambda}\beta_{\boldsymbol{\nu}}\prod_{q=1}^{d} S^\alpha_{\boldsymbol{\nu}_q}(y_q).$$

Therefore, using in sequence the Cauchy–Schwarz inequality in $\mathbb{R}^{\#\Lambda}$, (48) and Theorem 1, we have

$$\begin{aligned}
\|u(y)\|_{L^\infty(D)} &= \left\|\sum_{\boldsymbol{\nu}\in\Lambda}\beta_{\boldsymbol{\nu}}\prod_{q=1}^{d} S^\alpha_{\boldsymbol{\nu}_q}(y_q)\right\|_{L^\infty(D)} \\
&\le \sqrt{\sum_{\boldsymbol{\nu}\in\Lambda}|\beta_{\boldsymbol{\nu}}|^2}\sqrt{\sum_{\boldsymbol{\nu}\in\Lambda}\left(\prod_{q=1}^{d}\left\|S^\alpha_{\boldsymbol{\nu}_q}(y_q)\right\|_{L^\infty(-1,1)}\right)^2} \\
&\le \|u\|_{L^2_\rho(D)}\sqrt{W(\alpha,\alpha,d)^{-1}(\#\Lambda)^{2\alpha+2}}.
\end{aligned}$$

This completes the proof of (47) in the case $2\alpha - 1 \in \mathbb{N}_0$, with $W_\rho = W(\alpha,\alpha,d)$. The case $\alpha = 0$ has been proven in Theorem 6, and is included in (47) as well. □

Theorem 9. *For any d-variate polynomial $u \in \mathbb{P}_\Lambda(D)$ with Λ downward closed and any $\alpha,\beta \in \mathbb{N}_0$ it holds that*

$$\|u\|^2_{L^\infty(D)} \le (\#\Lambda)^{2\gamma_M+2} W_\rho^{-1}\|u\|^2_{L^2_\rho(D)}, \tag{49}$$

with $\rho = \rho_J^d$ being defined in (28) as the weight associated with tensorized Jacobi L^2_ρ-orthonormal polynomials, and with W_ρ being its integral defined in (2).

Proof. From (18) we have that the univariate Jacobi L^2_ρ-orthonormal polynomials with $\gamma_m + \gamma_M \geq 1$ satisfy

$$\begin{aligned}
\| J_n^{\alpha,\beta} \|^2_{L^\infty(-1,1)} &= \frac{(2n+\gamma_m+\gamma_M+1)(n+\gamma_m+\gamma_M)!(n+\gamma_M)!}{2^{\gamma_m+\gamma_M+1}(\gamma_M!)^2(n+\gamma_m)!n!} \\
&= \frac{(\gamma_m+\gamma_M+1)(\gamma_m+\gamma_M)!}{2^{\gamma_m+\gamma_M+1}\gamma_M!\gamma_m!} \left(\frac{2n}{\gamma_m+\gamma_M+1} + 1 \right) \\
&\quad \times \prod_{k=\gamma_m+1}^{\gamma_m+\gamma_M} \left(\frac{n}{k} + 1 \right) \prod_{k=1}^{\gamma_M} \left(\frac{n}{k} + 1 \right) \\
&\leq \frac{(\gamma_m+\gamma_M+1)(\gamma_m+\gamma_M)!}{2^{\gamma_m+\gamma_M+1}\gamma_M!\gamma_m!} (n+1)^{2\gamma_M+1} \\
&= \frac{(\gamma_m+\gamma_M+1)\,(\gamma_m+\gamma_M)!}{2^{\gamma_m+\gamma_M+1}\gamma_M!\gamma_m!} \sum_{l=0}^{2\gamma_M+1} \binom{2\gamma_M+1}{l} n^l \\
&= W(\alpha,\beta,1)^{-1} p(n).
\end{aligned} \tag{50}$$

In the last step of (50) we have introduced the polynomial $p(n) := \sum_{l=0}^{2\gamma_M+1} \binom{2\gamma_M+1}{l} n^l$ divided by the constant $W(\alpha,\beta,d)$ defined in (32) with $d=1$. The polynomial p has maximum degree $\eta = 2\gamma_M + 1$, and its coefficients satisfy the binomial condition (7) since $\binom{2\gamma_M+1}{l} \leq \binom{2\gamma_M+2}{l}$ for any $l = 0, \ldots, \eta$.

Any $u \in \mathbb{P}_\Lambda(D)$ can be expanded in series of tensorized Jacobi polynomials $(J_{\boldsymbol{\nu}}^{\alpha,\beta})_{\boldsymbol{\nu}\in\Lambda}$:

$$u(y) = \sum_{\boldsymbol{\nu}\in\Lambda} \beta_{\boldsymbol{\nu}} \prod_{q=1}^{d} J_{\boldsymbol{\nu}_q}^{\alpha,\beta}(y_q).$$

Proceeding as in the proof of Theorem 8, but using (50), we can apply Theorem 1 and obtain (49) in the case $\gamma_m + \gamma_M \geq 1$, with $W_\rho = W(\alpha,\beta,d)$. The case $\alpha = \beta = 0$ has been proven in Theorem 6, and is included in (49) as well. □

Remark 6 (*"Probabilistic" Orthonormalization Weight*)**.** In the weighted inner product (1) one can use an orthonormalization weight which integrates to one independently of the dimension d and of the shape parameters. Given any orthonormalization weight ρ and its integral W_ρ defined in (2), we define the "probabilistic" weighted L^2_ρ inner product as

$$\{\langle f_1, f_2 \rangle\}_{L^2_\rho(D)} := \int_D f_1(y) f_2(y) W_\rho^{-1} \rho(y) dy, \quad \forall f_1, f_2 \in L^2_\rho(D), \tag{51}$$

and the "probabilistic" weighted L^2_ρ norm as $\{\| \cdot \|\}_{L^2_\rho(D)} := \{\langle \cdot, \cdot \rangle\}^{1/2}_{L^2_\rho(D)}$. Of course it holds true that

$$\{\| f \|\}^2_{L^2_\rho(D)} = W_\rho^{-1} \| f \|^2_{L^2_\rho(D)}, \quad \forall f \in L^2_\rho(D).$$

Therefore we can rewrite the theses of Theorems 6–9 using the L^∞ norm and the "probabilistic" weighted L^2_ρ norm. The theses of Theorems 3–5 hold true also with the "probabilistic" weighted L^2_ρ norm, with the same constants of proportionality.

Equivalently, one might work directly with the "probabilistic" L^2_ρ-orthonormal Jacobi polynomials, which are orthonormal w.r.t. the inner product (51) with the orthonormalization weight $\rho = \rho_J^d$ defined in (28).

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