

Maximal Inequalities and Lebesgue's Differentiation Theorem for Best Approximant by Constant over Balls¹

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For $f \in L_p(\mathbb{R}^n)$, with $1 \leq p < \infty$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we denote by $T^\varepsilon(f)(x)$ the set of every best constant approximant to f in the ball $B(x, \varepsilon)$. In this paper we extend the operators T_p^ε to the space $L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, where L_0 is the set of every measurable functions finite almost everywhere. Moreover we consider the maximal operators associated to the operators T_p^ε and we prove maximal inequalities for them. As a consequence of these inequalities we obtain a generalization of Lebesgue's Differentiation Theorem. © 2001 Academic Press

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1. INTRODUCTION AND NOTATION

In this paper we consider a problem related to best local approximation. The notion may be stated as follows. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function in a normed linear space X with norm $\|\cdot\|$. Let V denote a subset of X . Let $B(x, \varepsilon)$ denote a net of sets containing x with diameters shrinking to 0 as $\varepsilon \rightarrow 0$. For each $\varepsilon > 0$ suppose that we have $f_\varepsilon \in V$ which minimizes $\|(f - g) \chi_{B(x, \varepsilon)}\|$ for $g \in V$, where $\chi_{B(x, \varepsilon)}$ is the characteristic function of $B(x, \varepsilon)$. If $f_\varepsilon \rightarrow f_x \in V$ then f_x is said to be the best local approximant of f at x . In [1] Chui, Diamond and Raphael proved that if f have $m + 1$ derivatives at x and the subspace $V \subset C^{m+1}(\mathbb{R}^n)$ is uniquely interpolating at x of order m then the best local approximant of f at x from V is the unique $f_x \in V$ whose derivatives up to order m match those of f at x .

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Clearly the condition $V \subset X$ can be substituted by $V\chi_{B(x, \varepsilon)} \subset X$ for every $\varepsilon > 0$. We observe that if $X = L_2(\mathbb{R}^n)$, V is the set of constant functions and $B(x, \varepsilon)$ is the ball of center in x and radius ε then

$$f_\varepsilon(x) = \frac{1}{m(B(x, \varepsilon))} \int_{B(x, \varepsilon)} f(t) dt.$$

As a consequence of the previous result we have that $f_\varepsilon(x) \rightarrow f(x)$ for $f \in C^1(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$. But it is well known that a more adequate version of this fact is given by the Lebesgue's Differentiation Theorem, which says that $f_\varepsilon(x) \rightarrow f(x)$ a.e. for every locally integrable function f over \mathbb{R}^n . This theorem is related to some inequalities satisfied by the Hardy–Littlewood maximal function, see [3].

In the present work we extend the maximal function of Hardy–Littlewood and the Lebesgue's Differentiation Theorem using best approximation by constants over balls in the $L_p(\mathbb{R}^n)$ spaces with $1 \leq p < \infty$.

In [2] Landers and Rogge have also considered problems related with maximal inequalities and almost everywhere convergence of best approximant. They studied these questions in L_p -spaces with $1 < p < \infty$. In particular they extend the operator of best approximation from L_p to L_{p-1} . In this paper we analyze the case $p = 1$.

Throughout this paper $B(x, \varepsilon)$ denotes the ball in \mathbb{R}^n with center in x and radius ε . For $1 \leq p < \infty$, $f \in L_p(\mathbb{R}^n)$ and $\varepsilon > 0$ we define $T_p^\varepsilon(f)(x)$ as the set of all constants a minimizing the expression

$$\int_{B(x, \varepsilon)} |f(t) - a|^p dt.$$

It is well known that $T_p^\varepsilon(f)(x) \neq \emptyset$ for every $f \in L_p(\mathbb{R}^n)$. Moreover if $1 < p < \infty$ then the set $T_p^\varepsilon(f)(x)$ has a unique element.

For our purposes we define the following maximal function over an open set $\Omega \subset \mathbb{R}^n$.

$$T_{\Omega, p}^*(f)(x) = \sup_{\varepsilon > 0} \{ |a| : a \in T_p^\varepsilon(f)(x) \text{ and } B(x, \varepsilon) \subset \Omega \}, \quad (1)$$

where $1 \leq p < \infty$.

2. MAXIMAL INEQUALITIES

For short we put L_p instead of $L_p(\mathbb{R}^n)$. Since for $f \in L_p + L_\infty$, $\int_{B(x, \varepsilon)} |f(x)|^p dx < \infty$ for any ball, we have that the operator T_p^ε is

defined for all functions in $L_p + L_\infty$. Moreover the following property holds:

(P1) If g is constant on $B(x, \varepsilon)$ then $T_p^\varepsilon(f+g)(x) = T_p^\varepsilon(f)(x) + g(x)$.

Next we shall see that T_p^ε admits a natural extension. For $p > 1$ we extend T_p^ε to $L_{p-1} + L_\infty$, so that it satisfies (P1). Notice that for $1 < p < 2$ the L_{p-1} space is not a normed space, however we still maintain the notation $\|f\|_p$ for $(\int |f(x)|^p dx)^{1/p}$.

For $f \in L_{p-1} + L_\infty$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we consider the following function

$$F(a) := \int_{B(x, \varepsilon)} |f(t) - a|^{p-1} \operatorname{sgn}(f(t) - a) dt, \quad a \in \mathbb{R}.$$

By the characterization theorem of best approximant we get $F(T^\varepsilon(f)(x)) = 0$ for $f \in L_p$.

Now we see that for every $f \in L_{p-1} + L_\infty$ there exists a unique $a \in \mathbb{R}$ such that $F(a) = 0$, so we define $T_p^\varepsilon(f)(x) = a$. In fact, as the function $\Phi(x) = |x|^{p-1} \operatorname{sgn}(x)$ is continuous and strictly increasing we have that the function F is continuous and strictly decreasing. Furthermore it satisfies that $\lim_{a \rightarrow +\infty} F(a) = -\infty$ and $\lim_{a \rightarrow -\infty} F(a) = +\infty$. Then there exists a unique number a such that $F(a) = 0$.

If $p = 1$ we extend the operator T_1^ε to the space L_0 of all measurable functions which are finite almost everywhere. In this case we consider the distribution function $\lambda(a) = m(\{t \in B(x, \varepsilon) : f(t) > a\})$. For every $f \in L_0$, $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we define

$$U^\varepsilon(f)(x) := \sup\{t : \lambda(t) \geq m(B(x, \varepsilon))/2\}$$

$$L^\varepsilon(f)(x) := \inf\{t : \lambda(t) \leq m(B(x, \varepsilon))/2\}.$$

For $f \in L_1$ it is well known, see [4, p. 199], that $T_1^\varepsilon(f)(x) = [L^\varepsilon(f)(x), U^\varepsilon(f)(x)]$. So that we extend the operator T_1^ε to the space L_0 by mean of this equality.

Clearly the extended operator T_p^ε satisfies (P1). Also we extend the maximal function $T_{\Omega, p}^*$ using (1). It is easy to show that the operator T_p^ε , with $1 \leq p < \infty$, satisfies:

(P2) for every $\alpha \in \mathbb{R}$, $T_p^\varepsilon(\alpha f) = \alpha T_p^\varepsilon(f)$

(P3) T_p^ε is monotone, i.e., for all f, g with $f \leq g$, a.e. we have $T_p^\varepsilon(f) \leq T_p^\varepsilon(g)$.

From (P2) and (P3) follows that

(P4) $|T_p^\varepsilon(f)| \leq T_p^\varepsilon(|f|)$.

Since the function $T_p^\varepsilon(f)$ is single valued for $p > 1$, we obtain from Lebesgue's Dominated Convergence Theorem that $T_p^\varepsilon(f)$ is continuous. Hence the maximal function $T_{p,\Omega}^*$ is lower semi-continuous, in particular is a measurable function. If $p = 1$ then $U^\varepsilon(f)$ ($L^\varepsilon(f)$) is upper (lower) semi-continuous. This affirmation follows easily from the fact that the function $g(x) := m(\{t \in B(x, \varepsilon) : f(t) > a\})$ is continuous for every a . So $U^\varepsilon(f)$ and $L^\varepsilon(f)$ are measurable functions. However we can not prove that the maximal function $T_1^*(f)$ is measurable. As a consequence we will use the outer measure m^* in some results. Now we prove maximal inequalities for the operator $T_{\Omega,p}^*$.

THEOREM 2.1. *Let f be a measurable function over Ω and let $1 \leq p < \infty$ then*

(a) *There exist constants A_{p-1} and k_{p-1} such that*

$$m^*(\{x \in \Omega : T_{\Omega,p}^*(f)(x) > \lambda\}) \leq \frac{A_{p-1}}{\lambda^{p-1}} \int_{\{|f| > k_{p-1}\lambda\}} |f(t)|^{p-1} dt$$

for every $f \in L_{p-1}$ and $\lambda > 0$.

(b) *Let $p > 1$. For $p-1 < p' \leq \infty$ there exists a constant $A_{p'}$ such that*

$$\|T_{\Omega,p}^*(f)\|_{p'} \leq A_{p'} \|f\|_{p'}.$$

Proof. Let $M_\lambda := \{x \in \Omega : T_{\Omega,p}^*(f)(x) > \lambda\}$. In order to prove part (a) we need to consider two cases.

Case 1. $1 < p < \infty$. In this case the set M_λ is measurable. For $x \in M_\lambda$ there is $\varepsilon = \varepsilon_x > 0$ such that $|T_p^\varepsilon(f)(x)| > \lambda$ and $B(x, \varepsilon) \subset \Omega$. Now we see that there exists a constant C such that

$$m(B(x, \varepsilon)) \leq \frac{C}{\lambda^{p-1}} \int_{B(x, \varepsilon)} |f(t)|^{p-1} dt. \quad (2)$$

In fact, it is easy to show that there are constants B_p and C_p with $B_p \leq 1 \leq C_p$ such that

$$B_p(a^{p-1} + b^{p-1}) \leq (a+b)^{p-1} \leq C_p(a^{p-1} + b^{p-1}), \quad a \geq 0, \quad b \geq 0. \quad (3)$$

For simplicity we put $N := B(x, \varepsilon) \cap \{f > T_p^\varepsilon(f)(x)\}$ and $L := B(x, \varepsilon) \cap \{f \leq T_p^\varepsilon(f)(x)\}$. Then

$$\begin{aligned}
m(B(x, \varepsilon)) \lambda^{p-1} &\leq m(B(x, \varepsilon)) |T_p^\varepsilon(f)(x)|^{p-1} \\
&= \int_N |T_p^\varepsilon(f)(x)|^{p-1} dt + \int_L |T_p^\varepsilon(f)(x)|^{p-1} dt \\
&\leq C_p \int_N |T_p^\varepsilon(f)(x)|^{p-1} dt + C_p \int_L |f(t)|^{p-1} dt \\
&\quad + C_p \int_L |T_p^\varepsilon(f)(x) - f(t)|^{p-1} dt \\
&\leq C \int_{B(x, \varepsilon)} |f(t)|^{p-1} dt,
\end{aligned}$$

where $C = \frac{C_p}{B_p}$. The last inequality follows from the equality

$$\int_L |T_p^\varepsilon(f)(x) - f(t)|^{p-1} dt = \int_N |T_p^\varepsilon(f)(x) - f(t)|^{p-1} dt$$

and from (3).

Thus we have proved that the diameters of the balls $B(x, \varepsilon_x)$ are bounded. Now, by [3, Lemma 1.6], we can select from this family of balls a sequence $B(x_i, \varepsilon_{x_i})$ of balls which are mutually disjoint and such that

$$\sum_i m(B(x_i, \varepsilon_{x_i})) \geq Dm(M_\lambda),$$

for some constant $D > 0$. Therefore, from (2) we obtain

$$\begin{aligned}
m(M_\lambda) &\leq \frac{C}{D\lambda^{p-1}} \sum_i \int_{B(x_i, \varepsilon_{x_i})} |f(t)|^{p-1} dt \\
&\leq \frac{C}{D\lambda^{p-1}} \int_\Omega |f(t)|^{p-1} dt.
\end{aligned}$$

Now we define $f_1(x) = f(x)$ if $|f(x)| > \lambda/2$ and $f_1(x) = 0$ otherwise. Then $|f(x)| \leq |f_1(x)| + \lambda/2$. From (i), (iii) and (1) follows that $T_{\Omega, p}^*(f)(x) \leq T_{\Omega, p}^*(f_1)(x) + \lambda/2$. Therefore

$$m(M_\lambda) \leq \frac{2^{p-1}C}{D\lambda^{p-1}} \int_\Omega |f_1(x)| dx = \frac{A_{p-1}}{\lambda^{p-1}} \int_{\{|f| > k_{p-1}\lambda\}} |f(x)|^{p-1} dx,$$

where $A_{p-1} = 2^{p-1}C/D$ and $k_{p-1} = 1/2$. This prove the case 1.

Case 2. $p = 1$. If $m(\{t \in \Omega : |f(t)| > \lambda\}) = \infty$ then the inequality in (a) is trivial. So that we can suppose $m(\{t \in \Omega : |f(t)| > \lambda\}) < \infty$. If $x \in M_\lambda$ then there exist $\varepsilon = \varepsilon_x$ and $a \in T_1^e(f)(x)$ such that $|a| > \lambda$ and $B(x, \varepsilon) \subset \Omega$. We shall prove that

$$m(B(x, \varepsilon)) \leq 2m(\{t \in B(x, \varepsilon) : |f(t)| > \lambda\}). \quad (4)$$

In fact, if $a > 0$ then $\lambda < U^e(f)(x)$. Hence $m(\{t \in B(x, \varepsilon) : |f(t)| > \lambda\}) \geq m(\{t \in B(x, \varepsilon) : f(t) > \lambda\}) \geq m(B(x, \varepsilon))/2$. Suppose $a < 0$ and let $a < s < -\lambda$. We have that

$$\begin{aligned} m(\{t \in B(x, \varepsilon) : |f(t)| > \lambda\}) &\geq m(\{t \in B(x, \varepsilon) : f(t) < -\lambda\}) \\ &= m(B(x, \varepsilon)) - m(\{t \in B(x, \varepsilon) : f(t) \geq -\lambda\}) \\ &\geq m(B(x, \varepsilon)) - m(\{t \in B(x, \varepsilon) : f(t) > s\}) \\ &\geq m(B(x, \varepsilon))/2. \end{aligned}$$

The last inequality follows from $L^e(f)(x) \leq a$. Then (4) follows.

Since $m(\{t \in \Omega : |f(t)| > \lambda\}) < \infty$ we obtain from (4) that the diameters of the balls $B(x, \varepsilon_x)$ are bounded. Let $G := \bigcup_{x \in M_\lambda} B(x, \varepsilon)$. The set G is open. Again, an application of [3, Lemma 1.6] give us a countable subcollection of balls mutually disjoint and $D > 0$ such that

$$Dm^*(M_\lambda) \leq Dm(G) \leq \sum_i m(B(x_i, \varepsilon_i)) \leq 2m(\{t \in \Omega : |f(t)| > \lambda\}).$$

This conclude the proof of case two if we take $k_0 = 1$ and $A_0 = 2/D$.

Now we prove part (b). Since $|f| \leq \|f\|_\infty$, a.e. we have that (P3) and (P4) imply $T_p^e(f) \leq \|f\|_\infty$ a.e. Hence we obtain (b) for $p' = \infty$ with $A_\infty = 1$. Using a similar argument as given in [3, p. 7] and using the properties (P1) and (P3) we obtain (b) for the case $p - 1 < p'$. ■

3. LEBESGUE'S DIFFERENTIATION THEOREM

In the present section we shall prove a generalization of the “Lebesgue’s Differentiation Theorem” for the operators $T_p^e f$. First we need to prove the following theorem. By $\mathcal{M} = \mathcal{M}(\Omega)$ we denote the set of all measurable functions over Ω .

THEOREM 3.1. *Let $\{T_\varepsilon^e\}_{\varepsilon > 0}$ be a family of operators, not necessarily linear, where $T_\varepsilon^e: L_p(\Omega) \rightarrow \mathcal{M}$ for some $0 \leq p < \infty$. We consider the maximal*

function $T^*(f)(x) := \sup_{\varepsilon > 0} |T^\varepsilon(f)(x)|$. Suppose that T^* satisfies the following weak inequality,

$$m^*(\{x \in \Omega : T^*(f)(x) > \lambda\}) \leq \frac{A}{\lambda^p} \int_{\{|f| > k\lambda\}} |f(x)|^p dx, \quad (5)$$

for some constants $A, k > 0$. Moreover assume that there exists a set $\mathcal{D} \subset L_p(\Omega)$ with the following properties:

(A1) For every $\lambda > 0$, $f \in L_p(\Omega)$ and $g \in \mathcal{D}$

$$m^*(\{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(f - g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda\}) = 0.$$

(A2) If $f \in L_p(\Omega)$ then for every $\varepsilon > 0$ and $\lambda > 0$ there exists $g \in \mathcal{D}$ such that

$$\int_{\{|f - g| > \lambda\}} |(f - g)(t)|^p < \varepsilon.$$

(A3) $\lim_{\varepsilon \rightarrow 0} T^\varepsilon(g)(x) = g(x)$ a.e. for every $g \in \mathcal{D}$.

Then we have that $\lim_{\varepsilon \rightarrow 0} T^\varepsilon(f)(x) = f(x)$ a.e. for every $f \in L_p(\Omega)$.

Proof. It is enough to prove that $m^*(\Omega_0) = 0$ for every $\lambda > 0$, where

$$\Omega_0 := \{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(f)(x) - f(x)| > \lambda\}$$

For $g \in \mathcal{D}$ we have that $\Omega_0 \subset \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ where

$$\Omega_1 := \{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(f - g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda/4\}$$

$$\Omega_2 := \{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(g)(x) - g(x)| > \lambda/4\}$$

$$\Omega_3 := \{x \in \Omega : |g(x) - f(x)| > \lambda/4\}$$

$$\Omega_4 := \{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(f - g)(x)| > \lambda/4\}.$$

By (A1) and (A3) we have that $m^*(\Omega_1) = m^*(\Omega_2) = 0$. Let $\delta > 0$ be arbitrary. From (A2) we get $g \in \mathcal{D}$ with $\int_{\{|f - g| > \alpha\}} |(f - g)(x)|^p dx < \delta$,

where $\alpha := \min\{\lambda/4, k\lambda/4\}$. Therefore we see that $m(\Omega_3) < \delta/\alpha^p$. Moreover inequality (5) implies that

$$\begin{aligned} m^*(\Omega_4) &\leq m^*(\{x \in \Omega : T^*(f-g)(x) > \lambda/4\}) \\ &\leq \frac{4^p A}{\lambda^p} \int_{\{|f-g| > k\lambda/4\}} |(f-g)(x)|^p dx \\ &\leq \frac{4^p A}{\lambda^p} \delta. \end{aligned}$$

Since the number δ is arbitrary we deduce that $m^*(\Omega_0) = 0$. ■

COROLLARY 3.2. *For $1 < p < \infty$ and $f \in L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ we have that*

$$\lim_{\varepsilon \rightarrow 0} T_p^\varepsilon(f)(x) = f(x) \text{ a.e.}$$

If $p = 1$ then

$$\lim_{\varepsilon \rightarrow 0} U^\varepsilon(f)(x) = \lim_{\varepsilon \rightarrow 0} L^\varepsilon(f)(x) = f(x) \text{ a.e.}$$

for every $f \in L_0(\mathbb{R}^n)$.

Proof. It is sufficient to prove that the convergence a.e. holds in all open set Ω of finite measure. Let Ω be an open set with $m(\Omega) < \infty$. We observe that if $f \in L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ then $f \in L_{p-1}(\Omega)$. Hence from Theorem 2.1 we have that T_p^ε , U^ε and L^ε satisfy (5). Let \mathcal{D} the set of all simple functions which are continuous almost everywhere, i.e. $g \in \mathcal{D}$ if $g = \sum_{i=1}^n a_i \chi_{A_i}$ with $m(\partial A_i) = 0$, for every $i = 1, \dots, n$, where ∂A denote the boundary of the set A . As a consequence of property (P1) we have that for $T^\varepsilon = T_p^\varepsilon$, U^ε or L^ε and $g = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{D}$

$$\{x \in \Omega : \limsup_{\varepsilon \rightarrow 0} |T^\varepsilon(f-g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda\} \subset \bigcup_{i=1}^n \partial A_i.$$

so that we have (A1) for every $1 \leq p < \infty$. The property (A2) is a consequence of the well known properties of density of the simple functions (observe that (A2) is not true if $m(\Omega) = \infty$ and $p = 1$). Moreover for $g \in \mathcal{D}$ it is easy to prove that (A3) holds for the operators T_p^ε , U^ε and L^ε . ■

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