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Szegő's condition on compact subsets of $\mathbb{C}$

Gökalp Alpan

Department of Mathematics, Rice University, Houston, TX 77005, USA

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Abstract

Let K be a non-polar compact subset of $\mathbb{C}$ and μ_K be its equilibrium measure. Let μ be a unit Borel measure supported on K. We prove that a Szegő condition in terms of the Radon–Nikodym derivative of μ with respect to μ_K implies that

$$\inf_n \frac{\|P_n(\cdot;\mu)\|_{L^2(\mathbb{C};\mu)}}{\mathrm{Cap}(K)^n} > 0.$$

We show that $\frac{\|P_n(\cdot;\mu_K)\|_{L^2(\mathbb{C};\mu_K)}}{\mathrm{Cap}(K)^n} \geq 1$ for any compact non-polar set K. We also prove that under an additional assumption, boundedness of the sequence $\left(\frac{\|P_n(\cdot;\mu_K)\|_{L^2(\mathbb{C};\mu_K)}}{\mathrm{Cap}(K)^n}\right)$ implies that K satisfies the Parreau–Widom condition.

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1. Introduction

Let μ be a unit Borel measure with an infinite compact support on $\mathbb{C}$. We denote by $P_n(z;\mu)$ the nth degree monic orthogonal polynomial associated with μ, i.e.,

$$\|P_n(\cdot;\mu)\|_{L^2(\mathbb{C};\mu)} = \inf_{Q\in\mathscr{P}_n} \|Q\|_{L^2(\mathbb{C};\mu)} \tag{1.1}$$

where $\mathscr{P}_n$ is the set of all nth degree monic (complex) polynomials and $\|\cdot\|_{L^2(\mathbb{C};\mu)}$ denotes the L^2 norm associated with μ.

E-mail address: alpan@rice.edu.

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For a general treatment of logarithmic potential theory, see e.g. [13,15]. Let us denote the logarithmic capacity by $\mathrm{Cap}(\cdot)$. For a non-polar compact subset K of $\mathbb{C}$, we denote the equilibrium measure of K by μ_K. For the component of $\overline{\mathbb{C}} \setminus K$ that contains ∞, we use Ω_K. By $g_{\Omega_K}(z) := g_{\Omega_K}(z;\infty)$, we mean the Green function for the domain Ω_K at infinity. If K is a regular compact subset of $\mathbb{C}$ with respect to the Dirichlet problem, let

$$\mathrm{PW}(K) := \sum g_{\Omega_K}(c_n)$$

where $\{c_n\}_n$ is the set of critical points of g_{Ω_K} counting multiplicity. If $\mathrm{PW}(K) < \infty$ then K is called a Parreau–Widom set and Ω_K is called a Parreau–Widom domain.

Let $W_n(\mu) := \frac{\|P_n(\cdot;\mu)\|_{L^2(\mathbb{C};\mu)}}{\mathrm{Cap}(\mathrm{supp}(\mu))^n}$ where $\mathrm{supp}(\cdot)$ denotes the support of μ. Since $\mathrm{Cap}(\mathrm{supp}(\mu_{\mathrm{K}})) = \mathrm{Cap}(K)$ (see Lemma 1.2.7 in [17]) we use these two expressions interchangeably.

The Szegő theorem on Parreau–Widom sets is as follows [5]:

Theorem 1.1. *Let K be a regular Parreau–Widom subset of $\mathbb{R}$. Let μ be unit Borel measure on $\mathbb{R}$ such that $d\mu = f(x)dx + d\mu_s$ where $d\mu_s$ is the singular part with respect to the Lebesgue measure and suppose that the support of μ, except possibly the isolated point masses, is equal to K. Denote the set of isolated points of the support by $\{x_k\}$. On condition that*

$$\sum_k g_{\Omega_K}(x_k) < \infty$$

we have

$$\limsup_{n\to\infty} W_n(\mu) > 0 \iff \int_K \log f(x)\, d\mu_K(x) > -\infty. \tag{1.2}$$

The main result of the paper is a generalization of one half of the Szegő theorem. This generalization is based on replacing the Szegő condition (the condition on the right hand side of (1.2)) by another condition which was suggested in [4] (see Section 6) . We refer the reader to [7,12,19] for the previous generalizations of the Szegő theorem. Our result reads as follows:

Theorem 1.2. *Let K be a non-polar compact subset of $\mathbb{C}$ and let μ be a unit Borel measure supported on K. Let μ_s denote the singular part of μ with respect to μ_K and h be a non-negative measurable function on $\partial\Omega_K$ such that*

- $d\mu = h\, d\mu_K + d\mu_s$.
- $M := \int \log h\, d\mu_K > -\infty$.

Then $\inf_{n\in\mathbb{N}} (W_n(\mu))^2 \geq e^M$.

For a given compact infinite set K in $\mathbb{C}$, the polynomial $T_{n,K}(z) = z^n + \cdots$ satisfying

$$\|T_{n,K}\|_K = \min\{\|Q_n\|_K : Q_n \text{ monic polynomial of degree } n\}$$

is called the nth Chebyshev polynomial for K where $\|\cdot\|_K$ is the sup norm on K.

For a non-polar compact set $K \subset \mathbb{C}$, let

$$M_{n,K} := \|T_{n,K}\|_K / \mathrm{Cap}(K)^n.$$

For a review of the recent results for these ratios we refer the reader to [8] and many basic results regarding the asymptotics of L^2 and L^∞ extremal polynomials can be found in [16].

The following result is a generalization of Theorem 3 in [3]:

Corollary 1.3. *Let K be a non-polar compact subset of $\mathbb{C}$. Then $\inf_{n\in\mathbb{N}} W_n(\mu_K) \geq 1$. The inequality is sharp: If $K = \mathbb{T}$ then $d\mu_K = d\theta/(2\pi)$ and $W_n(\mu_K) = 1$ for all $n \in \mathbb{N}$.*

Remark. We would like to draw the reader's attention to the similarity between the general results regarding the lower bounds of $W_n(\mu_K)$ and $M_{n,K}$: It is well known that $M_{n,K} \geq 1$, see Theorem 5.5.4 in [13] and the equality is obtained for all n on the unit circle.

Let K be a non-polar compact subset of $\mathbb{C}$. Following [11] (see p. 23–31), we say that F is a multiplicative analytic (resp. meromorphic) function on Ω_K if F is a multivalued analytic function on Ω_K with single valued absolute value $|F(z)|$. Each multiplicative analytic function determines a unique character: Let us fix a base point $\mathcal{O} \in \Omega_K$. Let $F_{\mathcal{O}}$ be a single valued branch of F at $\mathcal{O}$ and c be a closed curve in Ω_K issuing from $\mathcal{O}$. Then $F_{\mathcal{O}}$ can be analytically continued along c and the resulting function element at $\mathcal{O}$ is equal to $\zeta_F(c)F_{\mathcal{O}}$ where $|\zeta_F(c)| = 1$. Note that the value of ζ_F is the same for homotopic curves and it is independent of the base point. Besides, if c_1 and c_2 are two closed curves issuing from $\mathcal{O}$ then $\zeta_F(c_1c_2)F_{\mathcal{O}} = \zeta_F(c_1)\zeta_F(c_2)F_{\mathcal{O}}$. Thus, $\zeta_F(\cdot)$ is a character of the fundamental group $\Pi(\Omega_K)$. We denote the character group by $\Pi(\Omega_K)^\star$.

As in the proof of Theorem 1.4 in [6] we need the function B_{Ω_K} to prove Theorem 1.4. We can find a local harmonic conjugate to $-g_{\Omega_K}(z)$ for each z. Therefore, the equation $|B_{\Omega_K}(z)| = e^{-g_{\Omega_K}(z)}$ determines a multiplicative analytic function on Ω_K up to a multiplicative constant. We fix it by requiring

$$B_{\Omega_K}(z) = \mathrm{Cap}(K)/z + \mathrm{O}(|z|^{-2}) \tag{1.3}$$

near ∞.

Let c be a rectifiable curve on Ω_K such that c winds once around $L \subset K$ and around no other points of K, then the change of phase of B_{Ω_K} around c is given by $e^{-2\pi i\mu_K(L)}$, see Theorem 2.7 in [8]. Using this we can determine $\zeta_{B_{\Omega_K}}(\cdot)$. Let us denote the character of $B^n_{\Omega_K}$ by χ^n_K for simplicity.

Multiplication of two characters ζ_1 and ζ_2 in $\Pi(\Omega_K)^\star$ is defined as pointwise multiplication: $(\zeta_1\zeta_2)(c) = \zeta_1(c)\zeta_2(c)$. This makes $\Pi(\Omega_K)^\star$ an abelian group. Let us equip $\Pi(\Omega_K)$ with discrete topology. Then $\Pi(\Omega_K)^\star$ is a compact metrizable space with the topology of pointwise convergence since $\Pi(\Omega_K)$ is countable. The map $T\zeta := \chi_K\zeta$ is ergodic with respect to the Haar measure if and only if $\{\chi^n_K\}_{n=-\infty}^{\infty}$ is dense in $\Pi(\Omega_K)^\star$, see Theorem 1.9 in [18]. If $\{\chi^n_K\}_{n=-\infty}^{\infty}$ is dense then $\{\chi^n_K\}_{n=0}^{\infty}$ is also dense, see p. 132 in [18]. This fact is used in the proof of Theorem 1.4.

When $K \subset \mathbb{R}$, $\{\chi^n_K\}_{n=-\infty}^{\infty}$ is dense in $\Pi(\Omega_K)^*$ if and only if the following condition is satisfied, see Section 1 in [6]: Suppose that for each decomposition $K = E_1 \cup \cdots \cup E_l$ into closed disjoint sets and rational numbers $\{q_j\}_{j=1}^{l-1}$, not all of which are zero, we have

$$\sum_{j=1}^{l-1} q_j\mu_K(E_j) \neq 0 \ (\mathrm{mod}\, 1). \tag{1.4}$$

There are examples of Cantor sets $K(\gamma)$ such that $W_n(\mu_{K(\gamma)}) \to \infty$ as $n \to \infty$, see Example 5.3, [4]. We emphasize that $K(\gamma)$ does not satisfy (1.4), see Section 4 in [2]. The next result implies Theorem 1.4 in [6] in view of (2.10) and the proof is very similar.

Theorem 1.4. *Let K be a regular compact subset of $\mathbb{C}$. Suppose that $\{\chi^n_K\}_{n=-\infty}^{\infty}$ is dense in $\Pi(\Omega_K)^*$. If $(W_n(\mu_K))_{n=1}^{\infty}$ is bounded then K is a Parreau–Widom set.*

As a corollary of Theorem 1.4 we obtain the following result which complements Corollary 1.3 but the scope of Corollary 1.5 is much more limited. The proof of one of the implications is quite trivial and the inverse implication follows from Theorem 1.4 in [8] and Theorem 1.4.

Corollary 1.5. *Let K be a regular compact subset of $\mathbb{R}$ and $\{\chi_K^n\}_{n=-\infty}^{\infty}$ be dense in $\Pi(\Omega_K)^*$. Then $(W_n(\mu_K))_{n=1}^{\infty}$ is bounded if and only if $(M_{n,K})_{n=1}^{\infty}$ is bounded.*

In Section 2, we prove the new results stated in the introduction.

2. Proofs

Proof of Theorem 1.2. Let $P_n(z) := \prod_{j=1}^{n}(z-\tau_j)$. Then

$$\left(\int |P_n|^2 h\, d\mu_K + \int |P_n|^2 d\mu_s\right)^{1/2} \geq \left(\int |P_n|^2 h\, d\mu_K\right)^{1/2} \tag{2.1}$$

$$= e^{\log\left(\left(\int |P_n|^2 h\, d\mu_K\right)^{1/2}\right)} \tag{2.2}$$

$$\geq e^{\int \log(|P_n| h^{1/2})\, d\mu_K} \tag{2.3}$$

$$= e^{\frac{1}{2}\int \log h\, d\mu_K}\ e^{\int \sum_{j=1}^{n} \log|z-\tau_j|\, d\mu_K(z)} \tag{2.4}$$

$$\geq e^{\frac{1}{2}\int \log h\, d\mu_K}\ \mathrm{Cap(K)}^n. \tag{2.5}$$

Here, (2.3) follows from Jensen's inequality and (2.5) holds since

$$\int \log|z-\tau|\, d\mu_K(z) \geq \log \mathrm{Cap}(K)$$

for all $\tau \in \mathbb{C}$ by Frostman's theorem, see Theorem 3.3.4 (a) in [13]. Note that $\mathrm{Cap}(K) = \mathrm{Cap}(\mathrm{supp}(\mu))$ by our assumptions. We obtain the desired inequality by squaring the left hand side of (2.1) and (2.5). □

Proof of Corollary 1.3. We obtain $\inf_{n\in\mathbb{N}} W_n(\mu_K) \geq 1$ by letting $h \equiv 1$ and $\mu_s = 0$ in Theorem 1.2.

The proof of the second part of the corollary is quite straightforward. Since $|z| = 1$ on the unit circle, we get $\int |z|^{2n} d\mu_{\mathbb{T}}(z) = 1$ for all n. In addition, $P_n(z;\mu_{\mathbb{T}}) = z^n$ and $\mathrm{Cap}(\mathbb{T}) = 1$. Thus $W_n(\mu_{\mathbb{T}}) = 1$ for all $n \in \mathbb{N}$. □

We denote by $\mathscr{H}_q(\Omega_K,\zeta)$ the multiplicative analytic functions F whose character is ζ for which $|F|^q$ has a harmonic majorant and $\mathscr{H}_\infty(\Omega_K,\zeta)$ means $|F|$ is bounded. It is not difficult to see that, $1 \leq p \leq q \leq \infty$ implies that $\mathscr{H}_q(\Omega_K) \subset \mathscr{H}_p(\Omega_K)$.

The following characterization of the Parreau–Widom condition is due to Widom (see Theorem 1 in [20] and also Section 2B, in Ch. 5 in [11]):

Theorem 2.1. *Let K be a regular compact subset of $\mathbb{C}$. Then Ω_K is a Parreau–Widom domain if and only if $\mathscr{H}_2(\Omega_K,\zeta) \neq \{0\}$ for all $\zeta \in \Pi(\Omega_K)^*$.*

In the proof of Theorem 1.4 we use ideas from Theorem 1.4 in [6] and from the proof of Theorem in 5A, Ch. 5 (the main arguments of the proof can also be found in Theorem 3 in [20].) in [11].

Proof of Theorem 1.4. Let $M := \sup_n W_n(\mu_K)$ and $\chi \in \Pi(\Omega_K)^*$. Then there is a subsequence $n_j \to \infty$ such that $\chi_K^{n_j} \to \chi$. Let $F_j(z) := \frac{P_{n_j}(z;\mu_K) B_{\Omega_K}^{n_j}(z)}{\mathrm{Cap}(K)^{n_j}}$. For each j, $F_j \in \mathscr{H}_\infty(\Omega_K)$ because by the maximum principle

$$\|F_j\|_{\Omega_K} \leq \sup_{z \to \partial\Omega_K} |F_j(z)| = \sup_{z \in \partial\Omega_K} |P_{n_j}(z;\mu_K)| \mathrm{Cap}(K)^{-n_j}.$$

Hence we also have $F_j \in \mathscr{H}_2(\Omega_K)$.

We denote the harmonic measure for Ω_K at z by $w_{\Omega_K}(z;\cdot)$. The harmonic measure $w_{\Omega_K}(\infty;\cdot)$ at infinity is μ_K, see Theorem 4.3.14, [13]. If g is a Borel measurable function on $\partial\Omega_K$ such that

$$H_{\Omega_K}(z; g) := \int_{\partial\Omega_K} g \, dw_{\Omega_K}(z;\cdot) \tag{2.6}$$

is integrable for some $z \in \Omega_K$ then the integral in (2.6) is finite for all $z \in \Omega_K$, see Appendix A.3 in [15]. In this case $H_{\Omega_K}(z; g)$ is a harmonic function on Ω_K and it is called the solution of the Dirichlet problem corresponding to g and Ω_K. If additionally, g is continuous on $\partial\Omega_K$ and K is regular with respect to the Dirichlet problem then $\lim_{z\to u} H_{\Omega_K}(z; g) = g(u)$ for all $u \in \partial\Omega_K$, see Corollary 4.1.8 in [13]. Hence if g is continuous and K is regular then $H_{\Omega_K}(\cdot; g)$ can be extended continuously to $\overline{\Omega_K}$. We denote the extension by $H_{\overline{\Omega_K}}(z; g)$.

We say that $\gamma : [0,1] \to \mathbb{C}$ is a Jordan curve if it is simple and closed. A rectifiable Jordan curve γ is C^{2+} if γ is C^2 and the second derivative of γ satisfies a Lipschitz condition with some positive exponent.

We call $(K_n)_{n=1}^\infty$ a C^{2+} exhaustion of Ω_K if $(K_n)_{n=1}^\infty$ is increasing sequence of domains such that

(a) ∂K_n consists of finitely many non-intersecting C^{2+} Jordan curves.
(b) $\overline{K_n} \subset K_{n+1}$.
(c) $\cup K_n = \Omega_K$.

We can find a C^{2+} exhaustion for Ω_K, see VII. 4.4 in [9] or Ch.2, 12D in [1]. Let $(K_n)_{n=1}^\infty$ be a C^{2+} exhaustion of Ω_K. Then $w_{K_n}(z;\cdot) \to w_{\Omega_K}(z;\cdot)$ in the weak-star sense (see Theorem 10.9 in Section 21.11 in [10] for the proof).

For a multivalued function $F \in \mathscr{H}_2(\Omega_K)$ we denote the least harmonic majorant for $|F|^2$ by $\mathrm{LHM}(|F|^2)(\cdot)$. The function $|F|^2$ is subharmonic. Then as a consequence of Harnack's theorem (see Theorem 1.3.9 in [13]), we get (see e.g. Eq. (2.1.2) in [14])

$$\mathrm{LHM}(|F|^2)(z) = \lim_{n\to\infty} \int |F|^2 dw_{K_n}(z;\cdot).$$

In addition if $|F|^2$ can be extended continuously to $\overline{\Omega_K}$ then $\mathrm{LHM}(|F|^2)(z) = H_{\overline{\Omega_K}}(z; |F|^2)$. since $dw_{K_n}(z;\cdot) \to dw_{\Omega_K}(z;\cdot)$.

Note that $|F_j|^2$ can be extended continuously to $\overline{\Omega_K}$ since K is regular. Thus,

$$\mathrm{LHM}(|F_j|^2)(\infty) = \frac{\int |P_{n_j}(z;\mu_K)|^2 \, d\mu_K(z)}{\mathrm{Cap}(K)^{2n_j}} = (W_{n_j}(\mu_K))^2 \leq M^2. \tag{2.7}$$

Since $\mathrm{LHM}(|F_j|^2)(\cdot)$ is a positive harmonic function on Ω_K and the inequality (2.7) holds, in view of Harnack's inequality, $(\mathrm{LHM}(|F_j|^2)(\cdot))_{j=1}^\infty$ is uniformly bounded above by a positive number on each compact subset of Ω_K. By standard compactness argument based on Montel's theorem there is a convergent subsequence $(F_{j(k)})_{k=1}^\infty$. Let f be the limit of this subsequence. Then $\zeta_f(c) = \lim_{k\to\infty} \zeta_{F_{j(k)}}(c) = \chi(c)$ for each closed curve c in Ω_K issuing from $\mathscr{O}$.

Note that $|F_{j(k)}(\infty)| = 1$ for all k by (1.3). Therefore

$$|f(\infty)| = 1. \tag{2.8}$$

It remains to show that $|f|^2$ has a harmonic majorant. Note that $|F_{j(k)}(z)| \to |f(z)|$ uniformly on each compact subset of Ω_K. Fix a positive integer n. Let $\epsilon > 0$. Then there is a number k_0 such that

$$||F_{j(k)}(z)|^2 - |f(z)|^2| < \epsilon \tag{2.9}$$

is satisfied on ∂K_n for all $k \geq k_0$.

Let us denote the least harmonic majorant of a function G restricted to a region E by $\mathrm{LHM}_E(G)(\cdot)$. Then by (2.9), for each $k \geq k_0$,

$$\mathrm{LHM}_{K_n}(|f|^2)(z) \leq \mathrm{LHM}_{K_n}(|F_{j(k)}|^2 + \epsilon)(z) \leq \mathrm{LHM}_{K_n}(|F_{j(k)}|^2)(z) + \epsilon$$

for $z \in K_n$.

Clearly, $\mathrm{LHM}_{K_n}(|F_{j(k)}|^2)(z) \leq \mathrm{LHM}(|F_{j(k)}|^2)(z)$. Since ϵ is arbitrary, we get

$$\mathrm{LHM}_{K_n}(|f|^2)(z) \leq \limsup_{k\to\infty} \mathrm{LHM}(|F_{j(k)}|^2)(z).$$

By (2.7) and Harnack's inequality, there is a constant $C(z)$ depending only on z such that $\limsup_{k\to\infty} \mathrm{LHM}(|F_{j(k)}|^2)(z) \leq C(z)M^2$. Hence $\mathrm{LHM}_{K_n}(|f|^2)(z) \leq C(z)M^2$. Since n is arbitrary, $\mathrm{LHM}_{K_r}(|f|^2)(z) \leq C(z)M^2$ for all $r \in \mathbb{N}$.

For any fixed $z \in \Omega_K$, let l be an integer such that $z \in K_l$. Then $(\mathrm{LHM}_{K_n}(|f|^2)(z))_{n=l}^{\infty}$ is an increasing sequence bounded by $C(z)M^2$. Let $H(z) := \lim_{n\to\infty}(\mathrm{LHM}_{K_n}(|f|^2)(z))$. Then by Harnack's theorem, H is a harmonic function on Ω_K. Clearly $\mathrm{LHM}(|f|^2)(z) \leq H(z) \leq C(z)M^2$. Thus f is in $\mathscr{H}_2(\Omega_K, \chi)$. It is also non-zero by (2.8). Since χ is arbitrary, this proves that Ω_K is a Parreau–Widom domain by Theorem 2.1. □

Proof of Corollary 1.5. Suppose that $(W_n(\mu_K))_{n=1}^{\infty}$ is bounded. Then by Theorem 1.4, Ω_K is Parreau–Widom and by Theorem 1.4 in [8], $(M_{n,K})_{n=1}^{\infty}$ is bounded. This proves the first implication.

Suppose that $(M_{n,K})_{n=1}^{\infty}$ is bounded. Note that

$$\|P_n(\cdot;\mu_K)\|_{L^2(\mathbb{C};\mu_K)} \leq \|T_{n,K}\|_{L^2(\mathbb{C};\mu_K)} \leq \|T_{n,K}\|_K. \tag{2.10}$$

The inequality on the left in (2.10) follows from (1.1). Thus, $W_n(\mu_K) \leq M_n(K)$ and this implies that $(W_n(\mu_K))_{n=1}^{\infty}$ is also bounded. □

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References

[1] L.V. Ahlfors, L. Sario, Riemann Surfaces, Princeton University Press, 1960.

[2] G. Alpan, Spacing properties of the zeros of orthogonal polynomials on Cantor sets via a sequence of polynomial mappings, Acta Math. Hungar. 149 (2016) 509–522.

[3] G. Alpan, Orthogonal polynomials associated with equilibrium measures on $\mathbb{R}$, Potential Anal. 46 (2017) 393–401.
[4] G. Alpan, A. Goncharov, Orthogonal polynomials for the weakly equilibrium Cantor sets, Proc. Amer. Math. Soc. 144 (9) (2016) 3781–3795.
[5] J.S. Christiansen, Szegő's theorem on Parreau-Widom sets, Adv. Math. 229 (2012) 1180–1204.
[6] J.S. Christiansen, B. Simon, P. Yuditskii, M. Zinchenko, Asymptotics of Chebyshev Polynomials, II. DCT subsets of $\mathbb{R}$, Duke Math. J. 168 (2019) 325–349.
[7] J.S. Christiansen, B. Simon, M. Zinchenko, Finite Gap Jacobi Matrices, II. The Szegö Class, Constr. Approx. 33 (3) (2011) 365–403.
[8] J.S. Christiansen, B. Simon, M. Zinchenko, Asymptotics of Chebyshev polynomials, I: subsets of $\mathbb{R}$, Invent. Math 208 (2018) 217–245.
[9] J.B. Conway, A Course in Functional Analysis, Spronger-Verlag, New York, 1985.
[10] J.B. Conway, Functions of One Complex Variable II, Springer-Verlag, New York, 1995.
[11] M. Hasumi, Hardy Classes of Infinitely Connected Riemann Surfaces, in: Lecture Notes in Math, vol. 1027, Springer-Verlag, Berlin, 1983.
[12] F. Peherstorfer, P. Yuditskii, Asymptotic behavior of polynomials orthonormal on a homogeneous set, J. Anal. Math. 89 (2003) 113–154.
[13] T. Ransford, Potential Theory in the Complex Plane, Cambridge University Press, 1995.
[14] W. Rudin, Analytic functions of class H_p, Trans. Amer. Math. 78 (1955) 46–66.
[15] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, New York, 1997.
[16] B. Simon, Equilibrium measures and capacities in spectral theory, Inverse Probl. Imaging 1 (2007) 713–772.
[17] H. Stahl, V. Totik, General Orthogonal Polynomials, in: Encyclopedia of Mathematics, vol. 43, Cambridge University Press, New York, 1992.
[18] P. Walters, An Introduction to Ergodic Theory, in: Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.
[19] H. Widom, Extremal polynomials associated with a system of curves in the complex plane, Adv. Math. 3 (1969) 127–232.
[20] H. Widom, H_p Sections of vector bundles over Riemann surfaces, Ann. of Math. 94 (1971) 304–324.