Relative Asymptotics for Orthogonal Matrix Polynomials with Convergent Recurrence Coefficients

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The asymptotic behavior of $\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}$ and $P_n(x, d\beta)\,P_n^{-1}(x, d\alpha)$ is studied. Here $(\gamma_n(.))_n$ are the leading coefficients of the orthonormal matrix polynomials $P_n(x, .)$ with respect to the matrix measures $d\beta$ and $d\alpha$ which are related by $d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k)$, where M_k are positive definite matrices, δ is the Dirac measure and c_k lies outside the support of $d\alpha$ for $k = 1, ..., N$. Finally, we deduce the asymptotic behavior of $P_n(c, d\beta)\,MP_n^*(c, d\alpha)$ when $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with M a positive definite matrix and c outside the support of $d\alpha$.   © 2001 Academic Press

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1. INTRODUCTION

Let $\mathscr{B}$ be a σ-algebra of subsets of the space Ω, and let $\rho = [\rho_{i, j}]$ be a $N \times N$ non-negative matrix valued function on $\mathscr{B}$ which is countably additive, i.e., each entry function $\rho_{i, j}$ is countably additive on $\mathscr{B}$. We shall refer to ρ as a non-negative valued measure on $(\Omega, \mathscr{B})$.

Let Φ be a matrix-valued function on Ω and ν be a non-negative real valued measure on $\mathscr{B}$. If Φ is $\mathscr{B}$-measurable, i.e., if each of its entries $\varphi_{i, j}$

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is $\mathscr{B}$-measurable, then we say that Φ is integrable and we define $\int_{\Omega} \Phi \, dv = [\int_{\Omega} \varphi_{i,j} \, dv]$.

From the fact that ρ is a non-negative valued matrix, whose entries are non-negative real valued measures, we have $0 \leqslant \rho \leqslant \tau\rho \, I$ ($A \geqslant B$ means that $A - B$ is a positive semi-definite matrix, and $\tau\rho$ is the trace measure of ρ). It follows that each $\rho_{i,j}$ is absolutely continuous with respect to the measure $\tau\rho$, and the trace derivative $\rho'_{\tau} = [d\rho_{i,j}/d\tau\rho]_{1 \leqslant i, j \leqslant N}$ is $\mathscr{B}$-measurable, satisfying $0 \leqslant \rho'_{\tau} \leqslant I$ $\tau\rho$-almost everywhere. Consequently from the Radon-Nikodym theorem we have $\rho(A) = \int_A \rho'_{\tau}(w) \, d\tau\rho$, for each $A \in \mathscr{B}$. Let Φ and Ψ be two $N \times N$ real matrix-valued functions on Ω. We say that (Φ, Ψ) is integrable with respect to the non-negative valued measure ρ if $\Phi\rho'_{\tau}\Psi^*$ is integrable with respect to the real measure $\tau\rho$ and we define $\int_{\omega} \Phi \, d\rho \Psi^* = \int_{\omega} \Phi\rho'_{\tau}\Psi^* \, d\tau\rho$.

We shall consider in the linear space of the polynomials $\mathbb{C}^{N \times N}[t]$ in the variable t with matrix coefficients in $\mathbb{C}^{N \times N}$, an inner product defined in the following way

$$\langle P, Q \rangle_{d\alpha} \overset{\text{def}}{=} \int_{\Omega} P(t) \, d\alpha(t) \, Q^*(t), \tag{1}$$

with $\Omega = \mathbb{R}$, and α a positive definite valued matrix measure whose entries are Borel real measures (see [15]).

A generalization of the Gram-Schmidt orthonormalization procedure for the set $\{I, xI, x^2I, ...\}$ with respect to (1) will give a set of orthonormal matrix polynomials $(P_n(., d\alpha))_n$ which satisfies

$$\langle P_n(., d\alpha), P_m(., d\alpha) \rangle_{d\alpha} = \int_{\mathbb{R}} P_n(x, d\alpha) \, \alpha'(x) \, P_m^*(x, d\alpha) \, d\tau\alpha = \delta_{n,m}I.$$

Notice that the set $(U_nP_n(., d\alpha))_n$ is also a set of orthonormal matrix polynomials for every sequence of unitary matrices $(U_n)_n$.

Orthogonal matrix polynomials have been studied in the second half of this century. Krein obtained some results about matrix moment problems from the point of view of operator theory [11]. Recently, during the 80's they have been connected to scattering theory by Geronimo [7], and an analog of Favard's theorem has been established for three-term recurrence matrix relation by A. I. Aptekarev and E. M. Nikishin [1]. Some algebraic results and results concerning the zeros were found by D. Zhani [19]. More recently some results concerning zeros and quadrature formulae have been studied by A. Sinap and W. Van Assche [17] and finally some results concerning zeros, quadrature formulae, asymptotic behavior of orthogonal polynomials have been obtained by A. Durán and coworkers [3, 5, 6].

Matrix orthogonal polynomials appear in a natural way when we consider different kinds of non-standard inner products.

In [12] it is shown that a family of such orthogonal polynomials for matrix measures supported on the real line can be obtained from a sequence of polynomials orthogonal with respect to a scalar measure supported on an harmonic algebraic curve.

In [13] the same problem is studied when we consider polynomials orthogonal with respect to a scalar measure supported on a lemniscate. There, a class of a matrix orthogonal polynomials on the unit circle is obtained.

In [4], Sobolev-type polynomials are analyzed from the perspective of a decomposition of them. In such a way one can associate to them a matrix measure plus some matrix mass points. Again, matrix orthogonal polynomials on the real line appear. Thus, the knowledge of properties of matrix orthogonal polynomials, both analytic and algebraic, is a basic tool in order to obtain results for the non-standard scalar cases pointed out above.

In [9] Gonchar studies Padé approximation to Markov functions to which a rational function is added, which is the same as adding mass points to a given measure.

The aim of our work is to extend to the matrix case some asymptotic results obtained by P. Nevai [14, Lemma 16, p. 132] when some mass points are added to a measure such that the corresponding Jacobi matrix is a compact perturbation of the infinite tridiagonal matrix

$$\begin{pmatrix} b & a & & \\ a & b & a & \\ & a & b & a \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This kind of asymptotic results has been obtained, separately, in [9] in the framework of rational approximation. In fact, the approximation to some classes of meromorphic functions ("Markov functions" with a perturbation by a rational function with prescribed poles) is considered, and the relative asymptotics for the denominators of the corresponding Padé approximants is obtained (see [9]).

In this paper we shall consider two matrix measures $d\alpha$ and $d\beta$ such that $d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k)$, where M_k is a positive definite matrix, δ is the Dirac matrix measure and c_k lies outside the support of $d\alpha$ for $k = 1, ..., N$.

Let $(P_n(x, .) = \gamma_n(.) x^n + b_n(.) x^{n-1} + \textit{lower degree terms})_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$ and $d\beta$, respectively.

In Section 2 we will introduce orthogonal matrix polynomials on the real line and discuss some properties which we will need in the next sections. In Section 3 we deduce the behavior of $\gamma_n(d\beta)\, \gamma_n(d\alpha)^{-1}$. In Section 4 we give the relative asymptotics for some nth orthonormal matrix polynomials with respect to the matrix measure $d\beta$ and the nth orthonormal matrix polynomials with respect to the matrix measure $d\alpha$.

2. ORTHOGONAL MATRIX POLYNOMIALS ON THE REAL LINE

For each real matrix measure $d\alpha$, we introduce

$$\langle P, Q \rangle_{d\alpha} \overset{\text{def}}{=} \int_{\mathbb{R}} P(x)\, d\alpha Q^*(x). \tag{2}$$

This bilinear form satisfies

1. $\langle P, Q \rangle_{d\alpha} = \langle Q, P \rangle_{d\alpha}^*$.
2. $\langle xP, Q \rangle_{d\alpha} = \langle P, xQ \rangle_{d\alpha}$.

3. $\langle P, P \rangle_{d\alpha}$ is a non-negative definite matrix. If $\det P \neq 0$, it is a positive definite matrix.

In the following, we consider a real matrix measure $d\alpha$ for which $\langle P, P \rangle_{d\alpha} \neq 0$ for every matrix polynomial P with non-singular leading coefficient.

As in the scalar case, the orthonormal matrix polynomials $(P_n(x, d\alpha))_n$ with respect to the matrix measure $d\alpha$ are orthogonal to every matrix polynomial of degree less than n and they satisfy a three-term recurrence relation

$$xP_n(x, d\alpha) = D_{n+1}(d\alpha)\, P_{n+1}(x, d\alpha)$$
$$+ E_n(d\alpha)\, P_n(x, d\alpha) + D_n^*(d\alpha)\, P_{n-1}(x, d\alpha), \tag{3}$$

where $P_{-1} = 0$, $P_0 = \int d\alpha = I$, $D_n(d\alpha) = \gamma_{n-1}(d\alpha)\, \gamma_n^{-1}(d\alpha)$ is a positive definite matrix ($\gamma_n(d\alpha)$ is the leading coefficient of $P_n(x, d\alpha)$) and $E_n(d\alpha)$ is an hermitian matrix. The corresponding matrix polynomials of the second kind are defined by

$$Q_n(x, d\alpha) \overset{\text{def}}{=} \int \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t). \tag{4}$$

These matrix polynomials satisfy the following Liouville–Ostrogradski formula

$$Q_n(x, d\alpha)\, P_{n-1}^*(x, d\alpha) - P_n(x, d\alpha)\, Q_{n-1}^*(x, d\alpha) = D_n^{-1}(d\alpha). \tag{5}$$

Furthermore, defining

$$K_{n+1}(x, y, d\alpha) \overset{\text{def}}{=} \sum_{j=0}^{n} P_j^*(y, d\alpha) \, P_j(x, d\alpha),$$

we get the Christoffel-Darboux formula

$$(x - y) \, K_{n+1}(x, y, d\alpha) = P_n^*(y, d\alpha) \, D_{n+1}(d\alpha) \, P_{n+1}(x, d\alpha)$$
$$- P_{n+1}^*(y, d\alpha) \, D_{n+1}^*(d\alpha) \, P_n(x, d\alpha). \qquad (6)$$

By means of a straightforward computation we get the following equation

$$K_{n+1}(x, x, d\alpha) = P_{n+1}^*(x, d\alpha)' \, D_{n+1}^*(d\alpha) \, P_n(x, d\alpha)$$
$$- P_n^*(x, d\alpha)' \, D_{n+1}(d\alpha) \, P_{n+1}(x, d\alpha). \qquad (7)$$

The matrix $K_n(x, y, d\alpha)$ is called the nth reproducing kernel because of the following property. For every matrix polynomial $\Pi_m(x)$ of degree $m \leqslant n - 1$, we have

$$\langle \Pi_m, K_n(\cdot, y, d\alpha) \rangle_{d\alpha} = \int_{\mathbb{R}} \Pi_m(x) \, d\alpha K_n^*(x, y, d\alpha) = \Pi_m(y). \qquad (8)$$

In the next section, we will use the following results (see [6]) concerning ratio asymptotic properties for orthogonal matrix polynomials.

First we start with some definitions. Let Δ_n be the set of zeros of the matrix polynomial P_n, i.e.,

$$\Delta_n(d\alpha) \overset{\text{def}}{=} \{x_{n,k}; k = 1, ..., N: \det P_n(x_{n,k}, d\alpha) = 0\},$$

and setting

$$\Gamma = \bigcap_{N > 0} M_N \qquad \text{where} \qquad M_N = \overline{\bigcup_{n \geqslant N} \Delta_n(d\alpha)},$$

we have $supp(d\alpha) \subset \Gamma$. Here we define the support of $d\alpha$ as $supp(d\alpha) = supp(\tau d\alpha) = supp(d\alpha_{1,1} + d\alpha_{2,2} + \cdots + d\alpha_{N,N})$.

We recall that if H is a positive definite (resp. positive semidefinite) matrix then there is a unique square root $H_0 = H^{1/2}$ of H defined as follows. Writing $H = UDU^*$, where $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and $(\lambda_i)_{i=1,n}$ are the positive eigenvalues of H then $H_0 = UD_0U^*$, where $D_0 = diag(+\sqrt{\lambda_1}, +\sqrt{\lambda_2}, ..., +\sqrt{\lambda_n})$.

PROPOSITION 2.1 [6]. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials satisfying the three-term recurrence relation* (3). *Assume*

that $\lim_{n \to \infty} D_n(d\alpha) = D$ *and* $\lim_{n \to \infty} E_n(d\alpha) = E$, *with* D *a non-singular matrix then*

$$\lim_{n \to \infty} P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha) D_n^{-1}(d\alpha) = \int \frac{dW_{D, E}(t)}{z - t}; \quad z \in \mathbb{C} \backslash \Gamma, \quad (9)$$

where $W_{D, E}(t)$ *is the matrix weight for the Chebyshev matrix polynomials of the second kind. Moreover the convergence is uniform on compact subsets of* $\mathbb{C} \backslash \Gamma$.

The analytic function on the right-hand side of (9) *has the following explicit form*

- *If* D *is an hermitian positive definite matrix, then*

$$\int \frac{dW_{D, E}(t)}{x - t} = \frac{1}{2} D^{-1}(xI - E) D^{-1}$$

$$- \frac{1}{2} D^{-1/2} [\sqrt{D^{-1/2}(E - xI) D^{-1}(E - xI) D^{-1/2} - 4I}] D^{-1/2},$$

$$(10)$$

where $x \notin supp(dW_{D, E})$.

Here, $supp(dW_{D, E}) = \{x \in \mathbb{R}; D^{-1/2}(E - xI) D^{-1/2}$ *has at least one eigenvalue in* $[-2, 2]\}$.

- *If* D *is an hermitian matrix, then*

$$\int \frac{dW_{D, E}(t)}{x - t} = \frac{1}{2} D^{-1}(xI - E) D^{-1} - \frac{1}{2} D^{-1}(E - xI)^{1/2}$$

$$\times [\sqrt{I - 4(E - xI)^{-1/2} D(E - xI)^{-1} D(E - xI)^{-1/2}}]$$

$$\times (E - xI)^{1/2} D^{-1}, \quad (11)$$

where $x \notin supp(dW_{D, E})$.

Here, $supp(dW_{D, E}) = \{x \in \mathbb{R} \backslash [b_1, b_N]; (E - xI)^{1/2} D^{-1}(E - xI)^{1/2}$ *has at least one eigenvalue in* $[-2, 2]\}$, *and* $b_1 \leqslant b_2 \leqslant ... \leqslant b_N$ *are the eigenvalues of* E.

LEMMA 2.1. *Let* $(P_n(x, d\alpha))_n$ *be a sequence of orthonormal matrix polynomials with convergent recurrence coefficients which appear in* (3). *There exists a positive constant* $a > 0$ *such that if* $x_{n, k}$ *is a zero of* P_n *then* $|x_{n, k}| \leqslant a$, *and* $supp(d\alpha)$ *is contained in* $[-a, a]$.

Proof. Let

$$J \overset{\text{def}}{=} \begin{pmatrix} E_0 & D_1 & 0 & \cdots & \cdots \\ D_1^* & E_1 & D_1 & 0 & \cdots \\ 0 & D_2^* & E_2 & D_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

be the Jacobi matrix with convergent matrix parameters $\{E_i, D_{i+1}\}_{i=0}^{\infty}$. It is known that the zeros $(x_{n,k})_{k=1}^{nN}$ of P_n are eigenvalues of J_{nN} (J_{nN} is the truncated Jacobi matrix of dimension nN) (see [5, 17]). Using the Gershgorin disks for the location of eigenvalues, there is a positive number a such that $|x_{n,k}| \leq a$, and therefore $supp(d\alpha) \subset \Gamma \subset [-a, a]$. ∎

In the following, we denote by $\hat{\Gamma}$ the smallest closed interval which contains the support of $d\alpha$.

PROPOSITION 2.2. *The Markov matrix function $\int (dW_{D,E}(t))/(z-t)$ is positive or negative definite for each $z \in \mathbb{R} \backslash \hat{\Gamma}$. It is differentiable and its derivative $-\int (dW_{D,E}(t))/((z-t)^2)$ is negative definite when z is outside $supp(dW_{D,E})$.*

Proof. We consider

$$A = \{x \in \mathbb{R} : x > c; \, \forall c \in \hat{\Gamma}\}$$

and

$$B = \{x \in \mathbb{R} : x < c; \, \forall c \in \hat{\Gamma}\}.$$

The function $\frac{1}{z-t}$ is positive (resp. negative) when $z \in A$ (resp. $z \in B$) and $t \in supp(dW_{D,E}) = supp(d\alpha) \subset \hat{\Gamma}$. Since $dW_{D,E}$ is a positive definite matrix measure, the trace derivative $W'_{D,E}(t) = [dW_{D,E}/d\tau W_{D,E}]$ is positive definite ($d\tau W_{D,E}$ is the positive real trace measure of $W_{D,E}$), and for any real vector $u \in \mathbb{C}^N$, we have

$$u \left(\int \frac{dW_{D,E}(t)}{z-t} \right) u^* = u \left(\int \frac{W'_{D,E}(t)}{z-t} \, d\tau W_{D,E} \right) u^*$$

$$= \int \frac{(u W'_{D,E}(t) u^*)}{z-t} \, d\tau W_{D,E}.$$

Consequently the Markov matrix function is positive (resp. negative) definite in $z \in A$ (resp. $z \in B$).

In a similar way, we can show that the derivative of the Markov matrix function $\frac{d}{dz} \int (dW_{D,E}(t))/(z-t) = -\int (dW_{D,E}(t))/((z-t)^2)$ is negative definite when $z \notin \operatorname{supp}(dW_{D,E})$. ∎

In the next sections, we will need the asymptotics of the derivative of $(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha))_n$ where z belongs to a compact subset of $\mathbb{C}\backslash\Gamma$. To do that we use the well known theorem of Weierstrass concerning the differentiation of a sequence of holomorphic functions. We recall that $\{W_n\}_n$ is locally uniformly convergent in the open set G to a function W ($W_n(z) \looparrowright W(z)$) if $W_n(z)$ is uniformly convergent to $W(z)$ in every closed set contained in G.

THEOREM 2.1.   *If a sequence $\{W_n\}_n$ of functions holomorphic in an open set G is locally uniformly convergent in G to a function W, then the function W is also holomorphic in G, and if $\infty \notin G$, then*

$$W_n^{(k)}(z) \looparrowright W^{(k)}(z)$$

in the set G for $k = 1, 2, \ldots$.

Proof.   See [16]. ∎

COROLLARY 2.1.   *Under the hypothesis of Proposition 2.1 we have*

$$(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha))^{(k)} D_n^{-1} \looparrowright \left(\int \frac{dW_{D,E}(t)}{z-t} \right)^{(k)} \tag{12}$$

on compact subsets of $\mathbb{C}\backslash\Gamma$, for $k = 1, 2, \ldots$.

The local uniform convergence in (12) means that every entry of the left hand-side of (12) is locally uniformly convergent to its corresponding entry in the right hand-side of (12).

Proof.   Since $(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha))$ is an holomorphic matrix function in $\mathbb{C}\backslash\Gamma$, each one of its entries is an holomorphic function in $\mathbb{C}\backslash\Gamma$, and from the hypothesis we have that $(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha) D_n)_n$ converges uniformly on compact subsets of $\mathbb{C}\backslash\Gamma$, hence for each one of its entries. Therefore $(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha))_n$ is locally uniformly convergent in compact subsets of $\mathbb{C}\backslash\Gamma$ since each closed subset of a compact set is compact.

Applying Theorem 2.1 we get

$$(P_{n-1}(z, d\alpha) P_n^{-1}(z, d\alpha))^{(k)} D_n^{-1} \looparrowright \left(\int \frac{dW_{D,E}(t)}{z-t} \right)^{(k)}. \quad \blacksquare$$

3. THE RATIO ASYMPTOTICS FOR LEADING COEFFICIENTS

Let $d\alpha$ and $d\beta$ be two matrices of measures, such that $d\beta(u) = d\alpha(u) + M\delta(u - c)$, $c \in \mathbb{R} \backslash \hat{\Gamma}$, where M is a positive definite matrix. Let $(P_n(x,.) = \gamma_n(.) x^n + b_n(.) x^{n-1} + lower\ degree\ terms)_n$, be a sequence of orthonormal matrix polynomials with respect to the matrix measures $d\alpha$ and $d\beta$. We assume that

$$\lim_{n \to \infty} D_n(d\alpha) = D \qquad \text{and} \qquad \lim_{n \to \infty} E_n(d\alpha) = E, \qquad (13)$$

where $D_n(d\alpha)$ and $E_n(d\alpha)$ are the matrix coefficients in the recurrence formula (3) and D is a non-singular matrix.

In this section, we study the asymptotic behavior of the ratio of the leading coefficient of some nth orthonormal matrix polynomials associated to $d\beta$ and the leading coefficient of the nth orthonormal matrix polynomials associated $d\alpha$. We start with the following lemma which contains some formulas relating the sequences of orthonormal polynomials $(P_n(x, d\alpha))_n$ and $(P_n(x, d\beta))_n$.

LEMMA 3.1. *Let $d\alpha$ and $d\beta$ be two matrix measures, and M be a positive definite matrix such that $d\beta(u) = d\alpha(u) + M\delta(u - c)$, where c is a real number, Then*

$$P_n(x, d\beta) = \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \times [\, P_n(x, d\alpha)$$
$$- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} MK_{n+1}^*(c, x, d\alpha)]. \qquad (14)$$

Proof. Using the reproducing property (8) for the kernel polynomials, we have

$$P_n(x, d\beta) = \int P_n(u, d\beta)\, d\alpha(u)\, K_{n+1}^*(u, x, d\alpha)$$

$$= \int P_n(u, d\beta)\, d\beta(u)\, K_{n+1}^*(u, x, d\alpha)$$

$$- \int P_n(u, d\beta)\, MK_{n+1}^*(u, x, d\alpha)\, d\delta(u - c)$$

$$= \sum_{j=0}^{n} \left[\int P_n(u, d\beta)\, d\beta(u)\, P_j^*(u, d\alpha) \right] P_j(x, d\alpha)$$

$$- P_n(c, d\beta)\, MK_{n+1}^*(c, x, d\alpha)$$

$$= \int P_n(u, d\beta) \, d\beta(u) \, P_n^*(u, d\alpha) . P_n(x, d\alpha)$$

$$- P_n(c, d\beta) \, MK_{n+1}^*(c, x, d\alpha)$$

$$= \gamma_n(d\beta)^{-*} \, \gamma_n(d\alpha)^* \, P_n(x, d\alpha) - P_n(c, d\beta) \, MK_{n+1}^*(c, x, d\alpha).$$

If $x = c$, then

$$P_n(c, d\beta)\{I + MK_{n+1}(c, c, d\alpha)\} = \gamma_n(d\beta)^{-*} \, \gamma_n(d\alpha)^* P_n(c, d\alpha).$$

Since M and $K_{n+1}(c, c, d\alpha)$ are positive definite, we have

$$I + MK_{n+1}(c, c, d\alpha) = (K_{n+1}^{-1}(c, c, d\alpha) + M) \, K_{n+1}(c, c, d\alpha)$$

is non-singular because $I + MK_{n+1}(c, c, d\alpha)$ is the product of two positive definite matrices. Then

$$P_n(c, d\beta) = \gamma_n(d\beta)^{-*} \, \gamma_n(d\alpha)^* \, P_n(c, d\alpha)\{I + MK_{n+1}(c, c, d\alpha)\}^{-1}.$$

Finally

$$P_n(x, d\beta) = \gamma_n(d\beta)^{-*} \, \gamma_n(d\alpha)^* \, P_n(x, d\alpha) - \gamma_n(d\beta)^{-*} \, \gamma_n(d\alpha)^*$$

$$\times P_n(c, d\alpha)\{I + MK_{n+1}(c, c, d\alpha)\}^{-1} MK_{n+1}^*(c, x, d\alpha).$$

This means that

$$P_n(x, d\beta) = \gamma_n^{-*}(d\beta) \, \gamma_n^*(d\alpha) \times [P_n(x, d\alpha)$$

$$- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} MK_{n+1}^*(c, x, d\alpha)]. \quad \blacksquare$$

THEOREM 3.1. *Let $\gamma_n(d\beta)$ and $\gamma_n(d\alpha)$ be the leading coefficients of the nth orthonormal matrix polynomials associated to the matrix measures $d\beta$ and $d\alpha$ related by $d\beta(u) = d\alpha(u) + M\delta(u - c)$, where $c \in \mathbb{R}\backslash\hat{\Gamma}$ and M is a positive definite matrix. Assume that*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \qquad and \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where $D_n(d\alpha)$ and $E_n(d\alpha)$ are the matrix coefficients in the recurrence relation (3), *and D is a non-singular matrix. Then*

$$\lim_{n \to \infty} \, [\gamma_n(d\beta) \, \gamma_n(d\alpha)^{-1}]^* \, [\gamma_n(d\beta) \, \gamma_n(d\alpha)^{-1}]$$

$$= I + \left(\int \frac{dW_{D, E}(t)}{c - t} \right)\left\{ \frac{d}{dx} \left(\int \frac{dW_{D, E}(t)}{x - t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{D, E}(t)}{c - t} \right). \quad (15)$$

To prove this, we start with the following lemma.

LEMMA 3.2. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$. Let $D_n(d\alpha)$ and $E_n(d\alpha)$ be the matrix coefficients which appear in the recurrence relation* (3), *satisfying*

$$\lim_{n \to \infty} D_n(d\alpha) = D \qquad and \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where D is non-singular, then

$$\lim_{n \to \infty} P_n^{-1}(x, d\alpha) = 0, \tag{16}$$

for $x \in \mathbb{R} \backslash \hat{\Gamma}$.

Proof. Notice that from (4), and using (5), we have

$$\begin{aligned}
&P_n^{-1}(x, d\alpha)\, D_n^{-1}(d\alpha)\, P_{n-1}^{-*}(x, d\alpha) \\
&\quad = P_n^{-1}(x, d\alpha)(Q_n(x, d\alpha)\, P_{n-1}^*(x, d\alpha) \\
&\qquad - P_n(x, d\alpha)\, Q_{n-1}^*(x, d\alpha))\, P_{n-1}^{-*}(x, d\alpha) \\
&\quad = P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha) - (P_{n-1}^{-1}(x, d\alpha)\, Q_{n-1}(x, d\alpha))^*.
\end{aligned}$$

But from (4), we have

$$\begin{aligned}
&P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha) \\
&\quad = \int P_n^{-1}(x, d\alpha)\, \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t) \\
&\quad = \int P_n^{-1}(x, d\alpha)\, \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t) \\
&\qquad \times (P_n^*(x, d\alpha) - P_n^*(t, d\alpha))\, P_n^{-*}(x, d\alpha) \\
&\quad = \int d\alpha(t)\, \frac{P_n^*(x, d\alpha) - P_n^*(t, d\alpha)}{x - t}\, P_n^{-*}(x, d\alpha) \\
&\quad = (P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha))^*.
\end{aligned}$$

Since $(P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha))_n$ is a convergent sequence for $x \in \mathbb{C} \backslash \Gamma$ (see [3]), then we have

$$\lim_{n \to \infty} P_n^{-1}(x, d\alpha)\, D_n^{-1}(d\alpha)\, P_{n-1}^{-*}(x, d\alpha) = 0. \tag{17}$$

Let $(\varphi(n))_n$ be an increasing sequence of integer numbers such that the limit

$$L(x, d\alpha) = \lim_{n \to \infty} P_{\varphi(n)}^{-1}(x, d\alpha)$$

exists or is ∞, ($L(x, d\alpha) = \infty$ means that at least one of its entries is ∞). Then

$$P_{\varphi(n)}^{-1}(x, d\alpha)\, D_{\varphi(n)}^{-1}(d\alpha)\, P_{\varphi(n)-1}^{-*}(x, d\alpha)$$

$$= P_{\varphi(n)}^{-1}(x, d\alpha)\, D_{\varphi(n)}^{-1}(d\alpha)\, P_{\varphi(n)-1}^{-*}(x, d\alpha)\, P_{\varphi(n)}^{*}(x, d\alpha)\, P_{\varphi(n)}^{-*}(x, d\alpha)$$

$$= P_{\varphi(n)}^{-1}(x, d\alpha)\, D_{\varphi(n)}^{-1}(d\alpha).(P_{\varphi(n)}(x, d\alpha)\, P_{\varphi(n)-1}^{-1}(x, d\alpha))^*\, P_{\varphi(n)}^{-*}(x, d\alpha).$$

$$(18)$$

But from (9), we get

$$\lim_{n \to \infty} (P_{\varphi(n)}(x, d\alpha)\, P_{\varphi(n)-1}^{-1}(x, d\alpha))^* = \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-*} D^{-*}.$$

Using (17) and (18), we have

$$0 = \lim_{n \to \infty} P_{\varphi(n)}^{-1}(x, d\alpha)\, D_{\varphi(n)}^{-1}(d\alpha)\, P_{\varphi(n)-1}^{-*}(x, d\alpha)$$

$$= L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-*} D^{-*} L(x, d\alpha)^*$$

$$= (L(x, d\alpha)\, D^{-1}) \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-*} (L(x, d\alpha)\, D^{-1})^*$$

$$= (L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-1})$$

$$\times \int \frac{dW_{D,E}(t)}{x-t} \times \left(L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-1} \right)^*.$$

Since $x \in \mathbb{R} \setminus \hat{\Gamma}$ and $supp(dW_{D,E}) = supp(d\alpha) \subset \hat{\Gamma}$ we deduce from Proposition 2.2, that the Markov matrix function $\int (dW_{D,E}(t))/(x-t)$ is positive or negative definite. Hence $L(x, d\alpha)\, D^{-1}(\int (dW_{D,E}(t))/(x-t))^{-1} = 0$, and thus

$$L(x, d\alpha) = 0.$$

Hence $P_n^{-1}(x, d\alpha)$ has no subsequence that converges to a number (or ∞) other than 0. Hence (16) holds.

Proof (of Theorem 3.1).   We proceed in several steps

Step 3.1.1.   Writing $\Phi_n(c) = [\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}]^*\,[\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}]$, we have

$$\Phi_n(c) = I - [P_n^{-*}(c, d\alpha)\,M^{-1}P_n^{-1}(c, d\alpha)$$
$$+ P_n^{-*}(c, d\alpha)\,K_{n+1}(c, c, d\alpha)\,P_n^{-1}(c, d\alpha)]^{-1},$$
$$\sqrt{N} > \|\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}\|_E \tag{19}$$

where $\|.\|_E$ is the Frobenius norm.

Proof.   If we use Lemma 3.1, we have

$$\int P_n(x, d\beta)\,d\alpha(x)\,P_n^*(x, d\alpha)$$
$$= \gamma_n^{-*}(d\beta)\,\gamma_n^*(d\alpha)\Big\{I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\,M$$
$$\times \int K_{n+1}^*(c, x, d\alpha)\,d\alpha(x)\,P_n^*(x, d\alpha)\Big\}.$$

This means that

$$\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha) = \gamma_n^{-*}(d\beta)\,\gamma_n^*(d\alpha)$$
$$\{I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\,MP_n^*(c, d\alpha)\},$$

and so

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]$$
$$= I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\,MP_n^*(c, d\alpha).$$

Hence,

$$I - \Phi_n(c)$$
$$= P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\,MP_n^*(c, d\alpha)$$
$$= [P_n^{-1}(c, d\alpha)]^{-1}\,(I + MK_{n+1}(c, c, d\alpha))^{-1}\,[P_n^{-*}(c, d\alpha)\,M^{-1}]^{-1}$$
$$= [P_n^{-1}(c, d\alpha) + MK_{n+1}(c, c, d\alpha)\,P_n^{-1}(c, d\alpha)]^{-1}\,[P_n^{-*}(c, d\alpha)\,M^{-1}]^{-1}$$
$$= [P_n^{-*}(c, d\alpha)\,M^{-1}P_n^{-1}(c, d\alpha) + P_n^{-*}(c, d\alpha)\,K_{n+1}(c, c, d\alpha)\,P_n^{-1}(c, d\alpha)]^{-1}.$$

Now M and $K_n(c, c, d\alpha)$ are positive definite, hence $I - \Phi_n(c)$ is also positive definite and thus $\Phi_n(c) < I$.

Let $\|.\|_E$ and $\|.\|_s$ be respectively the Frobenius and the spectral norm defined by

$$\|A\|_E^2 = \sum_{i, j=1}^{N} |a_{i, j}|^2 \qquad \text{and} \qquad \|A\|_s^2 = \mu_{(A^* A)},$$

where μ_L is the spectral radius of L ($\mu_L = \max_{1 \leqslant j \leqslant N} |\lambda_j|$; λ_j is an eigen-value of L). These matrix norms satisfy

- $\|A\|_E^2 = tr(A^* A)$.
- If A is hermitian, then $\|A\|_s = \mu_A$.

Finally,

$$\|\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)\|_E^2 = tr(\Phi_n(c)) \leqslant N \mu_{\Phi_n(c)} = N \, \|\Phi_n(c)\|_s < N.$$

Hence (19) holds. ∎

Step 3.1.2.

$$\lim_{n \to \infty} P_n^{-*}(c, d\alpha) \, K_{n+1}(c, c, d\alpha) \, P_n^{-1}(c, d\alpha)$$
$$= -\left(\int \frac{dW_{D, E}(t)}{c-t} \right)^{-1} \left(\int \frac{dW_{D, E}(t)}{c-t} \right)' \left(\int \frac{dW_{D, E}(t)}{c-t} \right)^{-1}. \quad (20)$$

Proof. If we put $P_n^{-*}(c, d\alpha) \, K_{n+1}(c, c, d\alpha) \, P_n^{-1}(c, d\alpha) = \Upsilon_n(c)$, taking into account (7) and

$$P_n(x) \, P_n^{-1}(x) = I \Rightarrow (P_n(x))' \, P_n^{-1}(x) = -P_n(x)(P_n^{-1}(x))',$$

then we have

$$\begin{aligned}
\Upsilon_n(c) &= P_n^{-*}(c, d\alpha)[P_n^*(c, d\alpha) \, D_{n+1}(P_{n+1}(c, d\alpha))' \\
&\quad - P_{n+1}^*(c, d\alpha) \, D_{n+1}^*(P_n(c, d\alpha))'] \, P_n^{-1}(c, d\alpha) \\
&= D_{n+1}(P_{n+1}(c, d\alpha))' \, P_n^{-1}(c, d\alpha) \\
&\quad - P_n^{-*}(c, d\alpha) \, P_{n+1}^*(c, d\alpha) \, D_{n+1}^*(P_n(c, d\alpha))' \, P_n^{-1}(c, d\alpha) \\
&= D_{n+1}(P_{n+1}(c, d\alpha))' \, P_n^{-1}(c, d\alpha) \\
&\quad + P_n^{-*}(c, d\alpha) \, P_{n+1}^*(c, d\alpha) \, D_{n+1}^* \, P_n(c, d\alpha)(P_n^{-1}(c, d\alpha))' \\
&= D_{n+1}(P_{n+1}(c, d\alpha))' \, P_n^{-1}(c, d\alpha) \\
&\quad + P_n^{-*}(c, d\alpha) \, P_n^*(c, d\alpha) \, D_{n+1} \, P_{n+1}(c, d\alpha)(P_n^{-1}(c, d\alpha))'
\end{aligned}$$

$$= D_{n+1}((P_{n+1}(c, d\alpha))' \, P_n^{-1}(c, d\alpha) + P_{n+1}(c, d\alpha)(P_n^{-1}(c, d\alpha))')$$

$$= D_{n+1}(P_{n+1}(c, d\alpha) \, P_n^{-1}(c, d\alpha))'$$

$$= D_{n+1}[(P_n(c, d\alpha) \, P_{n+1}^{-1}(c, d\alpha))^{-1}]'$$

$$= - D_{n+1}(P_n(c, d\alpha) \, P_{n+1}^{-1}(c, d\alpha))^{-1}$$
$$\times (P_n(c, d\alpha) \, P_{n+1}^{-1}(c, d\alpha))' \, (P_n(c, d\alpha) \, P_{n+1}^{-1}(c, d\alpha))^{-1}.$$

Using (9) and (12), we have

$$\lim_{n \to \infty} P_n^{-*}(c, d\alpha) \, K_{n+1}(c, c, d\alpha) \, P_n^{-1}(c, d\alpha)$$
$$= - \left(\int \frac{dW_{D, E}(t)}{c - t} \right)^{-1} \left(\int \frac{dW_{D, E}(t)}{c - t} \right)' \left(\int \frac{dW_{D, E}(t)}{c - t} \right)^{-1}.$$

Thus, for the proof of Theorem 3.1, we can use Lemma 3.2 to find $\lim_{n \to \infty} P_n^{-1}(c, d\alpha) = 0$. Finally, from

$$\| P_n^{-*}(c, d\alpha) \, M^{-1} P_n^{-1}(c, d\alpha) \|_s \leqslant \| P_n^{-1}(c, d\alpha) \|_s^2 . \| M^{-1} \|_s,$$

where $\| . \|_s$ is the spectral norm, we get

$$\lim_{n \to \infty} P_n^{-*}(c, d\alpha) \, M^{-1} P_n^{-1}(c, d\alpha) = 0. \tag{21}$$

Thus (15) follows. ∎

Let $(P_n(x, d\alpha) = \gamma_n(d\alpha) \, x^n + b_n(d\alpha) \, x^{n-1} + lower \ degree \ terms)_n$ be a sequence of orthonormal matrix polynomials satisfying (13), then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta) = \gamma_n(d\beta) \, x^n + b_n(d\beta) \, x^{n-1} + lower \ degree \ terms)_n$ with respect to the matrix measure $d\beta$ for which $[\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements. In fact, if from a sequence $(P_n(x, d\beta))_n$ the ratio of leading coefficients $[\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]$ are not upper triangular matrices with positive diagonal elements. Then we can find a sequence of unitary matrices $(S_n)_n$ such that $(S_n^* P_n(x, d\beta) = \lambda_n(d\beta) \, x^n + lower \ degree \ terms)_n$ is a sequence of othonormal matrix polynomials for which $[\lambda_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements. To see this, we recall that the associated orthonormal matrix polynomials with respect to the matrix measure $d\alpha$ and $d\beta$ have, respectively, the form $(U_n P_n(x, d\alpha))_n$ and $(V_n P_n(x, d\beta))_n$ (U_n and V_n are unitary matrices). Then the leading coefficients are related by

$$\lambda_n(d\alpha) = U_n \gamma_n(d\alpha)$$
$$\lambda_n(d\beta) = V_n \gamma_n(d\beta).$$

Therefore

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,V_n^* = U_n^*[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]^* \tag{22}$$

and

$$[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]^*\,[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]$$
$$= U_n[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]\,U_n^*. \tag{23}$$

Taking $U_n = I$, (13) holds. Now, we consider the sequence of orthonormal matrix polynomials $(S_n^* P_n(x, d\beta))_n$ where $(S_n)_n$ are unitary matrices given by the QR factorization of Francis and Kublanovskaja [10] of $[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]$,

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)] = \tilde{S}_n\tilde{R}_n,$$

where $\tilde{S}_n$ is an unitary matrix and $\tilde{R}_n$ is an upper triangular matrix. Then taking $S_n = \tilde{S}_n J_n$ and $R_n = J_n\tilde{R}_n$, where

$$[J_n]_{i,j} = \begin{cases} \dfrac{[\tilde{R}_n]_{i,i}}{|[\tilde{R}_n]_{i,i}|} & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

we get that $[\lambda_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements, since from (22) we have

$$[\lambda_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^* = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,S_n = \tilde{R}_n^*J_n.$$

Now, using (19), let n_v be an increasing sequence of positive integer numbers such that the limit

$$\Lambda(c) = \lim_{v\to\infty}[\gamma_{n_v}(d\beta)\,\gamma_{n_v}^{-1}(d\alpha)]^*$$

exists. From (15), we have

$$\Lambda(c)\,\Lambda^*(c) = I + \int\frac{dW_{D,E}(t)}{c-t}\left\{\left(\int\frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1}\int\frac{dW_{D,E}(t)}{c-t}. \tag{24}$$

The matrix valued function

$$I + \int \frac{dW_{D,E}(t)}{c-t} \cdot \left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1} \cdot \int \frac{dW_{D,E}(t)}{c-t}$$

$$= I - \int \frac{dW_{D,E}(t)}{c-t} \cdot \left\{\int \frac{dW_{D,E}(t)}{(c-t)^2}\right\}^{-1} \cdot \int \frac{dW_{D,E}(t)}{c-t} \qquad (25)$$

is positive definite when $c \in \mathbb{R}\backslash\hat{\varGamma}$. In fact from (24), it is sufficient to prove that this matrix is non-singular.

We suppose that there is a non zero vector column x such that

$$\left(I + \int \frac{dW_{D,E}(t)}{c-t} \cdot \left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1} \cdot \int \frac{dW_{D,E}(t)}{c-t}\right) x = 0.$$

Then

$$\left(\int \frac{dW_{D,E}(t)}{c-t} \cdot \left\{\int \frac{dW_{D,E}(t)}{(c-t)^2}\right\}^{-1} \cdot \int \frac{dW_{D,E}(t)}{c-t}\right) x = x,$$

and since $\int (dW_{D,E}(t))/(c-t)$ and its derivative are non-singular when $c \in \mathbb{R}\backslash\hat{\varGamma}$, we have

$$\int \frac{dW_{D,E}(t)}{(c-t)^2} \cdot \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} x = \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^2 \cdot \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} x. \quad (26)$$

Writing

$$\int \frac{dW_{D,E}(t)}{c-t} = G_{D,E}(c)$$

$$\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} x = y,$$

then (26) becomes

$$G'_{D,E}(c)\, y = -(G_{D,E}(c))^2\, y; \qquad y \neq 0. \qquad (27)$$

From (3), we have

$$(tI - E_n(d\alpha))\, P_n(t, d\alpha) = D_{n+1}(d\alpha)\, P_{n+1}(t, d\alpha) + D_n^*(d\alpha)\, P_{n-1}(t, d\alpha)$$

and then

$$D_n^*(d\alpha)(P_{n-1}(t, d\alpha) \, P_n^{-1}(t, d\alpha) \, D_n^{-1}(d\alpha))$$
$$D_n(d\alpha)(P_n(t, d\alpha) \, P_{n+1}^{-1}(t, d\alpha) \, D_{n+1}^{-1}(d\alpha))$$
$$+ (E_n(d\alpha) - tI)(P_n(t, d\alpha) \, P_{n+1}^{-1}(t, d\alpha) \, D_{n+1}^{-1}(d\alpha)) + I = 0,$$

for $t \in \mathbb{R} \backslash \Gamma$. Using (13) and (9), we get

$$D^*G_{D, E}(t) \, DG_{D, E}(t) + (E - tI) \, G_{D, E}(t) + I = 0. \tag{28}$$

Taking derivatives in (28) at the point c, we get

$$D^*G'_{D, E}(c) \, DG_{D, E}(c) + D^*G_{D, E}(c) \, DG'_{D, E}(c)$$
$$- G_{D, E}(c) + (E - cI) \, G'_{D, E}(c) = 0.$$

Using (27), we have

$$D^*G'_{D, E}(c) \, DG_{D, E}(c) \, y - D^*G_{D, E}(c) \, D(G_{D, E}(c))^2 y$$
$$- G_{D, E}(c) \, y - (E - cI)(G_{D, E}(c))^2 y = 0.$$

This means that

$$(D^*G'_{D, E}(c) \, D) \, G_{D, E}(c) \, y - \{ D^*G_{D, E}(c) \, DG_{D, E}(c)$$
$$+ (E - cI) \, G_{D, E}(c) + I \} \, G_{D, E}(c) \, y = 0,$$

and therefore

$$(D^*G'_{D, E}(c) \, D) \, x = 0.$$

Hence (25) is positive definite when $c \in \mathbb{R} \backslash \hat{\Gamma}$. Using the Cholesky factorization of (25), the limit $\Lambda(c)$ is finite, with positive diagonal elements, and it is unique.

This means that every convergent subsequence of $[\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^*$ converges to $\Lambda(c)$, hence

$$\lim_{n \to \infty} \, [\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^* = \Lambda(c). \tag{29}$$

Notice that we can also find $(P_n(x, d\beta))_n$ such that $[\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^*$ are upper triangular matrices with positive diagonal elements. In fact, as before, it is sufficient to give the unitary matrices $(S_n)_n$ by using the QL factorization instead of the QR factorization, and since the matrix given in

the right hand side of (25) is positive definite, there is a unique upper triangular matrix $\tilde{\Lambda}(c)$, with positive diagonal elements satisfying

$$\tilde{\Lambda}(c).\tilde{\Lambda}(c)^* = I + \int \frac{dW_{D,E}(t)}{c-t} . \left\{ \left(\int \frac{dW_{D,E}(t)}{c-t} \right)' \right\}^{-1} . \int \frac{dW_{D,E}(t)}{c-t}$$

such that

$$\lim_{n \to \infty} [\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^* = \tilde{\Lambda}(c).$$

Now we consider the matrix measure

$$d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k), \tag{30}$$

where M_k are positive definite matrices and $c_k \in \mathbb{R} \backslash \hat{\Gamma}$ as well as the family of the matrix measures $d\beta_n$, $n = 0, ..., N$ defined by

$$d\beta_n(u) = d\alpha(u) + \sum_{k=1}^{n} M_k \delta(u - c_k), \qquad n = 1, ..., N-1,$$
$$d\beta_0 = d\alpha, \; d\beta_N = d\beta. \tag{31}$$

By repeated application of the previous results we obtain asymptotics and estimates for $\gamma_n(d\beta)$.

THEOREM 3.2. *Let $\gamma_n(d\alpha)$ be the leading coefficients of the orthonormal matrix polynomials $(P_n(x, d\alpha) = \gamma_n(d\alpha) x^n + lower \ degree \ terms)_n$ associated to $d\alpha$. Assume that*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \qquad and \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where D is a non-singular matrix. Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to $d\beta$ defined by (30) such that

$$\lim_{n \to \infty} [\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^* = \Lambda_1(c_1) \Lambda_2(c_2) \cdots \Lambda_N(c_N),$$

where

$$\Lambda_k(c_k) = \lim_{n \to \infty} [\gamma_n(d\beta_k) \gamma_n^{-1}(d\beta_{k-1})]^*, \qquad for \quad k = 1, ..., N. \tag{32}$$

Proof. The family of the matrix measures defined by (31) can be generated in the following way

$$d\beta_{m+1}(u) = d\beta_m(u) + M_{m+1}\delta(u - c_{m+1}); \quad m = 0, 1, ..., N-1.$$

Since

$$\begin{aligned}
\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha) &= \gamma_n(d\beta_N)\,\gamma_n^{-1}(d\beta_0) \\
&= \gamma_n(d\beta_N)\,\gamma_n^{-1}(d\beta_{N-1})\,\gamma_n(d\beta_{N-1})\cdots\gamma_n^{-1}(d\beta_1) \\
&\quad \gamma_n(d\beta_1)\,\gamma_n^{-1}(d\beta_0),
\end{aligned}$$

then from [18, Thm 4.1], if

$$\lim_{n\to\infty} D_n(d\beta_m) = D_m,$$

and

$$\lim_{n\to\infty} E_n(d\beta_m) = E_m; \quad D_m \text{ non-singular}$$

for each $m = 0, 1, ..., N-1$, then there is a sequence of orthonormal matrix polynomials with respect to $d\beta_{m+1}$, such that its associated matrix recurrence coefficients satisfy

$$\lim_{n\to\infty} D_n(d\beta_{m+1}) = \Lambda_{m+1}^*(c_{m+1})\, D_m \Lambda_{m-1}^{-*}(c_{m+1})$$

and

$$\lim_{n\to\infty} E_n(d\beta_{m+1}) = \Lambda_{m+1}^*(c_{m+1})$$

$$\left\{ E_m + D_m[\Lambda_{m-1}^{-*}(c_{m+1}).\Lambda_{m+1}^{-1}(c_{m+1}) - I_N]\, D_m^* \right.$$

$$\times \int \frac{dW_{D_m, E_m}(t)}{c_{m+1} - t}$$

$$- [\Lambda_{m-1}^{-*}(c_{m+1}).\Lambda_{m+1}^{-1}(c_{m+1}) - I_N]\, D_m^*$$

$$\left. \times \int \frac{dW_{D_m, E_m}(t)}{c_{m+1} - t}\, D_m \right\} \Lambda_{m-1}^{-*}(c_{m+1}).$$

Hence by repeated application of (29) and taking into account (32) we get

$$\lim_{n\to\infty} [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^* = \Lambda_1(c_1)\,\Lambda_2(c_2)\cdots\Lambda_N(c_N). \quad \blacksquare$$

EXAMPLE 1. We take

$$D = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{9} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A straightforward computation yields

$$\frac{1}{2} D^{-1}(cI - E) D^{-1} = \begin{pmatrix} \dfrac{-1 + c}{32} & 0 \\ 0 & \dfrac{81(-1 + c)}{2} \end{pmatrix}$$

and

$$D^{-1/2}(E - cI) D^{-1}(E - cI) D^{-1/2} - 4I = \begin{pmatrix} \dfrac{-63 - 2c + c^2}{16} & 0 \\ 0 & 77 - 162c + 81c^2 \end{pmatrix}.$$

From the equation (10), we get

$$\int \frac{dW_{D,E}(t)}{c - t} = \frac{1}{2} \begin{pmatrix} a(c) & 0 \\ 0 & b(c) \end{pmatrix}, \tag{33}$$

where

$$a(c) = \tfrac{1}{16}(-1 + c - \sqrt{-63 - 2c + c^2})$$
$$b(c) = (-81(1 - c) - 9\sqrt{77 - 162c + 81c^2}),$$

$supp(dW_{D,E}) = [-7, 9] \cup [\frac{7}{9}, \frac{11}{9}] = [-7, 9]$, and the square roots are chosen such that $\int (dW_{D,E}(t))/(z - t)$ is analytic in $\mathbb{C}\setminus[-7, 9]$.

Taking derivatives on the right hand-side of the equation (33) and computing its inverse, we get

$$\left(\frac{d}{dz} \int \frac{dW_{D,E}(t)}{z - t} \right)^{-1} (c) = \begin{pmatrix} x_1(c) & 0 \\ 0 & x_2(c) \end{pmatrix},$$

where

$$x_1(c) = -\frac{(-9 + c)(7 + c) + (-1 + c)\sqrt{(-9 + c)(7 + c)}}{2}$$
$$x_2(c) = -\frac{77}{162} + c - \frac{c^2}{2} - \frac{(-1 + c)\sqrt{(-11 + 9c)(-7 + 9c)}}{18}.$$

Computing the terms given in (24), we get

$$\Lambda(c)\,\Lambda^*(c) = \begin{pmatrix} y_1(c) & 0 \\ 0 & y_2(c) \end{pmatrix},$$

where

$$y_1(c) = \tfrac{1}{32}(-31 - 2\,c + c^2 - (-1+c)\sqrt{(-9+c)\,(7+c)})$$
$$y_2(c) = \tfrac{1}{2}(79 - 162\,c + 81\,c^2 - 9\,(-1+c)\sqrt{(-11+9\,c)\,(-7+9\,c)}).$$

Finally, using the Cholesky decomposition, we obtain

$$\Lambda(c) = \frac{1}{4\sqrt{2}} \begin{pmatrix} z_1(c) & 0 \\ 0 & z_2(c) \end{pmatrix}, \tag{34}$$

where

$$z_1(c) = \sqrt{-31 - 2\,c + c^2 - (-1+c)\sqrt{(-9+c)\,(7+c)}}$$
$$z_2(c) = 4\sqrt{79 - 162\,c + 81\,c^2 - 9\,(-1+c)\sqrt{(-11+9\,c)\,(-7+9\,c)}}.$$

4. RELATIVE ASYMPTOTICS FOR ORTHONORMAL MATRIX POLYNOMIALS

We will now give some asymptotic results for ratios and products of orthonormal matrix polynomials with respect to the matrix measures $d\alpha$ and $d\beta$ related by $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with $c \in \mathbb{R}\backslash\hat{\Gamma}$ and M a positive definite matrix.

First we will obtain the relative asymptotics for the ratio of some orthonormal matrix polynomials $(P_n(x, d\beta))_n$ and $(P_n(x, d\alpha))_n$ with respect to the matrix measure $d\beta$ and $d\alpha$ respectively.

THEOREM 4.1.   *Let* $(P_n(x, d\alpha))_n$ *be orthonormal matrix polynomials with respect to* $d\alpha$. *Let* $(D_n(d\alpha))_n$ *be positive definite and* $(E_n(d\alpha))_n$ *hermitian matrices which appear in* (3), *satisfying*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \ \lim_{n \to \infty} E_n(d\alpha) = E; \ D \text{ is non-singular.}$$

Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to $d\beta$ such that for $x \in \mathbb{R}\backslash\{\hat{\Gamma} \cup \{c\}\}$

$$
\lim_{n \to \infty} P_n(x, d\beta) P_n^{-1}(x, d\alpha) = \Lambda(c)^{-1} + \frac{1}{c - x} \{\Lambda(c)^* - \Lambda(c)^{-1}\}
$$
$$
\times \left\{ \left(\int \frac{dW_{D, E}(t)}{c - t} \right)^{-*} - \left(\int \frac{dW_{D, E}(t)}{x - t} \right)^{-1} \right\},
\tag{35}
$$

where $\Lambda(c)$ is given in (29).

Proof. If we multiply in (14) by $P_n^{-1}(x, d\alpha)$, we have

$$
P_n(x, d\beta) P_n^{-1}(x, d\alpha)
$$
$$
= \gamma_n^{-*}(d\beta) \gamma_n^*(d\alpha)[I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}
$$
$$
\times MK_{n+1}^*(c, x, d\alpha) P_n(x, d\alpha)^{-1}].
$$

But,

$$
(c - x) K_{n+1}^*(c, x, d\alpha)
$$
$$
= P_{n+1}^*(c, d\alpha) D_{n+1}^* P_n(x, d\alpha) - P_n^*(c, d\alpha) D_{n+1} P_{n+1}(x, d\alpha),
$$

so

$$
MK_{n+1}^*(c, x, d\alpha) P_n^{-1}(x, d\alpha) = \frac{1}{c - x} M[P_{n+1}^*(c, d\alpha) D_{n+1}^*
$$
$$
- P_n^*(c, d\alpha) D_{n+1} P_{n+1}(x, d\alpha) P_n^{-1}(x, d\alpha)],
$$

and hence

$$
P_n(x, d\beta) P_n^{-1}(x, d\alpha)
$$
$$
= \gamma_n^{-*}(d\beta) \gamma_n^*(d\alpha) \left\{ I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \frac{1}{c - x} M \right.
$$
$$
\times [P_{n+1}^*(c, d\alpha) D_{n+1}^* - P_n^*(c, d\alpha) D_{n+1} P_{n+1}(x, d\alpha) P_n^{-1}(x, d\alpha)] \bigg\}
$$
$$
= \gamma_n^{-*}(d\beta) \gamma_n^*(d\alpha) \left\{ I - \frac{1}{c - x} P_n(c, d\alpha) \right.
$$
$$
\times (I + MK_{n+1}(c, c, d\alpha))^{-1} MP_n^*(c, d\alpha)
$$
$$
\times [P_n^{-*}(c, d\alpha) P_{n+1}^*(c, d\alpha) D_{n+1}^* - D_{n+1} P_{n+1}(x, d\alpha) P_n^{-1}(x, d\alpha)] \bigg\}.
$$

Since

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]$$
$$= I - P_n(c,\,d\alpha)(I + MK_{n+1}(c,\,c,\,d\alpha))^{-1}\,MP_n^*(c,\,d\alpha)$$

we have

$$P_n(x,\,d\beta)\,P_n^{-1}(x,\,d\alpha)$$
$$= [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\left\{I - \frac{1}{c-x}\,I - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)])\right.$$
$$\left. \times [P_n^{-*}(c,\,d\alpha)\,P_{n+1}^*(c,\,d\alpha)\,D_{n+1}^* - D_{n+1}P_{n+1}(x,\,d\alpha)\,P_n^{-1}(x,\,d\alpha)]\right\}.$$
$$(36)$$

Writing

$$\varXi_n(x,\,d\alpha) = [D_{n+1}P_{n+1}(c,\,d\alpha)\,P_n^{-1}(c,\,d\alpha)]^*$$
$$- [D_{n+1}P_{n+1}(x,\,d\alpha)\,P_n^{-1}(x,\,d\alpha)],$$

(36) becomes

$$P_n(x,\,d\beta)\,P_n^{-1}(x,\,d\alpha)$$
$$= [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}$$
$$\times\left\{I - \frac{1}{c-x}\left(I - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]\right)\varXi_n(x,\,d\alpha)\right\}$$
$$= [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*} - \frac{1}{c-x}\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\,\varXi_n(x,\,d\alpha)$$
$$+ \frac{1}{c-x}\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\,\varXi_n(x,\,d\alpha)$$
$$= [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}$$
$$+ \frac{1}{c-x}\left\{[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)] - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\right\}\varXi_n(x,\,d\alpha).$$

From (9) we have

$$\lim_{n\to\infty}\varXi_n(x,\,d\alpha) = \left(\int\frac{dW_{D,\,E}(t)}{c-t}\right)^{-*} - \left(\int\frac{dW_{D,\,E}(t)}{x-t}\right)^{-1}.$$

By Theorem 3.2, there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ such that

$$\lim_{n \to \infty} [\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^* = \Lambda(c),$$

hence

$$\lim_{n \to \infty} P_n(x, d\beta) P_n^{-1}(x, d\alpha) = \Lambda(c)^{-1} + \frac{1}{c-x} \{\Lambda(c)^* - \Lambda(c)^{-1}\}$$
$$\times \left\{ \left(\int \frac{dW_{D, E}(t)}{c-t} \right)^{-*} - \left(\int \frac{dW_{D, E}(t)}{x-t} \right)^{-1} \right\}. \quad \blacksquare$$

Now we will generalize Theorem 4.1, assuming that $d\beta$ and $d\alpha$ are related by (30).

THEOREM 4.2. *Let* $(P_n(x, d\alpha))_n$ *be a sequence of orthonormal matrix polynomials with respect to the matrix measure* $d\alpha$. *Assume that the matrix coefficients in* (3) *satisfy*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \lim_{n \to \infty} E_n(d\alpha) = E; \ D \ is \ non\text{-}singular.$$

Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ *with respect to* $d\beta$ *such that for* $x \in \mathbb{R} \backslash \{\hat{\Gamma} \cup \{c_k; k = 1, ..., N\}\}$

$$\lim_{n \to \infty} P_n(x, d\beta) P_n^{-1}(x, d\alpha)$$
$$= \prod_{k=N}^{\overset{\curvearrowright}{1}} \{\Lambda_k(c_k)^{-1} + \frac{1}{c_k-x} [\Lambda_k(c_k)^* - \Lambda_k(c_k)^{-1}]$$
$$\times \left[\left(\int \frac{dW_{D, E}(t)}{c_k-t} \right)^{-*} - \left(\int \frac{dW_{D, E}(t)}{x-t} \right)^{-1} \right] \},$$

where $\prod_{k=r}^{\overset{\curvearrowright}{r+s}} T_k = T_r T_{r+1} \cdots T_{r+s}$ *and* $\Lambda_k(c_k)$ *is given by* (32).

Proof. This follows immediately from Theorem 4.1 using the proof given in Theorem 3.2. $\blacksquare$

Next we will obtain an asymptotic formula for the product of the orthonormal matrix polynomials $P_n(c, d\beta)$ and $P_n(c, d\alpha)$ at the mass point c, where $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with M a positive definite matrix and $c \in \mathbb{R} \backslash \hat{\Gamma}$.

THEOREM 4.3. *Let* $(P_n(x, d\alpha))_n$ *be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$. Assume that the matrix coefficients in* (3) *satisfy*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \ \lim_{n \to \infty} E_n(d\alpha) = E; \ D \text{ is non-singular.}$$

Then there exits a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ *with respect to the matrix measure $d\beta$, such that*

$$\lim_{n \to \infty} P_n(c, d\beta) \, MP_n^*(c, d\alpha)$$

$$= -\Lambda(c)^{-1} \left\{ \int \frac{dW_{D, E}(t)}{c - t} \right\} \left\{ \left(\int \frac{dW_{D, E}(t)}{c - t} \right)' \right\}^{-1} \left\{ \int \frac{dW_{D, E}(t)}{c - t} \right\}. \quad (37)$$

Proof. Multiplying $MP_n^*(c, d\alpha)$ to the right on the both hand-sides of (14), we have

$$P_n(c, d\beta) \, MP_n^*(c, d\alpha)$$
$$= [\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^{-*} \{ P_n(c, d\alpha) \, MP_n^*(c, d\alpha)$$
$$- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \, MK_{n+1}(c, c, d\alpha) \, MP_n^*(c, d\alpha) \}.$$

But

$$P_n(c, d\alpha) \, MP_n^*(c, d\alpha)$$
$$= P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \, MP_n^*(c, d\alpha)$$
$$+ P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} MK_{n+1}(c, c, d\alpha) \, MP_n^*(c, d\alpha),$$

so

$$P_n(c, d\beta) \, MP_n^*(c, d\alpha) = [\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^{-*}$$
$$\times \{ P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \, MP_n^*(c, d\alpha) \}.$$

This means that

$$P_n(c, d\beta) \, MP_n^*(c, d\alpha) = [\gamma_n(d\beta) \, \gamma_n^{-1}(d\alpha)]^{-*} \{ P_n^{-*}(c, d\alpha) \, M^{-1}P_n^{-1}(c, d\alpha)$$
$$+ P_n^{-*}(c, d\alpha) \, K_{n+1}(c, c, d\alpha) \, P_n^{-1}(c, d\alpha) \}^{-1}.$$

As in the proof of (21) we have

$$\lim_{n \to \infty} P_n^{-*}(c, d\alpha) \, M^{-1}P_n^{-1}(c, d\alpha) = 0,$$

and from (20)

$$\lim_{n \to \infty} P_n^{-*}(c, d\alpha) K_{n+1}(c, c, d\alpha) P_n^{-1}(c, d\alpha)$$

$$= -\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} \left(\int \frac{dW_{D,E}(t)}{c-t}\right)' \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}.$$

By Theorem 3.2, there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ such that

$$\lim_{n \to \infty} [\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^* = \Lambda(c),$$

hence

$$\lim_{n \to \infty} P_n(c, d\beta) M P_n^*(c, d\alpha)$$

$$= -\Lambda(c)^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\}\left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\}.$$

EXAMPLE 2. We compute the relative ratio asymptotic of orthogonal matrix polynomials whose coefficients of the recurrence relations converge to

$$D = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (33) in the Example 1, we have

$$\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1} = \begin{pmatrix} \alpha(c, x) & 0 \\ 0 & \beta(c, x) \end{pmatrix},$$

where

$$\alpha(c, x) = \frac{c + \sqrt{(-9+c)(7+c)} - x - \sqrt{(-9+x)(7+x)}}{2}$$

$$\beta(c, x) = \frac{9 c + \sqrt{(-11+9 c)(-7+9 c)} - 9 x - \sqrt{(-11+9 x)(-7+9 x)}}{18}.$$

Therefore

$$\lim_{n \to \infty} P_n(x, d\beta) P_n^{-1}(x, d\alpha) = \begin{pmatrix} \dfrac{\gamma(c, x)}{\lambda(c, x)} & 0 \\ 0 & \dfrac{\mu(c, x)}{v(c, x)} \end{pmatrix},$$

where

$$\gamma(c, x) = 64 - \frac{\left(\begin{array}{c}(63 + 2c - c^2 + (-1 + c)\sqrt{(-9 + c)(7 + c)}) \\ \times (c + \sqrt{(-9 + c)(7 + c)} - x - \sqrt{(-9 + x)(7 + x)})\end{array}\right)}{c - x}$$

$$\lambda(c, x) = \frac{16}{\sqrt{2}} \sqrt{-31 - 2c + c^2 - (-1 + c)\sqrt{(-9 + c)(7 + c)}}$$

$$\mu(c, x) = -9(-1 + c)(77 + 81c(-1 + x) - 81x)$$
$$+ \sqrt{(-11 + 9c)(-7 + 9c)}(77 + 81c(-1 + x) - 81x)$$
$$- (-11 + 9c)(-7 + 9c)\sqrt{(-11 + 9x)(-7 + 9x)} +$$
$$+ 9(-1 + c)\sqrt{(-11 + 9c)(-7 + 9c)}\sqrt{(-11 + 9x)(-7 + 9x)}$$

$$v(c, x) = \frac{36}{\sqrt{2}}(c - x)$$
$$\times \sqrt{79 - 162c + 81c^2 - 9(-1 + c)\sqrt{(-11 + 9c)(-7 + 9c)}}.$$

EXAMPLE 3. We compute the product of orthogonal matrix polynomials evaluated at the mass point when the coefficients in the recurrence relation converge to

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By computation of the terms given in (10), we have

$$\int \frac{dW_{D, E}(t)}{c - t} = \frac{1}{4}\begin{pmatrix} a(c) & b(c) \\ b(c) & a(c) \end{pmatrix},$$

where

$$a(c) = -2 + 2c - \sqrt{(-4 + c)c} - \sqrt{-4 + c^2}$$
$$b(c) = -2 - \sqrt{(-4 + c)c} + \sqrt{-4 + c^2},$$

$supp(dW_{D, E}) = [0, 4] \cup [-2, 2] = [-2, 4]$, and the square roots are chosen such that $\int (dW_{D, E}(t))/(z - t)$ is analytic in $z \in \mathbb{C}\backslash[-2, 4]$.

As in the example 1, we obtain

$$\Lambda(c) = \begin{pmatrix} u_1(c) & 0 \\ u_2(c) & u_3(c) \end{pmatrix},$$

where

$$u_1(c) = \frac{1}{2} \sqrt{2(-2+c)\,c - (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}}$$

$$u_2(c) = \frac{4 - 4c - (-2+c)\,\sqrt{(-4+c)\,c} + c\,\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c - (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}}}$$

$$u_3(c) = \frac{1}{2}\left(-4 - 2c + c^2 - \frac{(2+c)(2+(-4+c)\,c)}{\sqrt{-4+c^2}} \right.$$
$$\left. + \sqrt{(-4+c)\,c}\,\sqrt{-4+c^2} - \frac{\sqrt{(-4+c)\,c}\,(-2+c^2)}{c} \right)^{1/2}.$$

Then, we have

$$\lim_{n \to \infty} P_n(c, d\beta)\, MP_n^*(c, d\alpha) = \begin{pmatrix} w_1(c) & w_2(c) \\ w_3(c) & w_4(c) \end{pmatrix},$$

where

$$w_1(c) = \frac{4 - 2(-2+c)\,c + (-2+c)\,\sqrt{(-4+c)\,c} + c\,\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c - (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}}}$$

$$w_2(c) = \frac{4(-1+c) + (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c - (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}}}$$

$$4u_1^2(c)\,u_3(c)\,w_3(c) = 4(-1+c) + (-2+c)\,\sqrt{(-4+c)\,c} - c\,\sqrt{-4+c^2}$$

$$4u_1^2(c)\,u_3(c)\,w_4(c) = 4 - 12c + 2c^2 + 4c^3 - c^4$$
$$- (-2+c)\,c\,\sqrt{(-4+c)\,c}\,\sqrt{-4+c^2}$$
$$+ c(1 + (-4+c)\,c)\,\sqrt{-4+c^2}$$
$$+ (-2+c)\,\sqrt{(-4+c)\,c}\,(-3+c^2).$$

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REFERENCES

1. A. I. Aptekarev and E. M. Nikishin, The scattering problem for a discrete Sturm–Liouville operator, *Mat. USSR Sb.* **49** (1984), 325–355.

2. A. J. Durán, On orthogonal polynomials with respect to a positive definite matrix of measures, *Canad. J. Math.* **47** (1995), 88–112.

3. A. J. Durán, Markov theorem for orthogonal matrix polynomials, *Canad. J. Math.* **48** (1996), 1180–1195.

4. A. J. Durán and W. Van Assche, Orthogonal matrix polynomials and higher order recurrence relations, *Linear Algebra Appl.* **219** (1995), 261–280.

5. A. J. Durán and P. López-Rodríguez, Orthogonal matrix polynomials: zeros and Blumenthals theorem, *J. Approx. Theory* **84** (1996), 96–118.

6. A. J. Durán, Ratio asymptotic for orthogonal matrix polynomials, *J. Approx. Theory* **100** (1999), 304–344.

7. J. S. Geronimo, Scattering theory and the matrix orthogonal polynomials in the real line, *Circuits Systems Signal Process* **1** (1982), 471–495.

8. I. Gohberg, P. Lancaster, and L. Rodman, "Matrix Polynomials," Academic Press, New York, 1982.

9. A. A. Gonchar, On convergence of Padé approximants for some classes of meromorphic functions, *Mat. Sb.* **97** (139) (1975), 607–629; translated in *Math. USSR Sb.* **26** (1975), 555–575.

10. R. A. Horn and C. A. Johnson, "Topics in Matrix Analysis," Cambridge University Press, Cambridge, UK, 1991.

11. M. G. Krein, Fundamental aspect of the representation theory of Hermitian operators with deficiency index (m, m), *Amer. Math. Soc. Transl. (2)* **97** (1970), 75–143.

12. F. Marcellán and G. Sansigre, On a class of matrix orthogonal polynomials on the real line, *Linear Algebra and Appl.* **181** (1993), 97–109.

13. F. Marcellán and I. Rodriguez González, A class of matrix orthogonal polynomials on the Unit Circle, *Linear Algebra Appl.* **121** (1989), 233–241.

14. P. Nevai, "Orthogonal Polynomials," *Mem. Amer. Math. Soc.*, Vol. 213, Providence, Rhode Island, 1979.

15. M. Rosemberg, The square-integrability of matrix-valued functions with respect to a non-negative hermitian measure, *Duke Math. J.* **31** (1964) 291–298.

16. S. Saks and A. Zygmund, "Analytic Functions," PWN, Warsaw, 1971.

17. A. Sinap and W. Van Assche, Polynomial interpolation and Gaussian quadrature for matrix valued functions, *Linear Algebra Appl.* **207** (1994), 71–114.

18. H. O. Yakhlef, F. Marcellán, and M. Piñar, Perturbations in the Nevai matrix class of orthogonal matrix polynomials, *Linear Algebra Appl.*, in press.

19. D. Zhani, "Problème des moments matricielles sur la droite: construction d'une famille de solutions et questions d'unicité," thèse doctorale, Université Claude-Bernard, Lyon 1, 1983.