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Shape preserving properties of univariate Lototsky-Bernstein operators

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Abstract: The main goal of this paper is to study shape preserving properties of univariate Lototsky-Bernstein operators $L_n(f)$ based on Lototsky-Bernstein basis functions. The Lototsky-Bernstein basis functions $b_{n,k}(x)$ $(0 \le k \le n)$ of order n are constructed by replacing x in the i^{th} factor of the generating function for the classical Bernstein basis functions of degree n by a continuous nondecreasing function $p_i(x)$, where $p_i(0) = 0$ and $p_i(1) = 1$ for $1 \le i \le n$. These operators $L_n(f)$ are positive linear operators that preserve constant functions, and a non-constant function $\gamma_n^p(x)$. If all the $p_i(x)$ are strictly increasing and strictly convex, then $\gamma_n^p(x)$ is strictly increasing and strictly convex as well. Iterates $L_n^M(f)$ of $L_n(f)$ are also considered. It is shown that $L_n^M(f)$ converges to $f(0) + (f(1) - f(0))\gamma_n^p(x)$ as $M \to \infty$. Like classical Bernstein operators, these Lototsky-Bernstein operators enjoy many traditional shape preserving properties. For every $(1, \gamma_n^p(x))-$convex function $f \in C[0, 1]$, we have $L_n(f; x) \ge f(x)$; and by invoking the total positivity of the system $\{b_{n,k}(x)\}_{0\le k\le n}$, we show that if f is $(1, \gamma_n^p(x))-$convex, then $L_n(f; x)$ is also $(1, \gamma_n^p(x))-$convex. Finally we show that if all the $p_i(x)$ are monomial functions, then for every $(1, \gamma_{n+1}^p(x))-$convex function f, $L_n(f; x) \ge L_{n+1}(f; x)$ if and only if $p_1(x) = \cdots = p_n(x) = x$.

Keywords: Lototsky-Bernstein operators; Fixed point; Iterates; Shape preserving; Total positivity

MSC: 41A20; 41A36; 41A35; 41A50

1 Introduction

The primary goal of this paper is to study the shape preserving properties of the Lototsky-Bernstein operators. The Lototsky-Bernstein operators are generalizations of the classical polynomial Bernstein operators. The classical polynomial Bernstein operators are defined by

$$B_n(f; x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_k^n(x),$$

where $f \in C[0, 1]$ and $\{b_k^n(x), 0 \le k \le n\}$ denotes the Bernstein basis for the space of polynomials of degree at most n:

$$b_k^n(x) = \binom{n}{k} x^k (1 - x)^{n-k}, 0 \le k \le n.$$

The Bernsetin operators B_n have been the object of intense research, and have been generalized in several directions, for example, the $q-$Bernstein operators [26] and the $h-$Bernstein operators ([31]). The polynomials $B_n(f)$ converge uniformly to f, although the convergence might be very slow ([10], p.166). Moreover the operators B_n reduce the variation and preserve the shape of f. Also the derivative of $B_n(f)$ of a function of class C^1 converges uniformly to f' (see [10]). For all these reasons Bernstein bases and Bernstein operators are fundamental in approximation theory and computer aided geometric design (CAGD).

In Approximation Theory and CAGD one is often interested in approximating functions, and rendering curves and surfaces not given by polynomial functions. Thus it is natural to try to extend polynomial methods to nonpolynomial settings, while keeping as many of the good properties of Bernstein bases and Bernstein operators as possible.

From the binomial theorem, we can derive generating functions for the classical Bernstein basis functions

$$(xw + (1-x))^n = \sum_{k=0}^{n} b_k^n(x) w^k. \tag{1.1}$$

We can generalize these generating functions for the classical Bernstein basis functions $b_k^n(x)$ to generating functions for the Lototsky-Bernstein basis functions $b_{n,k}(x)$ $(0 \le k \le n)$. Let $\{p_i(x), 0 \le i \le n\}$ denote a sequence of real-valued functions on $[0,1]$. Define

$$b_{0,0}(x) = 1, b_{0,k}(x) = 0, k > 0, \tag{1.2}$$

$$\prod_{j=1}^{n} \Big(w p_j(x) + 1 - p_j(x) \Big) = \sum_{k=0}^{n} b_{n,k}(x) w^k. \tag{1.3}$$

By simple computations from (1.3) it is straightforward to confirm that

$$b_{n,k}(x) = \sum_{\substack{K \bigcup L = \{1,2,\cdots,n\} \\ |L| = n-k, |K| = k}} \prod_{m \in L} (1 - p_m(x)) \prod_{l \in K} p_l(x), \tag{1.4}$$

Thus the Lototsky-Bernstein basis functions $b_{n,k}(x)$ are generalizations of the classical Bernstein basis functions $b_k^n(x)$.

The Lototsky-Bernstein operators L_n are defined for each function $f \in C[0,1]$ by

$$L_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x). \tag{1.5}$$

Throughout this paper, we always assume that $p_i(x) \in C[0,1]$ $(1 \le i \le n)$ and that $0 < p_i(x) < 1$ for $x \in (0,1)$ and $p_i(0) = 0, p_i(1) = 1$. The canonical and most important examples of such functions are the monomials $p_i(x) = x^{N_i}$ $(1 \le i \le n)$, where N_i is a positive integer. We impose the assumption $p_i(0) = 0, p_i(1) = 1$ in order to guarantee that the operators $L_n(f)$ satisfy the endpoint interpolation property. Indeed we observe immediately from (1.5) that with these assumptions, $L_n(f; 0) = f(0)$ and $L_n(f; 1) = f(1)$. When it is necessary to emphasize the dependence of $L_n(f)$ upon $p_i(x)$, we will replace $L_n(f; x)$ by $L_n(f; x; \mathbb{P}_n)$, where $\mathbb{P}_n(x) := (p_1(x), \cdots, p_n(x))$.

Several authors have studied these operators. King [21] discusses conditions on the sequence of real-valued functions $p_j(x)$ that ensure the uniform convergence of $L_n(f; x)$ to $f(x)$. Eisenberg and Wood [12] discuss uniform approximation of analytic functions by means of Lototsky-Bernstein operators. Several other authors also discuss convergence properties of the operators $L_n(f)$ in special cases. Whenever all the $p_i(x)$ are equal to a suitably chosen function $r_n^*(x)$ such that $\lim_{n\to\infty} r_n^*(x) = x$, the operators $L_n(f)$ reduce to the King-type operators $K_n(f)$ [22]. The King-type operators $K_n(f)$ converge to the identity on $C[0,1]$ and preserve $e_i(x) = x^i, i = 0, 2$. For different choices of $r_n^*(x)$, many approximation properties of $K_n(f)$ such as rates of convergence, shape preservation, fixed points, asymptotic behavior and saturation have been investigated (see [5, 6, 7, 14, 15]). However, a systematic study of the general operators $L_n(f)$, especially shape preserving properties of $L_n(f)$, has yet to appear, except for some basic convergence results in [12, 21].

The present paper is devoted to investigating the shape preserving properties of the operators $L_n(f)$. A systematic treatment of the general convergence properties of the operators of $L_n(f)$ (eg., Voronowskaja Theorem, and convergence in terms of $p_i(x)(1 \le i \le n)$) will appear in a separate paper [34].

We proceed in the following fashion. In Section 2 we describe some basic properties of the Lototsky-Bernstein basis functions $b_{n,k}(x)$ $(0 \le k \le n)$. We study the first and second derivatives of the Lototsky-Bernstein operators $L_n(f)$ in Section 3. In Section 4, we find a non-constant fixed point $\gamma_n^p(x)$ of the operators $L_n(f)$, and we discuss approximation properties of the functions $\gamma_n^p(x)$. Limits of the iterates of $L_n(f)$ are also derived in terms of the functions $\gamma_n^p(x)$. In Section 5, we prove that for a strictly increasing and convex function φ on $[0, \infty)$, the function $L_n(f)$ is $\varphi-$total variation diminishing. We also show that the operators $L_n(f)$ enjoy many traditional shape preserving properties. For example, for every $(1, \gamma_n^p)$-convex function $f \in C[0, 1]$, we have $L_n(f) \ge f$. If f is $(1, \gamma_n^p)$-convex, then $L_n(f)$ is $(1, \gamma_n^p)$-convex as well, and if f is convex in the standard sense, then $L_n(f)$ is $(1, \sum\limits_{k=1}^{n} p_k(x))$-convex. In addition, in Section 5.2 we discuss total positivity and the variation diminishing property for special cases of the Lototsky-Bernstein basis functions. In Section 6, we show that if f is increasing, then the operators $L_n(f)$ behave in a natural way when we vary some $p_i(x)$, i.e., given $\mathbb{P}_n(x) = (p_1(x), \cdots, p_n(x))$ and $\mathbb{Q}_n(x) = (q_1(x), \cdots, q_n(x))$, if $p_i(x) \ge q_i(x)$ for all i, then $L_n(f; x; \mathbb{P}_n) \ge L_n(f; x; \mathbb{Q}_n)$; moreover $\gamma_n^p(x) \ge \gamma_n^q(x)$. In Section 7, we give a general result on the monotonicity of $\{L_n(f; x)\}_{n\ge1}$ with respect to n. If all the $p_i(x)$ are monomial functions, we show that for every $(1, \gamma_{n+1}^p(x))-$convex function f we have $L_n(f; x) \ge L_{n+1}(f; x)$ if and only if $p_1(x) = \cdots = p_n(x) = x$. Some additional results on the fixed points of $L_n(f; x)$ are also derived. In particular, if for all n, $\gamma_n^p(x) = \gamma_{n+1}^p(x)$, we show that all the $p_i(x)$ $(i \ge 2)$ must interpolate the same point $(1/2, 1/2)$ and the zeroes of all the $p_i(x), i \ge 1$ possess the same multiplicities at $x = 0$; in addition, $\sum\limits_{k=1}^{n} p_k(x)/n$ must converge to $\varrho(x) = x$. Furthermore if for all n, $p_1(x) = \cdots = p_n(x)$ are convex, then $\gamma_n^p(x) \ge \gamma_{n+1}^p(x)$.

2 Lototsky-Bernstein bases

We now present some basic properties of the Lototsky-Bernstein bases $b_{n,k}(x)$ $(0 \le k \le n)$, which we shall need later in this paper. We shall see shortly that the Lototsky-Bernstein bases $b_{n,k}(x)$ $(0 \le k \le n)$ are related to elementary symmetric functions.

Definition 2.1. *The rth* $(1 \le r \le n)$ *elementary symmetric function* $\sigma_r(x_1, \cdots, x_n)$*, is the sum of all products of r distinct variables chosen from n variables. That is*

$$\sigma_r(x_1, \cdots, x_n) = \sum_{1\le i_i<\cdots<i_r\le n} x_{i_1} \cdots x_{i_r}, \tag{2.1}$$

and we define $\sigma_0(x_1, \cdots, x_n) = 1$.

Now we list some basic properties of the Lototsky-Bernstein bases $b_{n,k}(x)$ $(0 \le k \le n)$.

i) If $x \in (0, 1)$, then $b_{n,k}(x) > 0, 0 \le k \le n$.

ii) Each $b_{n,k}(x)$ has a zero of order at least k $(1 \le k \le n)$ at 0 and a zero of order at least $n - k$ $(0 \le k \le n - 1)$ at 1, and $b_{n,0}(0) = 1, b_{n,n}(1) = 1$.

iii) $\sum\limits_{k=0}^{n} b_{n,k}(x) = 1$.

iv) The functions $\{b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x)\}$ are linearly independent.

v) The functions $\{b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x)\}$ form a non-negative, normalized basis for the space

$U_n := span\{\sigma_0(p_1(x), \cdots, p_n(x)), \cdots, \sigma_n(p_1(x), \cdots, p_n(x))\}$.

vi) The Lototsky-Bernstein basis of order $n+1$ may be generated from the Lototsky-Bernstein basis of order n through the recurrence relation

$$b_{n+1,k}(x) = p_{n+1}(x)b_{n,k-1}(x) + (1 - p_{n+1}(x))b_{n,k}(x), \tag{2.2}$$

for $0 \leq k \leq n+1$, where we define $b_{n,k}(x) = 0$ if $k < 0$ or $k > n$, and initiate the recursion with $b_{0,0}(x) \equiv 1$.

vii)

$$\sigma_k(p_1(x), \cdots, p_n(x)) = \sum_{j=k}^{n} \binom{j}{k} b_{n,j}(x), k = 0, \cdots, n. \tag{2.3}$$

viii)

$$b_{n,k}(x) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \sigma_j(p_1(x), \cdots, p_n(x)). \tag{2.4}$$

Properties i)-vi) are straightforward from (1.3) and (1.4); properties vii) and viii) can be derived from the Lototsky-Bernstein blossom (see [13] for an introduction to blossoming, for more details see [33]).

Remark 2.2. *Theorem 5.4 in Section 5 provides an example where the system* $(b_{n,0}(x), \cdots, b_{n,n}(x))$ *is a Chebyshev system. But by ii), the Lototsky-Bernstein basis function* $b_{n,k}(x)$ $(0 \leq k \leq n)$ *may have more than n zeros (counting multiplicities) in* $[0, 1]$*; therefore the system need not be an Extended Chebyshev system. Thus* $L_n(f)$ *are positive linear operators not necessarily in the framework of Extended Chebyshev systems. For Bernstein operators in the framework of Extended Chebyshev systems, see [1, 2, 3].*

3 Derivatives

We are now going to compute the first and second derivatives of $L_n(f; x)$. To simplify our notation, let

$$v_{1,k,i}(x) := \sum_{\substack{K \bigcup L = \{1,2,\cdots,n\} \setminus \{i\} \\ |L| = (n-1-k), |K| = k}} \prod_{m \in L} (1 - p_m(x)) \prod_{l \in K} p_l(x), 1 \leq i \leq n, 0 \leq k \leq n-1, \tag{3.1}$$

and

$$v_{2,k,i,j}(x) := \sum_{\substack{K \bigcup L = \{1,2,\cdots,n\} \setminus \{i,j\} \\ |L| = (n-2-k), |K| = k}} \prod_{m \in L} (1 - p_m(x)) \prod_{l \in K} p_l(x), 1 \leq i \neq j \leq n, 0 \leq k \leq n-2. \tag{3.2}$$

Proposition 3.1. *Let* $p_i(x)$ *be differentiable on* $[0, 1]$ *for* $1 \leq i \leq n$. *Then the first derivative of* $L_n(f; x)$ *is given by*

$$L_n'(f; x) = \sum_{k=0}^{n-1} \left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right) \sum_{i=1}^{n} p_i'(x) v_{1,k,i}(x). \tag{3.3}$$

Proof. By equation (1.3)

$$\sum_{k=0}^{n} b_{n,k}(x)w^k = \prod_{j=1}^{n}\left(wp_j(x) + 1 - p_j(x)\right).$$

Differentiating this equation with respect to x yields:

$$\begin{aligned}\sum_{k=0}^{n} b'_{n,k}(x)w^k &= \sum_{i=1}^{n} p'_i(x)\cdot(w-1)\prod_{j\neq i}(wp_j(x)+1-p_j(x))\\ &= \sum_{k=0}^{n-1}\sum_{i=1}^{n} p'_i(x)\nu_{1,k,i}(x)w^{k+1} - \sum_{k=0}^{n-1}\sum_{i=1}^{n} p'_i(x)\nu_{1,k,i}(x)w^k\\ &= \sum_{k=0}^{n}\sum_{i=1}^{n} p'_i(x)(\nu_{1,k-1,i}(x)-\nu_{1,k,i}(x))w^k.\end{aligned}$$

Noting that $\nu_{1,-1,i}(x) = \nu_{1,n,i}(x) = 0$ and equating the coefficient of w^k on both sides of this equality yields

$$b'_{n,k}(x) = \sum_{i=1}^{n} p'_i(x)(\nu_{1,k-1,i}(x) - \nu_{1,k,i}(x)). \tag{3.4}$$

Thus

$$\begin{aligned}L'(f;x) &= \sum_{k=0}^{n}\sum_{i=1}^{n} p'_i(x)(\nu_{1,k-1,i}(x)-\nu_{1,k,i}(x))f(k/n)\\ &= \sum_{k=0}^{n-1}(f((k+1)/n) - f(k/n))\sum_{i=1}^{n} p'_i(x)\nu_{1,k,i}(x).\end{aligned} \tag{3.5}$$

□

Remark 3.2. *Suppose that $f(x)$ is increasing and that $p_i(x)$ is differentiable and $p'_i(x)$ is nonnegative on $[0,1]$ for all $1 \le i \le n$. Then from (3.3) $L'_n(f;x) \ge 0$ on $[0,1]$. This inequality implies that $L_n(f;x)$ is increasing on $[0,1]$. Also, by assumption, $p_i(0) = 0$, $p_i(1) = 1$ ($1 \le i \le n$), so the first derivatives at the endpoints*

$$L'_n(f;0) = \sum_{i=1}^{n} p'_i(0)\left(f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right)\right), L'_n(f;1) = \sum_{i=1}^{n} p'_i(1)\left(f\left(\frac{n}{n}\right) - f\left(\frac{n-1}{n}\right)\right), \tag{3.6}$$

are proportional to the slopes of f at the two first and last two abscissae.

Proposition 3.3. *Let $p_i(x)$ be twice differentiable on $[0,1]$ for $1 \le i \le n$. Then the second derivative of $L_n(f,x)$ is given by*

$$\begin{aligned}L''_n(f;x) = &\sum_{k=0}^{n-1}\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right)\sum_{i=1}^{n} p''_i(x)\nu_{1,k,i}(x)\\ &+ \sum_{k=0}^{n-2}\left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right)\sum_{i=1}^{n} p'_i(x)\sum_{j\neq i} p'_j(x)\nu_{2,k,i,j}(x).\end{aligned} \tag{3.7}$$

Proof. The proof proceeds by differentiating (3.3), using arguments identical to those used in the proof of (3.3). □

Remark 3.4. *Suppose that $f(x)$ is increasing and convex, and that $p_i(x)$ is twice differentiable nondecreasing and convex on $[0,1]$ for $1 \le i \le n$. Then from (3.7), $L_n''(f;x) \ge 0$. This inequality implies that $L_n(f)$ is convex in $[0,1]$.*

Also, the second derivatives of $L_n(f;x)$ at the end points of $[0,1]$ are given by

$$L_n''(f;0) = \sum_{i=1}^{n} p_i''(0)\left(f\left(\frac{1}{n}\right) - f\left(\frac{0}{n}\right)\right) + \sum_{i=1}^{n} p_i'(0)\sum_{j\neq i} p_j'(0)\left(f\left(\frac{2}{n}\right) - 2f\left(\frac{1}{n}\right) + f\left(\frac{0}{n}\right)\right);$$

$$L_n''(f;1) = \sum_{i=1}^{n} p_i''(1)\left(f\left(\frac{n}{n}\right) - f\left(\frac{n-1}{n}\right)\right) + \sum_{i=1}^{n} p_i'(1)\sum_{j\neq i} p_j'(1)\left(f\left(\frac{n}{n}\right) - 2f\left(\frac{n-1}{n}\right) + f\left(\frac{n-2}{n}\right)\right).$$

The divided difference $f[0,1/n,2/n]$ is given by

$$f\left[0,\frac{1}{n},\frac{2}{n}\right] = n^2\left(f\left(\frac{2}{n}\right) - 2f\left(\frac{1}{n}\right) + f\left(\frac{0}{n}\right)\right).$$

Similarly,

$$f\left[\frac{n-2}{n},\frac{n-1}{n},\frac{n}{n}\right] = n^2\left(f\left(\frac{n}{n}\right) - 2f\left(\frac{n-1}{n}\right) + f\left(\frac{n-2}{n}\right)\right).$$

Therefore we have the following proposition.

Proposition 3.5. *Suppose that all the $p_i(x)$ $(1 \le i \le n)$ are differentiable of order 2, and $p_i''(0) = 0$, $p''(1) = 0$. Then the second derivatives of $L_n(f;x)$ at the endpoints of $[0,1]$ are given by*

$$L_n''(f;0) = \frac{\sum_{i=1}^{n} p_i'(0)\sum_{j\neq i} p_j'(0)}{n^2} f\left[0,\frac{1}{n},\frac{2}{n}\right],$$

$$L_n''(f;1) = \frac{\sum_{i=1}^{n} p_i'(1)\sum_{j\neq i} p_j'(1)}{n^2} f\left[\frac{n-2}{n},\frac{n-1}{n},\frac{n}{n}\right].$$

Thus the second derivatives at the endpoints are proportional to the second order finite differences of f at the first three and last three abscissae.

4 Fixed points and iteration

Iterates of the Lototsky-Bernstein operators $L_n(f)$ are defined recursively by

$$(L_n)^{M+1}(f;x) = L_n\left((L_n)^M(f;\cdot);x\right), M = 1,2,\cdots,$$

where $(L_n)^1(f;x) = L_n(f;x)$. For the classical Bernstein polynomials, the iterates converge to linear end point interpolation on $[0,1]$. Kelisky and Rivlin [20] consider this problem both when M depends on n and when M is independent of n. Iterates of Bernstein operators are also studied from the point of view of operator semigroups by Karlin and Ziegler [19] and by Micchelli [24]. Cooper and Waldron [9] investigate iterates of Bernstein operators using their eigenvalues and eigenvectors. Corresponding results for $q-$Bernstein operators can be found in [25]. In [29] the authors use a contraction principle to study the iterates of a class of positive linear operators preserving affine functions. The results in [29] can be applied to any finitely defined operators.

Before we state the main theorem of this section, we recall the Banach Fixed Point Theorem.

Lemma 4.1. *(Banach Fixed Point Theorem [35]) Let (X, d) be a complete metric space and let $T : X \to X$ be a contractive transformation. Then T has a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$; then $x_n \to x^*$ as $n \to \infty$.*

Theorem 4.2. *The operators L_n possess a unique nonconstant fixed point $\gamma_n^p(x)$ with respect to $p_i(x)$ $(1 \le i \le n)$, i.e.*

$$L_n(\gamma_n^p; x) = \gamma_n^p(x), \tag{4.1}$$

with $\gamma_n^p(0) = 0, \gamma_n^p(1) = 1$. Moreover, for $f \in C[0, 1]$

$$\lim_{M\to\infty} (L_n)^M (f; x) = f(0) + (f(1) - f(0))\,\gamma_n^p(x). \tag{4.2}$$

Proof. We shall prove this result by invoking the Banach Fixed Point Theorem. Consider the complete metric space $(C[0, 1], \|\cdot\|)$ ([11]), where $\|\cdot\|$ is the max norm on $[0, 1]$. Let $X_{\alpha,\beta} := \{f \in C[0, 1] : f(0) = \alpha, f(1) = \beta\}, \alpha, \beta \in R$. Then:

(a) $X_{\alpha,\beta}$ is a closed subset of $C[0, 1]$;

(b) $X_{\alpha,\beta}$ is an invariant subset of L_n for all $\alpha, \beta \in R$, i.e., if $f \in X_{\alpha,\beta}$, then $L_n(f) \in X_{\alpha,\beta}$;

(c) $C[0, 1] = \bigcup_{\alpha,\beta\in R} X_{\alpha,\beta}$.

Now we will prove that $L_n|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$ is a contraction.

First, note that for $0 \le x \le 1$

$$0 < m := \min_{0\le x\le 1}\{b_{n,0}(x) + b_{n,n}(x)\} < 1.$$

Now for all $f, g \in X_{\alpha,\beta}$

$$\begin{aligned} |L_n(f)(x) - L_n(g)(x)| = |L_n(f - g)(x)| &= \left|\sum_{k=1}^{n-1} b_{n,k}(x)(f - g)\left(\frac{k}{n}\right)\right| \\ &\le \left|1 - b_{n,0}(x) - b_{n,n}(x)\right| \|f - g\| \le |1 - m|\,\|f - g\|. \end{aligned}$$

Therefore

$$\|L_n(f)(x) - L_n(g)(x)\| \le |1 - m|\,\|f - g\|.$$

Thus L_n is a contraction from $X_{\alpha,\beta}$ to $X_{\alpha,\beta}$.

Since $L_n(f)$ interpolates at the endpoints 0 and 1, it follows by applying the Banach Fixed Point Theorem on the space $X_{0,1}$ that there exists a unique function $\gamma_n^p(x)$ such that

$$L_n(\gamma_n^p; x) = \gamma_n^p(x), \tag{4.3}$$

where $\gamma_n^p(0) = 0$ and $\gamma_n^p(1) = 1$. Moreover, by the Banach Fixed Point Theorem applied to the space $X_{\alpha,\beta}$, the operators $L_n(f)$ also have a unique fixed point function $\gamma_{n,\alpha,\beta}^p(x) = \alpha + (\beta - \alpha)\gamma_n^p(x)$ such that

$$L_n(\gamma_{n,\alpha,\beta}^p; x) = \gamma_{n,\alpha,\beta}^p(x),$$

with $\gamma_{n,\alpha,\beta}^p(0) = \alpha$, $\gamma_{n,\alpha,\beta}^p(1) = \beta$. Therefore since each $f \in C[0, 1]$ lies in some space $X_{\alpha,\beta}$, it follows by the Banach Fixed Point Theorem that the iterates $(L_n)^M(f; x)$ converge to $f(0) + (f(1) - f(0))\gamma_n^p(x)$ as $M \to \infty$. □

From (4.1), the fixed point function $\gamma_n^p(x)$ depends on the basis functions $b_{n,k}(x)$ $(0 \le k \le n)$. Moreover, we can compute an explicit expression for the function $\gamma_n^p(x)$ in the following way.

Note that $\gamma_n^p(0) = 0, \gamma_n^p(1) = 1$; therefore equation (4.3) becomes

$$\sum_{k=1}^{n-1} \gamma_n^p\left(\frac{k}{n}\right) b_{n,k}(x) + b_{n,n}(x) = \gamma_n^p(x). \tag{4.4}$$

To derive $\gamma_n^p(x)$ from (4.4), we assign x different values $k/n, k = 1, \cdots, n-1$. Thus we generate the following system of linear equations

$$\begin{cases} \left(b_{n,1}\left(\frac{1}{n}\right) - 1\right)\gamma_n^p\left(\frac{1}{n}\right) + b_{n,2}\left(\frac{1}{n}\right)\gamma_n^p\left(\frac{2}{n}\right) + \cdots + b_{n,n-1}\left(\frac{1}{n}\right)\gamma_n^p\left(\frac{n-1}{n}\right) = -b_{n,n}\left(\frac{1}{n}\right) \\ b_{n,1}\left(\frac{2}{n}\right)\gamma_n^p\left(\frac{1}{n}\right) + \left(b_{n,2}\left(\frac{2}{n}\right) - 1\right)\gamma_n^p\left(\frac{2}{n}\right) + \cdots + b_{n,n-1}\left(\frac{2}{n}\right)\gamma_n^p\left(\frac{n-1}{n}\right) = -b_{n,n}\left(\frac{2}{n}\right) \\ \vdots \\ b_{n,1}\left(\frac{n-1}{n}\right)\gamma_n^p\left(\frac{1}{n}\right) + b_{n,2}\left(\frac{n-1}{n}\right)\gamma_n^p\left(\frac{2}{n}\right) + \cdots + \left(b_{n,n-1}\left(\frac{n-1}{n}\right) - 1\right)\gamma_n^p\left(\frac{n-1}{n}\right) = -b_{n,n}\left(\frac{n-1}{n}\right). \end{cases} \tag{4.5}$$

Let Q_{n-1} denote the determinant $|q_{i,j}|$ where

$$q_{i,j} = \begin{cases} b_{n,j}\left(\frac{i}{n}\right), i \neq j, \\ \\ b_{n,j}\left(\frac{i}{n}\right) - 1, i = j. \end{cases} \tag{4.6}$$

In addition, let $Q_{n-1}^{(i)}$ denote the determinant identical to Q_{n-1}, except that the i-th column of Q_{n-1} is replaced by

$$\left(-b_{n,n}\left(\frac{1}{n}\right), -b_{n,n}\left(\frac{2}{n}\right), \cdots, -b_{n,n}\left(\frac{n-1}{n}\right)\right)^T.$$

Notice that the coefficient matrix $\{q_{i,j}\}$ of the system (4.5) is strictly diagonally dominant-that is, for every row of the matrix, the magnitude of the diagonal entry is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely

$$\left|b_{n,i}\left(\frac{i}{n}\right) - 1\right| = 1 - b_{n,i}\left(\frac{i}{n}\right) > \sum_{j=1, j\neq i}^{n-1} b_{n,j}\left(\frac{i}{n}\right).$$

Therefore by the Levy-Desplanques theorem [17], the coefficient matrix in (4.5) is nonsingular, and hence (4.5) has a unique solution.

Solving (4.5) for $\gamma_n^p\left(\frac{i}{n}\right)$ $(1 \le i \le n-1)$ by Cramer's rule, we find that

$$\gamma_n^p\left(\frac{i}{n}\right) = \frac{Q_{n-1}^{(i)}}{Q_{n-1}}. \tag{4.7}$$

Combining equations (4.4)-(4.7), we find an explicit expression for $\gamma_n^p(x)$

$$\gamma_n^p(x) = \sum_{k=1}^{n-1} \frac{Q_{n-1}^{(k)}}{Q_{n-1}} b_{n,k}(x) + b_{n,n}(x). \tag{4.8}$$

Remark 4.3. *Explicitly,* $\gamma_1^p(x) = p_1(x)$,

$$\gamma_2^p(x) = p_1(x)p_2(x) + \frac{p_1(\frac{1}{2})p_2(\frac{1}{2})}{1 - p_1(\frac{1}{2}) - p_2(\frac{1}{2}) + 2p_1(\frac{1}{2})p_2(\frac{1}{2})}\left[p_1(x) + p_2(x) - 2p_1(x)p_2(x)\right],$$

and

$$\gamma_3^p(x) = \frac{Q_2^{(1)}}{Q_2}b_{3,1}(x) + \frac{Q_2^{(2)}}{Q_2}b_{3,2}(x) + p_1(x)p_2(x)p_3(x),$$

where

$$\begin{aligned} Q_2 &= \left(b_{3,1}\left(\frac{1}{3}\right) - 1\right)\left(b_{3,2}\left(\frac{2}{3}\right) - 1\right) - b_{3,1}\left(\frac{2}{3}\right)b_{3,2}\left(\frac{1}{3}\right), \\ Q_2^{(1)} &= b_{3,3}\left(\frac{2}{3}\right)b_{3,2}\left(\frac{1}{3}\right) - b_{3,3}\left(\frac{1}{3}\right)\left(b_{3,2}\left(\frac{2}{3}\right) - 1\right), \\ Q_2^{(2)} &= b_{3,3}\left(\frac{1}{3}\right)b_{3,1}\left(\frac{2}{3}\right) - b_{3,3}\left(\frac{2}{3}\right)\left(b_{3,1}\left(\frac{1}{3}\right) - 1\right). \end{aligned}$$

It is straightforward to verify that when $p_i(x) = x$ $(i = 1, 2, 3)$ $(Q_2 = \frac{21}{81}, Q_2^{(1)} = \frac{7}{81}, Q_2^{(2)} = \frac{14}{81})$, $\gamma_3^p(x) = x$. This result coincides with the result for classical Bernstein operators. In fact, when $p_i(x) = x$ $(1 \leq i \leq n)$, we also have $\gamma_n^p(x) = x$. Indeed affine functions are the unique fixed points for the classical Bernstein operators. (This result can also be derived from the fact that $f(x) - B_n(f, x) = o_x(1/n)$ iff f is affine, see [4]). For the classical Bernstein operators $\gamma_n^p(0) = 0, \gamma_n^p(1) = 1$, and $\gamma_n^p(x) = x$, so

$$\gamma_n^p\left(\frac{i}{n}\right) = \frac{i}{n}. \tag{4.9}$$

Thus we have derived the seemingly nontrivial identities for the classical Bernstein basis functions:

$$nQ_{n-1}^{(i)} = iQ_{n-1}, i = 1, \cdots, n-1. \tag{4.10}$$

Almost all classical positive linear operators (such as the classical Bernstein, $q-$Bernstein, BBH, Baskakov, Szász, Stancu operators) preserve linear functions. Since properties of linear functions are clear, no literature is devoted to the study of fixed point functions of positive linear operators.

Next we discuss some approximation properties on $\gamma_n^p(x)$.

Proposition 4.4. *The fixed point functions $\gamma_n^p(x)$ for $L_n(f; x)$ satisfy the following properties:*
1) $0 \leq \gamma_n^p(x) \leq 1$;
2) if $p_i'(x) > 0$ on $(0, 1)$ for $1 \leq i \leq n$ (i.e., all the $p_i(x)$ are strictly increasing), then $\gamma_n^p(x)$ is strictly increasing as well, i.e. $(Q_{n-1}^{(k)} - Q_{n-1}^{(k-1)})/Q_{n-1} > 0$ $(2 \leq k \leq n-1)$;
3) if $p_i'(x) > 0, p_i''(x) > 0$ on $(0, 1)$ for $1 \leq i \leq n$ (i.e., all the $p_i(x)$ are strictly increasing and strictly convex), then $\gamma_n^p(x)$ is strictly convex as well, i.e. $(Q_{n-1}^{(k)} - Q_{n-1}^{(k-1)})/Q_{n-1} > (Q_{n-1}^{(k-1)} - Q_{n-1}^{(k-2)})/Q_{n-1}$ $(3 \leq k \leq n-1)$.

Proof. Since the matrix $\{q_{i,j}\}_{1 \leq i,j \leq n}$ is strictly diagonally dominant

$$sgn(Q_{n-1}) = sgn\prod_{i=1}^{n-1}\left[b_{n,i}\left(\frac{i}{n}\right) - 1\right] = (-1)^{n-1}. \tag{4.11}$$

By (4.7) to prove 1), it is enough to prove that $sgn(Q^{(i)}_{n-1}) = sgn(Q_{n-1})$ for $1 \leq i \leq n-1$. It is not difficult to derive the sign of $Q^{(i)}_{n-1}$. We shall only derive the sign of $Q^{(1)}_{n-1}$ (other cases may be similarly verified). Thus we need only show that

$$sgn(Q^{(1)}_{n-1}) = (-1)^{n-1}. \tag{4.12}$$

Indeed, subtracting column 2, column 3, $\cdots$, column $n-1$ from the 1st column, yields

$$Q^{(1)}_{n-1} = \begin{vmatrix} b_{n,0}(1/n) + b_{n,1}(1/n) - 1 & b_{n,2}(1/n) & b_{n,3}(1/n) & \cdots & b_{n,n-1}(1/n) \\ b_{n,0}(2/n) + b_{n,1}(2/n) & b_{n,2}(2/n) - 1 & b_{n,3}(2/n) & \cdots & b_{n,n-1}(2/n) \\ b_{n,0}(3/n) + b_{n,1}(3/n) & b_{n,2}(3/n) & b_{n,3}(3/n) - 1 & \cdots & b_{n,n-1}(3/n) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n,0}(n-1/n) + b_{n,1}(n-1/n) & b_{n,2}(n-1/n) & b_{n,3}(n-1/n) & \cdots & b_{n,n-1}(n-1/n) - 1 \end{vmatrix},$$

which is another strictly diagonally dominant matrix. Therefore,

$$sgn(Q^{(1)}_{n-1}) = sgn[b_{n,0}(1/n) + b_{n,1}(1/n) - 1] \cdot \prod_{i=2}^{n-1} \left[b_{n,i}\left(\frac{i}{n}\right) - 1 \right] = (-1)^{n-1}.$$

Moreover

$$\begin{aligned} & Q_{n-1} - Q^{(1)}_{n-1} \\ = & \begin{vmatrix} b_{n,1}(1/n) + b_{n,n}(1/n) - 1 & b_{n,2}(1/n) & b_{n,3}(1/n) & \cdots & b_{n,n-1}(1/n) \\ b_{n,1}(2/n) + b_{n,n}(2/n) & b_{n,2}(2/n) - 1 & b_{n,3}(2/n) & \cdots & b_{n,n-1}(2/n) \\ b_{n,1}(3/n) + b_{n,n}(3/n) & b_{n,2}(3/n) & b_{n,3}(3/n) - 1 & \cdots & b_{n,n-1}(3/n) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n,1}(n-1/n) + b_{n,n}(n-1/n) & b_{n,2}(n-1/n) & b_{n,3}(n-1/n) & \cdots & b_{n,n-1}(n-1/n) - 1 \end{vmatrix}. \end{aligned}$$

Since this matrix is also strictly diagonally dominant, again

$$sgn(Q_{n-1} - Q^{(1)}_{n-1}) = (-1)^{n-1}. \tag{4.13}$$

From (4.11) and (4.12), we derive that $Q^{(1)}_{n-1}/Q_{n-1} > 0$, and from (4.11) and (4.13), we conclude that $1 - Q^{(1)}_{n-1}/Q_{n-1} = (Q_{n-1} - Q^{(1)}_{n-1})/Q_{n-1} > 0$. Therefore, $0 < Q^{(1)}_{n-1}/Q_{n-1} < 1$. Similarly, $0 < Q^{(i)}_{n-1}/Q_{n-1} < 1 (2 \leq i \leq n-1)$; therefore from (4.7)

$$\gamma^p_n(0) = 0 < \gamma^p_n\left(\frac{i}{n}\right) = \frac{Q^{(i)}_{n-1}}{Q_{n-1}} < 1 = \gamma^p_n(1), 1 \leq i \leq n-1. \tag{4.14}$$

Thus, the normalization of the basis functions $b_{n,k}(x)$ ((see iii) in Section 1) and (4.8) yields $0 \leq \gamma^p_n(x) \leq 1$.

In order to prove 2), recall that $\gamma^p_n(x) = L_n(\gamma^p_n; x)$. Therefore by Proposition 3.1, we need to prove only that $\gamma^p_n\left(\frac{i}{n}\right) < \gamma^p_n\left(\frac{i+1}{n}\right)$ for all $0 \leq i \leq n-1$. Moreover from (4.14), we need only prove $\gamma^p_n\left(\frac{1}{n}\right) < \gamma^p_n\left(\frac{2}{n}\right) < \cdots < \gamma^p_n\left(\frac{n-1}{n}\right)$.

Choose an increasing function $f \in C[0,1]$ such that $f(0) = 0, f(1) = 1$. If $p'_i(x) > 0$ on $(0,1)$ for $1 \leq i \leq n$, then by Remark 3.2, $L_n(f; x)$ is also increasing. Repeatedly using Remark 3.2, we find that $(L_n)^2(f; x)$ is increasing as well. Continuing this process, we find that for any positive integer M, $(L_n)^M(f;x)$ is increasing. Since $\lim_{M\to\infty}(L_n)^M(f;x) = \gamma_n^p(x)$, it follows that $\gamma_n^p(x)$ is nondecreasing, i.e.

$$\gamma_n^p\left(\frac{1}{n}\right) \le \gamma_n^p\left(\frac{2}{n}\right) \le \cdots \le \gamma_n^p\left(\frac{n-1}{n}\right), \tag{4.15}$$

Next, we need to prove that $\gamma_n^p\left(\frac{i}{n}\right) \neq \gamma_n^p\left(\frac{j}{n}\right)$ whenever $i \neq j$. From (4.14) $\gamma_n^p\left(\frac{n}{n}\right) > \gamma_n^p\left(\frac{n-1}{n}\right)$. Moreover, since by assumption $p_i'(x) > 0$, we conclude from (3.3) and (4.15) that $L_n'(\gamma_n^p;x) > 0$ for $x \in (0,1)$; hence $\gamma_n^p(x) = L_n(\gamma_n^p;x)$ is strictly increasing. Thus none of the equalities in (4.15) hold. Substituting (4.7) into (4.15) yields $(Q_{n-1}^{(k)} - Q_{n-1}^{(k-1)})/Q_{n-1} > 0$ $(2 \le k \le n-1)$. This complete the proof of 2).

To prove 3), suppose that $p_i'(x) > 0, p_i''(x) > 0$ on $(0,1)$ for $1 \le i \le n$. Choose an increasing and convex function $f \in C[0,1]$ such that $f(0) = 0, f(1) = 1$. We know by Remark 3.4 that for any positive integer M, $L_n^M(f)$ is increasing and convex. Thus $\gamma_n^p(x)$ is convex as well by the same arguments as in the proof of 2), i.e.

$$\gamma_n^p\left(\frac{k+2}{n}\right) - \gamma_n^p\left(\frac{k+1}{n}\right) \ge \gamma_n^p\left(\frac{k+1}{n}\right) - \gamma_n^p\left(\frac{k}{n}\right) (0 \le k \le n-2). \tag{4.16}$$

Moreover, since $\gamma_n^p\left(\frac{n}{n}\right) > \cdots > \gamma_n^p\left(\frac{1}{n}\right) > \gamma_n^p\left(\frac{0}{n}\right)$ as in 2), we conclude from (4.16) and (3.7) by an argument similar to the proof of 2) (note that $L_n''(\gamma_n^p;x) > 0$ for $x \in (0,1)$) that $\gamma_n^p(x)$ is strictly convex, that is

$$\gamma_n^p\left(\frac{k+2}{n}\right) - \gamma_n^p\left(\frac{k+1}{n}\right) > \gamma_n^p\left(\frac{k+1}{n}\right) - \gamma_n^p\left(\frac{k}{n}\right) (0 \le k \le n-2). \tag{4.17}$$

Substituting (4.7) into (4.17) yields $(Q_{n-1}^{(k)} - Q_{n-1}^{(k-1)})/Q_{n-1} > (Q_{n-1}^{(k-1)} - Q_{n-1}^{(k-2)})/Q_{n-1}$ $(3 \le k \le n-1)$. This complete the proof of 3). □

Remark 4.5. *Notice that if $p_i'(x)$ (or even $p_i''(x)$) has a finite number of zeros in $[0,1]$ and $p_i'(x) \ge 0$ (or $p_i''(x) \ge 0$), then Proposition 4.4 2) and 3) may also be true. The most important such $p_i(x)$ are monomial functions ($p_i'(0) = 0$ and even $p_i''(0) = 0$ if possible). Another such $p_i(x)$ are the functions defined in Proposition 5.8 (in this case $p_i'(0) = 0, p_i'(1) = 0$ and even $p_i''(0) = 0, p_i''(1) = 0$ if possible).*

Corollary 4.6. *Suppose that all the $p_i'(x) > 0, x \in (0,1)$ and $n-1 \ge s > t \ge 1$. Then $sgn(Q_{n-1}^{(s)} - Q_{n-1}^{(t)}) = (-1)^{n-1}$. If, in addition, all the $p_i''(x) > 0, x \in (0,1)$, then $sgn(Q_{n-1}^{(k+2)} - 2Q_{n-1}^{(k+1)} + Q_{n-1}^{(k)}) = (-1)^{n-1}$ $(1 \le k \le n-3)$.*

An interesting question is what will be the corresponding result if $p_i^{(r)}(x) > 0, x \in (0,1)$ $(1 \le i \le n)$ for some integer $r \le n-1$. We mention in passing that we can prove by induction that

$$sgn(\Delta^r(Q_{n-1}^{(1)})) = (-1)^{n-1-r},$$

where Δ is the difference operator $\Delta(Q_{n-1}^{(k)}) = Q_{n-1}^{(k)} - Q_{n-1}^{(k+1)}$ and where the entries, for example, in $\Delta^3(Q_{n-1}^{(k)})$ are identical to the entries in $\Delta^2(Q_{n-1}^{(k)})$ except that the 4-th column in $\Delta^2(Q_{n-1}^{(k)})$ is replaced by

$$\left(-b_{n,k}\left(\frac{1}{n}\right), -b_{n,k}\left(\frac{2}{n}\right), \cdots, -b_{n,k}\left(\frac{n-1}{n}\right)\right)^T,$$

for some k. Continuing this process, we can derive the sign of the determinant $\Delta^r(Q_{n-1}^{(k)})$.

5 Shape preservation

In this section we study shape preserving properties of Lototsky-Bernstein operators from several different points of view, including bounded variation, total positivity and convexity.

5.1 Bounded variation

In order to study bounded variation of the Lototsky-Bernstein operators, we introduce a probabilistic interpretation for the Lototsky-Bernstein operators L_n similar to the probabilistic interpretation of the classical Bernstein operator B_n [23]. Let $\xi_1, \cdots, \xi_n$ be a sequence of independent, and on the interval $[0, 1]$, uniformly distributed random variables and let

$$\mathcal{S}_n(x) := I_{(\xi_1 \leq p_1(x))} + I_{(\xi_2 \leq p_2(x))} + \cdots + I_{(\xi_n \leq p_n(x))},$$

where I_C denotes the indicator function of the event C. Then $L_n(f)$ can be represented in the form

$$L_n(f; x) = Ef\left(\frac{\mathcal{S}_n(x)}{n}\right), \tag{5.1}$$

where E denotes expectation. Let Φ be the set of all real-valued strictly increasing convex functions φ defined on $[0, \infty)$ such that $\varphi(0) = 0$. For $\varphi \in \Phi$ and any real-valued function f defined on $[0, 1]$, the $\varphi-$variation of f on $[0, 1]$ is defined by

$$V_\varphi(f) := \sup \sum_{i=1}^{n} \varphi\left(|f(x_i) - f(x_{i-1})|\right),$$

where the supremum is taken over all finite sequences $0 = x_0 \leq x_1 \leq \cdots \leq x_n = 1$.

Theorem 5.1. *Let f have bounded $\varphi-$variation on $[0, 1]$ and suppose that all the $p_i(x)$ $(1 \leq i \leq n)$ are nondecreasing. Then*

$$V_\varphi(L_n(f)) \leq V_\varphi(f). \tag{5.2}$$

Proof. Let $0 \leq x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$ and set $Z_n^x := \dfrac{\mathcal{S}_n(x)}{n}$. Then

$$Z_n^{x_0} \leq Z_n^{x_1} \leq \cdots \leq Z_n^{x_n}.$$

Therefore, using the fact that φ is non-decreasing together with Jensen's inequality (i.e., if X is a random variable and ϕ is a convex function, then $\phi(E[X]) \leq E[\phi(X)]$, see [28], p.61), yields

$$\begin{aligned}
\sum_{i=1}^{n} \varphi\left(|L_n(f; x_i) - L_n(f; x_{i-1})|\right) &= \sum_{i=1}^{n} \varphi\left(|Ef(Z_n^{x_i}) - Ef(Z_n^{x_{i-1}})|\right) \\
&\leq \sum_{i=1}^{n} \varphi\left(E|f(Z_n^{x_i}) - f(Z_n^{x_{i-1}})|\right) \\
&\leq E\left(\sum_{i=1}^{n} \varphi\left(|f(Z_n^{x_i}) - f(Z_n^{x_{i-1}})|\right)\right) \\
&\leq V_\varphi(f).
\end{aligned}$$

□

Remark 5.2. *Observe that* (5.2) *is an extension of the total variation diminishing property ($\varphi(x) = x$).*

5.2 Total Positivity and the Variation Diminishing Property

Here we focus on shape preservation based on total positivity and the variation diminishing property.

A system of functions $(u_0(x), u_1(x), \cdots, u_n(x))$ is (strictly) totally positive in an interval I if every collocation matrix

$$M\begin{pmatrix} u_0 & ,\cdots, & u_n \\ t_0 & ,\cdots, & t_n \end{pmatrix} := \begin{pmatrix} u_0(t_0) & \cdots & u_n(t_0) \\ \vdots & & \vdots \\ u_0(t_n) & \cdots & u_n(t_n) \end{pmatrix},$$

with

$$t_0 < \cdots < t_n, t_i \in I, 0 \le i \le n,$$

is (strictly) totally positive, that is all its sub-determinants are (positive) nonnegative.

Totally positive systems lead in a natural way to the variation diminishing property. For any real sequence c, finite or infinite, we denote by $S^-(c)$ the number of strict sign changes in c. For $f \in C[0,1]$, we define $S^-(f)$ to be the number of sign changes of f, that is

$$S^-(f) = \sup S^-(f(x_0), \cdots, f(x_m)),$$

where the supremum is taken over all increasing sequences $0 \le x_0 < \cdots < x_m \le 1$ for all positive integers m. We say that a sequence of positive linear operators $\{L_n\}_{n\ge1}$ is variation diminishing if for all functions $f \in C[0,1]$

$$S^-(L_n(f)) \le S^-(f). \tag{5.3}$$

Theorem 5.3. *([16]) If $(b_{n,0}(x), \cdots, b_{n,n}(x))$ is totally positive on $[0,1]$, then the Lototsky-Bernstein operator L_n is variation diminishing.*

The connection between total positivity and shape preserving properties of bases is a classical subject that has been widely studied ([16, 18]). In Section 5.3, we shall use the total positivity of $\{b_{n,k}(x)\}$ to prove that the corresponding operators are generalized convexity preserving. Next we study the total positivity of certain Lototsky-Bernstein basis functions, where the functions $p_i(x) = x^{N_i} (1 \le i \le n)$ are monomials whose exponents N_i satisfy some special conditions.

Theorem 5.4. *Suppose that $p_i(x) = x^{N_i}$ $(1 \le i \le n)$ are monomial functions, such that $N_1 \le \cdots \le N_n$ and*

$$\begin{aligned} &N_n \le N_1 + N_2, \\ &N_n + N_{n-1} \le N_1 + N_2 + N_3, \\ &\qquad\vdots \\ &N_n + N_{n-1} + \cdots + N_2 \le N_1 + N_2 + \cdots + N_n. \end{aligned} \tag{5.4}$$

Then the corresponding set of Lototsky-Bernstein basis functions $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ is totally positive on $[0,1]$.

In order to prove Theorem 5.4, we need to invoke the following lemma.

Lemma 5.5. *(Fekete's Lemma) Suppose that C is an $m \times n$ $(m \le n)$ matrix such that all $(m-1)$st order minors are strictly positive and the mth order minors composed from consecutive columns are also strictly positive. Then all mth order minors of C are strictly positive. (see [27], Lemma 2.1)*

Proof of Theorem 5.4

Proof. First we prove that every collocation matrix on (0, 1) is strictly totally positive. We proceed by induction on the order of the minors of the collocation matrix $\left(b_{n,i}(x_j)\right)_{0\leq i,j\leq n}$. The fact that all minors of order 1 of the collocation matrix $\left(b_{n,i}(x_j)\right)_{0\leq i,j\leq n}$ are positive follows from the positivity of the system $(b_{n,0}(x), \cdots, b_{n,n}(x))$ (see i in Section 2). We now assume that all the minors of order $r \geq 1$ are positive, and proceed to show that all the minors of order $r+1$ are positive. By Lemma 5.5, it suffices to show that all minors of order $r+1$ with consecutive columns are positive, i.e., $f_k(x_0, x_1 \cdots, x_r) > 0$ for $k = 0, \cdots, n-r$, with $0 < x_0 < x_1 < \cdots < x_n < 1$, where

$$f_k(x, x_1, \cdots, x_r) := \begin{vmatrix} b_{n,k}(x) & b_{n,k+1}(x) & \cdots & b_{n,k+r}(x) \\ b_{n,k}(x_1) & b_{n,k+1}(x_1) & \cdots & b_{n,k+r}(x_1) \\ \cdots & \cdots & \vdots & \cdots \\ b_{n,k}(x_r) & b_{n,k+1}(x_r) & \cdots & b_{n,k+r}(x_r) \end{vmatrix}. \tag{5.5}$$

Note that $p_i(x) = x^{N_i}$, $1 \leq i \leq n$ and $N_1 \leq \cdots \leq N_n$. Removing common factors in (5.5), we get

$$f_k(x, x_1, \cdots, x_r) = x^{N_1+\cdots+N_k}(1-x)^{n-k-r} \begin{vmatrix} c_{n,k}(x) & c_{n,k+1}(x) & \cdots & c_{n,k+r}(x) \\ b_{n,k}(x_1) & b_{n,k+1}(x_1) & \cdots & b_{n,k+r}(x_1) \\ \cdots & \cdots & \vdots & \cdots \\ b_{n,k}(x_r) & b_{n,k+1}(x_r) & \cdots & b_{n,k+r}(x_r) \end{vmatrix}, \tag{5.6}$$

where $b_{n,j}(x) = x^{N_1+\cdots+N_k}(1-x)^{n-k-r}c_{n,j}(x)$ $(k \leq j \leq k+r)$. Denote the determinant in (5.6) by $g_k(x, x_1, \cdots, x_r)$. Now expand the determinant in (5.5) along the first row and denote by $a_{k+t}, 0 \leq t \leq r$ the determinant of the submatrix formed by deleting the first row and $(t+1)^{st}$ column. Since all the a_{k+t} are minors of order r, they are all positive by the inductive hypothesis. Thus

$$f_k(x, x_1, \cdots, x_r) = a_k b_{n,k}(x) - a_{k+1} b_{n,k+1}(x) + \cdots + (-1)^r a_{k+r} b_{n,k+r}(x). \tag{5.7}$$

Substituting (2.4) into (5.7), we derive

$$\begin{aligned} f_k(x, x_1, \cdots, x_r) = a_k\binom{k}{k}\sigma_k(p_1(x), \cdots, p_n(x)) - \left(a_k\binom{k+1}{k} + a_{k+1}\binom{k+1}{k+1}\right)\sigma_{k+1}(p_1(x), \cdots, p_n(x)) + \\ \cdots + (-1)^{n-k}\left(a_k\binom{n}{k} + a_{k+1}\binom{n}{k+1} + \cdots + a_{k+r}\binom{n}{k+r}\right)\sigma_n(p_1(x), \cdots, p_n(x)). \end{aligned} \tag{5.8}$$

Therefore we can write f_k as a polynomial

$$f_k(x, x_1, \cdots, x_r) = N_{k,r}(N_1 + \cdots + N_n)x^{N_1+\cdots+N_n} + \cdots + N_{k,r}(N_1 + \cdots + N_k)x^{N_1+\cdots+N_k}, \tag{5.9}$$

where $N_{k,r}(N_1 + \cdots + N_n)$ is the coefficient of the highest power of f_k, so

$$N_{k,r}(N_1 + \cdots + N_n) = (-1)^{n-k}\left(a_k\binom{n}{k} + a_{k+1}\binom{n}{k+1} + \cdots + a_{k+r}\binom{n}{k+r}\right), \tag{5.10}$$

and $N_{k,r}(N_1 + \cdots + N_k)$ is the coefficient of the lowest power of f_k, so

$$N_{k,r}(N_1 + \cdots + N_k) = C_0 \cdot a_k, C_0 > 0. \tag{5.11}$$

The constant C_0 appears in (5.11) since there may be integers $i_1 \leq \cdots \leq i_k$ that satisfy $N_1 + \cdots + N_k = N_{i_1} + \cdots + N_{i_k}$. The coefficients of the other powers of f_k are not important to our proof.

By equating the coefficients of the highest power and the lowest power on both sides of equation (5.6) and by invoking equation (5.9), we derive the following expression for $g_k(x, x_1, \cdots, x_r)$

$$g_k(x, x_1, \cdots, x_r) = N'_{k,r}(N_1 + \cdots + N_n)x^{(N_{k+1}-1)+\cdots+(N_n-1)+r} + \cdots + N'_{k,r}(N_1 + \cdots + N_k), \tag{5.12}$$

where

$$N'_{k,r}(N_1 + \cdots + N_n) = (-1)^r \left(a_k \binom{n}{k} + a_{k+1} \binom{n}{k+1} + \cdots + a_{k+r} \binom{n}{k+r} \right),$$
$$N'_{k,r}(N_1 + \cdots + N_k) = N_{k,r}[N_1 + \cdots + N_k].$$

From Descartes' law of signs, the number of positive roots (counting multiplicities) of the polynomial $f_k(x, x_1, \cdots, x_r)$ is less than or equal to the number of sign changes between consecutive nonzero coefficients. Now notice that $H_j := N_n + \cdots + N_{n-j+1}$ and $L_j := N_1 + \cdots + N_j$ are the highest power and the lowest power of x in $\sigma_j(p_1(x), \cdots, p_n(x))$ $(k \le j \le n)$. By (5.4) $H_j \le L_{j+1}$ $(k \le j \le n-1)$. If for all $k \le j \le n-1$ we have the strict inequalities $H_j < L_{j+1}$, then by (5.8) the number of sign changes between consecutive nonzero coefficients of f_k is $n-k$. Suppose that for some $k \le j_0 \le n-1$ we have $H_{j_0} = L_{j_0+1}$. In this case, set A_j equal to the absolute value of the coefficient of $\sigma_j(p_1(x), \cdots, p_n(x))$ $(k \le j \le n)$ in (5.8). Then

$$\begin{aligned} &f_k(x, x_1, \cdots, x_r) = \\ &\cdots + (-1)^{j_0-k} \left[A_{j_0}\sigma_{j_0}(p_1(x), \cdots, p_n(x)) - A_{j_0+1}\sigma_{j_0+1}(p_1(x), \cdots, p_n(x)) \right] + \cdots . \end{aligned}$$

Thus the coefficient of $x^{H_{j_0}}$ is $(-1)^{j_0-k}(C_1 A_{j_0} - C_2 A_{j_0+1})$, where the constants C_1, C_2 appear here for the same reason that the constant C_0 appears in (5.11). If $C_1 A_{j_0} - C_2 A_{j_0+1} > 0$, we merge the term $(-1)^{j_0-k+1} C_2 A_{j_0+1} x^{L_{j_0+1}}$ into $(-1)^{j_0-k} A_{j_0}\sigma_{j_0}$; otherwise, we merge the term $(-1)^{j_0-k} C_1 A_{j_0} x^{H_{j_0}}$ into $(-1)^{j_0-k+1} A_{j_0+1}\sigma_{j_0+1}$. In either case, the number of sign changes between consecutive nonzero coefficients of f_k does not change. If, however, for some $k \le j_0 \le n-2$ we have $H_{j_0} = L_{j_0+2}$, then $L_{j_0+1} = H_{j_0+1}$. Thus

$$\begin{aligned} &L_{j_0+1} = N_1 + \cdots + N_{j_0+1} \le N_2 + \cdots + N_{j_0+2} \le \cdots \le N_{n-j_0} + \cdots + N_n = H_{j_0+1} \\ &\Rightarrow (N_2 - N_1) + \cdots + (N_{j_0+2} - N_{j_0+1}) = \cdots = (N_{n-j_0} - N_{n-j_0-1}) + \cdots + (N_n - N_{n-1}) = 0. \end{aligned} \tag{5.13}$$

By (5.13) and the fact that $N_1 \le \cdots \le N_n$, it follows that $N_1 = \cdots = N_n$. In this case, the proof of total positivity is similar to the classical case (see [16]). Therefore the number of positive roots of $f_k(x, x_1, \cdots, x_r)$ is less than or equal to $n-k$. Furthermore, $f_k(x, x_1, \cdots, x_r) = x^{N_1+\cdots+N_k}(1-x)^{n-k-r} g_k(x, x_1, \cdots, x_r)$. Hence, the number of positive roots of the polynomial $g_k(x, x_1, \cdots, x_r)$ is less than or equal to r. From (5.5) and (5.6) we know that $g(x_1, x_1, \cdots, x_r) = \cdots = g_k(x_r, x_1, \cdots, x_r) = 0$. Therefore, $g_k(x, x_1, \cdots, x_r)$ has no positive root(s) other than $x_1, \cdots, x_r$. Thus for $0 < x_0 < x_1 < \cdots < x_r < 1$, it follows that $g_k(x_0, x_1, \cdots, x_r)$ is either strictly positive or strictly negative. Moreover since

$$f_k(x_0, x_1, \cdots, x_r) = x_0^{N_1+\cdots+N_k}(1-x_0)^{n-k-r} g_k(x_0, x_1, \cdots, x_r),$$

it follows that for $0 < x_0 < x_1 < \cdots < x_r < 1$, we also have that $f_k(x_0, x_1, \cdots, x_r)$ is either strictly positive or strictly negative. It remains then to show that these signs are positive. But by (5.12), we have $g_k(0, x_1, \cdots, x_r) > 0$. Thus for $0 < x_0 < x_1 < \cdots < x_r < 1$, we have $g_k(x_0, x_1, \cdots, x_r) > 0$, so $f_k(x_0, x_1, \cdots, x_r) > 0$.

Finally total positivity of the basis $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ in $[0, 1]$ follows by continuity from the strict total positivity in $(0, 1)$. □

Similarly, we can prove that the system of functions $\{\sigma_0(p_1(x), \cdots, p_n(x)), \cdots, \sigma_n(p_1(x), \cdots, p_n(x))\}$ is totally positive under the assumptions in (5.4).

Lemma 5.6. *Under the same hypotheses and with the same notation as in Theorem 5.4, the system of functions* $\{\sigma_0(p_1(x), \cdots, p_n(x)), \cdots, \sigma_n(p_1(x), \cdots, p_n(x))\}$ *are totally positive on* $[0, \infty)$.

Proof. The proof is an almost verbatim extension of the reasoning given in Theorem 5.4. For $0 \le x_0 < x_1 < \cdots < x_n < \infty$, set the determinant in (5.5) to $f_k^\sigma(x, x_1, \cdots, x_r)$ $(1 \le r \le n)$, where for all u in (5.5) the functions $b_{n,j}(u)$ are replaced by the functions $\sigma_j(p_1(u), \cdots, p_n(u))$. Then expanding this determinant, we get an equation similar to (5.7), i.e.,

$$\begin{aligned} &f_k^\sigma(x, x_1, \cdots, x_r) = \\ &a_k^\sigma \sigma_k(p_1(x), \cdots, p_n(x) - a_{k+1}^\sigma \sigma_{k+1}(p_1(x), \cdots, p_n(x)) + \cdots + (-1)^r a_{k+r}^\sigma \sigma_{k+r}(p_1(x), \cdots, p_n(x)), \end{aligned} \tag{5.14}$$

where a_v^σ $(k \le v \le k + r)$ have meaning similar to (i.e. cofactors) a_v $(k \le v \le k + r)$ in (5.7). Thus by (5.4), (5.14) and Descartes' law of signs, the number of positive roots of $f_k^\sigma(x, x_1, \cdots, x_r)$ is less than or equal to r. But $f_k^\sigma(x_i, x_1, \cdots, x_r) = 0$ $(1 \le i \le r)$. Therefore $f_k^\sigma(x, x_1, \cdots, x_r)$ has no positive root(s) other than $x_1, \cdots, x_r$. Moreover, we can rewrite $f_k^\sigma(x, x_1, \cdots, x_r)$ as

$$f_k^\sigma(x, x_1, \cdots, x_r) = x^{N_1 + \cdots + N_k} g_k^\sigma(x, x_1, \cdots, x_r). \tag{5.15}$$

It is not difficult to see that $g_k^\sigma(0, x_1, \cdots, x_r) > 0$. Therefore $g_k^\sigma(x_0, x_1, \cdots, x_r) > 0$, so from (5.15) we conclude that $f_k^\sigma(x_0, x_1, \cdots, x_r) > 0$. □

The following lemma can easily be derived from the definition of total positivity (see [16]).

Lemma 5.7. *Suppose that* $(\phi_0, \cdots, \phi_n)$ *is totally positive on an interval I. If f is an increasing function from an interval J into I, then* $(\phi_0 \circ f, \cdots, \phi_n \circ f)$ *is totally positive on J.*

Using Theorem 5.4, we can deduce

Proposition 5.8. *Let* N_i $(1 \le i \le n)$ *satisfy the conditions in* (5.4), *and set* $p_i(x) = \dfrac{x^{N_i}}{(1-x)^{N_i} + x^{N_i}}$. *Then the Lototsky-Bernstein system* $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ *is totally positive on* $[0, 1]$.

Proof. By observing that $p_i(x)/(1 - p_i(x)) = (x/(1 - x))^{N_i}$ and invoking the increasing function $f(x) = x/(1 - x)$, we can apply Lemma 5.6 and Lemma 5.7 to show that for all sequences of abscissae $X_n = \{0 \le x_0 < \cdots < x_n \le 1\}$ the following determinant is nonnegative:

$$\begin{aligned} det(b_{n,i}(x_j)) &= \prod_{k=0}^{n} ((1 - p_1(x_k)) \cdots (1 - p_n(x_k))) det(\sigma_i(\frac{p_1(x_j)}{1 - p_1(x_j)}, \cdots, \frac{p_n(x_j)}{1 - p_n(x_j)})) \\ &= \prod_{k=0}^{n} ((1 - p_1(x_k)) \cdots (1 - p_n(x_k))) det(\sigma_i(f(x_j^{N_1}), \cdots, f(x_j^{N_n}))). \end{aligned}$$

□

Remark 5.9. *It is evident that for every positive integer N, the functions* $p(x) = \dfrac{x^N}{(1-x)^N + x^N}$ *satisfy* $p'(x) = \dfrac{N x^{N-1}(1-x)^{N-1}}{[(1-x)^N + x^N]^2} \ge 0$ *on* $[0, 1]$, *i.e.,* $p(x)$ *is strictly increasing, and* $p(0) = 0, p(1) = 1$.

If $N_1 = \cdots = N_n = 1$, *i.e.* $p_i(x) = x, 1 \le i \le n$, *then* $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ *are the classical Bernstein basis functions. If* $N_1 = \cdots = N_{n-1} = 1, N_n = 2$, *then*

$$b_{n,k}(x) = \frac{x^k(1-x)^{n-k}}{(1-x)^2 + x^2} \left(\binom{n-1}{k}(1-x) + \binom{n-1}{k-1} x \right) (0 \le k \le n).$$

We believe that if all the $p_i(x)$ $(1 \le i \le n)$ are monomial functions, the Lototsky-Bernstein basis functions are totally positive even though we can not currently prove this result.

Conjecture 5.10. *Suppose that all the $p_i(x)$ $(1 \le i \le n)$ are monomial functions. Then the Lototsky-Bernstein basis functions $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ are totally positive on* $[0, 1]$.

Proposition 5.11. *Assume that Conjecture 5.10 is correct. Suppose $p_i(x) = x^{\rho_i}, 1 \le i \le n$ are power functions where all the ρ_i are positive rational numbers. Then the system $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ is totally positive on* $[0, 1]$.

Proof. Since all the ρ_i, $1 \le i \le n$ are positive rational numbers, there must exist a positive integer n_0 such that for all i, the numbers $n_0 \cdot \rho_i$ are positive integers. Thus by Conjecture 5.10, the system $(b^q_{n,0}(x), b^q_{n,1}(x), \cdots, b^q_{n,n}(x))$ with respect to $q_i(x) = x^{n_0 \cdot \rho_i}$ is totally positive on $[0, 1]$. Set $f(x) = x^{1/n_0}$. Then $f(x)$ is an increasing function on $[0, 1]$. Therefore by Lemma 5.7

$$(b^q_{n,0}(f(x)), b^q_{n,1}(f(x)), \cdots, b^q_{n,n}(f(x))) = (b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x)),$$

is totally positive on $[0, 1]$. □

Corollary 5.12. *Assume that Conjecture 5.10 is correct. Suppose $p_i(x) = x^{\alpha_i}, \alpha_i \in \mathbb{R}$ $(1 \le i \le n)$. Then the Lototsky-Bernstein system $(b_{n,0}(x), b_{n,1}(x), \cdots, b_{n,n}(x))$ is totally positive on* $[0, 1]$.

Proof. This result follows from Proposition 5.11 by taking the limit of rational powers for all the $p_i(x)$. □

5.3 Convexity

We begin our study of convexity by using the well known Abel transformation (5.16) to rewrite formula (1.5). Suppose $\{x_k\}$ and $\{y_k\}$ are two real sequences, then

$$\sum_{k=m}^{n} x_k(y_k - y_{k+1}) = (x_m y_m - x_{n+1} y_{n+1}) + \sum_{k=m}^{n} y_{k+1}(x_{k+1} - x_k). \tag{5.16}$$

Using (5.16) first with $x_k = f(k/n)$ and $y_k = \sum\limits_{j=k}^{n} b_{n,j}(x)$ and then with $x_k = f((k+1)/n) - f(k/n)$, and $y_k = \sum\limits_{j=k}^{n-1} \sum\limits_{i=j+1}^{n} b_{n,i}(x)$, it follows that

$$\begin{aligned} &L_n(f; x) \\ &= f(0) + \sum_{k=0}^{n-1} \left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right) \sum_{j=k+1}^{n} b_{n,j}(x) \\ &= f(0) + \left(f\left(\frac{1}{n}\right) - f(0) \right) \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} b_{n,i}(x) + \sum_{k=0}^{n-2} \left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right) \right) \sum_{j=k+1}^{n-1} \sum_{i=j+1}^{n} b_{n,i}(x), \end{aligned} \tag{5.17}$$

To simplify our notation, we define

$$V_{1,n,k}(x) := \sum_{j=k+1}^{n} b_{n,j}(x) \quad (0 \le k \le n-1),$$

and

$$V_{2,n,k}(x) := \sum_{j=k+1}^{n-1} \sum_{i=j+1}^{n} b_{n,i}(x) = \sum_{j=k+2}^{n} (j-k-1) b_{n,j}(x) \quad (-1 \le k \le n-2).$$

Remark 5.13. *It is not difficult to show that $V_{1,n,k}(x)$ is increasing if all the $p_i(x) \in C^1[0,1]$ are increasing. This result follows by setting $f(j/n) = 0$ $(0 \le j \le k)$, $f(j/n) = 1$ $(k+1 \le j \le n)$, so that $V_{1,n,k}(x) = L_n(f;x)$ and then using* (3.3). *Moreover $V_{2,n,k}(x)$ is convex if all the $p_i(x) \in C^2[0,1]$ are increasing and convex. This result follows by setting $f(j/n) = 0$ $(0 \le j \le k+1)$, $f(j/n) = j-k-1$ $(k+2 \le j \le n)$, so that $V_{2,n,k}(x) = L_n(f;x)$ and then using* (3.7). *Thus, if all the $p_i(x) \in C^2[0,1]$ are increasing and convex, and if $f(x)$ is increasing near the left end point of the interval* $[0,1]$ *and convex on* $[0,1]$, *then by* (5.17) *for n sufficiently large, $L_n(f)$ is convex.*

Next we are going to study general convexity preserving properties. To investigate this topic, we begin with a few definitions.

Definition 5.14. *The functions (f_0, f_1) form a Haar pair on $E \subset \mathbb{R}$ if $f_0 > 0$ and f_1/f_0 is strictly increasing on E.*

Definition 5.15. *A function $f : E \to \mathbb{R}$ is called convex on $E \subset \mathbb{R}$ with respect to a Haar pair (f_0, f_1), if for all x_0, x_1, x_2 in E with $x_0 \le x_1 \le x_2$, the determinant*

$$Det_{x_0,x_1,x_2}(f) := det\begin{pmatrix} f_0(x_0) & f_0(x_1) & f_0(x_2) \\ f_1(x_0) & f_1(x_1) & f_1(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{pmatrix},$$

is non-negative. We shall also use the shorter expression "(f_0, f_1)-convex". Likewise, we say that f is $(f_0, f_1)-$concave if $-f$ is (f_0, f_1)-convex.

From Definition 5.15, it follows that $(1, x)-$convex is standard convexity.

The following theorem can be proved in a manner similar to the proof of Theorem 15 in [1], see also Theorem 2 in [36].

Theorem 5.16. *Suppose that $p_i(x) \in C[0,1]$ $(1 \le i \le n)$ are strictly increasing and $p_i(0) = 0, p_i(1) = 1$. Then for every $(1, \gamma_n^p)$-convex function $f \in C[0,1]$, we have $L_n(f;x) \ge f(x)$.*

Moreover, by the positivity of L_n

$$f(0) + (f(1) - f(0))\gamma_n^p(x) \ge \cdots \ge (L_n)^M(f) \ge \cdots \ge (L_n)^2(f) \ge L_n(f) \ge f, \tag{5.18}$$

Remark 5.17. *One of the standard definitions of convexity stipulates that the graph of f between any two points must lie below the line segment joining these two points. From* (5.18), *an analogous characterization holds for $(1, \gamma_n^p(x))-$convex functions, but with affine functions replaced by $(1, \gamma_n^p(x))-$affine functions $f(0) + (f(1) - f(0))\gamma_n^p(x)$. This result is a straightforward consequence of iteration.*

In general, we have

Proposition 5.18. *Suppose that $p_i(x) \in C[0,1]$ $(1 \le i \le n)$ are strictly increasing and $p_i(0) = 0, p_i(1) = 1$. If f is $(1, \gamma_n^p(x))-$convex, then for all* $0 \le x_1 < x_2 \le 1$

$$f(x) \le f(x_1) + (f(x_2) - f(x_1))\frac{\gamma_n^p(x) - \gamma_n^p(x_1)}{\gamma_n^p(x_2) - \gamma_n^p(x_1)}, x \in [x_1, x_2]. \tag{5.19}$$

Proof. Let $L_n^*(f;x) = f(x_1) + (f(x_2) - f(x_1))\dfrac{\gamma_n^p(x) - \gamma_n^p(x_1)}{\gamma_n^p(x_2) - \gamma_n^p(x_1)}$. Then $L_n^*(f;x)$ interpolates f at the points $x_1 < x_2, x_1, x_2 \in [0,1]$. From Definition 5.15, if f is $(1, \gamma_n^p(x))-$convex

$$Det_{x_1,x,x_2}(f) = Det_{x_1,x,x_2}(f - L_n^*(f)) = -(f(x) - L_n^*(f;x))(\gamma_n^p(x_2) - \gamma_n^p(x_1)) \ge 0, \tag{5.20}$$

Thus $f(x) \le L_n^*(f;x)$. □

To state our next convexity result, let X and Y be two subsets of $\mathbb{R}$. A function $K : X \times Y \to \mathbb{R}$ is called *positive of order m* if $det(K(x_i, y_j))_{1\le i,j\le m} \ge 0$ for all $x_1 < \cdots < x_m$ in X and all $y_1 < \cdots < y_m$ in Y (see [18]). Now let $C(X)$ and $C(Y)$ denote the spaces of continuous functions on X and Y. Adopting the notation of [1], the following result is well known ([1], p.18); for more details see [18] (p. 284).

Theorem 5.19. *Suppose that X and Y are two subsets of $\mathbb{R}$. Let (F_0, F_1) form a Haar pair on Y and let $K : X \times Y \to \mathbb{R}$ be continuous and positive of order 3. Furthermore let μ be a non-negative sigma-finite measure and let $B_K : C(Y) \to C(X)$ be defined by*

$$B_K(F)(x) := \int_Y K(x, y)F(y)d\mu(y).$$

Then $(B_K F_0, B_K F_1)$ constitute a Haar pair on X. Moreover, if F is (F_0, F_1)-convex, then $B_K(F)$ is $(B_K F_0, B_K F_1)$-convex.

Put $X = [0, 1]$ and $Y = \{0, 1, \cdots, n\}$. Then every $F : Y \to \mathbb{R}$ is continuous. Set $\mu = \sum_{k=0}^{n} \delta_k$, where δ_k is the Dirac measure at the point $k \in \{0, 1, \cdots, n\}$. With $K(x, k) = b_{n,k}(x)$, define

$$B_K(F)(x) := \int_Y K(x, y)F(y)d\mu(y) = \sum_{k=0}^{n} F(k)b_{n,k}(x).$$

Now define the strictly increasing function $\psi : Y \to X$ by $\psi(k) := k/n$ ($k = 0, \cdots, n$). Then for $f : X \to \mathbb{R}$, we know $f \circ \psi : Y \to \mathbb{R}$ is continuous, and

$$B_K(f \circ \psi)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x) = L_n(f; x). \tag{5.21}$$

If f is $(f_0, f_1)-$convex, then $f \circ \psi$ is $(f_0 \circ \psi, f_1 \circ \psi)-$convex. (This result follows from Definition 5.15, since for every $x_0 < x_1 < x_2$, we have $\psi(x_0) < \psi(x_1) < \psi(x_2)$). Applying Theorem 5.19 it follows that if $\{b_{n,k}(x)\}$ is positive of order 3, then $B_K(f \circ \psi)$ is $(B_K(f_0 \circ \psi), B_K(f_1 \circ \psi))-$convex. Since f is $(f_0, f_1)-$convex, it follows by (5.21) that $L_n(f; x)$ is $(L_n(f_0), L_n(f_1))-$convex. Therefore, we have the following general shape preserving property for the operators L_n.

Theorem 5.20. *Suppose that all the $p_i(x)$, $1 \le i \le n$, satisfy one of the following two conditions:*

1) $p_i(x) = x^{N_i}$;

2) $p_i(x) = \dfrac{x^{N_i}}{(1-x)^{N_i} + x^{N_i}}$.

where all the N_i ($1 \le i \le n$) satisfy the conditions in (5.4).

If f is $(1, \gamma_n^p)$-convex, then $L_n(f; x)$ is $(1, \gamma_n^p)$-convex. More generally, for a Haar pair (f_0, f_1) in $C[0, 1]$, if f is (f_0, f_1)-convex, then $L_n(f; x)$ is $(L_n(f_0), L_n(f_1))$-convex. Thus, it follows that if f is convex in the standard sense, then $L_n(f; x)$ is $(1, \sum_{k=1}^{n} p_k(x))$-convex.

Proof. From Theorem 5.4 and Proposition 5.8, we know that $K(x, k) := b_{n,k}(x)$ is totally positive, so it is positive of order 3. Therefore these results follows from Theorem 5.19 (see also Theorem 22 in [1]). □

Remark 5.21. *In Theorem 5.20, we use $L_n(e_1; x) = \sum_{k=1}^{n} p_k(x)/n$, where $e_1(x) = x$. (see [21]).*

6 Dependence on $p_i(x)$

In order to study the dependence of the operators $L_n(f)$ on the functions $p_i(x)$, we introduce the following notation. Given two sequences $\mathbb{P}_n(x) = (p_1(x), \cdots, p_n(x))$ and $\mathbb{Q}_n(x) = (q_1(x), \cdots, q_n(x))$, if for all $i \in \{1, \cdots, n\}$, $p_i(x) \geq q_i(x)$ for all $x \in [0,1]$, then we write $\mathbb{P}_n(x) \geq \mathbb{Q}_n(x)$.

Theorem 6.1. *Consider two sequences of functions $\mathbb{P}_n(x)$ and $\mathbb{Q}_n(x)$ such that $\mathbb{P}_n(x) \geq \mathbb{Q}_n(x)$. If f is increasing, then*

$$L_n(f; x; \mathbb{P}_n) \geq L_n(f; x; \mathbb{Q}_n). \tag{6.1}$$

Proof. Since $L_n(f; x; \mathbb{P}_n)$ is a symmetric function in the functions $p_i(x), 1 \leq i \leq n$, we take, for example $p_n(x) \geq q_n(x)$ and $p_i(x) = q_i(x), 1 \leq i \leq n-1$ to explain the main idea of the proof.

By (2.2)

$$\begin{aligned}
L_n(f; x; \mathbb{P}_n) &= \sum_{k=0}^{n} b_{n,k}(x) f(k/n) \\
&= \sum_{k=0}^{n} (p_n(x) b_{n-1,k-1}(x) + (1 - p_n(x)) b_{n-1,k}(x)) f(k/n) \\
&= \sum_{k=0}^{n-1} b_{n-1,k}(x) f(k/n) + p_n(x) \sum_{k=0}^{n} (b_{n-1,k-1}(x) - b_{n-1,k}(x)) f(k/n) \\
&= \sum_{k=0}^{n-1} b_{n-1,k}(x) f(k/n) + p_n(x) \sum_{k=0}^{n-1} (f((k+1)/n) - f(k/n)) b_{n-1,k}(x) \\
&\geq \sum_{k=0}^{n-1} b_{n-1,k}(x) f(k/n) + q_n(x) \sum_{k=0}^{n-1} (f((k+1)/n) - f(k/n)) b_{n-1,k}(x) \\
&= \sum_{k=0}^{n} (q_n(x) b_{n-1,k-1}(x) + (1 - q_n(x)) b_{n-1,k}(x)) f(k/n) = L_n(f; x; \mathbb{Q}_n),
\end{aligned}$$

where the inequality follows because f is increasing. □

Corollary 6.2. *If $\mathbb{P}_n(x) \geq \mathbb{Q}_n(x)$, then $\gamma_n^p(x) \geq \gamma_n^q(x)$. Thus, if $p_i(x) \leq x$, (or $p_i(x) \geq x$) for all $1 \leq i \leq n$, then $\gamma_n^p(x) \leq x$ ($\gamma_n^p(x) \geq x$).*

Proof. Set $L_n^p(f; x) := L_n(f; x; \mathbb{P}_n)$. If $p_i(x) \geq q_i(x), 1 \leq i \leq n$, then by Theorem 6.1 $L_n^p(f; x) \geq L_n^q(f; x)$ for every increasing function f. Choose an increasing function f such that $f(0) = 0, f(1) = 1$. By repeatedly using (6.1) combined with the positivity of L_n, we find that

$$\begin{aligned}
&L_n^p(f; x) \geq L_n^q(f; x) \Rightarrow (L_n^p)^2(f; x) \geq L^p(L_n^q)(f; x) \geq (L_n^q)^2(f; x) \\
&\cdots \Rightarrow (L_n^p)^M(f; x) \geq (L_n^q)^M(f; x).
\end{aligned} \tag{6.2}$$

Taking the limit of both sides of 6.2 as $M \to \infty$, yields $\gamma_n^p(x) \geq \gamma_n^q(x)$. □

Remark 6.3. *Suppose that $p_i(x) \in C[0,1]$ $(i \geq 1)$ are strictly increasing with $p_i(x) \leq x$ for all i and $p_i(0) = 0, p_i(1) = 1$. If $f \in C[0,1]$ is strictly increasing, then $L_n(f; x; \mathbb{P}_n) \leq B_n(f; x)$. If, in addition, f is $(1, \gamma_n^p(x))-$convex, then by* (6.1) *and* (5.18)

$$f(x) \leq L_n(f; x; \mathbb{P}_n) \leq B_n(f; x). \tag{6.3}$$

Thus f is approximated better by the Lototsky-Bernstein operators $L_n(f; x, \mathbb{P}_n(x))$ than by the classical Bernstein operators. This conclusion generalizes Theorem 1 of [6] and Theorem 9 of [5]. Moreover, if for all n, f is $(1, \gamma_n^p(x))$−convex, then from (6.3) $\lim_{n\to\infty} L_n(f; x; \mathbb{P}_n) = f(x)$. *In this case, by the strict monotonicity of f, we will see later by using Theorem 7.6 that*

$$\lim_{n\to\infty} (p_1(x) + \cdots + p_n(x))/n = x.$$

7 Nesting

The space $U_n := span\{\sigma_0(p_1(x), \cdots, p_n(x)), \cdots, \sigma_n(p_1(x), \cdots, p_n(x))\}$ defined in Section 2 is not generally nested, ie., $U_n \subset U_{n+1}$ does not hold except when all the $p_i(x)$ $(1 \le i \le n)$ are identical. Therefore, in general, we do not have a degree elevation formula like

$$b_{n,k}(x) = \lambda_k b_{n+1,k}(x) + (1 - \lambda_{k+1}) b_{n+1,k+1}(x). \tag{7.1}$$

Temple shows in [32] that for every standard convex ((1, x)−convex) function f the monotonicity property

$$B_n(f; x) \ge B_{n+1}(f; x), \tag{7.2}$$

holds for the classical Bernstein operators $B_n(f; x)$. By using a degree elevation formula like (7.1) Aldaz, Kounchev and Render [1] show for Bernstein operators $B_n(f; x)$ satisfying $B_n(f_i; x) = f_i(x)$ with $f_i \in U_n$ $(i = 0, 1)$ in Extended Chebyshev spaces U_n which satisfy $U_n \subset U_{n+1} \subset C^{n+1}[0, 1]$ that if f is (f_0, f_1)−convex, then (7.2) holds as well.

However, for Lototsky-Bernstein operators $L_n(f; x)$ we have following proposition.

Proposition 7.1. *Suppose that $p_i(x) \in C[0, 1]$ $(i \ge 1)$ are strictly increasing and $p_i(0) = 0, p_i(1) = 1$. If for every $(1, \gamma_{n+1}^p(x))$−convex function f*

$$L_n(f; x) \ge L_{n+1}(f; x), \tag{7.3}$$

then $\gamma_n^p(x) = \gamma_{n+1}^p(x)$.

Proof. Suppose that (7.3) holds for every $(1, \gamma_{n+1}^p(x))$−convex function f. By Theorem 5.16 $L_{n+1}(f; x) \ge f$. Therefore

$$L_n(f; x) \ge L_{n+1}(f; x) \ge f. \tag{7.4}$$

So for every $(1, \gamma_{n+1}^p(x))$−convex function f, $L_n(f; x) \ge f$. Since both $\gamma_{n+1}^p(x)$ and $-\gamma_{n+1}^p(x)$ are $(1, \gamma_{n+1}^p(x))$−convex functions, (7.4) implies that

$$L_n(\gamma_{n+1}^p; x) \ge \gamma_{n+1}^p(x), \tag{7.5}$$
$$L_n(-\gamma_{n+1}^p; x) \ge -\gamma_{n+1}^p(x). \tag{7.6}$$

Inequalities (7.5) and (7.6) yield

$$L_n(\gamma_{n+1}^p; x) = \gamma_{n+1}^p(x). \tag{7.7}$$

But by Theorem 4.2 $\gamma_n^p(x)$ is the unique fixed point of $L_n(f; x)$. Thus, $\gamma_n^p(x) = \gamma_{n+1}^p(x)$. □

Remark 7.2. *From the proof of Proposition 7.1, we see that one necessary condition for* (7.3) *to hold is that for all n the fixed points of $L_n(f; x)$ must be identical.*

Next we discuss conditions on the sequence of real-valued functions $p_i(x)$ $(i \geq 1)$ which ensure that for all n the fixed points $\gamma_n^p(x)$ of $L_n(f;x)$ are identical.

Proposition 7.3. *Suppose that $p_1(x) \in C[0,1]$ is strictly increasing with $p_1(0) = 0, p_1(1) = 1$. Then all the $p_i(x)$ $(i \geq 2)$ are uniquely determined by the recursive formula*

$$p_{n+1}(x) = \frac{\sum_{k=0}^{n} \left[p_1(k/n) - p_1(k/(n+1))\right] b_{n,k}(x)}{\sum_{k=0}^{n} \left[p_1((k+1)/(n+1)) - p_1(k/(n+1))\right] b_{n,k}(x)}, \tag{7.8}$$

if and only if $\gamma_n^p(x) = \gamma_{n+1}^p(x)$ for all n. Moreover, in this case, all the $p_i(x)$ $(i \geq 2)$ satisfy $0 < p_i(x) < 1, x \in (0,1)$ and $p_i(0) = 0, p_i(1) = 1, p_i(1/2) = 1/2$.

Proof. Suppose that $\gamma_n^p(x) = \gamma_{n+1}^p(x)$ for all n. By Remark 4.3 and equation (4.4)

$$\gamma_n^p(x) = p_1(x) = \sum_{k=0}^{n} p_1(k/n) b_{n,k}(x). \tag{7.9}$$

Moreover, by (2.2),

$$\begin{aligned}
\gamma_{n+1}^p(x) &= \sum_{k=0}^{n+1} p_1(k/(n+1)) b_{n+1,k}(x) \\
&= \sum_{k=0}^{n+1} p_1(k/(n+1)) \left(p_{n+1}(x) b_{n,k-1}(x) + (1 - p_{n+1}(x)) b_{n,k}(x)\right) \\
&= p_{n+1}(x) \sum_{k=0}^{n} p_1((k+1)/(n+1)) b_{n,k}(x) + (1 - p_{n+1}(x)) \sum_{k=0}^{n} p_1(k/(n+1)) b_{n,k}(x).
\end{aligned} \tag{7.10}$$

Therefore,

$$\gamma_n^p(x) = \gamma_{n+1}^p(x) \Rightarrow$$
$$\sum_{k=0}^{n} p_1(k/n) b_{n,k}(x) = p_{n+1}(x) \sum_{k=0}^{n} p_1((k+1)/(n+1)) b_{n,k}(x) + (1 - p_{n+1}(x)) \sum_{k=0}^{n} p_1(k/(n+1)) b_{n,k}(x). \tag{7.11}$$

Now (7.8) follows since (7.8) is equivalent to (7.11).

Conversely, suppose that (7.8) holds, or equivalently for all n, (7.11) holds. We proceed by induction on n to prove that $\gamma_n^p(x) = \gamma_{n+1}^p(x)$ for all n. Recall from Remark 4.3 that $p_1(x) = \gamma_1^p(x)$. Now suppose that $p_1(x) = \gamma_1^p(x) = \cdots = \gamma_n^p(x), n \geq 1$. Then by the induction hypothesis and (7.11) we find that

$$p_1(x) = \gamma_n^p(x) = \sum_{k=0}^{n} p_1(k/n) b_{n,k}(x) = \sum_{k=0}^{n+1} p_1(k/(n+1)) b_{n+1,k}(x) = L_{n+1}(p_1; x). \tag{7.12}$$

Equation (7.12) implies that $p_1(x)$ is a fixed point of $L_{n+1}(f;x)$. But by Theorem 4.2 $\gamma_{n+1}^p(x)$ is the unique fixed point of $L_{n+1}(f;x)$. It follows that $p_1(x) = \gamma_n^p(x) = \gamma_{n+1}^p(x)$.

Since by assumption $p_1(x)$ is strictly increasing, $p_1((k+1)/(n+1)) \geq p_1(k/n)$ $(0 \leq k \leq n)$. Therefore by (7.8) $0 < p_i(x) < 1, x \in (0,1)$ and $p_i(0) = 0, p_i(1) = 1$ $(i \geq 2)$.

In order to prove $p_i(1/2) = 1/2$ $(i \geq 2)$, we also proceed by induction on n. For $n = 2$ since $\gamma_1^p(x) = \gamma_2^p(x)$, again by Remark 4.3,

$$p_1(x) = p_1(x)p_2(x) + \frac{p_1(\frac{1}{2})p_2(\frac{1}{2})}{1 - p_1(\frac{1}{2}) - p_2(\frac{1}{2}) + 2p_1(\frac{1}{2})p_2(\frac{1}{2})}\left[p_1(x) + p_2(x) - 2p_1(x)p_2(x)\right]. \quad (7.13)$$

Setting $x = \dfrac{1}{2}$, it follows that

$$p_1(\frac{1}{2})(1 - p_1(\frac{1}{2}) - 2p_2(\frac{1}{2}) + 2p_1(\frac{1}{2})p_2(\frac{1}{2})) = p_1(\frac{1}{2})(1 - p_1(\frac{1}{2}))(1 - 2p_2(\frac{1}{2})) = 0.$$

Since $p_1(x)$ is strictly increasing with $p_1(0) = 0, p_1(1) = 1$, we have $p_1(1/2) \neq 0, p_1(1/2) \neq 1$; therefore $p_2(\frac{1}{2}) = \frac{1}{2}$. Thus equation (7.13) simplifies to

$$p_1(x) = p_1(x)p_2(x) + p_1(\frac{1}{2})\left[p_1(x) + p_2(x) - 2p_1(x)p_2(x)\right]. \quad (7.14)$$

Now suppose that $p_i(1/2) = 1/2, 2 \leq i \leq n$. Then by the induction hypothesis and (1.4)

$$b_{n,k}(1/2) = \left(p_1(1/2)\binom{n-1}{k-1} + (1 - p_1(1/2))\binom{n-1}{k}\right)2^{-n+1}, \quad 0 \leq k \leq n. \quad (7.15)$$

Since $p_{n+1}(1/2)$ is uniquely determined by (7.8), it suffices to prove that substituting $x = 1/2$ in the right-hand side of (7.8) yields $1/2$. Equivalently, since (7.8) is equivalent to (7.11), it suffices to prove that

$$\begin{aligned} p_1(1/2) &= 1/2\sum_{k=0}^{n}\left(p_1((k+1)/(n+1))b_{n,k}(1/2) + p_1(k/(n+1))b_{n,k}(1/2)\right) \quad (7.16)\\ &= 1/2\sum_{k=0}^{n} p_1((k+1)/(n+1))(b_{n,k}(1/2) + b_{n,k+1}(1/2)). \end{aligned}$$

But, by (7.15), for $0 \leq k \leq n$,

$$\begin{aligned} &b_{n,k}(1/2) + b_{n,k+1}(1/2)\\ &= \left(p_1(1/2)\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] + (1 - p_1(1/2))\left[\binom{n-1}{k} + \binom{n-1}{k+1}\right]\right)2^{-n+1}\\ &= 2^{-n+1}\left(p_1(1/2)\binom{n}{k} + (1 - p_1(1/2))\binom{n}{k+1}\right) = 2b_{n+1,k+1}(1/2). \end{aligned}$$

Substituting this result into the right-hand side of (7.16), we obtain

$$1/2\sum_{k=0}^{n} p_1((k+1)/(n+1))2b_{n+1,k+1}(1/2) = p_1(1/2), \quad (7.17)$$

as required. □

Proposition 7.4. *Suppose that all the $p_i(x)$ $(i \geq 1)$ are monomial functions. Then for all n*

$$\gamma_n^p(x) = \gamma_{n+1}^p(x) \quad (7.18)$$

if and only if $p_i(x) = x$ $(i \geq 1)$.

Proof. " $\Leftarrow$ " is straightforward.

" $\Rightarrow$ ". We shall prove that all the $p_i(x), i \geq 1$ must be identical. We proceed by induction on n. Setting $p_1(x) = x^m, p_2(x) = x^r, m, r \in \mathbb{N}$, it follows from (7.14) that

$$x^m = x^{m+r} + p_1(\frac{1}{2})[x^m + x^r - 2x^{m+r}]$$
$$\Rightarrow (1 - p_1(\frac{1}{2}))x^m - p_1(\frac{1}{2})x^r - (1 - 2p_1(\frac{1}{2}))x^{m+r} = 0. \tag{7.19}$$

If $m \neq r$, then the system $\{x^m, x^r, x^{m+r}\}$ is linearly independent. From (7.19) it would then follow that $p_1(\frac{1}{2}) = 1, p_1(\frac{1}{2}) = 0$ and $p_1(\frac{1}{2}) = \frac{1}{2}$, yielding a contradiction. Thus $p_1(x) = p_2(x) = x^m$. Combined with $p_2(\frac{1}{2}) = \frac{1}{2}$, it follows that $p_1(x) = p_2(x) = x$.

Now assume that $p_1(x) = \cdots = p_n(x) = x$. Since for all $n, \gamma_n^p(x) = \gamma_{n+1}^p(x) = p_1(x)$, it follows by equation (2.3) with $k = 1$ that

$$\sum_{k=1}^{n} p_1\left(\frac{k}{n}\right) b_{n,k}(x) = \sum_{k=1}^{n+1} p_1\left(\frac{k}{n+1}\right) b_{n+1,k}(x)$$
$$\Rightarrow \frac{p_1(x) + \cdots + p_n(x)}{n} = \frac{p_1(x) + \cdots + p_n(x) + p_{n+1}(x)}{n+1}$$
$$\Rightarrow p_{n+1}(x) = x. \tag{7.20}$$

□

Combining Equation (7.2), Proposition 7.1, Remark 7.2 and Proposition 7.4, we have

Proposition 7.5. *Suppose that all the $p_i(x)$ $(1 \leq i \leq n+1)$ are monomial functions. For every $(1, \gamma_{n+1}^p(x))-$convex function f*

$$L_n(f; x) \geq L_{n+1}(f; x) \tag{7.21}$$

for all n if and only if $p_1(x) = \cdots = p_{n+1}(x) = x$.

To prove part (ii) of Proposition 7.7, we shall need to use the following theorem. To prove this theorem, we shall make use of the following identities (see [21] ((9),(10), see also (2.3) with $k = 1$): Let $e_1(x) = x$ and $e_2(x) = x^2$, then

$$L_n(e_1; x) = \sum_{i=1}^{n} p_i(x)/n, \qquad L_n(e_2; x) = \left(\sum_{i=1}^{n} p_i(x)\right)^2 /n^2 - \sum_{i=1}^{n} p_i^2(x)/n^2 + \sum_{i=1}^{n} p_i(x)/n^2. \tag{7.22}$$

Theorem 7.6. *Suppose that $p_i(x) \in C[0, 1]$, $0 < p_i(x) < 1$ for $x \in (0, 1)$ and $p_i(0) = 0, p_i(1) = 1$ $(i \geq 1)$. If $(p_1(x) + \cdots + p_n(x))/n$ converges, then for $f \in C[0, 1]$ the sequence $\{L_n(f; x)\}_{n=1}^{\infty}$ converges to a bounded function $L_\infty(f; x)$, where*

$$L_\infty(f; x) = \lim_{n\to\infty} L_n(f; x) = f\left(\lim_{n\to\infty} \frac{p_1(x) + \cdots + p_n(x)}{n}\right). \tag{7.23}$$

Moreover, if $\lim_{n\to\infty}(p_1(x) + \cdots + p_n(x))/n$ converges uniformly on $[0, 1]$, then the convergence in (7.23) is uniform on $[0, 1]$.

Conversely, if for some strictly monotone function $f_0 \in C[0, 1]$, the sequence $\{L_n(f_0; x)\}_{n=1}^{\infty}$ converges, then $(p_1(x) + \cdots + p_n(x))/n$ also converges.

Proof. The following proof is standard. Since $f \in C[0,1]$, there exists a positive number $M > 0$ such that $|f(x)| \leq M$ for $x \in [0,1]$. Moreover, given any $\varepsilon > 0$, we can always find a $\delta > 0$ such that for any $x, y \in [0,1]$, we have $|f(x)-f(y)| < \varepsilon$ whenever $|x-y| < \delta$. If $\varrho_n(x) := (p_1(x)+\cdots+p_n(x))/n$ converges, set $\mu(x) := \lim_{n\to\infty} \varrho_n(x)$; then $\psi_n(x) := \varrho_n(x) - \mu(x) \to 0$ as $n \to \infty$. Therefore, for a fixed $x \in [0,1]$

$$
\begin{aligned}
&|L_n(f;x) - f(\mu(x))| \\
&\leq \sum_{|k/n-\mu(x)|<\delta} |f(k/n) - f(\mu(x))| b_{n,k}(x) + \sum_{|k/n-\mu(x)|\geq\delta} |f(k/n) - f(\mu(x))| b_{n,k}(x) \\
&\leq \varepsilon \sum_{|k/n-\mu(x)|<\delta} b_{n,k}(x) + \frac{2M}{\delta^2} \sum_{|k/n-\mu(x)|\geq\delta} (k/n - \mu(x))^2 b_{n,k}(x) \\
&\leq \varepsilon + \frac{2M}{\delta^2} \sum_{k=0}^{n} (k/n - \mu(x))^2 b_{n,k}(x) \\
&= \varepsilon + \frac{2M}{\delta^2} \left(\left(\sum_{k=1}^{n} (p_k(x)/n) - \mu(x) \right)^2 + \sum_{k=1}^{n} p_k(x)(1 - p_k(x))/n^2 \right) \\
&\leq \varepsilon + \frac{2M}{\delta^2} (\psi_n(x)^2 + 1/4n),
\end{aligned}
$$

where the last two steps are derived from (7.22) and from the fact that $0 \leq p_k(x) \leq 1$ implies $p_k(x)(1-p_k(x)) \leq 1/4$. From this inequality, we see that $|L_n(f;x) - f(\mu(x))| \leq 2\varepsilon$ for n sufficiently large. Since ε is arbitrary, (7.23) follows.

Suppose now that $\lim_{n\to\infty}(p_1(x) + \cdots + p_n(x))/n$ converges uniformly on $[0,1]$. Then $\theta_n := \sup_{x\in[0,1]} |\psi_n(x)| \to 0$ as $n \to \infty$. Thus the previous inequality is independent of x and the convergence to $f(\mu(x))$ is uniform in $[0,1]$.

Conversely, suppose that $\lim_{n\to\infty} L_n(f_0;x)$ exists for some strictly monotone function $f_0 \in C[0,1]$. If the sequence $\varrho_n(x) = (p_1(x) + \cdots + p_n(x))/n$ does not converge for every $x \in [0,1]$ as $n \to \infty$, then since $0 \leq \varrho_n(x) \leq 1$ is bounded for all n, it follows that for some $x_0 \in (0,1)$, the sequence $\varrho_n(x_0)$ has at least two limit points, say $\varrho^1(x_0) \neq \varrho^2(x_0)$. Therefore, there exist two sequences of natural numbers n_k^1, n_k^2 such that $\lim_{k\to\infty} \varrho_{n_k^i}(x_0) = \varrho^i(x_0), i = 1,2$. By performing the same arguments as in the first paragraph, we find that $\lim_{k\to\infty} L_{n_k^i}(f_0;x_0) = f_0(\varrho^i(x_0))$ $(i = 1,2)$. But, by assumption, $\lim_{n\to\infty} L_n(f_0;x)$ exists for every $x \in [0,1]$; hence $f(\varrho^1(x_0)) = f(\varrho^2(x_0))$. Since f_0 is strictly monotone $\varrho^1(x_0) = \varrho^2(x_0)$, a contradiction. Therefore if $\lim_{n\to\infty} L_n(f_0;x)$ exist, then $\lim_{n\to\infty} \varrho_n(x) = \lim_{n\to\infty}(p_1(x) + \cdots + p_n(x))/n$ also exists.

□

Proposition 7.7. *Suppose that* $p_1(x) \in C[0,1]$ *is strictly increasing, and all the* $p_i(x)$ $(i \geq 2)$ *are determined recursively by* (7.8)*, or equivalently for all n,* $\gamma_n^p(x) = \gamma_{n+1}^p(x)$*. Then*

(i) if $p_1(x) \in C^\infty[0,1]$*, the roots at zero of all the* $p_i(x), i \geq 1$ *have the same multiplicities, i.e.* $p_i^{(k)}(0) = 0, 0 \leq k \leq m-1, p_i^{(m)}(0) \neq 0, i \geq 1$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{n} p_k(x)/n = x$.

Proof. First we prove that the roots at zero of all the $p_i(x), i \geq 1$ have the same multiplicities. We proceed by induction on n. By assumption $\gamma_n^p(x) = \gamma_1^p(x) = p_1(x)$. For $n = 2$, set $p_1(x) = x^m r_1(x), p_2(x) = x^l r_2(x), m, l \geq 1$, with $r_1(0) \neq 0, r_2(0) \neq 0$. Invoking (7.14), we find that

$$x^m r_1(x) = x^{m+l} r_1(x) r_2(x) + p_1(\frac{1}{2})[x^m r_1(x) + x^l r_2(x) - 2x^{m+l} r_1(x) r_2(x)]. \tag{7.24}$$

If $m < l$, dividing both sides of (7.24) by x^m and then setting $x = 0$, we derive $p_1(\frac{1}{2}) = 1$, a contradiction, since by assumption $p_1(x)$ is strictly increasing. If $m > l$, dividing both sides of (7.24) by x^l and then setting $x = 0$, we derive $p_1(\frac{1}{2}) = 0$, yielding yet another contradiction. Thus $m = l$.

Now assume that $p_i(x) = x^m r_i(x), 1 \le i \le n$, with $r_i(0) \neq 0$, and let $p_{n+1}(x) = x^s r_{n+1}(x)$ with $r_{n+1}(0) \neq 0$. Again since $\gamma_n^p(x) = \gamma_{n+1}^p(x) = p_1(x)$, it follows that

$$\begin{aligned} & p_1\left(\frac{1}{n}\right) b_{n,1}(x) + \sum_{k=2}^{n} p_1\left(\frac{k}{n}\right) b_{n,k}(x) \\ &= p_1\left(\frac{1}{n+1}\right) b_{n+1,1}(x) + \sum_{k=2}^{n+1} p_1\left(\frac{k}{n+1}\right) b_{n+1,k}(x). \end{aligned} \tag{7.25}$$

Therefore, by (2.4), (7.25) implies that there must exist suitable $P(x), Q_1(x), Q_2(x)$ with $P(0) \neq 0, Q_i(0) \neq 0, i = 1, 2$ such that

$$\begin{aligned} & p_1(\frac{1}{n})(x^m r_1(x) + \cdots + x^m r_n(x)) + x^{2m} P(x) \\ &= p_1(\frac{1}{n+1})(x^m r_1(x) + \cdots + x^m r_n(x) + x^s r_{n+1}(x)) + x^{2m} Q_1(x) + x^{m+s} Q_2(x). \end{aligned} \tag{7.26}$$

If $m < s$, dividing both sides of (7.26) by x^m and then setting $x = 0$, we derive $p_1(\frac{1}{n}) = p_1(\frac{1}{n+1})$, a contradiction. If $m > s$, dividing both sides of (7.26) by x^s and then setting $x = 0$, we derive $r_{n+1}(0) = 0$, yielding yet another contradiction. Thus $m = s$.

Again since $\gamma_n^p(x) = p_1(x)$

$$\gamma_n^p(x) = \sum_{k=0}^{n} p_1\left(\frac{k}{n}\right) b_{n,k}(x) = L_n(p_1; x). \tag{7.27}$$

Therefore, taking the limit of both sides of (7.27), we get

$$p_1(x) = \lim_{n\to\infty} \gamma_n^p(x) = L_\infty(p_1; x). \tag{7.28}$$

Since, by assumption, $p_1(x)$ is strictly increasing, it follows by the last part of Theorem 7.6 that $\mu(x) := \lim_{n\to\infty}(p_1(x) + \cdots + p_n(x)/n$ exists. Furthermore, again by Theorem 7.6

$$p_1(x) = p_1(\mu(x)). \tag{7.29}$$

Since $p_1(x)$ is strictly increasing, we conclude that $\mu(x) = x$. □

Remark 7.8. *Set $p_1(x) = x^2$. Then from* (7.14)*, we can derive $p_2(x) = \dfrac{3x^2}{2x^2+1}$. Similarly, by using* (7.8) *recursively we can also derive $p_3(x) = \dfrac{10x^4 + 5x^2}{4x^4 + 10x^2 + 1}$, $p_4(x) = \dfrac{7x^2(2x^2+1)(4x^4+10x^2+1)}{16x^8 + 120x^6 + 148x^4 + 30x^2 + 1}$. In general, if $p_1(x) = x^2$, we have the recursive formula*

$$\begin{aligned} p_{n+1}(x) &= \frac{\sum_{k=0}^{n} \left(k^2/n^2 - k^2/(n+1)^2\right) b_{n,k}(x)}{\sum_{k=0}^{n} \left((k+1)^2/(n+1)^2 - k^2/(n+1)^2\right) b_{n,k}(x)} \\ &= \frac{(2n+1)\sum_{k=0}^{n} k^2/n^2 b_{n,k}(x)}{\sum_{k=0}^{n} (2k+1) b_{n,k}(x)} = \frac{(2n+1)x^2}{1 + 2\sum_{k=1}^{n} p_k(x)}, \end{aligned} \tag{7.30}$$

where the first equality follows by (7.8)*, and the third equality follows by* (7.27) *and by equation* (2.3) *with* $k = 1$*. Moreover by Proposition 7.7* $\lim_{n\to\infty}(p_1(x) + \cdots + p_n(x))/n = x$*. Combining this result with* (7.30)*, we conclude that* $\lim_{n\to\infty} p_n(x) = x$*.*

Corollary 7.9. *i) Suppose that* $p_1(x) \in C[0,1]$ *is strictly increasing and all the* $p_i(x)$ $(i \geq 2)$ *are determined recursively by* (7.8)*, or equivalently for all n,* $\gamma_n^p(x) = \gamma_{n+1}^p(x)$*. If* $f \in C[0,1]$*, then* $\lim_{n\to\infty} L_n(f;x) = f(x)$ *uniformly on* $[0,1]$*.*

ii) Given any $p(x) \in C[0,1]$ *such that* $p(x)$ *is strictly increasing and* $p(0) = 0, p(1) = 1$*, there exist Lototsky-Bernstein operators* $L_n(f;x)$ *that fix* 1 *and* $p(x)$*, and* $\lim_{n\to\infty} L_n(f;x) = f(x)$ *uniformly on* $[0,1]$ *for any* $f \in C[0,1]$*.*

Proof. i) is straightforward by Proposition 7.7 (ii) and Korovkin's Theorem (see [21]). ii) follows by setting $p_1(x) = p(x)$ and computing $p_i(x)$ $(i \geq 2)$ recursively using (7.8). □

Proposition 7.10. *Let* $r(x) \in C[0,1]$ *be increasing and convex in* $[0,1]$ *with* $r(0) = 0, r(1) = 1$*. Suppose that all the* $p_i(x) = r(x), i \geq 1$*. Then*

$$\gamma_n^p(x) \geq \gamma_{n+1}^p(x). \tag{7.31}$$

Moreover, $\gamma_n^p(x) = \gamma_{n+1}^p(x)$ *for all n if and only if* $r(x) = x$*.*

Proof. From the assumptions of the Proposition, the operators $L_n(f)$ are King-type operators [22], i.e.

$$L_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} r^k(x)(1 - r(x))^{n-k}. \tag{7.32}$$

By applying the same method as in [32], we can prove that for any convex function f

$$L_n(f;x) \geq L_{n+1}(f;x). \tag{7.33}$$

Let f be increasing and convex in $[0,1]$ with $f(0) = 0, f(1) = 1$. Since $r(x)$ is increasing and convex, it follows from Remark 3.2 and Remark 5.13 that both $L_n(f)$ and $L_{n+1}(f)$ are increasing and convex on $[0,1]$. Repeatedly using 7.33 and the positivity of $L_n(f)$, we have

$$\begin{aligned} &L_n(f) \geq L_{n+1}(f) \Rightarrow (L_n)^2(f) \geq L_{n+1}(L_n(f)) \geq (L_{n+1})^2(f) \\ &\cdots \Rightarrow (L_n)^M(f) \geq (L_{n+1})^M(f). \end{aligned} \tag{7.34}$$

Taking the limit of both sides of (7.34) as $M \to \infty$, yields $\gamma_n^p(x) \geq \gamma_{n+1}^p(x)$.

If for all n, $\gamma_n^p(x) = \gamma_{n+1}^p(x)$, then by Proposition 7.7 (ii), $r(x) = x$. □

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