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Isotonic Approximation in L_1[1]

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Let $(\Omega, \mathscr{A}, P)$ be a measurable space and $\mathscr{L} \subset \mathscr{A}$ a sub-σ-lattice of the σ-algebra $\mathscr{A}$. For $X \in L_1(\Omega, \mathscr{A}, P)$ we denote by $P_{\mathscr{L}}X$ the set of conditional 1-mean (or best approximants) of X given $L_1(\mathscr{L})$ (the set of all $\mathscr{L}$-measurable and integrable functions). In this paper, we obtain characterizations of the elements in $P_{\mathscr{L}}X$, similar to those obtained by Landers and Rogge for conditional s-means with $1<s<\infty$. Moreover, using these characterizations we can extend the operator $P_{\mathscr{L}}$ to a bigger space $L_0(\Omega, \mathscr{A}, P)$. When, in certain sense, $\mathscr{L}_n$ goes to $\mathscr{L}_\infty$, we will be able to prove theorems about convergence and we will obtain bounds for the maximal function $|\sup_n P_{\mathscr{L}_n}X|$. A sharper characterization of conditional 1-means for certain particular σ-lattice was proved in previous papers. In the last section of this paper we generalize those results to all totally ordered σ-lattices.

Key Words: best approximants; maximal inequalities; a.e. convergence.

1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $0<s<\infty$. As is usual, we denote by $M(\Omega, \mathscr{A}, P)$ the set of the equivalence classes of $\mathscr{A}$-measurable functions and by $L_s(\Omega, \mathscr{A}, P)$ the classical Lebesgue spaces. Moreover, the space $L_0(\Omega, \mathscr{A}, P)$ is defined as the system of all equivalence classes of $\mathscr{A}$-measurable functions that are finite a.e.

Let $\mathscr{L} \subset \mathscr{A}$ be a σ-lattice, that is, $\mathscr{L}$ is closed under countable unions and intersections and $\emptyset, \Omega \in \mathscr{L}$. By $\bar{\mathscr{L}}$ we denote the dual σ-lattice of $\mathscr{L}$, i.e., $\bar{\mathscr{L}} := \{D : \Omega \backslash D \in \mathscr{L}\}$. We say that $X : \Omega \to \overline{\mathbb{R}}$ is $\mathscr{L}$-measurable if $\{X > a\} \in \mathscr{L}$ for every $a \in \mathbb{R}$. As in [7] we let $L_s(\mathscr{L})$ denote the system of all equivalence classes in $L_s(\Omega, \mathscr{A}, P)$ containing a $\mathscr{L}$-measurable function.

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Let $\mathscr{L}$ be a σ-lattice and $X \in L_s(\Omega, \mathscr{A}, P)$. An element $Y \in L_s(\mathscr{L})$ is called a *conditional s-mean* of X given the σ-lattice $\mathscr{L}$ if

$$\|X - Y\|_s \leqslant \|X - Z\|_s \qquad \text{for every } Z \in L_s(\mathscr{L}).$$

It is well known, see [6], that for $s \geqslant 1$ there exists a conditional s-mean for every $X \in L_s$. Moreover it is unique for $s > 1$. But for $s = 1$ the uniqueness can fail. In any case we let $P^s_{\mathscr{L}}X$ denote the set of all conditional s-means. Henceforth, for simplicity, we write $P_{\mathscr{L}}X$ instead of $P^1_{\mathscr{L}}X$.

In several papers Landers and Rogge studied conditional means, see for example [6–9]. They studied the above situation in more general settings, for example in [6, 8, 9]. In [7] Landers and Rogge established results about characterization, a.e. convergence, and boundedness of maximal functions for conditional s-means with $1 < s < \infty$. In the present paper we are interested in considering the case $s = 1$. The non-uniqueness of conditional 1-means presents an additional difficulty. However, as we will see, the conditional 1-mean has advantages over any other s-mean. For example, we can define the conditional 1-mean for any measurable finite almost everywhere function.

Conditional 1-means were also studied for particular σ-lattices, see for example [2–4, 10, 11]. We denote by $\mathbb{B}^n$ the Borel σ-algebra of $[0, 1]^n$ and by $\mathbb{L}^n$ the σ-lattice defined by the condition $C \in \mathbb{L}^n$ iff $\boldsymbol{x} \in C$ and $\boldsymbol{x} \leqslant \boldsymbol{y}$ imply $\boldsymbol{y} \in C$, ($\boldsymbol{x} \leqslant \boldsymbol{y}$ means that $x_i \leqslant y_i$, $i = 1, \ldots, n$ with x_i and y_i denoting the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively). When $n = 1$ by we write $\mathbb{B}$ and $\mathbb{L}$ instead of $\mathbb{B}^1$ and $\mathbb{L}^1$ for short. In [4] Huotari, Meyerowitz and Sheard gave a sharper characterization of conditional 1-means for the measurable space $([0, 1], dx)$, where dx denotes the Lebesgue measure, and for the σ-lattice $\mathbb{L}$. In the last section of this paper we prove a generalization of this characterization to an arbitrary probability measurable space and a totally ordered σ-lattice. In [11, p. 185] Marano and Quesada generalized the previous result obtained in [4] to L_Φ spaces. However, the underlying space and the σ-lattice considered in that paper are the same as the ones considered in [4].

We want to point out that the techniques in this paper are different from the ones used in the previous papers already mentioned. Moreover, they are elementary. For example, we do not use arguments about approximate continuity, as was done in [4, 11]. Finally, we want to mention that conditional s-means have applications to statistical inference, see for example [1] and the most recent paper [5].

The organization of the paper is as follows. In Section 2 we prove a characterization of conditional 1-mean. As a corollary of this we obtain previous results of Shintani and Ando [12] about characterization of the conditional 1-mean given a σ-algebra. In Section 3 we extend the operator $P_{\mathscr{L}}$ to the space $L_0(\Omega, \mathscr{A}, P)$ and we obtain results about the structure of the set $P_{\mathscr{L}}X$. We also prove results about a.e. convergence and obtain boundedness for certain maximal functions in Section 4. Except the last one result, these are immediate consequences of Proposition 3.5, which is a corollary of our characterization given in Theorem 2.1 and the analogous property for the operator $P_{\mathscr{L}} : L_1 \to L_1$. In Section 5 we consider a totally ordered σ-lattice and we extend previous results of Huotari, Meyerowitz and Sheard about monotone approximation.

2. A CHARACTERIZATION OF $P_{\mathscr{L}}X$

The aim of this section is to prove a characterization of the conditional 1-mean similar to the one proved in [7], for $s > 1$, see for example Theorem 3.4 and Corollary 3.5. Here the main difficulty to overcome is the non-uniqueness which was important in the arguments used in [7, Theorem 3.4]. The main result of this section is the following theorem.

THEOREM 2.1 *Let $X \in L_1(\Omega, \mathscr{A}, P)$ and $Y \in L_1(\mathscr{L})$. The following facts are equivalent*:

(1) $Y \in P_{\mathscr{L}}X$.
(2) *For every $Z \in L_1(\mathscr{L})$ we have*

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y)(Z - Y)\, dP \leqslant \int_{X=Y} |Z - Y|\, dP.$$

(3) *For every $\alpha \in \mathbb{R}$, $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$ we get* (a)$P(\{X \leqslant Y\} \cap C \cap \{Y < \alpha\}) \geqslant P(\{X > Y\} \cap C \cap \{Y < \alpha\})$, (b)$P(\{X \geqslant Y\} \cap D \cap \{Y > \alpha\}) \geqslant P(\{X < Y\} \cap D \cap \{Y > \alpha\})$.

(4) *For every $\alpha \in \mathbb{R}$, $D \in \bar{\mathscr{L}}$ and $C \in \mathscr{L}$*, (a)$P(\{X < \alpha\} \cap C \cap \{Y < \alpha\}) \geqslant \frac{1}{2}P(C \cap \{Y < \alpha\})$, (b)$P(\{X > \alpha\} \cap D \cap \{Y > \alpha\}) \geqslant \frac{1}{2}P(D \cap \{Y > \alpha\})$.

(5) *For every $C \in \mathscr{L}$, $D \in \bar{\mathscr{L}}$ and $B \in \mathbb{B}$ we have* (a)$P(\{X \leqslant Y\} \cap C \cap Y^{-1}(B)) \geqslant \frac{1}{2}P(C \cap Y^{-1}(B))$, (b)$P(\{X \geqslant Y\} \cap D \cap Y^{-1}(B)) \geqslant \frac{1}{2}P(D \cap Y^{-1}(B))$.

(6) *For every $\mathbb{B}$-measurable function $\phi : \mathbb{R} \to [0, \infty)$ and $Z \in L_1(\mathscr{L})$*:

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y) Z \phi(Y)\, dP \leqslant \int_{\{X=Y\}} |Z| \phi(Y)\, dP,$$

if the integral exists.

The structure of the proof is the following: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) and finally (3) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Leftrightarrow$ (2): This equivalence is a classical result about best approximants. For completness sake we include the short proof of it. We consider the following function:

$$\varphi(t) := \int |X - (Y + t(Y - Z))| \, dP.$$

We observe that for $0 \leqslant t \leqslant 1$ we have $Y + t(Y - Z) \in L_1(\mathscr{L})$ and that $\varphi : [0, 1] \to$ is a convex function. Therefore, $Y \in P_{\mathscr{L}}X$ iff φ has a minimum at 0, for every $Z \in L_1(\mathscr{L})$. Hence $Y \in P_{\mathscr{L}}X$ iff $\varphi^+(0) \geqslant 0$ (where φ^+ denotes the right derivative of φ). This implies that (1) is equivalent to (2).

(2) $\Rightarrow$ (3): It is sufficient to prove inequality (b) because inequality (a) is a consequence of applying (b) to the functions $-X$, $-Y$ and the lattice $\bar{\mathscr{L}}$ (note that $P_{\mathscr{L}}X = -P_{\bar{\mathscr{L}}}(-X)$). Let $n \in$, $A_n := \{Y > \alpha + 1/n\}$ and $A := \{Y > \alpha\} = \bigcup_{n=1}^{\infty} A_n$. Now for $D \in \bar{\mathscr{L}}$ we define the following functions:

$$Z_n(w) := \begin{cases} Y(w) & \text{if } w \notin A \cap D, \\ \alpha & \text{if } w \in (A - A_n) \cap D, \\ Y(w) - \frac{1}{n} & \text{if } w \in A_n \cap D. \end{cases}$$

It is easy to check that $Z_n \in L_1(\mathscr{L})$. We denote by E, F, G and H the sets $\{X \neq Y\} \cap A_n \cap D$, $\{X \neq Y\} \cap (A \backslash A_n) \cap D$, $\{X = Y\} \cap A_n \cap D$ and $\{X = Y\} \cap (A \backslash A_n) \cap D$, respectively. Then applying (2) to Z_n we obtain

$$-\frac{1}{n} \int_E \operatorname{sgn}(X - Y) \, dP + \int_F \operatorname{sgn}(X - Y)(\alpha - Y) \, dP \leqslant \frac{1}{n} \int_G dP + \int_H |\alpha - Y| \, dP.$$

We have that $-1 \leqslant n(\alpha - Y(w)) < 0$, for every $w \in A - A_n$ and $P(A \backslash A_n) \to 0$, for $n \to \infty$. Therefore, multiplying the above inequality by n and taking the limit as $n \to \infty$ we get the inequality

$$-\int_{\{X \neq Y\} \cap A \cap D} \operatorname{sgn}(X - Y) \, dP \leqslant \int_{\{X = Y\} \cap A \cap D} dP.$$

So this inequality implies the inequality in (3)(b).

(3) $\Leftrightarrow$ (4): We observe that inequalities (3)(a) and (b) are trivially equivalent to the following inequalities:

$$P(\{X \leqslant Y\} \cap C \cap \{Y < \alpha\}) \geqslant \tfrac{1}{2} P(C \cap \{Y < \alpha\}) \tag{2.1}$$

and

$$P(\{X \geqslant Y\} \cap D \cap \{Y > \alpha\}) \geqslant \tfrac{1}{2} P(D \cap \{Y > \alpha\}). \tag{2.2}$$

Clearly (2.1) and (2.2) imply (4)(a) and (b), respectively. Now consider the reverse implications. We assume that X and Y satisfy (4)(a) and (b). Let $D \in \bar{\mathscr{L}}$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, as a consequence of (4)(b) we get

$$P\left(\{X > \alpha\} \cap D \cap \left\{Y \leqslant \alpha + \frac{1}{n}\right\} \cap \{Y > \alpha\}\right) \geqslant \frac{1}{2} P\left(D \cap \left\{\alpha < Y \leqslant \alpha + \frac{1}{n}\right\}\right).$$

This implies

$$P\left(\left\{X > Y - \frac{1}{n}\right\} \cap D \cap \left\{\alpha < Y \leqslant \alpha + \frac{1}{n}\right\}\right) \geqslant \frac{1}{2} P\left(D \cap \left\{\alpha < Y \leqslant \alpha + \frac{1}{n}\right\}\right).$$

By applying the above inequality to $\alpha + k/n$ with $k \in \mathbb{N} \cup \{0\}$ instead of α, we obtain for $A_k := \{\alpha + k/n < Y \leqslant \alpha + (k+1)/n\}$,

$$P\left(\left\{X > Y - \frac{1}{n}\right\} \cap D \cap A_k\right) \geqslant \frac{1}{2} P(D \cap A_k).$$

Now since $\{Y > \alpha\} = \bigcup_{k \geqslant 0} A_k$, we get

$$P\left(\left\{X > Y - \frac{1}{n}\right\} \cap D \cap \{\alpha < Y\}\right) \geqslant \frac{1}{2} P(D \cap \{\alpha < Y\}).$$

Taking the limit as $n \to \infty$ we obtain (2.2). Hence, we have proved that (4)(b) implies (2.2) for every $\mathscr{L}$. Applying this fact to the lattice $\bar{\mathscr{L}}$ and to the functions $-X$, $-Y$ and $-\alpha$ we get (2.1).

(4) $\Rightarrow$ (5): Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Since $C \cap \{\beta < Y\} \in \mathscr{L}$, from (4)(a) we obtain

$$P(\{X \leqslant Y\} \cap C \cap \{\beta < Y < \alpha\}) \geqslant \tfrac{1}{2} P(C \cap \{\beta < Y < \alpha\})$$

for every $C \in \mathscr{L}$, and $\alpha, \beta \in \mathbb{R}$. Let $B \in \mathbb{B}$. We consider two measures over $\mathbb{B}: P_1(B) := P(\{X \leqslant Y\} \cap C \cap Y^{-1}(B))$ and $P_2(B) := 1/2P(C \cap Y^{-1}(B))$. From the last inequality we have that $P_1(B) \geqslant P_2(B)$, where B is an interval of $\mathbb{R}$. Therefore, the inequality holds for every $B \in$ and thus we obtain (5)(a). By repeating the arguments already used we obtain (5)(b).

(5) $\Rightarrow$ (6): We can assume that ϕ is a simple function, otherwise ϕ is the uniform and non-decreasing limit of simple, positive and $\mathbb{B}$-measurable

functions. We observe that from (5)(a) we obtain

$$\int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)1_C 1_B(Y)\,dP \leqslant \int_{\{X=Y\}} 1_C 1_B(Y)\,dP$$

for every $C \in \mathscr{L}$ and $B \in \mathbb{B}$. Therefore, we get

$$\int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)1_C \phi(Y)\,dP \leqslant \int_{\{X=Y\}} 1_C \phi(Y)\,dP \tag{2.3}$$

for every $C \in \mathscr{L}$. Let $Z \in L_1(\mathscr{L})$. We suppose that Z is a simple and positive function. Then $Z = \sum_{i=1}^{n} a_i 1_{C_i}$, with $C_i \in \mathscr{L}$, $C_1 \subset \cdots \subset C_n$ and $a_i \geqslant 0$. So from (2.3) we obtain

$$\int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)Z\phi(Y)\,dP \leqslant \int_{\{X=Y\}} Z\phi(Y)\,dP. \tag{2.4}$$

Since every positive $Z \in L_1(\mathscr{L})$ is a non-decreasing limit of simple, positive and $\mathscr{L}$-measurable functions, we have (2.4) for every $Z \geqslant 0$ and $Z \in L_1(\mathscr{L})$.

Let $Z \leqslant 0$ be a $\mathscr{L}$-measurable function. If we apply (2.4) to $-X$, $-Y$, $-Z$, $\bar{\mathscr{L}}$ and $\phi_0(x) = \phi(-x)$ we deduce

$$\int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)Z\phi(Y)\,dP \leqslant \int_{\{X=Y\}} |Z|\phi(Y)\,dP. \tag{2.5}$$

Now, let $Z \in L_1(\mathscr{L})$ be such that the integrals in (6) exist. We put $Z = Z^+ - Z^-$. Since $0 \leqslant Z^+ \in L_1(\mathscr{L})$ and $0 \geqslant -Z^- \in L_1(\mathscr{L})$ from (2.4) and (2.5) we obtain

$$\begin{aligned}
\int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)Z\phi(Y)\,dP &\leqslant \int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)Z^+\phi(Y)\,dP \\
&\quad + \int_{\{X\neq Y\}} \operatorname{sgn}(X-Y)(-Z^-)\phi(Y)\,dP \\
&\leqslant \int_{\{X=Y\}} Z^+\phi(Y)\,dP \\
&\quad + \int_{\{X=Y\}} |Z^-|\phi(Y)\,dP \\
&= \int_{\{X=Y\}} |Z|\phi(Y)\,dP.
\end{aligned}$$

So (6) is proved.

(6) $\Rightarrow$ (3): We obtain immediately (3)(a) putting $\phi = 1_{(-\infty,\alpha)}$ and $Z = 1_C$. Inequality (3)(b) is obtained in a similar way.

(3) $\Rightarrow$ (2): Let $Z \in L_1(\mathscr{L})$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define the set $A_{k,n} := \{k/n < Y \leqslant (k+1)/n\}$ and for $m \in n\mathbb{N}$ (i.e. m is a multiple of n) and $j \in \mathbb{Z}$ we put $B_{j,m} := \{Z < j/m\}$. Moreover, we define $E_{j,m} := B_{j,m} - B_{(j-1),m}$ and the following functions:

$$Y_n := \sum_{k=-\infty}^{\infty} \frac{k}{n} 1_{A_{k,n}} \qquad \text{and} \qquad Z_m := \sum_{j=-\infty}^{\infty} \frac{j}{m} 1_{E_{j,m}}.$$

We have that $Y_n, Z_m \in L_1(\mathscr{L})$ and $Y_n \to Y$ as $n \to \infty$ uniformly and $Z_m \to Z$ as $m \to \infty$ uniformly.

Fix n, k, m and let $j^* := km/n$ ($j^* \in \mathbb{Z}$). We have the following equality:

$$(Y_n - Z_m) 1_{A_{k,n}} 1_{B_{j^*,m}} = \left(\frac{1}{m} \sum_{j=-\infty}^{j^*-1} 1_{B_{j,m}} \right) 1_{A_{k,n}} 1_{B_{j^*,m}}. \tag{2.6}$$

Now putting $D = B_{j,m} \cap \{Y \leqslant (k+1)/n\}$ and $\alpha = k/n$ we obtain from (3)(b)

$$P(\{X \geqslant Y\} \cap B_{j,m} \cap A_{k,n}) \geqslant P(\{X < Y\} \cap B_{j,m} \cap A_{k,n}),$$

or equivalently

$$-\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y) 1_{B_{j,m}} 1_{A_{k,n}} \, dP \leqslant \int_{\{X = Y\}} 1_{B_{j,m}} 1_{A_{k,n}} \, dP.$$

Now multiplying by m^{-1} and summing over $j < j^*$ we obtain from (2.6)

$$\begin{aligned} -\int_{\{X \neq Y\}} & \operatorname{sgn}(X - Y)(Y_n - Z_m) 1_{A_{k,n}} 1_{\{Z < Y_n\}} \, dP \\ & \leqslant \int_{\{X = Y\}} (Y_n - Z_m) 1_{A_{k,n}} 1_{\{Z < Y_n\}} \, dP. \end{aligned}$$

Taking the limit as $m \to \infty$ we get

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y)(Z - Y_n) 1_{A_{k,n}} 1_{\{Z < Y_n\}} \, dP \leqslant \int_{\{X = Y\}} (Y_n - Z) 1_{A_{k,n}} 1_{\{Z < Y_n\}} \, dP.$$

Summing over $k \in \mathbb{Z}$ we have

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y)(Z - Y_n) 1_{\{Z < Y_n\}} \, dP \leqslant \int_{\{X = Y\}} (Y_n - Z) 1_{\{Z < Y_n\}} \, dP. \tag{2.7}$$

Since $1_{\{Z<Y_n\}} \to 1_{\{Z<Y\}}$ a.e. when $n \to \infty$ we obtain from (2.7)

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y)(Z - Y)1_{\{Z<Y\}} \, dP \leqslant \int_{\{X=Y\}} (Y - Z)1_{\{Z<Y\}} \, dP. \qquad (2.8)$$

Hence, we have shown that for every σ-lattice the inequality in (3)(b) implies (2.8). We can apply this fact to the lattice $\bar{\mathscr{L}}$ and to the functions $-X, -Y$ and $-Z$. Therefore, we get

$$\int_{\{X \neq Y\}} \operatorname{sgn}(X - Y)(Z - Y)1_{\{Z>Y\}} \, dP \leqslant \int_{\{X=Y\}} (Z - Y)1_{\{Z>Y\}} \, dP. \qquad (2.9)$$

Now (2.8) and (2.9) imply (2). ∎

We finish this section with some consequences of Theorem 2.1. When $\mathscr{L}$ is a σ-algebra, a characterization of the conditional 1-mean was obtained by Shintani and Ando [12, Theorem 2, Corollary 3]. We can get this characterization as a consequence of our Theorem 2.1. As is usual we denote by $E_{\mathscr{L}}$ the *conditional expectation operator* $E_{\mathscr{L}} : L_1(\Omega, \mathscr{A}, P) \to L_1(\Omega, \mathscr{A}, P)$ which is defined, for $X \in L_1(\Omega, \mathscr{A}, P)$, by the following two conditions: (i) $E_{\mathscr{L}}X$ is a $\mathscr{L}$-measurable function and (ii) for every $C \in \mathscr{L}$ we have

$$\int_C E_{\mathscr{L}}X \, dP = \int_C X \, dP. \qquad (2.10)$$

It is well known that for $X \in L_2(\Omega, \mathscr{A}, P)$, $E_{\mathscr{L}}X$ is the conditional 2-mean of X. The mentioned result of Shintani and Ando is the following.

Corollary 2.2. *Let $\mathscr{L}$ be a σ-algebra and let $X \in L_1(\Omega, \mathscr{A}, P)$ and $Y \in L_1(\mathscr{L})$. Then $Y \in P_{\mathscr{L}}X$ iff the following inequalities holds*:

$$E_{\mathscr{L}}(\{X \leqslant Y\}) := E_{\mathscr{L}}(1_{\{X \leqslant Y\}}) \geqslant \tfrac{1}{2},$$
$$E_{\mathscr{L}}(\{X \geqslant Y\}) := E_{\mathscr{L}}(1_{\{X \geqslant Y\}}) \geqslant \tfrac{1}{2}. \qquad (2.11)$$

Proof. We use inequalities (2.1) and (2.2), which are equivalent to all items in Theorem 2.1. Since $\mathscr{L}$ is a σ-algebra, (2.1) and (2.2) are equivalent to

$$P(\{X \leqslant Y\} \cap C) \geqslant \tfrac{1}{2} P(C),$$
$$P(\{X \geqslant Y\} \cap C) \geqslant \tfrac{1}{2} P(C) \qquad (2.12)$$

for every $C \in \mathscr{L}$. The "if" part of the corollary is obtained by integrating inequalities (2.11) over C and using equality (2.10). For the "only if" part, we suppose that (2.11) is not true. We can also assume that there exists a $\mathscr{L}$ measurable set C with $P(C) > 0$ such that for every $\omega \in C$,

$$E_{\mathscr{L}}(\{X \leqslant Y\})(\omega) < \tfrac{1}{2}.$$

Now, integrating this inequality over C and using (2.10) we obtain $P(\{X \leqslant Y\} \cap C) < 1/2P(C)$. This contradicts (2.12). ∎

The following is an immediate consequence of Corollary 2.2.

COROLLARY 2.3. *Let $\mathscr{L}$ be a σ-algebra and let $X \in L_1(\Omega, \mathscr{A}, P)$ and $Y \in L_1(\mathscr{L})$. Suppose that $Y_1 \leqslant Y \leqslant Y_2$, where $Y_1, Y_2 \in P_{\mathscr{L}}X$. Then $Y \in P_{\mathscr{L}}X$.*

3. EXTENSION OF THE OPERATOR $P_{\mathscr{L}}$

In this section we will extend the operator $P_{\mathscr{L}}$ to the space $L_0(\Omega, \mathscr{A}, P)$. We recall that in [7] Landers and Rogge extended the operator $P_s^{\mathscr{L}} : L_s \to L_s$ to the space L_{s-1}. They used a density argument. More precisely, they took a function $X \in L_{s-1}$ bounded from below by a function in L_s and then they defined $P_s^{\mathscr{L}}X := \lim_{n\in\mathbb{N}} P_s^{\mathscr{L}}(X \wedge n)$. If $X \in L_s$ is an arbitrary function then $X \vee n$ is bounded from below by a function in L_s. As consequence of all this, they could define $P_s^{\mathscr{L}}X := \lim_{n\in\mathbb{N}} P_s^{\mathscr{L}}(X \vee n)$. This technique is not appropriate for the case $s = 1$ since the operator $P_{\mathscr{L}}$ is set valued. However, we can use Theorem 2.1 for our objective.

DEFINITION 3.1. Let $\mathscr{L}$ be a σ-lattice and $X \in L_0(\Omega, \mathscr{A}, P)$. Then $Y \in \tilde{P}_{\mathscr{L}}X$ iff Y is a $\mathscr{L}$-measurable function and satisfies for every $\alpha \in \mathbb{R}$, $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$

(1) $P(\{X < \alpha\} \cap C \cap \{Y < \alpha\}) \geqslant \frac{1}{2} P(C \cap \{Y < \alpha\})$,
(2) $P(\{X > \alpha\} \cap D \cap \{Y > \alpha\}) \geqslant \frac{1}{2} P(D \cap \{Y > \alpha\})$.

Remark. In Theorem 3.4 we will prove that for every $X \in L_0(\Omega, \mathscr{A}, P)$ the conditional 1-mean always exists (probably it is non-unique). Notice that the proof of Theorem 2.1 is still valid if we change $X \in L_1(\Omega, \mathscr{A}, P)$ to $X \in L_0(\Omega, \mathscr{A}, P)$. (The statement of item (2) should be modified adding to the conditions the existence of the integral.) Also Corollaries 2.2 and 2.3 remain true when $\mathscr{L}$ is a σ-algebra. Throughout the rest of this paper we will use these extended version of Theorem 2.1, Corollaries 2.2 and 2.3.

Now note that if $X \in L_1(\Omega, \mathscr{A}, P)$ then $P_{\mathscr{L}}X = \tilde{P}_{\mathscr{L}}X$. This is an immediate consequence of Theorem 2.1 and the following theorem.

THEOREM 3.2. *Let $X \in L_0(\Omega, \mathscr{A}, P)$ and $Y \in \tilde{P}_{\mathscr{L}}X$. Then*

(1) *$P(\{|Y| > \alpha\}) \leqslant 2P(\{|X| > \alpha\})$, for every $\alpha > 0$.*
(2) *If $X \in L_s(\Omega, \mathscr{A}, P)$, for $s \geqslant 0$, then $Y \in L_s(\mathscr{L})$.*
(3) *In particular, if $X \in L_1(\Omega, \mathscr{A}, P)$ then $Y \in L_1(\mathscr{L})$. Therefore by Theorem (2.1) Y is a conditional 1-mean.*

Proof. Property (1) is an immediate consequence of Definition 3.1 applied to $C = D = \Omega$, α and $-\alpha$, with $\alpha > 0$. Property (2) is easily obtained from (1) for $s > 0$ by using the equality

$$\int_\Omega |Z|^s \, dP = s \int_0^\infty \alpha^{s-1} P(\{|Z| > \alpha\} \, d\alpha \qquad \text{for every } Z \in M(\Omega, \mathscr{A}, P), \quad (3.1)$$

and for $s = 0$, it follows from the equality $\{|Y| = \infty\} = \bigcap_{n \in \mathbb{N}} \{|Y| > n\}$. ∎

As a consequence of (3), we may write $P_{\mathscr{L}}X$ for every $X \in L_0(\Omega, \mathscr{A}, P)$ instead of $\tilde{P}_{\mathscr{L}}X$. In the rest of this section, we prove some properties of the extended operator $P_{\mathscr{L}}$.

LEMMA 3.3. *Let $X_n \in L_0(\Omega, \mathscr{A}, P)$ be a sequence of functions with $X_n \uparrow X \in L_0(\Omega, \mathscr{A}, P)$ ($X_n \downarrow X \in L_0(\Omega, \mathscr{A}, P)$). Let $Y_n \in P_{\mathscr{L}}X_n$ be such that $Y_n \uparrow Y$ ($Y_n \downarrow Y$). Then $Y \in P_{\mathscr{L}}X$.*

Proof. We will establish the increasing case. The decreasing case runs similarly. The function Y is obviously $\mathscr{L}$-measurable. Now, let us prove that Y satisfies (1) of Definition 3.1. Let $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $C \in \mathscr{L}$. If $A_{n,k} := \{X_n < \alpha - 1/k\}$ and $B_{n,k} := \{Y_n < \alpha - 1/k\}$ then

$$P(A_{n,k} \cap C \cap B_{n,k}) \geqslant \tfrac{1}{2} P(C \cap B_{n,k}). \quad (3.2)$$

We have that, for $n \to \infty$, $A_{n,k} \downarrow A_k$ for some $\mathscr{A}$-measurable set with $\{X < \alpha - 1/k\} \subset A_k \subset \{X \leqslant \alpha - 1/k\}$. Similarly, $B_{n,k} \downarrow B_k \in \mathscr{L}$ with $\{Y < \alpha - 1/k\} \subset B_k \subset \{Y \leqslant \alpha - 1/k\}$. Therefore, from (3.2) we obtain $P(A_k \cap C \cap B_k) \geqslant \frac{1}{2} P(C \cap B_k)$. We observe that $1_{A_k} \to 1_A$ a.e., where $A := \{X < \alpha\}$, and $1_{B_k} \to 1_B$ a.e., where $B := \{Y < \alpha\}$. Taking the limit as $k \to \infty$ and using the Dominated Convergence Theorem we obtain (1) of Definition 3.1. Inequality (2) in Definition 3.1 is proved similarly. ∎

THEOREM 3.4. *Let $\mathscr{L}$ be a σ-lattice and $X \in L_0(\Omega, \mathscr{A}, P)$. Then $P_{\mathscr{L}}X \neq \emptyset$.*

Proof. We suppose first that we have two functions $X_i \in L_0(\Omega, \mathscr{A}, P)$, $i = 1, 2$, which are bounded from below by functions in $L_1(\Omega, \mathscr{A}, P)$ and

$X_1 \leqslant X_2$. For $n \in \mathbb{N}$ we define $X_i^n := X_i \wedge n$ for $i = 1, 2$. Then $X_i^n \in L_1(\Omega, \mathscr{A}, P)$. Therefore from [6, Theorem 4] we can take $Y_i^n \in P_{\mathscr{L}} X_i^n$. Moreover, since the operator $P_{\mathscr{L}}$ is monotone over $L_1(\Omega, \mathscr{A}, P)$ (see [6, Theorem 18]), we can assume $Y_i^n \leqslant Y_i^{n+1}$ for $i = 1, 2$, and $Y_1^n \leqslant Y_2^n$ for every $n \in \mathbb{N}$. We put $Y_i = \lim Y_i^n$. As a consequence of Lemma 3.3 we obtain $Y_i \in P_{\mathscr{L}} X_i$ for $i = 1, 2$. Moreover $Y_1 \leqslant Y_2$.

Now we suppose that $X \in L_0(\Omega, \mathscr{A}, P)$ is an arbitrary function. We define $X_n := X \vee (-n)$. These functions are bounded from below by functions in $L_1(\Omega, \mathscr{A}, P)$. As a consequence of the first part of this proof, we can get $Y_n \in P_{\mathscr{L}} X_n$ with $Y_n \geqslant Y_{n+1}$. If $Y = \lim Y_n$, then Lemma 3.3 implies that $Y \in P_{\mathscr{L}} X$. ∎

The following property seems to be unknown even if $X \in L_1(\Omega, \mathscr{A}, P)$. It is very important for our purposes, since it simplifies the proof of many of our statements by allowing a reduction to the case $X \in L_1(\Omega, \mathscr{A}, P)$.

Proposition 3.5. *Let $X \in L_0(\Omega, \mathscr{A}, P)$ and $Y \in P_{\mathscr{L}} X$. Then for every $\beta \in \mathbb{R}$ we have $Y \wedge \beta \in P_{\mathscr{L}}(X \wedge \beta)$ and $Y \vee \beta \in P_{\mathscr{L}}(X \vee \beta)$.*

Proof. It is easy to see that the functions $X \wedge \beta$ and $Y \wedge \beta$ ($X \vee \beta$ and $Y \vee \beta$) satisfy Definition 3.1. ∎

Remark. The property in Proposition 3.5 does not hold for other conditional s-means, with $s > 1$, as the following simple example shows. Let $\Omega := \{0, 1\}$, P defined by $P(\{0\}) = P(\{1\}) = 1/2$ and $\mathscr{L} := \{\emptyset, \Omega\}$. We consider $X := 1_{\{1\}}$. Then it is easy to see that $P_s^{\mathscr{L}}(\beta X) = \beta/2$. Therefore for $1/2 < \beta < 1$ we have $P_s^{\mathscr{L}}(\beta \wedge X) = P_s^{\mathscr{L}}(\beta X) = \beta/2 \neq 1/2 = P_s^{\mathscr{L}} X \wedge \beta$.

Proposition 3.6. *Let $X \in L_0(\Omega, \mathscr{A}, P)$. Then*

(1) *The set $P_{\mathscr{L}} X$* is a *σ-lattice* (*i.e.*, *it is closed under suprema and infima of countable subsets*).

(2) *It has a maximum* (*minimum*) *element $U_{\mathscr{L}} X$($V_{\mathscr{L}} X$).*

(3) *$P_{\mathscr{L}} X$ is a convex set.*

Proof. For (1), let $Y_1, Y_2 \in P_{\mathscr{L}} X$ and $n, m \in \mathbb{N}$. Then from Proposition 3.5 we get $Y_i^{m,n} := (-m) \vee Y_i \wedge n \in P_{\mathscr{L}}((-m) \vee X \wedge n)$ for $i = 1, 2$. Since $X_{m,n} := (-m) \vee X \wedge n \in L_1(\Omega, \mathscr{A}, P)$ then $P_{\mathscr{L}}(X_{m,n})$ is a lattice (see [6]). So $Y_1^{m,n} \vee Y_2^{m,n} = (-m) \vee (Y_1 \vee Y_2) \wedge n \in P_{\mathscr{L}}(X_{m,n})$. Therefore by Lemma 3.3 taking the limit as $n, m \to \infty$ we obtain $Y_1 \vee Y_2 \in P_{\mathscr{L}} X$. Applying this last fact to the lattice $\bar{\mathscr{L}}$ and the functions $-X$, $-Y_1$ and $-Y_2$, we obtain $Y_1 \wedge Y_2 \in P_{\mathscr{L}} X$.

Now given $Y_n \in P_{\mathscr{L}}X$ we consider the increasing sequence $Y_1 \vee \cdots \vee Y_n \in P_{\mathscr{L}}X$. From Lemma 3.3 we obtain $\sup_n Y_n \in P_{\mathscr{L}}X$. Analogously $\inf_n Y_n \in P_{\mathscr{L}}X$. All of this proves that $P_{\mathscr{L}}X$ is a σ-lattice.

The proof of (2) and (3) follows as in [6, Theorem 14] and the proof of (1) in this theorem, respectively. ∎

4. CONVERGENCE A.E. AND MAXIMAL INEQUALITIES

The a.e. convergence of the extended conditional 1-means is a consequence of Proposition 3.5 and results previously known of convergence, see [8].

We consider $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N} \cup \{\infty\}$, σ-lattices, such that $\mathscr{L}_n$, $n \in \mathbb{N}$, is increasing (decreasing) to $\mathscr{L}_\infty$, i.e. $\mathscr{L}_n \subset \mathscr{L}_{n+1}(\mathscr{L}_{n+1} \subset \mathscr{L}_n)$ and $\mathscr{L}_\infty$ is the σ-lattice generated by $\bigcup_n \mathscr{L}_n(\mathscr{L}_\infty = \bigcap_n \mathscr{L}_n)$.

THEOREN 4.1. *Let $X \in L_0(\Omega, \mathscr{A}, P)$ and $\mathscr{L}_n$, $n \in \mathbb{N}$, be σ-lattices increasing or decreasing to the σ-lattice $\mathscr{L}_\infty$. Suppose that $Y_n \in P_{\mathscr{L}_n}X$. Then*

(1) $\underline{\lim}_{n\to\infty} Y_n := Y_\infty \in P_{\mathscr{L}_\infty}X$ *and* $\overline{\lim}_{n\to\infty} Y_n := Y^\infty \in P_{\mathscr{L}_\infty}X$.

(2) $P(\{\sup_n |Y_n| > \alpha\}) \leqslant 2P(\{\sup_n |Y_n| > \alpha\} \cap \{|X| > \alpha\})$ *for every* $\alpha > 0$. *By analogy to the classical case, we call* (2) *a "weak inequality of type* (0,0)".

(3) *If $X \in L_s(\Omega, \mathscr{A}, P)$ with $s > 0$ then*

$$\int_\Omega \sup_n |Y_n|^s \, dP \leqslant 2 \int_\Omega |X|^s \, dP.$$

This is the strong inequality of type (s, s).

(4) *If $X \in L_s(\Omega, \mathscr{A}, P)$, $s > 0$, and $P_{\mathscr{L}_\infty}X$ has only one element, namely Y, then*

$$\int_\Omega |Y - Y_n|^s \, dP \to 0 \qquad \textit{when } n \to \infty.$$

Proof. Let $k, m \in \mathbb{N}$. Define the functions $X_{k,m} := (-k) \vee X \wedge m$ and $Y_n^{k,m} := (-k) \vee Y_n \wedge m$. As a consequence of Proposition 3.5 we have $Y_n^{k,m} \in P_{\mathscr{L}_n}X_{k,m}$. Since $X_{k,m} \in L_1(\Omega, \mathscr{A}, P)$ we obtain from [8, Theorem 1] $(-k) \vee Y_\infty \wedge m \in P_{\mathscr{L}_\infty}X_{k,m}$ and $(-k) \vee Y^\infty \wedge m \in P_{\mathscr{L}_\infty}X_{k,m}$. Applying Lemma 3.3 we get (1). Inequality (3) is consequence of (2) and (3.1). The convergence result (4) is obtained from (1), (3) and the Dominanted Convergence Theorem. Hence, it only remains to prove inequality (2). Let $\alpha > 0$. As was shown in [7] it suffices to prove the finite version of the maximal inequality, i.e. we will prove that for $n \in \mathbb{N}$,

$$P\left(\left\{\max_{1 \leqslant i \leqslant n} |Y_i| > \alpha\right\}\right) \leqslant 2P\left(\left\{\max_{1 \leqslant i \leqslant n} |Y_i| > \alpha\right\} \cap \{|X| > \alpha\}\right). \tag{4.1}$$

Now, it is sufficient to prove (4.1) in the increasing case because if $\mathscr{L}_i$, $1 \leqslant i \leqslant n$, is a decreasing finite sequence then $\mathscr{L}_n, \mathscr{L}_{n-1}, \cdots, \mathscr{L}_1$ is an increasing sequence. Also we can suppose that X and Y_i are positive, since if $X \in L_0(\Omega, \mathscr{A}, P)$ is an arbitrary function, from Proposition 3.5 we have that $Y_i^+ \in P_{\mathscr{L}} X^+$ and $Y_i^- \in P_{\bar{\mathscr{L}}} X^-$ (where $Z^+ = Z \vee 0$ and $Z^- = -Z \wedge 0$). Therefore, assuming that the inequality is true for positive functions we obtain

$$\begin{aligned}
P\left(\left\{\max_i |Y_i| > \alpha\right\}\right) &\leqslant P\left(\left\{\max_i Y_i^+ > \alpha\right\}\right) + P\left(\left\{\max_i Y_i^- > \alpha\right\}\right) \\
&\leqslant 2P\left(\left\{\max_i Y_i^+ > \alpha\right\} \cap \{X^+ > \alpha\}\right) \\
&\quad + 2P\left(\left\{\max_i Y_i^- > \alpha\right\} \cap \{X^- > \alpha\}\right) \\
&\leqslant 2P\left(\left\{\max_i |Y_i| > \alpha\right\} \cap \{X^+ > \alpha\}\right) \\
&\quad + 2P\left(\left\{\max_i |Y_i| > \alpha\right\} \cap \{X^- > \alpha\}\right) \\
&= 2P\left(\left\{\max_i |Y_i| > \alpha\right\} \cap \{|X| > \alpha\}\right).
\end{aligned}$$

So we suppose $X \geqslant 0$ and $Y_i \geqslant 0$ and $\{\mathscr{L}_i\}$ increasing for $1 \leqslant i \leqslant n$. We define $A_i := \{Y_1 \leqslant \alpha, \ldots, Y_{i-1} \leqslant \alpha, Y_i > \alpha\}$. We observe the following facts about A_i:

(i) Each A_i is of the form $\{Y_i > \alpha\} \cap D_i$ with $D_i \in \bar{\mathscr{L}}$.
(ii) The A_i's are mutually disjoint sets.
(iii) $\bigcup_i A_i = \{\max_i Y_i > \alpha\}$.

Therefore from (2) of Definition 3.1 we obtain $P(\{X > \alpha\} \cap A_i) \geqslant 1/2P(A_i)$. Finally summing over i we get (4.1). ∎

5. TOTALLY ORDERED σ-LATTICES

We say that $\mathscr{L}$ is *totally ordered* if for each pair of sets $C_1, C_2 \in \mathscr{L}$ we have $C_1 \subset C_2$ or $C_2 \subset C_1$. In this section, we will prove the results about the conditional 1-mean when the σ-lattice $\mathscr{L}$ is totally ordered. When $\Omega = [0,1]$ and $\mathscr{A}$ is the σ-algebra of Lebesgue, an important example of a totally ordered σ-lattice is $\mathbb{L} := \{(a,1] : 0 \leqslant a \leqslant 1\} \cup \{[a,1] : 0 \leqslant a \leqslant 1\}$. In this case, $L_1(\mathbb{L})$ is the set of all integrable and non-decreasing functions. In [4] Huotari, Meyerowitz and Sheard studied the conditional 1-mean by monotone functions. They obtained a charaterization of which functions between $V_{\mathscr{L}}X$ and $U_{\mathscr{L}}X$ are conditional 1-means. It is our purpose to generalize those results to every probability measurable space and totally ordered σ-lattice and every $X \in L_0(\Omega, \mathscr{A}, P)$. In [10] Marano considered totally ordered σ-lattices. He worked with an apparently more restrictive class of σ-lattices, namely σ-lattices $\mathscr{L}$ such that there exists a map $F : I \to \mathscr{L}$, where I is a closed subset of $[0,1]$, such that F is one to one, $F(I) = \mathscr{L}$, and F is monotone and continuous in a certain sense. However, his definition is equivalent to the statement that $\mathscr{L}$ is totally ordered. In fact, to see this we can consider the set $I := P(\mathscr{L})$ because if $\mathscr{L}$ is totally ordered then $P(C_1) = P(C_2)$, with $C_1, C_1 \in \mathscr{L}$, implies that $C_1 = C_2$ a.e. Therefore, we can consider $F = P_{|\mathscr{L}}^{-1}$, where $P_{|\mathscr{L}}$ denotes the restriction of P to $\mathscr{L}$. In [10] Marano proved (in the discrete case) that best Φ-approximants given $\mathscr{L}$ are best Φ-approximants with respect to an appropriate σ-algebra. We shall prove (see Theorem 5.3) that conditional 1-means given $\mathscr{L}$ are conditional 1-means given an appropriate σ-algebra $\mathscr{A}(\mathscr{L}^*)$, where $\mathscr{L}^*$ depends on X. Our results do not totally include the one given in [10] because we are considering a particular norm. However, we hope that the techniques developed here may be adapted to more general settings.

If $\mathscr{G} \subset \mathscr{A}$ we denote by $\mathscr{A}(\mathscr{G})$ and $\mathscr{L}(\mathscr{G})$ the least σ-algebra, σ-lattice, respectively, containing $\mathscr{G}$. If for a certain $\mathscr{A}$-measurable function Y, we have $\mathscr{G} = \{\{Y > a\},\ a \in \mathbb{R}\}$ then we write $\mathscr{A}(Y)$ and $\mathscr{L}(Y)$ instead of $\mathscr{A}(\mathscr{G})$ and $\mathscr{L}(\mathscr{G})$, respectively. Throughout this section we assume that $\mathscr{L}$ is a totally ordered σ-lattice. Let $X \in L_0(\Omega, \mathscr{A}, P)$ and $E \in \mathscr{A}$. We denote by $\mathscr{L}_E$ the σ-lattice induced by $\mathscr{L}$ on the set E, i.e. $\mathscr{L}_E := \{A \cap E : A \in \mathscr{L}\}$. Let $Y : E \to \mathbb{R}$ be a $\mathscr{L}_E$-measurable function. We say that $Y \in P_{\mathscr{L}}(X, E)$ if Y is a conditional 1-mean with respect to the set E. When $Y \in P_{\mathscr{L}}(X, E)$ and $\mathscr{L}$ is the trivial lattice $\{\emptyset, E\}$, we say that Y is a *constant conditional 1-mean* (note that Y is a constant function in this case). Moreover, we define $V(X, E)$ $(U(X, E))$ as the minimum (maximum) of all constants conditional 1-mean over E. These numbers satisfy the inequality

$$V(X, E) \leqslant U(X, E). \tag{5.1}$$

Furthermore as a consequence of Definition 3.1, we have

$$P(\{X \geqslant U(X, E)\} \cap E) \geqslant \tfrac{1}{2} P(E) \tag{5.2}$$

and

$$P(\{X \leqslant V(X,E)\} \cap E) \geqslant \tfrac{1}{2} P(E). \tag{5.3}$$

Remark. The number $U(X,E)$ ($V(X,E)$) is the maximum (minimum) number satisfying (5.2) ((5.3)).

DEFINITION 5.1. For $X \in L_0(\Omega, \mathscr{A}, P)$ we define $\mathscr{L}^* = \mathscr{L}^*_X$ as the set of all the sets $C^* \in \mathscr{L}$ such that for every $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$ we have

$$V(X, C \backslash C^*) \leqslant U(X, C^* \cap D). \tag{5.4}$$

This definition is equivalent to the statement that there exists a number α such that

$$P(\{X \geqslant \alpha\} \cap C^* \cap D) \geqslant 1/2 P(C^* \cap D), \tag{5.5}$$

$$P(\{X \leqslant \alpha\} \cap C \backslash C^*) \geqslant 1/2 P(C \backslash C^*) \tag{5.6}$$

for every $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$. Using this fact, one can easily check that $\mathscr{L}^*$ is a totally ordered σ-lattice (since $\mathscr{L}$ is totally ordered, it is sufficient to prove that $\mathscr{L}^*$ is closed for increasing unions and decreasing intersections).

If $Y \in P_{\mathscr{L}} X$ and $\beta \in \mathbb{R}$ then the set $C^* = \{Y > \beta\}$ is $\mathscr{L}^*$-measurable. Inequality (5.5) follows directly from (2) of Definition 3.1 with $\alpha = \beta$. Inequality (5.6) follows considering $\beta_n \downarrow \beta$, using (1) of Definition 3.1 and taking $n \to \infty$. Therefore, every $Y \in P_{\mathscr{L}} X$ is a $\mathscr{L}^*$-measurable function. This implies that for every $X \in L_0(\Omega, \mathscr{A}, P)$ we have that

$$P_{\mathscr{L}} X = P_{\mathscr{L}^*} X. \tag{5.7}$$

DEFINITION 5.2. Let $\mathscr{B}$ be a sub-σ-algebra of the σ-algebra $\mathscr{A}$. We define the set **Atom**($\mathscr{B}$) as the set of all atoms of the σ-algebra $\mathscr{B}$, i.e. $B \in$ **Atom**($\mathscr{B}$) iff B is $\mathscr{B}$-measurable and for every $\mathscr{B}$-measurable set $E \subset B$ we have that $P(E) = 0$ or $P(E) = P(B)$.

If Y is a $\mathscr{L}$-measurable function then $A \in$ **Atom**($\mathscr{A}(Y)$) iff there exists a number α with $A = \{Y = \alpha\}$ a.e. and $P(\{Y = \alpha\}) > 0$. Since Ω has finite measure then **Atom**($\mathscr{B}$) is a countable set for every sub-σ-algebra $\mathscr{B}$. Therefore, for every $\mathscr{L}$-measurable function Y there exists a sequence α_k such that for the sets $B_k = \{Y = \alpha_k\}$ we have that

$$\mathbf{Atom}(\mathscr{A}(Y)) \backslash \mathbf{Atom}(\mathscr{A}(\mathscr{L})) = \{B_k : k \in \mathbb{N}\}. \tag{5.8}$$

The following is the main result of this section.

THEOREM 5.3. *Let $X \in L_0(\Omega, \mathscr{A}, P)$, $Y \in P_{\mathscr{L}}X$ and B_k as in equality* (5.8). *We define the set $E := \Omega \setminus \bigcup_k B_k$. Then we have*

(1) $Y \in P_{\mathscr{A}(\mathscr{L})}(X, E)$.
(2) $P_{\mathscr{L}}X = P_{\mathscr{L}^*}X = P_{\mathscr{A}(\mathscr{L}^*)}X \cap L_0(\mathscr{L}^*)$.

Remark 1. Part (1) of the above theorem generalizes Corollary 2 in [4] because in that case $\mathscr{A}(\mathbb{L})$ is the Borel σ-algebra, with $\Omega = [0, 1]$. Therefore if $Y \in P_{\mathscr{A}(\mathbb{L})}(X, E)$ then $X = Y$ a.e. over E.

Remark 2. If $\mathscr{L}$ is non-totally ordered then Theorem 5.3 is not true in general, as we show in the following simple example. Let $\Omega = [0, 1]^2$ with P being the Lebesgue measure. Let us consider the lattice $\mathscr{L} = \mathbb{L}^2$. It is easy to check that Z is a $\mathbb{L}^2$-measurable function iff Z is monotone non-decreasing in each variable separately. Now, we consider the following function $X : \Omega \to \mathbb{R}$ defined by $X(\boldsymbol{\omega}) := (\omega_2 - 1)(\omega_1 - 1) + \omega_2$. Fix $\omega_2 \in [0, 1]$, then the function $\tilde{X}(\omega_1) := X(\boldsymbol{\omega})$ is a monotone decreasing function. Therefore, the conditional 1-mean of $\tilde{X}$ with respect to $\mathbb{L}$ is constant. To see this fact, we can apply (2) of Theorem 5.3 because if $\tilde{X}$ is non-increasing then the σ-lattice $\mathbb{L}^*$ is the trivial lattice $\mathbb{L}^* = \{\emptyset, [0, 1]\}$. Now, it is easy to check that the constant $(1 + \omega_2)/2$ is a conditional 1-mean of $\tilde{X}$. We define the function $Y(\boldsymbol{\omega}) = (1 + \omega_2)/2$ and let $Z \in L_1(\mathbb{L}^2)$. Then for every fixed $\omega_2 \in [0, 1]$ we have

$$\int_0^1 |X - Y| \, d\omega_1 \leqslant \int_0^1 |X - Z| \, d\omega_1.$$

By Fubbini's Theorem we obtain $\|X - Y\|_1 \leqslant \|X - Z\|_1$. Hence, Y is a conditional 1-mean of X with respect to the lattice $\mathbb{L}^2$. Now notice that $\mathbf{Atom}(\mathscr{A}(Y)) = \emptyset$. Therefore, the set E in (1) of Theorem 5.3 is equal to Ω. We observe that the σ-algebra $\mathscr{A}(\mathbb{L}^2)$ is the Borel σ-algebra $\mathbb{L}^2$ of Ω. Therefore if $Y \in P_{\mathbb{B}^2}X$ then we would have $Y = X$ a.e., but this is not true.

Remark 3. Whether or not (2) of Theorem 5.3 is true for every σ-lattice it remains an open question.

Before proving Theorem 5.3 we give a corollary of it. This corollary together with Theorem 5.3 generalizes [4, Theorem 8].

COROLLARY 5.4. *Let $X \in L_0(\Omega, \mathscr{A}, P)$ then $Y \in P_{\mathscr{L}}X$ iff $V_{\mathscr{L}}X \leqslant Y \leqslant U_{\mathscr{L}}X$ and Y is $\mathscr{L}^*$-measurable.*

Proof. The "only if" part was already established. Let us see the "if" part. We have that $V_{\mathscr{L}}X \leqslant Y \leqslant U_{\mathscr{L}}X$ and, as a consequence of (2) of Theorem 5.3, that $V_{\mathscr{L}}X, U_{\mathscr{L}}X \in P_{\mathscr{A}(\mathscr{L}^*)}X$. Since $\mathscr{A}(\mathscr{L}^*)$ is a σ-algebra it follows, from

Corollary 2.3 that $Y \in P_{\mathscr{A}(\mathscr{L}^*)}X$. Now, since Y is $\mathscr{L}^*$-measurable, again from (2) of Theorem 5.3 we obtain that $Y \in P_{\mathscr{L}}X$. ∎

Proof (1) *of Theorem* 5.3. By the inequalities in (2.12) and the remark following Definition 3.1, it is sufficient to prove

$$P(\{X \leqslant Y\} \cap E \cap A) \geqslant \tfrac{1}{2} P(E \cap A) \tag{5.9}$$

and

$$P(\{X \geqslant Y\} \cap E \cap A) \geqslant \tfrac{1}{2} P(E \cap A) \tag{5.10}$$

for every $A \in \mathscr{A}(\mathscr{L})$. We prove (5.10). The proof of (5.9) is a consequence of inequality (5.10) applied to $-X$ and $-Y$, using the fact that $\mathscr{A}(\mathscr{L}) = \mathscr{A}(\bar{\mathscr{L}})$.

The σ-algebra $\mathscr{A}(\mathscr{L})$ is generated by the algebra of sets of the form $\bigcup_{i=1}^n C_i \cap D_i$, with $C_i \in \mathscr{L}$ and $D_i \in \bar{\mathscr{L}}$. By elementary facts of measure theory, one can get, for every $A \in \mathscr{A}(\mathscr{L})$ and $\varepsilon > 0$, a collection of sets $C_i \in \mathscr{L}$ and $D_i \in \bar{\mathscr{L}}$, $i = 1, \ldots, n$ such that $P(A \triangle \bigcup_{i=1}^n C_i \cap D_i) < \varepsilon$. Since $\mathscr{L}$ is a totally ordered σ-lattice we can suppose $C_n \subset \cdots \subset C_1$. Now since $\bar{\mathscr{L}}$ is also totally ordered, we can assume that $D_1 \subset \cdots \subset D_n$. Notice that if for some i we have $D_{i+1} \subset D_i$, then $C_i \cap D_i \cup C_{i+1} \cap D_{i+1} = C_i \cap D_i$. Hence the set $C_{i+1} \cap D_{i+1}$ can be removed from the union. We put $\tilde{D}_i := D_i \cap \overline{C_{i+1}}$ (where $C_{n+1} := \emptyset$). Then $\bigcup_i C_i \cap D_i = \bigcup_i C_i \cap \tilde{D}_i$. So we can suppose that the sets $C_i \cap D_i$ are mutually disjoint. Thus, it is sufficient to prove (5.10) for $A = C \cap D$ with $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$.

We define

$$E_m := \Omega \setminus \bigcup_{i=1}^m \{Y = \alpha_i\}.$$

For each $m \in \mathbb{N}$ we have a permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $\alpha_{\sigma(i)} \leqslant \alpha_{\sigma(i+1)}$. We put $\alpha_{\sigma(0)} := -\infty$ and $\alpha_{\sigma(m+1)} := \infty$. Then

$$E_m = \bigcup_0^m \{\alpha_{\sigma(i)} < Y < \alpha_{\sigma(i+1)}\}. \tag{5.11}$$

Let $C \in \mathscr{L}$ and $D \in \bar{\mathscr{L}}$. From (5.11) we obtain

$$C \cap D \cap E_m := \bigcup_{i=0}^m C_i \cap D_i, \tag{5.12}$$

where $C_i := C \cap \{Y > \alpha_{\sigma(i)}\}$ and $D_i := D \cap \{Y < \alpha_{\sigma(i+1)}\}$. Suppose $C_i \cap D_i \neq \emptyset$ for every i. We define

$$\beta_i := \sup\{\alpha : C_i \subset \{Y > \alpha\}\}.$$

Trivially, $C_i \subset \{Y \geqslant \beta_i\}$ and $\alpha_{\sigma(i)} \leqslant \beta_i < \alpha_{\sigma(i+1)}$. Since for $\alpha > \beta_i$ we have $C_i \not\subset \{Y > \alpha\}$ and $\mathscr{L}$ is a totally ordered set then $\{Y > \alpha\} \subset C_i$. Now taking $\alpha \downarrow \beta_i$ we get

$$\{Y > \beta_i\} \subset C_i \subset \{Y \geqslant \beta_i\}. \tag{5.13}$$

Now we define sets A_i according to the following cases:

(1) If $P(\{Y = \beta_i\}) = 0$ then $A_i := \{Y > \beta_i\}$. Here $P(C_i \triangle A_i) = 0$.

(2) $P(\{Y = \beta_i\}) > 0$ and $\{Y = \beta_i\} \in \mathbf{Atom}(\mathscr{A}(\mathscr{L}))$. In this case, we have $C_i = \{Y > \beta_i\}$ a.e. or $C_i = \{Y \geqslant \beta_i\}$ a.e. then we put $A_i := \{Y > \beta_i\}$ or $A_i := \{Y \geqslant \beta_i\}$, respectively. Again we get $P(C_i \triangle A_i) = 0$. We observe that the relation given in (3)(b) of Theorem 2.1 holds with $\{Y \geqslant \alpha\}$ in the place of $\{Y > \alpha\}$.

(3) $P(\{Y = \beta_i\}) > 0$ and $\{Y = \beta_i\} \notin \mathbf{Atom}(\mathscr{A}(\mathscr{L}))$ (we observe that in this case β_i is some α_{k_i}). Here defining $A_i := \{Y > \beta_i\}$ we get $P(C_i \triangle A_i) \leqslant P(\{Y = \beta_i\})$. In this case if $\beta_i = \alpha_{\sigma(i)}$ then $P(C_i \triangle A_i) = 0$.

In any case $P(C_i \triangle A_i) = 0$ or $P(C_i \triangle A_i) \leqslant P(\{Y = \alpha_{k_i}\})$ with $k_i > m$. By virtue of the inequality $\alpha_{\sigma(i)} \leqslant \beta_i < \alpha_{\sigma(i+1)}$ we can affirm that $k_i \neq k_j$ if $i \neq j$. Therefore, we obtain the inequality

$$P\left(C \cap D \cap E_m \triangle \bigcup_{i=1}^{m} A_i \cap D_i\right) \leqslant \sum_{k \geqslant m} P(\{Y = \alpha_k\}). \tag{5.14}$$

From (3)(b) of Theorem 2.1 we obtain

$$P\left(\{X \geqslant Y\} \cap \bigcup_{i=1}^{m} D_i \cap A_i\right) \geqslant \frac{1}{2} P\left(\bigcup_{i=1}^{m} D_i \cap A_i\right).$$

Given $\varepsilon > 0$ we choose $M \in \mathbb{N}$ such that the right member of (5.14) is less than ε for every $m > M$. Now since $C_i \cap D_i \supset D_i \cap A_i$, by (5.14) we have

$$\begin{aligned} P(\{X \geqslant Y\} \cap C \cap D \cap E_m) &\geqslant P\left(\{X \geqslant Y\} \cap \bigcup_{i=1}^{m} D_i \cap A_i\right) \\ &\geqslant \frac{1}{2} P\left(\bigcup_{i=1}^{m} D_i \cap A_i\right) \\ &\geqslant \frac{1}{2} P(C \cap D \cap E_m) - \varepsilon \end{aligned}$$

for every $m > M$. Since ε is arbitrary, taking the limit as m goes to ∞ we obtain (5.10) with $A = C \cap D$. ∎

In order to prove (2) of Theorem 5.3 we will need the following lemmas.

LEMMA 5.5. *Let $E \in \mathscr{A}$ and $\gamma \in \mathbb{R}$. Suppose that $V(X,E) \leqslant \gamma < U(X,E)$ or $V(X,E) < \gamma \leqslant U(X,E)$. Then*

$$P(\{X \geqslant \gamma\} \cap E) = P(\{X \leqslant \gamma\} \cap E) = \tfrac{1}{2}P(E).$$

Proof. Suppose that $V(X,E) \leqslant \gamma < U(X,E)$. The other case is similar. Then from (5.2) and (5.3) we obtain $P(\{X > \gamma\} \cap E) \geqslant 1/2P(E)$ and $P(\{X \leqslant \gamma\} \cap E) \geqslant 1/2P(E)$. These inequalities imply the statement of the lemma. ∎

LEMMA 5.6. *Suppose $X \in L_0(\Omega, \mathscr{A}, P)$ and $Y \in P_{\mathscr{L}^*}X$. Let $A \in$ **Atom** $(\mathscr{A}(Y)) \backslash$ **Atom**$(\mathscr{A}(\mathscr{L}))$. Let $\mathscr{L}'$ be a sub-σ-lattice of $\mathscr{L}^*$ with the following properties*:

(1) *If $C^* \in \mathscr{L}^*$ satisfies $P(A \cap C^*) = 0$ or $P(A \cap C^*) = P(A)$ then $C^* \in \mathscr{L}'$.*

(2) *The set $\{C \in \mathscr{L}' : 0 < P(A \cap C) < P(A)\}$ is finite.*

Then $Y \in P_{\mathscr{A}(\mathscr{L}')}(X, A)$.

Proof. We recall that $A = \{Y = \alpha\}, P(A) > 0$ and $A \notin$ **Atom** $(\mathscr{A}(\mathscr{L}))$. Suppose that the lattice induced by $\mathscr{L}'$ over A is equal to $\{C_0 \cap A, \ldots, C_n \cap A\}$ with $\emptyset = C_0 \cap A \subset \cdots \subset C_n \cap A = A$. Let $E_i = A \cap (C_i - C_{i-1})$, $i = 1, \ldots, n$. We have that the σ-algebra induced by $\mathscr{A}(\mathscr{L}')$ over A is equal to the σ-algebra generated by the sets E_i and the sets E_i are atoms of this σ-algebra. Therefore, it is sufficient to prove that

$$V(X, E_i) \leqslant \alpha \leqslant U(X, E_i) \qquad \text{for } i = 1, \ldots, n. \tag{5.15}$$

Before proving this fact, we will show that the following inequalities hold:

$$V(X, A \backslash C_i) \leqslant \alpha \leqslant U(X, A \backslash C_i) \qquad \text{for } i = 1, \ldots, n \tag{5.16}$$

and

$$V(X, A \cap C_i) \leqslant \alpha \leqslant U(X, A \cap C_i) \qquad \text{for } i = 1, \ldots, n. \tag{5.17}$$

Let us first consider these inequalities in the case $i = n$. In this case just inequality (5.17) has sense. In virtue of inequalities (5.2), (5.3) and the remark following them, we have to prove that

$$P(\{X \geqslant \alpha\} \cap A) \geqslant \tfrac{1}{2}P(A) \qquad \text{and} \qquad P(\{X \leqslant \alpha\} \cap A) \geqslant \tfrac{1}{2}P(A). \tag{5.18}$$

These inequalities are consequences of (5) of Theorem 2.1 putting $B = \{\alpha\}$ and $C = D = \Omega$.

We suppose $i < n$. Putting $B := \{\alpha\}$, $D := \Omega \setminus C_i$ and $C := C_i$ in (5)(a) and (b) of Theorem 2.1 we obtain

$$P(\{X \geqslant \alpha\} \cap (A \setminus C_i)) \geqslant \tfrac{1}{2} P(A \setminus C_i) \tag{5.19}$$

and

$$P(\{X \leqslant \alpha\} \cap (A \cap C_i)) \geqslant \tfrac{1}{2} P(A \cap C_i). \tag{5.20}$$

Now from inequalities (5.2), (5.3) and the remark following them, we get

$$V(X, A \cap C_i) \leqslant \alpha \leqslant U(X, A \setminus C_i). \tag{5.21}$$

Since $C_i \in \mathscr{L}^*$, $i = 1, \ldots, n$, we have from Definition 5.1 that

$$V(X, A \setminus C_i) \leqslant U(X, A \cap C_i). \tag{5.22}$$

If we can prove that $V(X, A \setminus C_i) \leqslant \alpha \leqslant U(X, A \cap C_i)$, then we would have proved (5.16) and (5.17). Suppose that this inequality is not true. We can assume that $\alpha < V(X, A \setminus C_i)$ (the other case is similar). Then from (5.21) and (5.22), we get $V(X, A \cap C_i) \leqslant \alpha < U(X, A \cap C_i)$. Lemma 5.5 implies that

$$P(\{X \leqslant \alpha\} \cap A \cap C_i)) = \tfrac{1}{2} P(A \cap C_i). \tag{5.23}$$

Since $\alpha < V(X, A \setminus C_i)$, from the remark following (5.3) we have

$$P(\{X \leqslant \alpha\} \cap (A \setminus C_i)) < \tfrac{1}{2} P(A \setminus C_i). \tag{5.24}$$

Adding (5.23) and (5.24), we obtain $P(\{X \leqslant \alpha\} \cap A) < 1/2P(A)$ which contradicts (5.18).

From inequalities (5.16) and (5.17) we obtain (5.15) in the particular case $i = 1$ or $i = n$. Therefore in order to prove (5.15), we can suppose that $1 < i < n$. Suppose that there exists i such that $U(X, E_i) < \alpha$ (the case $\alpha < V(X, E_i)$ is similar). Then by virtue of Definition 5.1 (note that $A \setminus C_i = \{Y \geqslant \alpha\} \setminus C_i$) and inequality (5.16), we get

$$V(X, A \setminus C_i) \leqslant U(X, E_i) < \alpha \leqslant U(X, A \setminus C_i). \tag{5.25}$$

Let β be any number with $U(X, E_i) < \beta < U(X, A \setminus C_i)$. Then $V(X, A \setminus C_i) < \beta < U(X, A \setminus C_i)$ and $P(\{X \geqslant \beta\} \cap E_i) < 1/2P(E_i)$. Therefore from Lemma 5.5

we obtain $P(\{X \geqslant \beta\} \cap A \backslash C_i) = 1/2P(A \backslash C_i)$. Hence,

$$\begin{aligned} P(\{X \geqslant \beta\} \cap (A \backslash C_{i-1})) &= P(\{X \geqslant \beta\} \cap (A \backslash C_i)) \\ &\quad + P(\{X \geqslant \beta\} \cap E_i) \\ &< \tfrac{1}{2} P(A \backslash C_{i-1}). \end{aligned}$$

Thus $U(X, A \backslash C_{i-1}) < \beta$, since β is an arbitrary number with $U(X, E_i) < \beta < U(X, A \backslash C_i)$ we get $U(X, A \backslash C_{i-1}) \leqslant U(X, E_i)$. Therefore,

$$V(X, A \cap C_{i-1}) \leqslant U(X, A \backslash C_{i-1}) \leqslant U(X, E_i) < \alpha \leqslant U(X, A \cap C_{i-1}). \tag{5.26}$$

Now, from (5.25), (5.26) and Lemma 5.5 we deduce the following two inequalities:

$$P(\{X \geqslant \alpha\} \cap (A \backslash C_i)) = \tfrac{1}{2} P(A \backslash C_i), \tag{5.27}$$

$$P(\{X \geqslant \alpha\} \cap A \cap C_{i-1}) = \tfrac{1}{2} P(A \cap C_{i-1}). \tag{5.28}$$

Moreover, since $U(X, E_i) < \alpha$, we obtain

$$P(\{X \geqslant \alpha\} \cap E_i) < \tfrac{1}{2} P(E_i). \tag{5.29}$$

Finally, adding (5.27)–(5.29), we obtain $P(\{X \geqslant \alpha\} \cap A) < \frac{1}{2} P(A)$ which contradicts (5.18). ∎

LEMMA 5.7. *Let X, Y and A as in Lemma* 5.6. *Then* $Y \in P_{\mathscr{A}(\mathscr{L}^*)}(X, A)$.

Proof. We consider the set $I = \{P(C^*) : C^* \in \mathscr{L}^*$ and $0 < P(C^* \cap A) < P(A)\}$. Let $\{r_k : k \in \mathbb{N}\}$ be a dense and countable subset of I. Now we define the following sequence of sub-σ-lattices of $\mathscr{L}^*$. For every n we define $\mathscr{L}_n$ by the condition $C \in \mathscr{L}_n$ iff $C \in \mathscr{L}^*$ and $P(C \cap A) = 0$ or $P(C \cap A) = P(A)$ or $P(C) = r_k$ for some $k = 1, \ldots, n$. It is easy to check the following facts about $\mathscr{L}_n$: (i) $\mathscr{L}_n$'s are σ-lattices; (ii) $\mathscr{A}(\mathscr{L}_n) \uparrow \mathscr{A}(\mathscr{L}^*)$. Therefore, the lemma is a consequence of Lemma 5.6 and Theorem 4.1. ∎

LEMMA 5.8. *Let $\mathscr{L}$ be a sub-σ-algebra of $\mathscr{A}$. Suppose $E \in \mathscr{L}$ and $F = \Omega \backslash E$. Then $Y \in P_{\mathscr{L}} X$ iff $Y \in P_{\mathscr{L}}(X, E)$ and $Y \in P_{\mathscr{L}}(X, F)$.*

Proof. It is a consequence of Corollary 2.2 and the well known properties of the conditional expectation operator. ∎

Proof of (2) *of Theorem* 5.3. It is sufficient (by virtue of (5.6)) to prove that $P_{\mathscr{L}^*} X = P_{\mathscr{A}(\mathscr{L}^*)} X \cap L_0(\mathscr{L}^*)$. Notice that the inclusion $P_{\mathscr{L}^*} X \supset P_{\mathscr{A}(\mathscr{L}^*)} X$

$\cap L_0(\mathscr{L}^*)$ is clear. It only remains to prove that $P_{\mathscr{L}^*}X \subset P_{\mathscr{A}(\mathscr{L}^*)}X \cap L_0(\mathscr{L}^*)$. If $Y \in P_{\mathscr{L}^*}X$ then, from (1) of Theorem 5.3 and (5.6) we obtain $Y \in P_{\mathscr{A}(\mathscr{L})}(X, E)$. We recall that $E = \Omega \setminus \bigcup_k B_k$, where the $B_k's$ are defined in (5.7). Since Y is a $\mathscr{A}(\mathscr{L}^*)$-measurable function we have $Y \in P_{\mathscr{A}(\mathscr{L}^*)}(X, E)$. On the other hand, since E is a $\mathscr{A}(\mathscr{L}^*)$-measurable set, in order to prove (2) of Theorem 5.3 it is sufficient to show that $Y \in P_{\mathscr{A}(\mathscr{L}^*)}(X, F)$, where $F := \Omega \setminus E = \bigcup_k B_k$ (see Lemma 5.7). Since each set B_k is $\mathscr{A}(\mathscr{L}^*)$-measurable, again by virtue of Lemma 5.7, we only need to prove that $Y \in P_{\mathscr{A}(\mathscr{L}^*)}(X, B_k)$ for every k and that was already proved in Lemma 5.7. ∎

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