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# On a Bivariate Interpolation Problem

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In this paper we solve the poisedness problem for a bivariate interpolation introduced by B. Bojanov and Y. Xu. The authors were informed the problem was solved earlier, by a different method, also by B. Bojanov and Y. Xu (to appear, On a Hermite interpolation by polynomials of two variables, *SIAM J. Numer. Anal.*). Parameters of the original interpolation from $\Pi_{2k}$ are the values of a function and its radial derivatives up to some order $k$ at $2k+1$ equidistant nodes of the unit circumference. We also consider other closely related polynomial interpolations and prove their poisedness. Meanwhile the poisedness of several general univariate Birkhoff interpolation problems is proved to which the above problem is reduced. At the end we consider the corresponding useful cubature formulas. © 2002 Elsevier Science (USA)

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## 1. MAIN RESULTS

Denote by $Z_0 = \{z_\nu\}_{\nu=1}^{2k+1} = \{(x_\nu, y_\nu)\}_{\nu=1}^{2k+1}$, a set of equidistant nodes on the unit circumference

$$S^1 := \{z \in R^2 : |z| = 1\}.$$

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The directional derivative of $F$ along $\xi = (a, b) \in R^2$ will be denoted by

$$D_\xi F := aF_x + bF_y.$$

The radial derivative at $z_0 \in S^1$ is $D_{z_0} F(z_0)$.

The space of univariate polynomials of degree $\leqslant m$, and the corresponding space of bivariate polynomials of total degree $\leqslant m$ we will denote by $\pi_m$ and $\Pi_m$, respectively.

The following theorem gives an affirmative answer to the question formulated by B. Bojanov and Y. Xu.

THEOREM 1. *For any set of given numbers* $\{c_{ij}\}$ *there exists a unique polynomial* $p \in \Pi_{2k}$ *such that*

$$(D_{z_i})^j P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, ..., 2k+1; \quad j = 0, ..., k. \tag{1}$$

This is an even degree polynomial interpolation (see the odd case in the next theorem).

It should be noted that the poisedness of this interpolation is not simple at all because of the following consideration. All known poised bivariate interpolations (perhaps except the one arised from the Pascal theorem for $\Pi_2$ and 6 points) have a characteristic trait that there are enough many points on algebraic curves, in particular on straight lines, which enables to use the Bezout's theorem for the factorization (see [1]). In this case, both on a line and on circumference, there are only about the half of number of points needed:

(i) On circumference we have $2k+1$ points while the corresponding number in Bezout's theorem is $2 \times (2k) + 1 = 4k + 1$ (i.e., the order of the curve times the degree of the polynomial and plus one).

(ii) On radial lines we have $k+1$ conditions while the corresponding number is $2k+1$.

The next theorem states the poisedness of another closely related interpolation. This is the corresponding odd degree interpolation. Let us mention that the formulation of this theorem is a result of discussions with H. Gevorgian.

THEOREM 2. *For any set of given numbers* $\{c_{ij}\}$ *there exists a unique polynomial* $p \in \Pi_{2k-1}$ *such that*

$$(D_{z_i})^j P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, ..., 2k+1; \quad j = 0, ..., k-1. \tag{2}$$

In the next theorem we show that the above type radial derivative conditions are linearly independent for all polynomial spaces. Namely the conditions

$$(D_{z_i})^j\, P(z_i) = 0, \qquad i = 1, \dots, 2k+1; \quad j = 0, \dots, s, \tag{3}$$

are independent for $\Pi_{k+s}$, where $s \geqslant 0$. Independence here means (see [2]) that the dimension of the space of polynomials satisfying (3) equals

$$\dim \Pi_{k+s} - (s+1)(2k+1) = (k-s)(k-s-1)/2 =: l.$$

Or, equivalently, one can add $l$ simple nodes to $Z_0$ such that the resulted set produces a poised interpolation. Here we are able also to specify the conditions which are to be added. In the case $s \leqslant k-2$ those conditions are quite arbitrary, while in the case $s \geqslant k+1$ they are in the form of radial derivatives.

We are thus led to the following two strengthenings of the previous theorems.

THEOREM 3. *Assume that a set of nodes* $Z_1 = \{z_\nu^1\}_{\nu=1}^{m(m-1)/2} \in R^2 \setminus S^1$, *which is poised for* $\Pi_{m-2}$, *is given, where* $1 \leqslant m \leqslant k$. *Then for any given sets of numbers* $\{c_{ij}\}$ *and* $\{d_i\}$ *there exists a unique polynomial* $P \in \Pi_{2k-m}$ *such that*

$$(D_{z_i})^j\, P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, \dots, 2k+1; \quad j = 0, \dots, k-m, \tag{4}$$

*and*

$$P(z_i^1) = d_i \qquad \textit{for any} \quad i = 1, \dots, m(m-1)/2. \tag{5}$$

THEOREM 4. *Assume that a set of nodes* $Z_2 = \{z_\nu^2\}_{\nu=1}^{m} \in S^1 \setminus Z_0$ *is given, such that* $Z_0 \cup Z_1$ *does not contain opposite pairs. Then for any given sets of numbers* $\{c_{ij}\}$ *and* $\{d_{ij}\}$ *there exists a unique polynomial* $P \in \Pi_{2k+m} (m \geqslant 0)$ *such that*

$$(D_{z_i})^j\, P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, \dots, 2k+1; \quad j = 0, \dots, k+m, \tag{6}$$

*and*

$$(D_{z_i})^j\, P(z_i^2) = d_{ij} \qquad \textit{for any} \quad i = 1, \dots, m; \quad j = 0, \dots, i-1. \tag{7}$$

Note that one can replace conditions (7) by $m(m+1)/2$ conditions at simple nodes, near the $S^1$, by using the continuity property of the poisedness of the polynomial interpolation with respect to the nodes. As we mentioned above this is a requirement in independence of conditions (6).

Let us mention the following two special cases of Theorems 3 and 4, where just one point $z_0$ is added to $Z_0$.

COROLLARY 5. *Let $z_0 \notin S^1$. Then for any given set of numbers $\{c_{ij}\}$ and $c_{00}$ there exists a unique polynomial $P \in \Pi_{2k-2}$ such that*

$$P(z_0) = c_{00},$$

*and*

$$(D_{z_i})^j P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, ..., 2k+1; j = 0, ..., k-2.$$

Note that in the following second case the restriction on the additional point in the corollary is slightly less than in the respective Theorem 4.

COROLLARY 6. *Let the node $z_0$ be outside the lines passing through the origin and one of $z_i$, $i = 1, ..., 2k+1$. Then for any set of given numbers $\{c_{ij}\}$ and $c_{00}$ there exists a unique polynomial $P \in \Pi_{2k+1}$ such that*

$$P(z_0) = c_{00},$$

*and*

$$(D_{z_i})^j P(z_i) = c_{ij} \qquad \textit{for any} \quad i = 1, ..., 2k+1; \quad j = 0, ..., k+1.$$

We will see that the crucial step in proving all the previous results is the following Factorization theorem:

THEOREM 7 (On Factorization). *Let $P$ be a bivariate polynomial of degree $k \leqslant m \leqslant 2k$. Assume that*

$$(D_{z_i})^j P(z_i) = 0 \qquad \textit{for any} \quad i = 1, ..., 2k+1; \quad j = 0, ..., m-k. \tag{8}$$

*Then we have that*

$$P(x, y) = (x^2 + y^2 - 1)^{m+1-k} Q(x, y),$$

*where $Q \in \Pi_{2k-m-2}$. In particular $Q = 0$, if $m = 2k$ or $m = 2k-1$, and $Q = c = \text{const.}$, if $m = 2k-2$.*

We will use the following standard tool in the polynomial interpolation.

*Assertion* 8. To prove Theorems 1–4 it is enough to establish only the uniqueness there when the given number sets are null; i.e., it is enough to check that if the conditions (1)–(2) and (4)–(7) hold with $c_{ij} = 0$, $d_i = 0$, and

$d_{ij} = 0$, respectively, where $P$ is from the respective polynomial space, then $P = 0$.

In view of this, Theorem 7 implies Theorems 1 and 2, immediately. To check this for Theorem 3 we apply Theorem 7 for this case and get the following factorization

$$P(x, y) = (x^2 + y^2 - 1)^{k-m+1} Q(x, y), \tag{9}$$

where $Q \in \Pi_{m-2}$. We have that $P$ satisfies the conditions (5) of Theorem 3 with $d_i = 0$, i.e.,

$$P(z_i^1) = 0 \qquad \text{for any} \quad i = 1, \ldots, m(m-1)/2,$$

where $z_i^1 \notin S^1$. Hence we get from (9) that

$$Q(z_i^1) = 0 \qquad \text{for any} \quad i = 1, \ldots, m(m-1)/2.$$

Now the poisedness of nodes implies $Q = 0$ and therefore $P = 0$.

The proof of Theorem 4 (and Corollary 6) based on Theorem 7 needs some preliminaries which will be developed later. We will present the proof at the end of the following section.

## 2. THE REDUCTION OF THE FACTORIZATION THEOREM TO A UNIVARIATE RESULT

We start this section with the formulation of the above mentioned univariate result.

THEOREM 9. *Let $k$ be a nonnegative integer, $\delta = 0$ or $1$. Let also $p$ be a univariate polynomial of degree $m$, $k - \delta + 1 \leqslant m \leqslant 2(k - \delta) + 1$:*

$$p(x) = \sum_{i=0}^{m} a_{m-i} x^i.$$

*Assume that*

$$p(t_i) = 0 \qquad \textit{for} \quad i = \delta, \ldots, k, \tag{10}$$

*where $t_i = \cos(2i\pi/(2k+1))$. Also assume that*

$$a_{2i-1} = 0 \qquad \textit{for} \quad i = 1, \ldots, m - k + \delta. \tag{11}$$

*Then $p = 0$.*

Note that, in view of Assertion 8, this is a statement of the poisedness of a Birkhoff interpolation (see Theorem 12 and Remark 13). Let us mention also that the forthcoming Corollary 18 is a strengthening of Theorem 9. We will prove Theorem 9 in the next section.

To reduce the proof of the Factorization theorem to the above result we introduce some notations. *The homogeneous part of degree* $i$ of a polynomial $P \in \Pi_s$ we denote by $Hom_i\ P$, $i = 0\ldots, s$. For the polynomial $P \in \Pi_\nu$ we consider the operator $D_*: \Pi_\nu \to \Pi_{\nu-1}$ given by the following formula:

$$D_* P(x, y) = \nu P(x, y) - D_{(x, y)} P(x, y).$$

We have that $D_*^\mu: \Pi_\nu \to \Pi_{\nu-\mu}$, where $D_*^\mu P = D_*^{\mu-1}(D_* P)$, and we accept that $D_* P \in \Pi_{\nu-1}$ if $P \in \Pi_\nu$. Note that the constant $\nu$ in the definition of $D_*$ depends on the degree of the polynomial class.

We call the condition (3) the *"zero s"* condition.

Let us mention the following two properties of the operator $D_*$.

(i) If $P$ satisfies the "zero $s$" condition, then $D_* P$ satisfies the "zero $s-1$" condition and $D_*^\mu P$ satisfies the "zero $s-\mu$" condition.

(ii) We have

$$Hom_i\ D_* P = \lambda\ Hom_i\ P, \qquad \text{for} \quad P \in \Pi_s,$$

where $\lambda = s - i \neq 0$ and $i = 1, \ldots, s-1$. Thus the corresponding homogeneous parts of $D_* P$ and $P$ are constant proportional.

We will often use the following Bezout's theorem:

*Assertion* 10. Assume that

$$P(z_i) = 0 \qquad \text{for any} \quad i = 1, \ldots 2k+1 \quad (z_i \in S^1),$$

where $P$ is a bivariate polynomial of degree $k$. Then

$$P(x, y) = (x^2 + y^2 - 1)\ Q(x, y),$$

where $Q \in \Pi_{k-2}$.

We will use also the following lemma concerning the operator $D_*$. Let us note that the univariate analog of this, with the usual differentiation operator, is a well-known statement on the multiplicity of root.

LEMMA 11. *Let* $P = (x^2 + y^2 - 1)\ Q_1$ *and* $D_* P = (x^2 + y^2 - 1)^s\ Q_2$. *Then* $P = (x^2 + y^2 - 1)^{s+1}\ Q_3$, *where* $Q_1$, $Q_2$, *and* $Q_3$ *are polynomials.*

*Proof.* We use induction on $s \geqslant 0$. The case $s=0$ is obvious. Assume that the lemma is valid for $s-1$. We will prove it for $s$. By using the induction hypothesis we get

$$P=(x^2+y^2-1)^s\, R,$$

where $R$ is a polynomial. Now we have

$$(x^2+y^2-1)^s\, Q_2 = D_* P = (x^2+y^2-1)^s\, D_* R + 2s(x^2+y^2-1)^{s-1}\,(x^2+y^2)\, R.$$

Therefore

$$(x^2+y^2-1)\, Q_2 = (x^2+y^2-1)\, D_* R + 2s(x^2+y^2)\, R.$$

We get from here that

$$R|_{S^1}=0,$$

where $|_{s^1}$ means the restriction on $S^1$. To complete the proof it remains to use Assertion 10.

Now we can start

*Proof of Factorization theorem* 7 *Based on Theorem* 9. First we rotate the set of points $Z_0=\{z_i\}$ and reorder them so that

$$z_i = \left( cos\, \frac{2i\pi}{2k+1},\, sin\, \frac{2i\pi}{2k+1} \right), \qquad i=0, ..., 2k.$$

This can be done in view of the respective invariance of the polynomial space $\Pi_m$. Now we have that

$$(x, y) \in Z_0 \text{ implies } (x, -y) \in Z_0. \tag{12}$$

This implies that it is enough to prove Theorem 7 for polynomials which are *even or odd with respect to the variable* $y$. Indeed, assume that the theorem is proved for such polynomials and let $P \in \Pi_m$ satisfy the condition (8) of Theorem 7. Then, in view of (12), the following polynomials

$$Q(x, y) = P(x, y) + P(x, -y), \qquad R(x, y) = P(x, y) - P(x, -y),$$

which are even and odd, respectively, in above sense, also satisfy them and therefore they can be factorized. Now, in view of the equality $P=(Q+R)/2$, we get the desired factorization for $P$.

Let us now prove Theorem 7 by induction on $m \geqslant k$, in two cases: for polynomials which are either even or odd in $y$. Assume that $P$ satisfies the condition (8); i.e., "zero $m-k$" condition. Then, in view of property (i) of the operator $D_*$, we have that $D_*^{m-k} P \in \Pi_k$ satisfies "zero 0" condition. Now, by taking into account Assertion 10, we get that

$$D_*^{m-k} P = (x^2 + y^2 - 1)\, Q,$$

for some polynomial $Q \in \Pi_{k-2}$. This completes the first step of induction. Now assume that Theorem 7 is true for $m-1$ we shall prove it for $m$. Let $P \in \Pi_m$ satisfy the assumptions in the theorem. Then $D_* P$ belongs to $\Pi_{m-1}$ and satisfies "zero $m-k-1$" condition. Thus by the induction hypothesis we have that

$$D_* P = (x^2 + y^2 - 1)^{m-k}\, Q, \tag{13}$$

for some polynomial $Q \in \Pi_{2k-m-1}$. This implies for $i \leqslant m-1$ that

$$Hom_i\, D_* P(x, y) = \sum_{j=0}^{i} c_j\, Hom_j\, Q(x, y),$$

where $c_j = 0, -1, 1$, or a power of $x^2 + y^2$. We get from here, in view of property (ii) of $D_*$, that

$$Hom_i\, P|_{S^1} \in \Pi_{2k-m-1}, \qquad \text{for} \quad i \leqslant m-1. \tag{14}$$

*Case* 1. Consider first the case when $P$ is even with respect to $y$:

$$P(x, y) = \sum_{i=0}^{m} Hom_i P(x, y) = \sum_{i=0}^{m} Q_i(x, y^2). \tag{15}$$

Let us check now, using Theorem 9, that

$$P|_{S^1} = 0. \tag{16}$$

We get from (14)–(15) that

$$p_1(x) := P(x, \sqrt{1-x^2}) = P(x, y)|_{S^1} = q_1(x) + Q_m(x, 1-x^2),$$

where $q_1 \in \pi_{2k-m-1}$ and $Q_m \in \Pi_m$. Since $Q_m(x, 1-x^2)$ contains only monomials $x^{m-2i}$, it follows that $p_1$ satisfies the conditions of Theorem 9 with $\delta = 0$. Whence $p_1 = 0$. Thus (16) is established. Now to complete the proof, it remains to use Assertion 10 and then Lemma 11.

*Case* 2. Consider now the case when $P$ is odd with respect to $y$:

$$P(x, y) = \sum_{i=0}^{m} Hom_i\, P(x, y) = y \sum_{i=0}^{m-1} R_i(x, y^2). \tag{17}$$

Let us check now the condition (16). We get from (14) and (17) that

$$p_2(x) := (1/y)\, P(x, \sqrt{1-x^2}) = q_2(x) + R_{m-1}(x, 1-x^2),$$

where $q_2 \in \pi_{2k-m-2}$ and $R_{m-1} \in \Pi_{m-1}$. This means that $p_2$ satisfies the conditions of Theorem 9, with $\delta = 1$. This completes the proof as in the previous case.

We end the section, as was promised, by

*Proof of Theorem* 4 *Based on Theorem* 7. We will prove Theorem 4 by induction on $m \geqslant 0$. The case $m = 0$ coincides with Theorem 1, which is proved already. Assuming that Theorem 4 is true for $m-1$ we will prove it for $m$. Let $P \in \Pi_{2k+m}$ satisfy the conditions (6)–(7), where $c_{ij} = d_{ij} = 0$. In view of property (i) of $D_*$, we have that $D_* P \in \Pi_{2k+m-1}$ satisfies these conditions (6)–(7) with $m$ replaced by $m-1$. From the induction hypothesis we obtain

$$D_* P = 0.$$

Therefore, in view of the property (ii) of $D_*$, we get

$$Hom_i\, P = 0, \qquad \text{for} \quad i \leqslant 2k+m-1.$$

Thus we have

$$P(z) = Hom_{2k+m}\, P(z) = 0 \qquad \text{for all} \quad z \in Z_0 \cup Z_2 \subset S^1.$$

Now since the set $\{Z_0 \cup Z_2\}$ does not contain opposite pairs and its cardinality is $2k+m+1$, we conclude that $P = 0$. The condition on $z_0$ in Corollary 6, can be verified in the same way.

## 3. THE PROOF OF THE UNIVARIATE RESULT

In this section we will prove Theorem 9 which, in view of Assertion 8, implies the following result on the poisedness of a Birkhoff interpolation, for $\delta = 0, 1$.

THEOREM 12. *Let $\delta = 0$ or $1$ and the integers $m, k, k-\delta+1 \leqslant m \leqslant 2(k-\delta)+1$, be given. Let also $t_i = \cos(2i\pi/(2k+1))$, $i = 0, ..., k$. Then for any given sets of numbers $\{c_i\}$ and $\{d_i\}$ there exists a unique polynomial $p \in \pi_m$ such that*

$$p(t_i) = c_i \qquad \textit{for any} \quad i = \delta, ..., k, \qquad \textit{and}$$
$$p^{(m-2i+1)}(0) = d_i \qquad \textit{for any} \quad i = 1, ..., m-k+\delta.$$

*Remark* 13. (i) Note that for a fixed $k=0, 1, 2\ldots$, the above theorem states the poisedness of $k+1$ and $k$ Birkhoff interpolations connected with the point sets $\{t_i\}_{i=0}^k$ and $\{t_i\}_{i=1}^k$ $(t_0=1)$, respectively.

(ii) Let us mention that, since so far we did not use the values of $\{t_i\}_{i=1}^k$, all the previous results will remain true also in the following general symmetric (not necessarily equidistant) case of the point set

$$Z_0=\{(t_\nu, \pm\sqrt{1-t_\nu^2})\}_{\nu=0}^k, \qquad \text{where} \quad -1<t_i<1, \quad i=1, 2, ..., k,$$

provided that the Birkhoff interpolations listed in (i) are poised.

(iii) Moreover, one can hope that the Factorization Theorem 7 is true (and hence all the previous results are true) for the symmetric point set described in (ii) if and only if the Birkhoff interpolations listed in (i) are all poised.

Let us mention that Theorem 9 (or 12) is not true for $m \geqslant 2(k-\delta)+2$. To see this we consider

$$p(x)=\prod_{i=\delta}^k (x^2-t_i^2).$$

Note also that the cases of degree $m=k-\delta+1$, $2(k-\delta)$, and $2(k-\delta)+1$ are the easiest ones in Theorem 9 (or 12). Indeed, in the first case we are to show only that $\sum_{i=\delta}^k t_i \neq 0$, while in the other two cases conditions (11) imply that the polynomial $p$ is either even or odd. In the even case $p$ has zeros $\{t_i, -t_i\}_{i=\delta}^k$ and one more: 0 in the odd case. In both cases the number of zeros is the degree of $p$ plus one. Hence $p=0$.

We start now

*Proof of Theorem* 9, which consists of two parts corresponding to values $\delta=0, 1$, and includes a theorem and several lemmas.

*Outline of Proof of Theorem 9*. The general scheme for proving the theorem is the translation of the conditions (11) into a linear system where number of equations is one more than the number of unknowns (which are the forthcoming $\gamma_i$, $i=1, ..., m-k-1+\delta$) and the proving that the resulted system is inconsistent.

**Part 1.** In this part we consider the case $\delta=0$. Suppose, on the contrary, that Theorem 9 (in this case) is not valid; i.e., there exists a nonzero polynomial $p\in\pi_m$ $(k+1\leqslant m\leqslant 2k+1)$,

$$p(x)=\sum_{i=0}^m a_{m-i}x^i,$$

with

$$p(t_i) = 0 \qquad \text{for} \quad i = 0, \ldots k, \tag{18}$$

and

$$a_{2i-1} = 0 \qquad \text{for} \quad i = 1, \ldots, m-k. \tag{19}$$

Note that we have $m \leqslant 2k+1$ which implies that $2(m-k)-1 \leqslant m$; i.e., the indices of $a$ in conditions (19) are within the correct limits.

Without loss of generality we can assume that $a_0 = 1$. Denote by $q_0$ (the index 0 here corresponds to the case $\delta = 0$) the polynomial

$$q_0(x) = \prod_{i=0}^{k} (x - t_i) =: \sum_{i=0}^{k+1} \alpha_{k+1-i} x^i.$$

According to (18) there is a polynomial

$$r = \sum_{i=0}^{m-k-1} \gamma_{k-i} x^i,$$

such that $p = q_0 r$, i.e.,

$$p(x) = \sum_{i=0}^{k+1} \alpha_{k+1-i} x^i \sum_{i=0}^{m-k-1} \gamma_{m-k-1-i} x^i,$$

where $\alpha_0 = \gamma_0 = 1$.

Now conditions (19) are translating into the following linear system, which was mentioned above, in the outline of the proof,

$$\gamma_1 = -\alpha_1,$$

and

$$\sum_{j=1}^{2i-1} \gamma_j \alpha_{2i-j-1} = -\alpha_{2i-1}, \qquad \text{for} \quad i = 2, \ldots, m-k, \tag{20}$$

where $\alpha_l = 0$, if $l \notin [0, k+1]$ and $\gamma_l = 0$, if $l \notin [0, m-k-1]$.

In what follows, in this section, we will prove that this linear system (and the linear system corresponding to the case $\delta = 1$, in Part 2) of $m-k$ equations and with $m-k-1$ unknowns: $\gamma_1, \ldots, \gamma_{m-k-1}$ is always inconsistent, thus proving the theorem. It may seem that it will be difficult to carry out such a plan since we have a system of general dimension, where the coefficients depend on $k$ and $m$, but as we will see the *fine* set of roots $\{t_i\}$ will enable us to succeed.

*Step* 1. Here we develop a representation of coefficients $\{\alpha_i\}$ of the polynomial $q_0$.

THEOREM 14. *Let*

$$q_0(x) = \Pi_{i=0}^{k}(x - t_i) =: \sum_{i=0}^{k+1} \alpha_{k+1-i} x^i,$$

*where* $t_i = \cos(2i\pi/(2k+1))$, $i = 0, \dots k$. *Then we have*

$$\alpha_{2\nu} = \frac{(-1)^\nu}{4^\nu} \frac{k+1}{\nu} \binom{k-\nu}{\nu-1},$$

*and*

$$\alpha_{2\nu+1} = \frac{(-1)^{\nu+1}}{2 \cdot 4^\nu} \frac{k}{\nu} \binom{k-\nu-1}{\nu-1}, \tag{21}$$

$\nu = 0, \dots, [(k+1)/2]$.

In particular we have

$$\alpha_0 = 1, \qquad \alpha_1 = -1/2, \qquad \alpha_2 = -(k+1)/4, \qquad \text{and} \qquad \alpha_3 = k/8.$$

To prove this theorem we need some lemmas. The following lemma presents an identity involving the coefficients and roots of a polynomial, which can be readily checked.

LEMMA 15. *Let*

$$q(x) = \Pi_{i=0}^{k}(x - t_i) =: \sum_{i=0}^{k+1} \alpha_{k+1-i} x^i.$$

*Then we have for* $1 \leqslant \mu \leqslant k+1$,

$$\mu \alpha_\mu = \sum_{i=1}^{\mu} (-1)^i \alpha_{\mu-i} \beta_i, \tag{22}$$

*where* $\beta_i = \sum_{j=0}^{k} t_j^i$.

LEMMA 16. *Under the assumptions of Theorem* 14 *we have*

$$\beta_{2\nu-1} = \tfrac{1}{2},$$

*and*

$$\beta_{2\nu} = \frac{2k+1}{2 \cdot 4^\nu} \binom{2\nu}{\nu} + \frac{1}{2} =: \eta_{2\nu} + \frac{1}{2},$$

$\nu = 1, \ldots, k$.

*Proof.* The following equalities

$$\sum_{i=0}^{k} \cos \frac{2i\pi\mu}{2k+1} = 1/2 \qquad \text{for all} \quad \mu = 0, \ldots, 2k, \tag{23}$$

can be checked readily, in view of the formula

$$\sum_{i=1}^{k} \cos i\varphi = \left[ \cos \frac{k+1}{2} \varphi \sin \frac{k\varphi}{2} \right] \Big/ \sin \frac{\varphi}{2}.$$

We will use also the following expansion

$$\cos^q x = \sum_{\nu=0}^{q} c_\nu \cos(\nu x). \tag{24}$$

By taking $x = 0$ here we get

$$\sum_{\nu=0}^{q} c_\nu = 1.$$

Also we have that

$$c_0 = \frac{1}{\pi} \int_0^\pi \cos^q x \, dx = \begin{cases} 0, & \text{if} \quad q = 2\nu+1, \\ 1/4^\nu \binom{2\nu}{\nu}, & \text{if} \quad q = 2\nu. \end{cases} \tag{25}$$

Now, in view of (23), we get from (24)

$$\beta_q = \sum_{j=0}^{k} t_j^q = (k+1)\, c_0 + (1/2) \sum_{\nu=1}^{q} c_\nu = (k+1)\, c_0 + (1/2)(1 - c_0).$$

To finish the proof it remains to use (25).

The following lemma can be checked readily by induction.

LEMMA 17. *Let $S_n := \sum_{i=0}^{n} \alpha_i$, for $n = 0 \ldots, k$. Then we have*

$$S_{2\nu} = \frac{(-1)^\nu}{4^\nu} \binom{k-\nu}{\nu},$$

*and*

$$S_{2\nu+1} = \frac{(-1)^\nu}{2 \cdot 4^\nu} \binom{k-\nu-1}{\nu}, \tag{26}$$

*for* $\nu = 0, ..., [(k+1)/2]$, *provided that Theorem* 14 *is valid.*

In particular we have

$$s_0 = 1, \qquad s_1 = 1/2, \qquad s_2 = -(k-1)/4, \qquad \text{and} \qquad s_3 = -(k-2)/8.$$

Now we are in a position to present

*Proof of Theorem* 14. We use induction on $\nu$. Assuming the formulas hold for $\nu - 1$; we will prove them for $\nu$. By making use of Lemma 16 we can simplify (22) to

$$\mu \alpha_\mu = -(1/2)\, S_{\mu-1} - \sum_{i=1}^{[\mu/2]} \alpha_{\mu-2i} \eta_{2i}. \tag{27}$$

Now, on the basis of the induction hypothesis, Lemma 17 is valid. Using it and considering separately the cases $\mu = 2\nu + 1$ and $\mu = 2\nu$ reduces (27) to the equality

$$\binom{k-\nu-1}{\nu-1} + (-1)^\nu (1/2) \binom{2\nu}{\nu} = \sum_{i=1}^{\nu-1} (-1)^{\nu+i+1} \frac{k}{2i} \binom{k-i-1}{i-1} \binom{2(\nu-i)}{\nu-i}.$$

We need to prove that this is an identity, which can be done readily by induction on $k \geqslant 0$. Indeed, in the case $k = 0$ we are to check just that the left hand side of the above equality is 0. Then assuming the equality is true for $k$ and subtracting it from the corresponding equality for $k+1$ we arrive again to the case of $k$. This means that the equality is true also for $k+1$, and the proof is complete.

*Step* 2. Now we turn to the linear system (20). We will prove that the system is inconsistent by showing that:

(i) The subsystem consisting of all equations but the first has a unique solution.

(ii) This unique solution is not a solution of the first equation.

The same we will do in Part 2. Notice that by this we actually establish the following result of independent interest.

COROLLARY 18. *Let $\delta=0$ or 1 and $k$ be a nonnegative integer. Then there exists a unique nonzero polynomial of degree $=m$, $k-\delta+1 \leqslant m \leqslant 2(k-\delta)+1$,*

$$p(x)=\sum_{i=0}^{m} a_{m-i} x^i, \qquad a_0=1,$$

*such that*

$$p(t_i)=0 \qquad \text{for} \quad i=\delta, \ldots, k,$$

*and*

$$a_{2i-1}=0 \qquad \text{for} \quad i=2, \ldots, m-k+\delta,$$

*where $t_i=\cos(2i\pi/(2k+1))$.*

*Moreover we have that $a_1 \neq 0$.*

We will prove the above mentioned point (i) by showing that the determinant of the following matrix does not vanish

$$A=\begin{pmatrix} \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 & \cdots & 0 \\ & & & \cdots & & \cdots & & & & \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \alpha_{n-5} & \alpha_{n-6} & \alpha_{n-7} & \cdots & \alpha_{m-k} \end{pmatrix},$$

where $n=2m-2k-2$ and $\alpha_l=0$, if $l \notin [0, k+1]$. Note that here we have

$$a_{i,j}=\alpha_{2i-j+1}. \tag{28}$$

By the Cramer Rule, (ii) is equivalent to

$$\det A_1+\alpha_1 \det A \neq 0, \tag{29}$$

where $A_1$ is obtained from $A$ by replacing its first column with the free terms in the linear system:

$$-(\alpha_3, \alpha_5, \ldots, \alpha_{2(m-k)-1})^T.$$

Since $\alpha_1=-1/2$, we can rewrite (29) in the form

$$\det A_{1*} \neq 0,$$

where $A_{1*}$ is obtained from $A$ by replacing its first column with the following column:

$$(2\alpha_3+\alpha_2,\, 2\alpha_5+\alpha_4,\, \ldots,\, 2\alpha_{2(m-k)-1}+\alpha_{2(m-k)-2})^T.$$

In view of (21), we have for the general term here

$$2\alpha_{2\nu+1}+\alpha_{2\nu} = \frac{(-1)^{\nu}}{4^{\nu}}\frac{k-1}{\nu-1}\binom{k-\nu-1}{\nu-2}. \tag{30}$$

Thus, it remains to show that

$$\det A \neq 0 \qquad \text{and} \qquad \det A_{1*} \neq 0.$$

We evaluate each of this determinants by the Gauss Elimination Method. The results are

$$\det A = \pm 2^{(k-m+1)(m-k)/2}\binom{m-1}{k}, \qquad \text{and} \qquad \det A_{1*} = \pm 2^{(1-m)\,m/2},$$

where $\pm = (-1)^{(m-k)/2}$, if $m-k$ is even, and $(+1)$, otherwise.

These equalities follow from the forthcoming reduced upper triangular forms $B$ and $B_{1*}$ of the matrices $A$ and $A_{1*}$ in the Gauss elimination. Namely, in the right hand sides of the above equalities we have the products of diagonal elements of matrices $B$ and $B_{1*}$, respectively. Here we have the matrix $B$,

$$B = B(k)$$
$$= \begin{pmatrix} b_{11} & -1/2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_{22} & b_{23} & -1/2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & b_{35} & -1/2 & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_{m-k-1,\,m-k-1} \end{pmatrix},$$

where

$$\begin{aligned} b_{i,\,2j} &= \frac{(-1)^{i-j+1}}{i2\cdot 4^{i-j}}\left[\binom{j-1}{i-j-1}k+\binom{j}{i-j}i\right], \\ b_{i,\,2j+1} &= \frac{(-1)^{i-j}}{i4^{i-j}}\left[\binom{j}{i-j-1}k+\binom{j+1}{i-j}i\right]. \end{aligned} \tag{31}$$

It is interesting that

$$B_{1*} = B(0). \tag{32}$$

To prove that in fact these are the above mentioned reduced forms we will use induction on $n$, which is the number of steps (or rows) in the Gauss elimination. The case $n=0$ is obvious since the first rows of the matrices $A$ and $B$ (or $A_{1*}$ and $B_{1*}$, respectively) coincide. We assume that after the step $n-1$ in the Gauss elimination of $A$ ($B$) we have a matrix whose first $n-1$ rows coincide with the corresponding rows of $B$ ($B_{1*}$). We will show that after the step $n$ here, the $n$th row will coincide with the corresponding row of $B$ ($B_{1*}$). Let us do this separately for the cases of matrices $A$ and $B$.

(i) Case of the matrix $A$. In this case we show that

$$n\text{th row of } A - \sum_{i=0}^{n-1} c_{i,n} \times i\text{th row of } B = n\text{th row of } B, \tag{33}$$

where

$$c_{i,n} = \frac{(-1)^{n-i}\, i}{n 4^{n-i}} \binom{k-n}{n-i}.$$

The relation (33), for the odd column $2j+1$, in view of (21), (28), and (31), reduces to

$$\frac{k+1}{n-j}\binom{k-n+j}{n-j-1} = \frac{1}{n}\sum_{i=1}^{n}\left[\binom{j}{i-j-1}k + \binom{j+1}{i-j} i\right]\binom{k-n}{n-i}.$$

This equality can be checked readily by using the following identity of combinations:

$$\binom{r+s}{q} = \sum_{i=0}^{q}\binom{r}{i}\binom{s}{q-i}.$$

Similarly the case of even columns can be checked.

(ii) Case of the matrix $A_{1*}$. In this case we show that

$$n\text{th row of } A_{1*} - \sum_{i=0}^{n-1} d_{i,n} \times i\text{th row of } B_{1*} = n\text{th row of } B_{1*}, \tag{34}$$

where

$$d_{i,n} = \frac{(-1)^{n-i}}{4^{n-i}}\,\frac{k-i}{k-n}\binom{k-n}{n-i}.$$

The relation (34), for the odd column $2j+1>1$, in view of (21), (28), (32), and (31), reduces to

$$\frac{k+1}{n-j}\binom{k-n+j}{n-j-1}=\sum_{i=1}^{n}\frac{k-i}{k-n}\binom{j+1}{i-j}\binom{k-n}{n-i}.$$

This relation can be checked in the same way as the above one. Similarly can be checked the cases of the first column (in view of (30)) and even columns.

**Part 2.** In this part we consider the case $\delta=1$ (of Theorem 9). In this case we have

$$q_1(x)=\prod_{i=1}^{k}(x-t_i)=:\sum_{i=0}^{k}\alpha_{k-i}^{*}x^i.$$

Hence $q_0(x)=q_1(x)(x-1)$, which implies that

$$\alpha_n^{*}=\sum_{i=0}^{n}\alpha_i=s_n.$$

Thus at once we get that **Step 1** of Part 2 is completed already in Lemma 17, where we have a representation for $s_\mu$.

*Step* 2. Here the matrices $\bar{A}$ and $\bar{A}_{1*}$, which correspond to $A$ and $A_{1*}$ of Part 1, have elements $s_i$ instead of $\alpha_i$ and have dimension $m-k$ instead of $m-k-1$ there. Thus we are to prove that

$$\det\bar{A}\neq 0\qquad\text{and}\qquad\det\bar{A}_{1*}\neq 0,$$

where $\bar{A}_{1*}$ is obtained from $\bar{A}$ by replacing its first column with the column:

$$(2s_3-s_2,\,2s_5-s_4,\,\ldots,\,2s_{2(m-k)+1}-s_{2(m-k)})^T.$$

In view of (26) we have for the general term here

$$2s_{2\nu+1}-s_{2\nu}=\frac{(-1)^{\nu+1}}{4^\nu}\binom{k-\nu-1}{\nu-1}. \tag{35}$$

In this part we will just evaluate these determinants since it is hard to find the reduced forms of the Gauss elimination. First we simplify the determinants, by means of elementary operations.

We start by factoring out the coefficients of the combinations in the representation of $s_\mu$ in (26) from the considered determinants. This can be done obviously for the coefficient $1/2$, which results with the final factor $1/2^n$, where $n=m-k$ is the dimension of the determinant in this part.

Next, for the case of $(-1)^\nu$, we multiply the column nos. 2, 3, 6, 7, 10, 11 and so on by $(-1)$ after which elements in each row have the same sign. In this way finally we factor out $\pm 1$, namely $(-1)$, if $m-n=4i+1$ for some $i$; and $(+1)$ otherwise. In the case of coefficient $1/4^\nu$ we multiply column nos. 2 and 3 by $1/4$; nos. 4 and 5 by $1/4^2$; nos. 6 and 7 by $1/4^3$, and so on. After this we get that such factors are the same in each row. Thus we can factor out all of them from the first row. This will result to the following final such factor $4^{-n(n+2)/4}$. Summarizing all the above factors we get the factor $\pm 2^{-n(n+3)/2}$. Thus we have

$$-\det \bar{A} = \pm 2^{(k-m)(m-k+3)/2} \det \hat{A}, \tag{36}$$

where $\pm = (-1)^{(m-k+1)/2}$, if $m-k$ is odd, and $(+1)$, otherwise. Here $\hat{A}$ contains only the combinations in representation (26) of $s_\mu$, namely it is the following matrix,

$$\begin{pmatrix} \binom{k-1}{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{k-2}{2} & \binom{k-2}{1} & \binom{k-1}{1} & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \binom{k-3}{3} & \binom{k-3}{2} & \binom{k-2}{2} & \binom{k-2}{1} & \binom{k-1}{1} & 1 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \cdots & & & & \\ \binom{r}{n} & \binom{r}{n-1} & \binom{r-1}{n-1} & \binom{r-1}{n-2} & \binom{r-2}{n-2} & \binom{r-2}{n-3} & \binom{r-3}{n-3} & \binom{r-3}{n-4} & \cdots & \binom{*}{*} \end{pmatrix},$$

where $r=k-n$, $n=m-k$, $\binom{*}{l}=0$, if $l<0$, and $\binom{*}{*}$ is the combination in the representation of $s_{m-k+1}$.

Note that the matrices $\bar{A}$ and $\bar{A}_1*$ differ only in the first column and the coefficients of the combinations there, in view of (26) and (35), are opposite. Thus, we have the following equality, which corresponds to (36),

$$\det \bar{A}_1 * = \pm 2^{(k-m)(m-k+3)/2} \det \hat{A}_1 *. \tag{37}$$

Here $\hat{A}_{1*}$ is obtained from $\hat{A}$ by replacing its first column with the combinations from (35), i.e.,

$$\left[\binom{k-2}{0}, \binom{k-3}{1}, \binom{k-4}{2}, \ldots, \binom{2k-m-1}{m-k-1}\right]^T. \tag{38}$$

In the second step of simplification we are using the identity

$$\binom{\mu}{\nu}-\binom{\mu-1}{\nu}=\binom{\mu-1}{\nu-1}, \tag{39}$$

and perform the following elementary operations to evaluate $\det \bar{A}$, and $\det \bar{A}_{1*}$: Choose two neighbour columns where the bottom items in the corresponding combinations equal (such as 2nd and 3rd, 4th and 5th) and subtract from the right column the left one. By doing this operation, each time, first for the rightest possible pair one will arrive to the following matrix $\hat{A}$, in the case of $\bar{A}$,

$$\begin{pmatrix} \binom{k-1}{1} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \binom{k-2}{2} & \binom{k-2}{1} & 1 & 0 & 0 & \cdots & 0 \\ \binom{k-3}{3} & \binom{k-3}{2} & \binom{k-3}{1} & 1 & 0 & \cdots & 0 \\ & & & \cdots & & \cdots & \\ \binom{k-n}{n} & \binom{k-n}{n-1} & \binom{k-n}{n-2} & \binom{k-n}{n-3} & \binom{k-n}{n-4} & \cdots & \binom{k-n}{1} \end{pmatrix},$$

where $n=m-k$.

In the above elementary operations we did not use the first column. Therefore, the corresponding matrix $\hat{A}_{1*}$, in the case of $\bar{A}_{1*}$, differs from $\hat{A}$ only with the first column which is (38) and we have that

$$\det \bar{A}=\det \hat{A}, \qquad \text{and} \qquad \det \bar{A}_{1*}=\det \hat{A}_{1*}. \tag{40}$$

Now, we evaluate $\det \hat{A}$. To do this we subtract from the first column the $c_i \times i$th column, $i=2, \ldots, m-k$, succesively, to make the element in the first column and in the row $i-1$ to become zero each time.

Let us prove that after doing this with the $\nu$th column the first column equals

$$(-1)^{\nu-1}\frac{k-i}{i}\binom{k+\nu-1}{\nu-1}\binom{k-i-1}{i-\nu}=: d(\nu, i), \qquad i=1, \ldots m-k.$$

We use induction on $\nu$, i.e., assuming that it is true for $\nu$ we prove it for the case of column $\nu+1$. Since the top nonzero element in that columns is 1 so one has

$$c_{\nu+1}=d(\nu,\nu)=(-1)^{\nu-1}\frac{k-\nu}{\nu}\binom{k+\nu-1}{\nu-1}.$$

Thus we arrive to the following identity

$$d(v+1, i) = d(v, i) - c_{v+1} \binom{k-i}{i-v},$$

which can be cheked easely. After the finishing this procedure the only nonzero element in the first column is the last one, which equals $d(m-k, m-k)$. It is easely seen then that we have, at the same time,

$$\det \hat{A} = (-1)^{m-k+1}\, d(m-k, m-k).$$

Thus we get, in view of (36) and (40), that

$$\det \bar{A} = \pm 2^{(k-m)(m-k+3)} \frac{m-2k}{m} \binom{m}{k},$$

where $\pm = (-1)^{(m-k+1)/2}$, if $m-k$ is odd, and $(-1)$, otherwise.

Since in the case of $\delta = 1$ we have, in Theorem 9, that $m \leqslant 2k-1$ hence $2k-m \neq 0$ and we get that $\det \bar{A} \neq 0$.

At the end we turn to $\det \hat{A}_{1*}$, evaluate which will be easier. We subtract from the first column the $(-1)^i \times i$th column, $i = 2, ..., m-k$. In view of identity (39) after this first column equals

$$(-1)^{n+1} \binom{k-v-1}{v-n}, \qquad v = 1, ..., n,$$

with $n = m-k$; i.e., the last element is the only nonzero one and equals to $(-1)^{m-k+1}$. Therefore

$$\det \hat{A}_{1*} = 1.$$

Finally, we get, in view of (37) and (40), that

$$\det \bar{A}_{1*} = \pm 2^{(k-m)(m-k+3)/2} \neq 0.$$

This completes the proof of Theorem 9.

## 4. CUBATURE FORMULAS

In this section we consider three cubature formulas associated with interpolations described in Theorems 1, 2, and Corollary 5.

Let us start with the interpolation described in Theorem 1. Denote by $P_{i_0 j_0} \in \Pi_{2k}$, $i_0 = 0, \ldots, 2k$; $j_0 = 0, \ldots, k$ the fundamental polynomials of this interpolation, i.e.,

$$P_{i_0 j_0}^{(j)}(z_i) = \delta_{i_0, j_0}^{i, j},$$

where $\delta$ is the Kronecker symbol. In view of the rotational symmetry of these fundamental polynomials with fixed $j_0$, we get that the integral

$$\int_{x^2+y^2 \leqslant 1} P_{i_0 j_0}(x, y)\, dx\, dy = \lambda_{j_0}; \tag{41}$$

i.e., it does not depend on $i_0$.

Consider the following formula for the interpolating polynomial $P_f$ of a function $f$,

$$P_f(x, y) = \sum_{i=0}^{2k} \sum_{j=0}^{k} f^{(j)}(z_i)\, P_{ij}(x, y).$$

By integrating this formula, having into account (41), we get, in a standard way, the cubature formula,

$$\int_{x^2+y^2 \leqslant 1} f(x, y)\, dx\, dy \doteq \sum_{i=0}^{k} \lambda_j L_j(f), \tag{42}$$

where

$$L_j(f) = \sum_{j=0}^{2k} (D_{z_i})^j f(z_i).$$

Since this is an interpolatory cubature formula we have that $d_1 \geqslant 2k$, where $d_1$ is its algebraic degree of precision.

This cubature rule contains very small number of coefficients—the same number as Gauss quadrature formulas of the same algebraic precision.

Now let us find these coefficients. We have that (42) is precise for $\Pi_{2k}$. Hence we get by taking $f(x, y) = (x^2+y^2)^v$,

$$\frac{2\pi}{2v+2} = (2k+1)[\lambda_0 + 2v\lambda_1 + \cdots + (2v)^k \lambda_k], \qquad v = 0, \ldots, k.$$

Interestingly, these conditions just mean that the polynomial $p_1(x) := \sum_{i=0}^{k} \lambda_i x^i$ is the interpolation polynomial of the function $c_0/(x+2)$

($c_0 = 2\pi/(2k+1)$) with the knot system 0, 2, ..., $2k$. Therefore, by using the Newton form we get

$$p_1(x) = \frac{2\pi}{2k+1}\left[\frac{1}{2} + \sum_{v=0}^{k-1} x(x-2)\cdots(x-2v)\,\frac{(-1)^{v+1}}{2^{v+2}(v+2)!}\right].$$

Or, we have

$$p_1(2x) = \frac{\pi}{2k+1}\left[1 + \sum_{v=0}^{k-1} x(x-1)\cdots(x-v)\,\frac{(-1)^{v+1}}{(v+2)!}\right].$$

This gives a representation for the coefficients $\lambda_j$.

Now, let us consider the interpolation described in Theorem 2. By repeating each step above, for this case, we arive to the following cubature formula

$$\int_{x^2+y^2\leqslant 1} f(x, y)\, dx\, dy \doteq \sum_{i=0}^{k-1} \theta_j L_j(f),$$

with the degree of precision $d_2 \geqslant 2k-1$.

For finding the coefficients this time we have that $p_2(x) := \sum_{i=0}^{k-1} \theta_i x^i$ is the interpolation polynomial of the function $c_0/(x+2)$ with the knot system 0, 2, ..., $2(k-1)$, and we get as above

$$p_2(2x) = \frac{\pi}{2k+1}\left[1 + \sum_{v=0}^{k-2} x(x-1)\cdots(x-v)\,\frac{(-1)^{v+1}}{(v+2)!}\right].$$

Finally let us consider the interpolation described in Corollary 5, where, for the rotational symmetry, we take $z_0 = (0, 0)$. Then in the same way as above we get the cubature formula

$$\int_{x^2+y^2\leqslant 1} f(x, y)\, dx\, dy \doteq \eta f(0, 0) + \sum_{i=0}^{k-2} \zeta_j L_j(f),$$

with the degree of precision $d_3 \geqslant 2k-2$.

In this case we get that $\eta = \pi$, while for the other coefficients we have that $p_3(x) := \sum_{i=0}^{k-2} \theta_i x^i$ is the interpolation polynomial of the function $c_0/(x+2)$ with the knot system 2, ..., $2(k-1)$, and therefore

$$p_3(2x) = \frac{\pi}{2k+1}\left[1/2 + \sum_{v=1}^{k-2} (x-1)\cdots(x-v)\,\frac{(-1)^{v+1}}{(v+2)!}\right].$$

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