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1-greedy renormings of Garling sequence spaces

Fernando Albiac[a,*], José L. Ansorena[b], Ben Wallis[c]

[a] Mathematics Department, Universidad Pública de Navarra, Campus de Arrosadía, Pamplona, 31006, Spain
[b] Department of Mathematics and Computer Sciences, Universidad de La Rioja, Logroño, 26004, Spain
[c] Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, USA

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Abstract

We show that all Garling sequence spaces admit a renorming with respect to which their standard unit vector basis is 1-greedy. We also discuss some additional properties of these Banach spaces related to uniform convexity and superreflexivity. In particular, our approach to the study of the superreflexivity of Garling sequence spaces provides an example of how essentially non-linear tools from greedy approximation can be used to shed light on the linear structure of these spaces.

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1. Introduction and background

A semi-normalized basis $(\mathbf{x}_n)_{n=1}^{\infty}$ of a Banach space $(X, \|\cdot\|)$ is said to be *C-greedy under renorming* (C-GUR, for short) if there is an equivalent norm $|||\cdot|||$ on X (i.e., a renorming of X)

* Corresponding author.
E-mail addresses: fernando.albiac@unavarra.es (F. Albiac), joseluis.ansorena@unirioja.es (J.L. Ansorena), benwallis@live.com (B. Wallis).

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with respect to which $(\mathbf{x}_n)_{n=1}^{\infty}$ is C-greedy, i.e.,

$$\left|\left|\left| f - \sum_{n\in A} a_n \mathbf{x}_n \right|\right|\right| \le C\, |||f - g|||$$

for any $f = \sum_{n=1}^{\infty} a_n \mathbf{x}_n \in X$, any $A \subseteq \mathbb{N}$ finite such that $|a_n| \ge |a_k|$ whenever $n \in A$ and $k \in \mathbb{N} \setminus A$, and any $g \in X$ with $|\text{supp}(g)| \le |A|$.

A problem that goes back to [5] is to determine if a given (greedy) basis is 1-GUR. For symmetric bases the answer to this problem is positive and quite simple because C-symmetric bases are C-greedy and every symmetric basis becomes 1-symmetric under a suitable renorming; thus any symmetric basis is 1-GUR.

For subsymmetric bases the situation is different. Recall that a basis $(\mathbf{x}_n)_{n=1}^{\infty}$ is said to be *subsymmetric* if it is unconditional and equivalent to $(\mathbf{x}_{\phi(n)})_{n=1}^{\infty}$ for every increasing map $\phi : \mathbb{N} \to \mathbb{N}$. It can be shown that then there is a constant C such that

$$C^{-1} \left\| \sum_{n=1}^{\infty} a_n \mathbf{x}_n \right\| \le \left\| \sum_{n=1}^{\infty} \varepsilon_n a_n \mathbf{x}_{\phi(n)} \right\| \le C \left\| \sum_{n=1}^{\infty} a_n \mathbf{x}_n \right\|$$

for every $\sum_{n=1}^{\infty} a_n \mathbf{x}_n \in X$, every increasing map $\phi : \mathbb{N} \to \mathbb{N}$, and every $(\varepsilon_n)_{n=1}^{\infty}$ sequence of signs. In this case the basis is said to be C-*subsymmetric*.

Every 1-symmetric basis is 1-subsymmetric. In turn, taking into account the relation between the constants involved (see e.g. [4, Chapter 10]), one immediately sees that 1-subsymmetric bases are always 2-greedy. Hence, since any subsymmetric basis becomes 1-subsymmetric under a suitable renorming (see [7]), we have that any subsymmetric basis is 2-GUR.

Let us now put our problem into context by summarizing its background. Albiac and Wojtaszczyk exhibited in [5] an example of a 1-subsymmetric basis that is not 1-greedy. Later on, Dilworth et al. constructed in [9] an example of a subsymmetric basis which, in spite of not being symmetric, was 1-greedy. Therefore a natural question in the theory is to determine if a particular subsymmetric (and non-symmetric) basis is 1-GUR.

Recently, the authors have investigated in [3] the geometric properties of a class of Banach spaces, called Garling sequence spaces, in which the canonical basis is subsymmetric but not symmetric. In this note we further the study of the greedy behavior of subsymmetric bases and investigate Garling sequence spaces from the point of view of the greedy algorithm. To be precise, in Section 3 we prove that the canonical basis of Garling sequence spaces is 1-GUR. In Section 2 we use the properties of the democracy functions of these spaces to give a necessary condition for them to be super-reflexive. In addition, we prove that Garling sequence spaces are never uniformly convex.

It is worth pointing out that investigating greedy renormings of non-subsymmetric bases is also of interest. Indeed, the starting problem of this theory, posed in [5] and still unsolved as of the writing of this article, is to determine if the Haar system in $L_p[0, 1]$, $1 < p < \infty$, is a 1-GUR basis. Recall that the Haar system in $L_p[0, 1]$ is greedy [16] but it is not subsymmetric [12]. The most significant advances in the study of greedy renormings of non-subsymmetric bases were made in [9]. Here the authors found examples of non-subsymmetric greedy bases which are not 1-GUR (like the Haar basis in the dyadic Hardy space H_1 and the canonical basis of the Tsirelson space), and an example of a non-subsymmetric greedy basis which is 1-GUR (namely, the canonical basis of the space $\ell_2 \oplus \ell_{2,1}$).

Throughout this article we use standard facts and notation from Banach spaces and approximation theory. We refer the reader to [4,13,14] for the necessary background. Next we single out the notation that it is more heavily employed. We will denote by $\mathbb{F}$ the real or complex field. We denote by $(\mathbf{e}_k)_{k=1}^\infty$ the canonical basis of $\mathbb{F}^{\mathbb{N}}$, i.e., $\mathbf{e}_k = (\delta_{k,n})_{n=1}^\infty$, were $\delta_{k,n} = 1$ if $n = k$ and $\delta_{k,n} = 0$ otherwise. The domain of a function f will be denoted by $\mathrm{D}(f)$, while $\mathrm{R}(f)$ denotes its range. Given families of positive real numbers $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that $\sup_{i\in I}\alpha_i/\beta_i < \infty$, while $\alpha_i \approx \beta_i$ for $i \in I$ means that $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$.

2. Superreflexivity in Garling sequence spaces

Let us consider the set of weights

$$\mathcal{W} := \left\{(w_n)_{n=1}^\infty \in c_0 \setminus \ell_1 : 1 = w_1 \geq w_2 > \cdots w_n \geq w_{n+1} \geq \cdots > 0\right\}.$$

Given $1 \leq p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$ the *Garling sequence space* $g(\mathbf{w}, p)$ is defined as the Banach space consisting of all scalar sequences $f = (a_n)_{n=1}^\infty$ such that

$$\|f\|_{g(\mathbf{w},p)} = \sup_{\phi\in\mathcal{O}_\infty} \left(\sum_{n=1}^\infty |a_{\phi(n)}|^p w_n\right)^{1/p},$$

where $\mathcal{O}_\infty$ denotes the set of all increasing functions from $\mathbb{N}$ to $\mathbb{N}$. If $\mathbf{w}$ and p are clear from context, the norm of the space will be shortened to $\|\cdot\|_g$. The isomorphic structure of these Banach spaces, which generalize an example of Garling from [11], has been recently studied in [3].

Theorem 2.1 gathers a few properties of Garling sequence spaces that are of interest for the purposes of this paper.

Recall that given a basis $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ for a Banach space X, the *lower democracy function* $(\varphi_l[\mathcal{B}, X](m))_{m=1}^\infty$ and the *upper democracy function* $(\varphi_u[\mathcal{B}, X](m))_{m=1}^\infty$ of $\mathcal{B}$ are defined, respectively, by

$$\varphi_l[\mathcal{B}, X](m) = \inf_{|A|\geq m} \left\|\sum_{n\in A} \mathbf{x}_n\right\|,$$

and

$$\varphi_u[\mathcal{B}, X](m) = \sup_{|A|\leq m} \left\|\sum_{n\in A} \mathbf{x}_n\right\|,$$

and that $\mathcal{B}$ is Δ-democratic if and only if $\varphi_u[\mathcal{B}, X](m) \leq \Delta\varphi_l[\mathcal{B}, X](m)$ for all $m \in \mathbb{N}$.

Theorem 2.1 (*See [3]*). *Let $1 \leq p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$. Then:*

(i) *The canonical basis $\mathcal{E} = (\mathbf{e}_n)_{n=1}^\infty$ is a 1-subsymmetric basis of $g(\mathbf{w}, p)$.*
(ii) *$\varphi_l[\mathcal{E}, g(\mathbf{w}, p)](m) = \varphi_u[\mathcal{E}, g(\mathbf{w}, p)](m) = (\sum_{n=1}^m w_n)^{1/p}$ for all $m \in \mathbb{N}$.*
(iii) *$g(\mathbf{w}, p)$ is reflexive if and only if $p > 1$.*
(iv) *Any subsymmetric basis of $g(\mathbf{w}, p)$ is equivalent to its canonical basis.*
(v) *For every $\varepsilon > 0$ there is a sublattice of $g(\mathbf{w}, p)$ that is $(1 + \varepsilon)$-lattice isomorphic to ℓ_p and $(1 + \varepsilon)$-lattice complemented in $g(\mathbf{w}, p)$.*

Let us get started by using the democracy functions to obtain some embedding results. Recall that a weight $(w_n)_{n=1}^\infty$ is said to be *regular* if there is a constant $C \geq 1$ such that

$$\frac{1}{m}\sum_{n=1}^m w_n \leq C w_m, \quad m \in \mathbb{N}.$$

Proposition 2.2. *Let $1 \le p < \infty$. Let $\mathbf{v} = (v_n)_{n=1}^\infty$, and $\mathbf{w} = (w_n)_{n=1}^\infty$ be weights in $\mathcal{W}$ with $\mathbf{w}$ regular. Then $g_p(\mathbf{w}) \subseteq g_p(\mathbf{v})$ if and only if $v_n \lesssim w_n$ for $n \in \mathbb{N}$.*

Proof. If $g_p(\mathbf{w}) \subseteq g_p(\mathbf{v})$ the inclusion is continuous and so

$$\varphi_u[\mathcal{E}, g(\mathbf{v}, p)](m) \lesssim \varphi_u[\mathcal{E}, g(\mathbf{w}, p)](m), \quad m \in \mathbb{N}.$$

Appealing to Theorem 2.1(ii) we get

$$v_m \le \frac{1}{m} \sum_{n=1}^{m} v_n \lesssim \frac{1}{m} \sum_{n=1}^{m} w_n \lesssim w_m, \quad m \in \mathbb{N}.$$

The converse is obvious. $\square$

Corollary 2.3. *Let $1 \le p < \infty$. Let $\mathbf{v} = (v_n)_{n=1}^\infty$, and $\mathbf{w} = (w_n)_{n=1}^\infty$ be weights in $\mathcal{W}$.*

(i) *$g_p(\mathbf{v}) \approx g_p(\mathbf{w})$ if and only if $g_p(\mathbf{v}) = g_p(\mathbf{w})$.*
(ii) *Assume that both $\mathbf{v}$ and $\mathbf{w}$ are regular and that $g_p(\mathbf{v}) = g_p(\mathbf{w})$. Then $v_n \approx w_n$ for $n \in \mathbb{N}$.*

Proof. Part (i) is a consequence of Theorem 2.1(iv), and part (ii) is straightforward from Proposition 2.2. $\square$

Proposition 2.4. *The space $g(\mathbf{w}, p)$ fails to be uniformly convex for any $1 \le p < \infty$ and any $\mathbf{w} \in \mathcal{W}$.*

Proof. For $j \in \mathbb{N}$ put

$$\alpha_j = \left(\frac{1 - w_{j+1}}{\sum_{n=1}^{j} w_n} \right)^{1/p},$$

and consider the vectors

$$u^{(j)} = (\underbrace{\alpha_j, \dots, \alpha_j}_{j \text{ times}}, 1, 0, 0, 0, \dots)$$

and

$$v^{(j)} = (\underbrace{\alpha_j, \dots, \alpha_j}_{j \text{ times}}, 0, 1, 0, 0, 0, \dots).$$

Observe that

$$\begin{aligned} \frac{1}{2}\|u^{(j)} + v^{(j)}\|_g &\ge \frac{1}{2}\left[\sum_{n=1}^{j} (2\alpha_j)^p w_n + w_{j+1} + w_{j+2} \right]^{1/p} \\ &= \left[\alpha_j^p \sum_{n=1}^{j} w_n + \frac{1}{2^p}(w_{j+1} + w_{j+2}) \right]^{1/p} \\ &= \left[1 - w_{j+1} + \frac{1}{2^p}(w_{j+1} + w_{j+2}) \right]^{1/p} \\ &:= N_j. \end{aligned}$$

Since $\lim_j N_j = 1$, in order to show that $g(\mathbf{w}, p)$ fails to be uniformly convex, it suffices to find an increasing sequence of integers $(j_k)_{k=1}^{\infty}$ such that $\|u^{(j_k)}\|_g = \|v^{(j_k)}\|_g = 1$ and $\|u^{(j_k)} - v^{(j_k)}\|_g > 1$ for all $k \in \mathbb{N}$.

Due to the fact that $(w_j)_{j=1}^{\infty} \in c_0 \setminus \ell_1$ we have

$$\lim_{j\to\infty} \alpha_j = 0.$$

Hence, we can find a subsequence $(\alpha_{j_k})_{k=1}^{\infty}$ such that

$$\alpha_{j_k} \le \min_{i \le j_k} \alpha_i, \quad k \in \mathbb{N}.$$

Now, fix $k \in \mathbb{N}$. By definition of $g(w, p)$, since $w_1 = 1$, either $\|u^{(j_k)}\|_g = 1$, or else we can pick $i \in \{1, \dots, j_k\}$ with

$$1 \le \|u^{(j_k)}\|_g^p = \sum_{n=1}^{i} w_n \alpha_{j_k}^p + w_{i+1} \le \sum_{n=1}^{i} w_n \alpha_i^p + w_{i+1} = 1$$

so that $\|u^{(j_k)}\|_g = \|v^{(j_k)}\|_g = 1$. Observing that

$$\|u^{(j_k)} - v^{(j_k)}\|_g = (w_1 + w_2)^{1/p} > 1$$

finishes the proof. $\square$

Enflo proved in [10] that a Banach space is superreflexive if and only if it is uniformly convex under a suitable renorming. Having shown that $g(\mathbf{w}, p)$ is never uniformly convex, and in light of the identification between superreflexivity and uniform convexifiability, the next natural question to ask is: Given $1 < p < \infty$, can we choose $\mathbf{w} \in \mathcal{W}$ so that $g(\mathbf{w}, p)$ is superreflexive?

We tackle this issue by using well-known properties of the democracy functions of bases in Banach spaces. Following [8] we say that a sequence $(s_n)_{n=1}^{\infty}$ of positive numbers has the *lower regularity property* (LRP for short) if there is an integer $r \ge 2$ with

$$s_{rn} \ge 2 s_n, \quad n \in \mathbb{N}.$$

Our next Proposition establishes the close relation between a weight $(w_n)_{n=1}^{\infty}$ being regular and its *primitive weight* $(s_n)_{n=1}^{\infty}$ given by $s_n = \sum_{k=1}^{n} w_k$ having the LRP. Recall that $(w_n)_{n=1}^{\infty}$ is *essentially decreasing* if there is a constant $C \ge 1$ with $w_k \le C w_n$ for $k \ge n$.

Proposition 2.5. *Let $(s_n)_{n=1}^{\infty}$ be the primitive weight of an essentially decreasing weight $(w_n)_{n=1}^{\infty}$. The following are equivalent.*

(a) *There is $C > 1$ such that $s_{2n} \ge C s_n$ for all $n \in \mathbb{N}$.*
(b) *For every $C > 1$ there is $r \in \mathbb{N}$ with $s_{rn} \ge C s_n$ for all $n \in \mathbb{N}$.*
(c) *$(s_n)_{n=1}^{\infty}$ has the LRP.*
(d) *There is $C > 1$ and $r \in \mathbb{N}$ with $s_{rn} \ge C s_n$ for all $n \in \mathbb{N}$.*
(e) *There exists $a > 0$ such that $(n^{-a} s_n)_{n=1}^{\infty}$ is essentially increasing.*
(f) *$(n^{-1} s_n)_{n=1}^{\infty}$ is a regular weight.*
(g) *$(w_n)_{n=1}^{\infty}$ is a regular weight.*

Proof. Taking into account [6, Theorem 1] and [1, Lemma 2.12], we must only prove (a) ⇒ (g). Assume that $s_{rn} \ge C s_n$ for some $C > 1$, some $r \ge 2$ and all $n \in \mathbb{N}$. Let $D = \sup_{k \le n} w_n / w_k$. We have

$$\frac{nw_n}{s_n} \geq \frac{1}{D(r-1)} \frac{s_{rn} - s_n}{s_n} \geq \frac{C-1}{D(r-1)}$$

for all $n \in \mathbb{N}$. $\square$

Lemma 2.6. *Let $1 \leq p < \infty$ and $\mathbf{w} \in \mathcal{W}$. Then $g(\mathbf{w}, p)$ is p-convex and it is not q-convex for any $q > p$.*

Proof. By Theorem 2.1(v), the space $g(\mathbf{w}, p)$ contains ℓ_p as a sublattice hence it is not q-convex for any $q > p$. Showing that $g(\mathbf{w}, p)$ is p-convex is straightforward. $\square$

Proposition 2.7. *Let $1 < p < \infty$ and $\mathbf{w} \in \mathcal{W}$. The following are equivalent.*

(a) *$g(\mathbf{w}, p)$ is superreflexive.*
(b) *$g(\mathbf{w}, p)$ has non-trivial cotype.*
(c) *$g(\mathbf{w}, 1)$ has non-trivial cotype.*

Proof. (b) $\Rightarrow$ (c) Assume that $g(\mathbf{w}, p)$ has cotype q for some $q < \infty$. Then, $g(\mathbf{w}, p)$ satisfies a lower q-estimate. Since

$$\|f\|_{g(\mathbf{w},1)} = \||f|^{1/p}\|^p_{g(\mathbf{w},p)},$$

it follows that $g(\mathbf{w}, 1)$ satisfies a lower q/p-estimate. By [14, Proposition 1.f.3 and Theorem 1.f.7], $g(\mathbf{w}, 1)$ has cotype r whenever $r \geq 2$ and $r > q/p$.

(c) $\Rightarrow$ (a) Assume that $g(\mathbf{w}, 1)$ has cotype $r < \infty$. Arguing as before, we claim that $g(\mathbf{w}, p)$ satisfies a lower pr-estimate. Taking into account Lemma 2.6, we infer from [14, Theorem 1.f.10] that $g(\mathbf{w}, p)$ is superreflexive.

(a) $\Rightarrow$ (b) is a well known consequence of [15, Theorem 1.1]. $\square$

The key ingredient in the proof of the next theorem is the link between the (Rademacher) type/cotype of a space and the regularity properties of the democracy functions of its almost greedy bases (see [8]).

Theorem 2.8. *Let $1 < p < \infty$ and let $\mathbf{w} \in \mathcal{W}$ be such that $g(\mathbf{w}, p)$ is superreflexive. Then $\mathbf{w}$ is a regular weight.*

Proof. The space $g(\mathbf{w}, 1)$ has finite cotype by Proposition 2.7. Combining [8, Proposition 4.1] and Theorem 2.1(ii) yields that $(\sum_{n=1}^{m} w_n)_{m=1}^{\infty}$ has the LRP. Then, by Proposition 2.5, $\mathbf{w}$ is a regular weight. $\square$

Corollary 2.9. *Let $1 \leq p < \infty$ and let $\mathbf{w} \in \mathcal{W}$ be non-regular. Then ℓ_∞ is finitely representable in $g(\mathbf{w}, p)$.*

Proof. By Theorem 2.8, $g(\mathbf{w}, 2)$ is not superreflexive. Then, by Proposition 2.7, $g(\mathbf{w}, p)$ has trivial cotype. The proof is over by appealing to [15, Theorem 1.1]. $\square$

Remark 2.10. Corollary 2.9 could alternatively be shown by following the steps of the proof from [6] that $d(\mathbf{w}, p)$ is not superreflexive if $\mathbf{w}$ fails to be regular. Altshuler's method leads to the following result: for each $p > 1$, each non-regular weight $\mathbf{w}$, each $\varepsilon > 0$, and each $k \in \mathbb{N}$

there is a constant-coefficient finite block basic sequence of the canonical basis of $g(\mathbf{w}, p)$ that is $(1+\varepsilon)$-equivalent to the canonical basis of ℓ_∞^k. We would also like to point out that the fact that $d(\mathbf{w}, p)$ is superreflexive only if $\mathbf{w}$ is regular can be obtained using intrinsic ideas from this article.

3. Greedy renormings of Garling sequence spaces

Given a basis $(\mathbf{x}_n)_{n=1}^\infty$ for X and f, g in X we say that g is a *greedy permutation* of f if we can write

$$f = h + t \sum_{n\in A} \varepsilon_n \mathbf{x}_n \quad \text{and} \quad g = h + t \sum_{n\in B} \theta_n \mathbf{x}_n \tag{3.1}$$

for some $h \in X$, some sets of integers A and B of the same finite cardinality with $\operatorname{supp}(h) \cap (A \cup B) = \emptyset$, some signs $(\varepsilon_n)_{n\in A}$ and $(\theta_n)_{n\in B}$, and some scalar t such that $\sup_n |\mathbf{x}_n^*(h)| \le t$. If, in addition, $A \cap B = \emptyset$, we say that g is a *disjoint greedy permutation* of f. In other words, g is a disjoint greedy permutation of f if g is obtained from f by moving those terms of f (or some of them) whose coefficients are maximum in absolute value to gaps in the support of f. We are also allowed to change the sign of (some of) the terms we move. Then, the basis $(\mathbf{x}_n)_{n=1}^\infty$ is said to satisfy *Property (A)* if $\|f\| = \|g\|$ whenever g is a disjoint greedy permutation of f. Actually, $(\mathbf{x}_n)_{n=1}^\infty$ has Property (A) if and only if whenever g is a greedy permutation of f then $\|g\| = \|f\|$ (which is the way Property (A) was originally defined in [5]). Property (A) is stronger than democracy. Albiac and Wojtaszczyk [5] proved that a basis is 1-greedy if and only if it is 1-suppression unconditional and has Property (A).

As an immediate consequence of Theorem 2.1(i) we obtain that the canonical basis of $g(\mathbf{w}, p)$ is 2-greedy. However, it is never 1-greedy as we see next.

Lemma 3.1. *The canonical basis of $g(\mathbf{w}, p)$, $1 \le p < \infty$ and $\mathbf{w} \in \mathcal{W}$, is not 1-greedy.*

Proof. Choose $k \in \mathbb{N}$ and $v \in (0, \infty)$ with $w_n = 1$ for $1 \le n \le k$ and $w_{k+1} = v < 1$. Pick $t > 1$ and put $f = t\mathbf{e}_1 + \sum_{n=2}^{k+1} \mathbf{e}_k$ and $g = t\mathbf{e}_{k+2} + \sum_{n=2}^{k+1} \mathbf{e}_k$. Consider for each $j \in \mathbb{N} \cup \{0\}$ the translation map $\phi_j \in \mathcal{O}$ given by $\phi_j(n) = n + j$. Let

$$x := \|f\|_g^p = \max_{j\in\{0,k\}} \|f \circ \phi_j\|_{p,\mathbf{w}}^p = \max\{t + k - 1 + v, tv\},$$

and

$$y := \|g\|_g^p = \max_{j\in\{1,2\}} \|g \circ \phi_j\|_{p,\mathbf{w}}^p = \max\{k + tv, t + k - 1\}.$$

Notice that g is a greedy rearrangement of f. Hence, assuming that $(\mathbf{e}_n)_{n=1}^\infty$ is 1-greedy, yields $x = y$. We infer that $x = t + k - 1 + v$ and $y = k + tv$. Then we reach the absurdity $v = 1$. $\square$

In order to give more relevance to Theorem 3.2, it is convenient to recall that the canonical basis is never a symmetric basis for $g(\mathbf{w}, p)$ (see [2, Theorem 3]).

Theorem 3.2. *Let $1 \le p < \infty$ and let $\mathbf{w} \in \mathcal{W}$ be a regular weight. Then there is a renorming of $g(\mathbf{w}, p)$ with respect to which the canonical basis is 1-greedy and 1-subsymmetric.*

Before proving Theorem 3.2 we shall introduce some additional notation. Suppose $1 \le p < \infty$, and let $\mathbf{w} = (w_i)_{i\in I}$ be a family of positive scalars. Given a family of scalars $f = (a_i)_{i\in A}$, where $A \subseteq I$, we put

$$\|f\|_{p,\mathbf{w}} = \left(\sum_{i\in A} |a_i|^p w_i\right)^{1/p}.$$

Given $r \in \mathbb{N}$, denote by $\mathcal{O}_r$ the set of all increasing functions from the integer interval $[0,r]\cap\mathbb{Z}$ into $\mathbb{N}$. Put $\mathcal{O}_f = \cup_{r=1}^{\infty}\mathcal{O}_r$ and $\mathcal{O} = \mathcal{O}_f \cup \mathcal{O}_\infty$. Note that for all $f \in \mathbb{F}^{\mathbb{N}}$,

$$\|f\|_g = \sup_{\phi\in\mathcal{O}_f} \|f\circ\phi\|_{p,\mathbf{w}} = \sup_{\phi\in\mathcal{O}_\infty} \|f\circ\phi\|_{p,\mathbf{w}} = \sup_{\phi\in\mathcal{O}} \|f\circ\phi\|_{p,\mathbf{w}},$$

where $f\circ\phi = (a_{\phi(n)})_{n\in \mathrm{D}(\phi)}$. Let $\mathcal{H}$ be the set of all increasing functions from a subset of $\mathbb{N}$ into $\mathbb{N}$. Given $\beta\in\mathcal{H}$ consider the linear operator $U_\beta : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ defined by $U_\beta(f) = (b_n)_{n=1}^{\infty}$, where, if $f = (a_n)_{n=1}^{\infty}$,

$$b_n = \begin{cases} a_{\beta(n)} & \text{if } n \in \mathrm{D}(\beta),\\ 0 & \text{otherwise.}\end{cases}$$

Note that if the canonical basis $(\mathbf{e}_n)_{n=1}^{\infty}$ of a sequence space X is 1-unconditional and verifies $\sup_{\beta\in\mathcal{H}}\|U_\beta : X \to X\| \le C$ then $(\mathbf{e}_n)_{n=1}^{\infty}$ is a C-subsymmetric basic sequence in X.

Proof of Theorem 3.2. Let $\mathbf{w} = (w_n)_{n=1}^{\infty}$ and put

$$D = \sup_m \frac{1}{m w_m}\sum_{n=1}^{m} w_n.$$

Note that D is finite in view of Proposition 2.5. For $m\in\mathbb{N}$ define

$$t_m = \frac{2}{m}\sum_{n=1}^{m} w_n.$$

We have that $(t_m)_{m=1}^{\infty}$ is non-increasing, that $t_m \le 2Dw_r$ for $r \le m$, and that $(m+1)t_{m+1} - m t_m = 2w_{m+1}$.

Given $m \in \mathbb{N}\cup\{0\}$, let us denote by $\mathcal{F}_m$ the set of pairs (A,α), where $A\subseteq\mathbb{N}$ and $\alpha\in\mathcal{H}$ verify $|A| = m$, $\mathrm{D}(\alpha)\subseteq[m+1,\infty)$, and $\mathrm{R}(\alpha)\cap A = \emptyset$.

Consider also the set $\mathcal{F}'_m$ of triads (ρ,ϕ,ψ), where $\rho\in\mathcal{O}_m$, $\phi,\psi\in\mathcal{O}_r$ for some $r\in\mathbb{N}\cup\{\infty\}$, $\mathrm{R}(\psi)\subseteq[m+1,\infty)$ and $\mathrm{R}(\rho)\cap\mathrm{R}(\phi)=\emptyset$. Note that the mapping $(\rho,\phi,\psi)\mapsto(A,\alpha)$ where A and α are determined by

$$A = \mathrm{R}(\rho), \quad \alpha(\psi(n)) = \phi(n) \text{ for all } n\in\mathrm{D}(\phi), \tag{3.2}$$

is a bijection from $\mathcal{F}'_m$ onto $\mathcal{F}_m$.

Given $(A,\alpha)\in\mathcal{F}_m$ and $f\in\mathbb{F}^{\mathbb{N}}$, we define

$$\|f\|_{A,\alpha} = \left(t_m\|f|_A\|_p^p + \|f\circ\alpha\|_{p,\mathbf{w}}^p\right)^{1/p}.$$

Let (ρ,ϕ,ψ) be the element in $\mathcal{F}'_m$ that corresponds to (A,α) by the relation (3.2). We have

$$\begin{aligned}\|f\|_{A,\alpha} &= \left(t_m\|f\circ\rho\|_p^p + \|f\circ\phi\|_{p,\mathbf{w}\circ\psi}^p\right)^{1/p}\\ &\le \left(2D\|f\circ\rho\|_{p,\mathbf{w}}^p + \|f\circ\phi\|_{p,\mathbf{w}}^p\right)^{1/p}\\ &\le (2D+1)^{1/p}\|f\|_g.\end{aligned}$$

Put $\mathcal{F} = \cup_{m=0}^{\infty} \mathcal{F}_m$ and define

$$\|f\| = \sup_{(A,\alpha)\in\mathcal{F}} \|f\|_{A,\alpha}, \quad f \in \mathbb{F}^{\mathbb{N}}.$$

We have

$$\|f\| \le (2D+1)^{1/p} \|f\|_g$$

and

$$\|f\| \ge \sup_{(A,\alpha)\in\mathcal{F}_0} \|f\|_{A,\alpha} = \sup_{\alpha\in\mathcal{H}} \|f\circ\alpha\|_{p,\mathbf{w}} \ge \sup_{\alpha\in\mathcal{O}} \|f\circ\alpha\|_{p,\mathbf{w}} = \|f\|_g.$$

Hence $(g(\mathbf{w}, p), \|\cdot\|)$ is a renorming of $g(\mathbf{w}, p)$.

In the next step we substantiate the following Claim:

Claim. *Let $f = (a_n)_{n=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ and let $k \in \mathbb{N}$ be such that $|a_k| \ge |a_n|$ for every $n \in \mathbb{N}$. Then*

$$\|f\| = \sup\{\|f\|_{A,\alpha} : (A,\alpha) \in \mathcal{F}, k \in A, A \cup \mathrm{R}(\alpha) \subseteq \mathrm{supp}(f)\}.$$

Assume without loss of generality that $|a_k| = 1$. Pick $(A,\alpha) \in \mathcal{F}_m$ for some $m \in \mathbb{N}$. Let $B = A \cap \mathrm{supp}\, f$ and β be the restriction of α to $\alpha^{-1}(\mathrm{supp}(f))$. We have $(B,\beta) \in \mathcal{F}_r$ for some $r \le m$, that

$$x := \|f|_A\|_p = \|f|_B\|_p,$$

and that

$$\|f\circ\alpha\|_{p,\mathbf{w}} = \|f\circ\beta\|_{p,\mathbf{w}}.$$

Hence $\|f\|_{A,\alpha} \le \|f\|_{B,\beta}$.

If $k \in B$ we are done. Assume that $k \notin B$. Let $E = B \cup \{k\}$ and γ be the restriction of β to $\beta^{-1}(\mathbb{N} \setminus \{k\}) \cap [r+2, \infty)$. Notice that $(E,\gamma) \in \mathcal{F}_{r+1}$. If $r+1 \in \mathrm{D}(\beta)$ put

$$u = w_{r+1} \text{ and } y = |a_{\beta(r+1)}|,$$

and, otherwise, put $y = u = 0$. If there is a (unique) $j \ge r+2$ with $\beta(j) = k$, put

$$v = w_j \text{ and } z = |a_{\beta(j)}|,$$

and, otherwise, put $z = v = 0$. Taking into account that $x^p \le p$ and that $y^p \le 1$, $z^p \le 1$, and that $t_r \le t_{r+1}$, yields

$$\begin{aligned}
\|f\|_{B,\beta}^p - \|f\|_{E,\gamma}^p &= t_r\, x^p - t_{r+1}(1+x^p) + y^p u + z^p v \\
&\le t_r\, p - t_{r+1}(1+p) + u + v \\
&= -2w_{r+1} + u + v \\
&\le -2w_{r+1} + w_{r+1} + w_{r+2} \\
&\le 0,
\end{aligned}$$

as desired.

Now we are ready to prove that $(\mathbf{e}_n)_{n=1}^{\infty}$ is 1-greedy with respect to the norm $\|\cdot\|$. Since it is 1-unconditional, we must only show that it has Property (A). To that end it suffices to see that

$$\|\mathbf{e}_k + f\| \le \|\varepsilon\mathbf{e}_j + f\| \tag{3.3}$$

for every $f \in \mathbb{F}^{\mathbb{N}}$ with $\|f\|_\infty \le 1$, every sign ε, and every $j, k \notin \mathrm{supp}(f)$ with $j \ne k$.

In order to compute $\|\mathbf{e}_k + f\|$, taking into account the Claim, we can restrict our attention to $(A, \alpha) \in \mathcal{F}$ with $k \in A$ and $A \cup \mathrm{R}(\alpha) \subseteq \{k\} \cup \mathrm{supp}(f)$. In particular, we have $j \notin A \cup \mathrm{R}(\alpha)$. Choose $B = (A \cup \{j\}) \setminus \{k\}$. We have $(B, \alpha) \in \mathcal{F}$ and

$$\|(\mathbf{e}_k + f)|_A\|_p = \|(\varepsilon \mathbf{e}_j + f)|_B\|_p.$$

Hence,

$$\|\mathbf{e}_k + f\|_{A,\alpha} = \|\varepsilon \mathbf{e}_j + f\|_{B,\alpha} \leq \|\varepsilon \mathbf{e}_j + f\|.$$

We obtain (3.3) by taking the supremum on (A, α).

Let us prove that the canonical basis is 1-subsymmetric with respect to the norm $\|\cdot\|$. Let $\beta \in \mathcal{H}$, $f \in \mathbb{F}^{\mathbb{N}}$ and $(A, \alpha) \in \mathcal{F}$. Since $|\beta(A)| \leq |A|$, we have $(\beta(A), \beta \circ \alpha) \in \mathcal{F}$. Moreover

$$\|U_\beta(f)|_A\|_p = \|f|_{\beta(A)}\|_p,$$

and

$$\|U_\beta(f) \circ \alpha\|_{p,\mathbf{w}} = \|f \circ \beta \circ \alpha\|_{p,\mathbf{w}},$$

so that $\|U_\beta(f)\|_{A,\alpha} \leq \|f\|$. Consequently, $\|U_\beta(f)\| \leq \|f\|$. $\square$

This note exhibits the first known examples of spaces that possess a subsymmetric basis which, despite not being symmetric, is 1-GUR. Hence our work naturally yields the following question:

Problem 3.3. Does every Banach space with a subsymmetric basis admit a 1-greedy renorming?

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References

[1] F. Albiac, J.L. Ansorena, Lorentz spaces and embeddings induced by almost greedy bases in Banach spaces, Constr. Approx. 43 (2) (2016) 197–215.
[2] F. Albiac, J.L. Ansorena, D. Leung, B. Wallis, Optimality of the rearrangement inequality with applications to Lorentz-type sequence spaces, Math. Inequal. Appl. 21 (1) (2018) 127–132.
[3] F. Albiac, J.L. Ansorena, B. Wallis, Garling sequence spaces, J. London Math. Soc. (2018). http://dx.doi.org/10.1112/jlms.12120.
[4] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, second revised and updated ed., in: Graduate Texts in Mathematics, vol. 233, Springer International Publishing, 2016.
[5] F. Albiac, P. Wojtaszczyk, Characterization of 1-greedy bases, J. Approx. Theory 138 (1) (2006) 65–86.
[6] Z. Altshuler, Uniform convexity in Lorentz sequence spaces, Israel J. Math. 20 (3–4) (1975) 260–274.
[7] J.L. Ansorena, A note on subsymmetric renormings of Banach spaces, Quaest. Math. (2018). http://dx.doi.org/10.2989/16073606.2017.1393704.
[8] S.J. Dilworth, N.J. Kalton, Denka Kutzarova, V.N. Temlyakov, The thresholding greedy algorithm, greedy bases, and duality, Constr. Approx. 19 (4) (2003) 575–597.
[9] S.J. Dilworth, E. Odell, Th. Schlumprecht, A. Zsák, Renormings and symmetry properties of 1-greedy bases, J. Approx. Theory 163 (9) (2011) 1049–1075.
[10] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, in: Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), Vol. 1972, 1973, pp. 281–288.

[11] D.J.H. Garling, Symmetric bases of locally convex spaces, Studia Math. 30 (1968) 163–181.
[12] M.I. Kadec, A. Pełczyński, Bases, lacunary sequences and complemented subspaces in the spaces L_p, Studia Math. 21 (1961/1962) 161–176.
[13] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces. I, in: Sequence Spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin-New York, 1977.
[14] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces. II, in: Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin-New York, 1979. Function spaces.
[15] B. Maurey, G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, Studia Math. 58 (1) (1976) 45–90 (French).
[16] V.N. Temlyakov, The best m-term approximation and greedy algorithms, Adv. Comput. Math. 8 (3) (1998) 249–265.