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Ratio and relative asymptotics of polynomials orthogonal with respect to varying Denisov-type measures

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Abstract

Let μ be a finite positive Borel measure with compact support consisting of an interval $[c, d] \subset \mathbb{R}$ plus a set of isolated points in $\mathbb{R} \backslash [c, d]$, such that $\mu' > 0$ almost everywhere on $[c, d]$. Let $\{w_{2n}\}$, $n \in \mathbb{Z}_+$, be a sequence of polynomials, $\deg w_{2n} \leqslant 2n$, with real coefficients whose zeros lie outside the smallest interval containing the support of μ. We prove ratio and relative asymptotics of sequences of orthogonal polynomials with respect to varying measures of the form $d\mu/w_{2n}$. In particular, we obtain an analogue for varying measures of Denisov's extension of Rakhmanov's theorem on ratio asymptotics. These results on varying measures are applied to obtain ratio asymptotics for orthogonal polynomials with respect to fixed measures

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on the unit circle and for multi-orthogonal polynomials in which the measures involved are of the type described above.

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1. Introduction

1.1. Motivation

Two main developments in the general theory of orthogonal polynomials over the past 25 years are E.A. Rakhmanov's theorem on ratio asymptotics of orthogonal polynomials (see [20–22]) and the extension of Szegő's theory, by A. Maté, P. Nevai and V. Totik, concerning the comparison of two systems of orthogonal polynomials whose measures do not satisfy Szegő's condition (see [15–18]). We recommend the reader to look at Chapters 9 and 13 of Barry Simon's recent excellent monograph [24]. Besides the proofs you will find at the end of each section historical notes with original sources and later developments.

Last year, S. Denisov [7] established an important extension of Rakhmanov's theorem. It includes all measures μ whose support consists of an interval $[c, d]$ on the real line on which $\mu' > 0$ a.e. plus a set of isolated mass points on $\mathbb{R} \setminus [c, d]$. He used operator theoretic arguments. Later, P. Nevai and V. Totik [19] found an alternative proof that does not involve operator theory.

In connection with applications to rational approximation, we have extended these theorems on ratio and relative asymptotics to polynomials orthogonal with respect to varying measures (the measure of orthogonality depends on the degree of the polynomial) with no mass points outside the continuous part of their support. Such results are relevant for the proof of asymptotic properties of orthogonal polynomials with respect to fixed measures as well (see [1–3,5,12,14]).

In this paper, we obtain a version of the Denisov–Rakhmanov theorem on ratio asymptotics for varying measures containing infinitely many mass points outside the continuous part of their support. We also give a result on relative asymptotics for such measures. This is new even when the measures are fixed. Finally, we apply these theorems to obtain some results for polynomials orthogonal with respect to fixed measures on the unit circle and for so-called multi-orthogonal polynomials which share their orthogonality conditions with a system of measures.

1.2. Definitions and statements

Let $\{w_{2n}\}_{n\in\mathbb{N}}$ be a sequence of polynomials with real coefficients such that, for each $n \in \mathbb{N}$, $\deg(w_{2n}) = i_n$, $0 \leqslant i_n \leqslant 2n$. We denote by $\{x_{n,i}\}_{i=1}^{2n}$ the set of zeros of w_{2n} whenever $i_n = 2n$. If $i_n < 2n$, we define $x_{n,i} = \infty$ for $i = 1, \ldots, 2n - i_n$ and denote by $\{x_{n,i}\}_{i=2n-i_n+1}^{2n}$ the set of zeros of w_{2n}. We assume that the zeros are enumerated so that $|x_{n,i}| > |x_{n,i+1}|$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures whose supports $\operatorname{supp}(\mu_n)$ contain infinitely many points and are all contained in a compact set $S \subset \mathbb{R}$. For each $n \in \mathbb{N}$, the polynomial w_{2n} is non-negative on S and

$$\int_S \frac{d\mu_n}{w_{2n}} < \infty.$$

We can construct the table of polynomials $\{l_{n,j}\}$, $\deg l_{n,j} = j$, $j \in \mathbb{Z}^+$, that are orthonormal with respect to $d\mu_n/w_{2n}$; that is, these polynomials have positive leading coefficient and satisfy

$$\int_S l_{n,k} l_{n,s} \frac{d\mu_n}{w_{2n}} = \delta_{k,s}, \quad k, s \in \mathbb{Z}^+,$$

where $\delta_{k,s}$ denotes the Kronecker delta.

Given a finite positive Borel measure μ supported on $\mathbb{R}$, $\mu'(x)$ will stand for the Radon–Nikodym derivative of μ with respect to the Lebesgue measure dx. By $\mu_n \overset{*}{\longrightarrow} \mu$, $n \to \infty$, we denote the weak* convergence of $\{\mu_n\}$ to μ. This means that for every real continuous function f with compact support

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, d\mu_n(x) = \int_{\mathbb{R}} f(x)\, d\mu(x).$$

It is obvious that the support of the measure μ will also be contained in the compact set S. Let $[a, b]$ be any interval of the real line, we will denote by $\Psi_{[a,b]}$ the conformal mapping of $\overline{\mathbb{C}} \setminus [a, b]$ onto $\{|z| > 1\}$, such that $\Psi_{[a,b]}(\infty) = \infty$ and $\Psi'_{[a,b]}(\infty) > 0$, i.e.

$$\Psi_{[a,b]}(x) = \frac{2x-a-b}{b-a} + \sqrt{\left(\frac{2x-a-b}{b-a}\right)^2 - 1},$$

where the square root is taken so that $\sqrt{t} > 0$ for $t > 0$. As an abbreviation, we will denote by $\Psi(x)$ the function $\Psi_{[-1,1]}(x) = x + \sqrt{x^2-1}$.

Let f be a Borel measurable function on $[0, 2\pi]$, such that $\log f \in L^1[0, 2\pi]$. The Szegő function $D(f, \cdot)$ associated with f is given by

$$D(f, z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \log f(t) \frac{e^{it}+z}{e^{it}-z}\, dt \right\}, \quad |z| < 1.$$

Keeping in mind the definitions and notations above, we introduce the following connections between the measures $\{\mu_n\}$ and the polynomials $\{w_{2n}\}$.

Definition 1. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{d\mu_n\}, \{w_{2n}\}, k)$ is admissible on S if

(i) There exists a finite Borel measure μ on $\mathbb{R}$, such that $\mu_n \overset{*}{\longrightarrow} \mu$, $n \to \infty$.

(ii) In case that k is negative, then $\int_S \prod_{i=1}^{-k} \left|1 - x/x_{n,i}\right|^{-1} d\mu_n(x) \leqslant M_k < \infty$, $n \in \mathbb{N}$, where $x/x_{n,i} = 0$ if $x_{n,i} = \infty$.

(iii) $\lim_{n\to\infty} \sum_{i=1}^{2n} \left(1 - |\Psi_{[a,b]}(x_{n,i})|^{-1}\right) = \infty$, where $[a, b]$ is the convex hull of S.

Definition 2. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S if $(\{d\mu_n\}, \{w_{2n}\}, k)$ is admissible on S and

(iv) $\lim_{n\to\infty} \int_S \left|\mu_n'(x) - \mu'(x)\right| \, dx = 0.$

We need to impose certain additional restrictions on the measures μ_n as well as on the set S.

Definition 3. Let $\{\mu_n\}$, $n \in \mathbb{N}$, be a sequence of finite positive Borel measures supported on the compact set $S \subset \mathbb{R}$. We say that $\{\mu_n\}$ is a Denisov-type sequence on S if

(a) There exists a finite positive Borel measure μ, such that $\mathrm{supp}(\mu) = S$ and $\mu_n \xrightarrow{*} \mu,\ n \to \infty$.
(b) $[-1, 1] \subset S$ and for each $\varepsilon > 0$, $S \setminus (-1-\varepsilon, 1+\varepsilon)$ is a finite set.
(c) $\mu'(x) > 0$ a.e. on $[-1, 1]$ and for all sufficiently large n, $\mu_n'(x) > 0$ a.e. on $[-1, 1]$.

In many applications $d\mu_n = h_n\, d\mu$, where $\mathrm{supp}(\mu) = S$ is as in (b) of Definition 3, $\lim_{n\to\infty} h_n = h > 0$ uniformly on S, and the zeros of $\{w_{2n}\}$ lie on a compact set disjoint from S. In this case all the assumptions in these definitions are satisfied if $\mu'(x) > 0$ a.e. on $[-1, 1]$.

From the classical theory of orthogonal polynomials it follows that the polynomials $\{l_{n,n+j}\}$, n, $j \in \mathbb{N}$, are related by the recurrence relations

$$\left.\begin{array}{l} a_{n,n+k-1}\, l_{n,n+k}(x) = (x - b_{n,n+k-1})\, l_{n,n+k-1}(x) - a_{n,n+k-2}\, l_{n,n+k-2}(x), \\ n \geqslant -k+1, \quad l_{n,0} \equiv 1, \quad l_{n,-1} \equiv 0 \end{array}\right\} \tag{1}$$

(notice that the three polynomials appearing in the formula correspond to the same measure). The so-called Jacobi parameters verify $b_{n,j} \in \mathbb{R},\ a_{n,j} > 0$. The monic polynomials are

$$L_{n,j}(x) = \left(a_{n,0} \cdots a_{n,j-1}\right) l_{n,j}(x), \quad n, \quad j \in \mathbb{N}. \tag{2}$$

The following result extends to varying measures Denisov's theorem (see [7,19]) on ratio asymptotics. When the measures have no mass points outside of $[-1, 1]$ it appears as Theorem 6 in [5].

Theorem 1. *Suppose that, for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S and $\{\mu_n\}$ is a Denisov-type sequence on S. Then, for each fixed $k \in \mathbb{Z}$*

$$\left.\begin{array}{rcl} \lim_{n\to\infty} a_{n,n+k} & = & 1/2, \\ \lim_{n\to\infty} b_{n,n+k} & = & 0 \end{array}\right\} \tag{3}$$

and

$$\lim_{n\to\infty} \frac{l_{n,n+k}(x)}{l_{n,n+k-1}(x)} = \Psi(x), \tag{4}$$

uniformly on each compact subsets of $\mathbb{C} \setminus S$.

(4) is a direct consequence of (3) (see [11, Theorem 2.1]). Therefore, we limit ourselves to the proof of (3).

Regarding relative asymptotics, the next result extends Theorem 3.2 of [6] and is new even for the case of fixed measures ($\mu_n = \mu$, $w_{2n} \equiv 1$, $n \in \mathbb{Z}_+$). If there are no mass points outside $[-1, 1]$ the result for fixed measures appears in [18].

Theorem 2. *Suppose that for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, 2k)$ is strongly admissible on S and $\{\mu_n\}$ is a Denisov-type sequence on S. Let h be a non-negative Borel measurable function on S verifying*:

(1) *There exists an algebraic polynomial Q, such that $Qh^{\pm 1} \in L^\infty(S)$.*
(2) *For each $k \in \mathbb{Z}$, $(\{h\, d\mu_n\}, \{w_{2n}\}, 2k)$ is strongly admissible on S.*

Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on S which converges to $g > 0$ uniformly on S. For each $n \in \mathbb{N}$, set $h_n = hg_n$ and let $\{q_{n,m}\}_{m\in\mathbb{N}}$, be the sequence of orthonormal polynomials with respect to $h_n\, d\mu_n/w_{2n}$. Then, for each fixed $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{q_{n,n+k}(x)}{l_{n,n+k}(x)} = \frac{1}{D(\widetilde{h}\widetilde{g}, 1/\Psi(x))},$$

uniformly on compact subsets of $\mathbb{C} \setminus S$, where $\widetilde{h}(\theta) = h(\cos\theta)$ and $\widetilde{g}(\theta) = g(\cos\theta)$, $\theta \in [0, 2\pi]$.

One can obtain the following corollaries on ratio and relative asymptotics of orthogonal polynomials with respect to fixed Denisov-type measures on the unit circle. Corollary 1 is a version of Theorem 13.4.4 of [24]. Corollary 2 is new.

Corollary 1. *Let σ be a finite positive Borel measure on the unit circle Γ whose support S consists of an arc γ plus isolated mass points in $\Gamma \setminus \gamma$. Assume that $\sigma' > 0$ a.e. on γ. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials and $\phi_n(z) = \beta_n z^n + \cdots$, $\beta_n > 0$. Assume that $\sigma' > 0$ a.e. on γ, then*

$$\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = \frac{1}{\mathrm{Cap}(\gamma)},$$

where $\mathrm{Cap}(\gamma)$ *denotes the logarithmic capacity of γ. Moreover*,

$$\lim_{n\to\infty} \frac{\phi_{n+1}(\zeta)}{\phi_n(\zeta)} = G(\zeta),$$

uniformly on each compact subset of $\mathbb{C} \setminus S$, where G denotes the conformal mapping of $\mathbb{C} \setminus \gamma$ onto the exterior of the unit circle, such that $G(\infty) = \infty$ and $G'(\infty) > 0$.

Corollary 2. *Let σ be a finite positive Borel measure on the unit circle Γ whose support S consists of an arc γ plus isolated mass points in $\Gamma\setminus\gamma$. Assume that $\sigma' > 0$ a.e. on γ and let h be a non-negative measurable function on S, such that there exists a polynomial Q for which Qh, $Qh^{-1} \in L_\infty(\sigma)$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n(h;\cdot)\}_{n\in\mathbb{N}}$ be the sequences of orthonormal polynomials with respect to σ and $h\, d\sigma$, where $\phi_n(z) = \beta_n z^n + \cdots$, $\beta_n > 0$, and $\phi_n(h; z) = \beta_n(h) z^n + \cdots$, $\beta_n(h) > 0$. Then*

$$\lim_{n\to\infty} \frac{\phi_n(h;\zeta)}{\phi_n(\zeta)} = D_\gamma(h;\zeta),$$

uniformly on each compact subset of $\mathbb{C} \setminus S$ and

$$\lim_{n\to\infty} \frac{\beta_n(h)}{\beta_n} = D_\gamma(h;\infty),$$

where $D_\gamma(h;\zeta)$ is the unique function which satisfies the conditions:

(i) $D_\gamma(h;\zeta) \in H(\overline{\mathbb{C}} \setminus \gamma)$ *and*

$$\lim_{r\to 1^+} D_\gamma(h; r\zeta) = D_\gamma(h;\zeta_+), \qquad \lim_{r\to 1^-} D_\gamma(h; r\zeta) = D_\gamma(h;\zeta_-),$$

for almost every $\zeta \in \gamma$,

(ii) $D_\gamma(h;\zeta) \neq 0$, $\zeta \in \overline{\mathbb{C}} \setminus \gamma$, $D_\gamma(h;\infty) > 0$, *and*

(iii) $|D_\gamma(h;\zeta_+)|^2 = |D_\gamma(h;\zeta_-)|^2 = \frac{1}{h(\zeta)}$ *almost everywhere on* γ.

The assumptions of Corollary 2 imply that $\log h$ is integrable with respect to the Lebesgue measure on γ. The construction of $D_\gamma(h;\zeta)$ and its uniqueness is easy to reduce by conformal mapping to the case of the unit circle.

We will not prove these two corollaries since they are obtained following step by step the proofs of Theorems 1 and 2 in [2] and using at appropriate places Theorems 1 and 2 stated above, instead of the weaker versions employed in [2]. The basic idea is to translate the problem to the real line by using a bilinear change of variables. The orthogonal polynomials with respect to the measure given on the unit circle are connected with orthogonal polynomials with respect to varying measures on the real line whose varying part depends on the bilinear transformation used.

Another application of Theorems 1 and 2 is to obtain ratio asymptotics of multiple orthogonal polynomials for the so-called Nikishin systems of measures in which the measures involved in the construction are of Denisov type. When the measures do not have mass points outside the interval containing the continuous part of their support the corresponding result was proved in [1]. To avoid introducing at this stage more notation, we leave the statement of these results for the final section.

Section 2 is dedicated to the proof of some auxiliary results for varying measures on the unit circle. In Sections 3 and 4, we prove Theorems 1 and 2, respectively. Section 5 is devoted to some applications. In the sequel, we maintain the notations introduced above.

2. Auxiliary results on the unit circle

In order to prove Theorems 1 and 2, we start out from the unit circle. Here, we give definitions analogous to those of Section 1. Let $\{d\rho_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures on the interval $[0, 2\pi]$, such that for each $n \in \mathbb{N}$ the support of $d\rho_n$ contains an infinite set of points. Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials such that, for each $n \in \mathbb{N}$, W_n has degree n ($\deg W_n = n$) and all its zeros $\{w_{n,i}\}$, $1 \leqslant i \leqslant n$, lie in the closed unit disk. We assume that the indices are taken so that if $w = 0$ is a zero of W_n of degree m then $w_{n,1} = w_{n,2} = \ldots = w_{n,m} = 0$. Set

$$d\sigma_n(\theta) = \frac{d\rho_n(\theta)}{|W_n(z)|^2}, \qquad z = e^{i\theta}.$$

Assume that, for each natural number n, $\int_0^{2\pi} d\sigma_n(\theta) < +\infty$. This assumption guarantees that for each pair (n, m) of natural numbers we can construct a polynomial $\varphi_{n,m}(z) = \beta_{n,m} z^m + \cdots$ that is uniquely determined by the relations of orthogonality

$$\frac{1}{2\pi}\int \bar{z}^j \varphi_{n,m}(z)\, d\sigma_n(\theta) = 0, \quad j = 0, 1, \dots, m-1, \quad z = e^{i\theta},$$
$$\frac{1}{2\pi}\int |\varphi_{n,m}(z)|^2\, d\sigma_n(\theta) = 1, \quad \deg \varphi_{n,m} = m, \quad \beta_{n,m} > 0.$$

Definition 4. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{d\rho_n\}, \{W_n\}, k)$ is admissible on $[0, 2\pi]$ if

(I) There exists a finite Borel measure ρ on $[0, 2\pi]$, such that $\rho_n \xrightarrow{*} \rho$, $n \to \infty$.

(II) In case that k is negative, we have $\int \prod_{i=1}^{-k} |e^{i\theta} - w_{n,i}|^{-2}\, d\rho_n(\theta) \leqslant M_k < +\infty$, for each $n \in \mathbb{N}$.

(III) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty$.

Definition 5. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{d\rho_n\}, \{W_n\}, k)$ is strongly admissible on $[0, 2\pi]$ if $(\{d\rho_n\}, \{W_n\}, k)$ is admissible on $[0, 2\pi]$ and

(IV) $\lim_{n\to\infty} \int_0^{2\pi} |\rho_n'(\theta) - \rho'(\theta)|\, d\theta = 0$.

Let $\Phi_{n,m}(z) = z^m + \cdots = (\beta_{n,m})^{-1}\varphi_{n,m}(z)$ and set $\Phi^*_{n,m}(z) = z^m \overline{\Phi_{n,m}}(1/z)$. The next formula is a simple reformulation of a known result (notice that n is fixed) and its proof may be found in [15]. For all $n, m \in \mathbb{N}$ we have

$$|\Phi_{n,m+1}(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left| \left| \frac{\varphi_{n,m}(z)}{\varphi_{n,m+1}(z)} \right|^2 - 1 \right| d\theta, \quad z = e^{i\theta}. \tag{5}$$

The next lemma is Theorem 1 of [5].

Lemma 1. *Let* $(\{d\rho_n\}, \{W_n\}, k)$ *be admissible on* $[0, 2\pi]$, *then*

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\, d\theta \xrightarrow{*} d\rho(\theta), \quad n \in \mathbb{N}, \quad z = e^{i\theta}. \tag{6}$$

Given a Borel set $B \subset \mathbb{R}$, $|B|$ stands for the Lebesgue measure of B. In the proof of the following lemma, we follow the arguments used in [7] to prove a statement similar to (7).

Lemma 2. *Assume that* $(\{d\rho_n\}, \{W_n\}, k)$ *is strongly admissible on* $[0, 2\pi]$ *for all* $k \in \mathbb{Z}$. *Set* $\widetilde{\mathcal{K}} = \{\theta \in [0, 2\pi] : \rho'(\theta) > 0\}$. *If* $|\widetilde{\mathcal{K}}| \geqslant 2\pi - \widetilde{\delta}$, *then, for each fixed integer k*

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left| \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 - 1 \right| d\theta \leqslant \widetilde{\mathcal{L}}_1(\widetilde{\delta}), \tag{7}$$

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left| \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right|^2 \rho_n'(\theta) - 1 \right| d\theta \leqslant \widetilde{\mathcal{L}}_2(\widetilde{\delta}) \tag{8}$$

and for each $f \in L^\infty([0,2\pi])$

$$\limsup_{n\to\infty} \left| \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right|^2 d\rho_n(\theta) - \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta \right| \leqslant \|f\|_{[0,2\pi]} \widetilde{\mathcal{L}}_3(\widetilde{\delta}), \tag{9}$$

where $\widetilde{\mathcal{L}}_i(\widetilde{\delta})$ *tend to* 0 *as* $\widetilde{\delta}$ *tends to* 0, $i = 1, 2, 3$, *and* $z = e^{i\theta}$.

Proof. Set $z = e^{i\theta}$. Notice that, for each fixed $k \in \mathbb{Z}$, we have

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left| \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 - 1 \right| d\theta \right)^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left(\left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| - 1 \right)^2 d\theta \times \frac{1}{2\pi}\int_0^{2\pi} \left(\left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| + 1 \right)^2 d\theta.$$

Since

$$\left(\left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| \pm 1 \right)^2 = \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 \pm 2 \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| + 1, \tag{10}$$

integrating (10) and using (see [9, (1.20)])

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left| \varphi_{n,n+k}(z) \right|^2 d\sigma_n(\theta) = 1,$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| \pm 1 \right)^2 d\theta = 2 \pm \frac{2}{2\pi}\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| d\theta, \tag{11}$$

where

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| d\theta \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 d\theta \right)^{1/2} = 1. \tag{12}$$

Taking (11)–(12) into account and using the Cauchy–Schwarz inequality, it follows that

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left| \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right|^2 - 1 \right| d\theta \right)^2 \leqslant 8 \left(1 - \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| d\theta \right). \tag{13}$$

Using twice the Cauchy–Schwarz inequality, we have

$$\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right| \rho_n'(\theta)^{1/2}\, d\theta \leqslant \left(\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)} \right| d\theta \right)^{1/2} \times \left(\int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right|^2 \rho_n'(\theta)\, d\theta \right)^{1/4} \times \left(\int_0^{2\pi} \left| \frac{\varphi_{n,n+k+1}(z)}{W_n(z)} \right|^2 \rho_n'(\theta)\, d\theta \right)^{1/4}.$$

From (13) and

$$\int_0^{2\pi} \left|\frac{\varphi_{n,m}(z)}{W_n(z)}\right|^2 \rho_n'(\theta)\,d\theta \leqslant \int_0^{2\pi} \left|\varphi_{n,m}(z)\right|^2 \frac{d\rho_n(\theta)}{|W_n(z)|^2} = 2\pi, \tag{14}$$

it follows that

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|\left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+1}(z)}\right|^2 - 1\right| d\theta$$
$$\leqslant 8^{1/2}\left(1-\liminf_{n\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\right)^2\right)^{1/2}. \tag{15}$$

Consider an arbitrary non-negative continuous function f defined on $[0, 2\pi]$. Then

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left(f(\theta)\rho_n'(\theta)\right)^{1/4}\,d\theta\right)^2 \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\right)$$
$$\times\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right| f(\theta)^{1/2}\,d\theta\right).$$

Using (6) and the condition (IV) of strong admissibility, we obtain

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left(f(\theta)\rho'(\theta)\right)^{1/4}\,d\theta\right)^4 \leqslant \liminf_{n\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\right)^2$$
$$\times\left(\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\rho(\theta)\right).$$

From [16, Theorem 3.2] (cf. also Corollary 3.3), we get

$$\liminf_{n\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\right)^2 \geqslant \left(\frac{|\widetilde{\mathcal{K}}|}{2\pi}\right)^3 \geqslant \left(1-\frac{\widetilde{\delta}}{2\pi}\right)^3. \tag{16}$$

Taking into account (15), we have proved (7) with

$$\widetilde{\mathcal{L}}_1(\widetilde{\delta}) := 8^{1/2}\left(1-\left(1-\frac{\widetilde{\delta}}{2\pi}\right)^3\right)^{1/2}.$$

To prove (8), we use that

$$\left(\int_0^{2\pi}\left|\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)-1\right|d\theta\right)^2 \leqslant \int_0^{2\pi}\left(\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}-1\right)^2 d\theta$$
$$\times\int_0^{2\pi}\left(\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}+1\right)^2 d\theta,$$

where

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\pm 1\right)^2 d\theta \leqslant 2 \pm \frac{2}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta$$

and (see (14))

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\leqslant 1.$$

Hence,

$$\frac{8^{-\frac{1}{2}}}{2\pi}\int_0^{2\pi}\left|\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)-1\right|d\theta\leqslant\left(1-\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\rho_n'(\theta)^{1/2}\,d\theta\right)^{\frac{1}{2}}.$$

Taking limit, as n tends to infinity, and using (16), we obtain (8) with

$$\widetilde{\mathcal{L}}_2(\widetilde{\delta}):=8^{1/2}\left(1-\left(1-\frac{\widetilde{\delta}}{2\pi}\right)^{3/2}\right)^{1/2}.$$

Finally, we prove (9). Taking $m=n+k$ on the right-hand side of (14), we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\rho_n^{(s)}(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\left(1-\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)\right)d\theta,$$

where $d\rho_n^{(s)}(\theta)$ stands for the singular part of $d\rho_n(\theta)$ with respect to the Lebesgue measure. Letting n tend to infinity, and using (8), we obtain

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\rho_n^{(s)}(\theta)\leqslant\widetilde{\mathcal{L}}_2(\widetilde{\delta}). \tag{17}$$

Then, for any $f\in L^\infty([0,2\pi])$,

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\rho_n(\theta)-\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta\right|\\&\quad\leqslant\limsup_{n\to\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\rho_n^{(s)}(\theta)\right|\\&\qquad+\limsup_{n\to\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)\,d\theta-\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta\right|,\end{aligned}$$

which proves (9) using (8) and (17). □

Lemma 3. *Under the same conditions as in Lemma* 2, *for each fixed* $k\in\mathbb{Z}$, *we have*

(i) *On* $\{z:|z|<1\}$

$$\limsup_{n\to\infty}\left|\frac{\varphi_{n,n+k}(z)}{\varphi^*_{n,n+k}(z)}\right|\leqslant\widetilde{\mathcal{L}}_4(z,\widetilde{\delta}), \tag{18}$$

where $\widetilde{\mathcal{L}}_4(z,\widetilde{\delta})$ *tends to* 0 *as* $\widetilde{\delta}$ *tends to* 0, *uniformly on compact subsets of* $\{z:|z|<1\}$.

(ii)

$$\limsup_{n\to\infty} \left\| \left| \frac{\varphi_{n,n+k+m}(z)}{\varphi_{n,n+k}(z)} \right|^2 - 1 \right\|_{|z|=1} \leqslant \widetilde{\mathcal{L}}_6(m, \widetilde{\delta}), \tag{19}$$

where for each fixed $m \in \mathbb{Z}$, $\widetilde{\mathcal{L}}_6(m,\widetilde{\delta})$ *tends to* 0 *as* $\widetilde{\delta}$ *tends to* 0 *and* $\|\cdot\|_{|z|=1}$ *denotes the uniform norm on the unit circle.*

Proof. Consider the well-known formulas

$$\Phi_{n,n+k+1}(z) = z\,\Phi_{n,n+k}(z) + \Phi_{n,n+k+1}(0)\Phi^*_{n,n+k}(z), \quad n \geqslant -k \tag{20}$$

and

$$\Phi^*_{n,n+k+1}(z) = \Phi^*_{n,n+k}(z) + \overline{\Phi_{n,n+k+1}(0)}z\Phi_{n,n+k}(z), \quad n \geqslant -k.$$

Dividing one by the other, it follows that

$$\frac{\Phi_{n,n+k+1}(z)}{\Phi^*_{n,n+k+1}(z)} := \xi_{n,n+k+1}(z) = \frac{z\xi_{n,n+k}(z) + \Phi_{n,n+k+1}(0)}{1 + \overline{\Phi_{n,n+k+1}(0)}z\xi_{n,n+k}(z)}, \quad |z| \leqslant 1.$$

Since $|\Phi_{n,m}(0)| \leqslant 1$ and $|\xi_{n,m}(z)| \leqslant 1$, $m \in \mathbb{N}$, for $|z| \leqslant \frac{1}{4}$ we obtain

$$|\xi_{n,n+k+1}(z)| \leqslant \tfrac{1}{3}|\xi_{n,n+k}(z)| + \tfrac{4}{3}|\Phi_{n,n+k+1}(0)|.$$

Applying this inequality N times, we obtain

$$\begin{aligned} |\xi_{n,n+k+1}(z)| \leqslant{}& \left(\tfrac{1}{3}\right)^N |\xi_{n,n+k-N+1}(z)| + \left(\tfrac{1}{3}\right)^{N-1} \tfrac{4}{3}|\Phi_{n,n+k-N+2}(0)| \\ &+ \left(\tfrac{1}{3}\right)^{N-2} \tfrac{4}{3}|\Phi_{n,n+k-N+3}(0)| + \cdots + \tfrac{4}{3}|\Phi_{n,n+k+1}(0)|, \quad |z| \leqslant 1/4. \end{aligned}$$

Take N sufficiently large so that $(\frac{1}{3})^N \leqslant \widetilde{\mathcal{L}}_1(\widetilde{\delta})$. Let $N_1 \geqslant \max\{N, -k\}$ be such that for all $n \geqslant N_1$ and $i = 2, \ldots, N+1$

$$|\Phi_{n,n+k-N+i}(0)| \leqslant 2\widetilde{\mathcal{L}}_1(\widetilde{\delta}).$$

Then, for all $n \geqslant N_1$ and $|z| \leqslant \frac{1}{4}$

$$|\xi_{n,n+k+1}(z)| \leqslant \widetilde{\mathcal{L}}_1(\widetilde{\delta}) + \left((\tfrac{1}{3})^{N-1} + (\tfrac{1}{3})^{N-2} + \cdots + 1 \right) 8\widetilde{\mathcal{L}}_1(\widetilde{\delta})/3 \leqslant 5\widetilde{\mathcal{L}}_1(\widetilde{\delta})$$

which gives (18) if the compact set is contained in $\{z : |z| \leqslant \frac{1}{4}\}$. Since $|\xi_{n,n+k+1}(z)| \leqslant 1$ on $\{z : |z| \leqslant 1\}$ and is analytic in the ring $\{z : \frac{1}{4} \leqslant |z| \leqslant 1\}$, from the two constants theorem, it follows that

$$|\xi_{n,n+k+1}(z)| \leqslant \left(5\widetilde{\mathcal{L}}_1(\widetilde{\delta})\right)^{-\log|z|/\log 4}$$

which completes the proof of part (i).

Let us prove (ii). Rewrite (20) as

$$\frac{\Phi_{n,n+k+1}(z)}{z\Phi_{n,n+k}(z)} - 1 = \frac{\Phi^*_{n,n+k}(z)}{z\Phi_{n,n+k}(z)}\Phi_{n,n+k+1}(0), \quad |z| \geqslant 1.$$

Use (5), (7), and the fact that $\Phi^*_{n,n+k}(z)/(z\,\Phi_{n,n+k}(z))$ is an analytic function in the region $\{z : |z| \geqslant 1\}$, such that

$$\left|\frac{\Phi^*_{n,n+k}(z)}{z\Phi_{n,n+k}(z)}\right| = 1 \quad \text{whenever } |z| = 1,$$

to obtain

$$\limsup_{n\to\infty}\left|\frac{\Phi_{n,n+k+1}(z)}{z\,\Phi_{n,n+k}(z)} - 1\right| \leqslant \limsup_{n\to\infty}\left|\Phi_{n,n+k+1}(0)\right| \leqslant \widetilde{\mathcal{L}}_1(\widetilde{\delta}), \quad |z| \geqslant 1.$$

In addition to this, we have

$$\begin{aligned}\frac{\varphi_{n,n+k+1}(z)}{z\varphi_{n,n+k}(z)} &= \frac{1}{\sqrt{1-\left|\Phi_{n,n+k+1}(0)\right|^2}}\left(\frac{\Phi_{n,n+k+1}(z)}{z\Phi_{n,n+k}(z)} - 1\right)\\ &\quad + \frac{1}{\sqrt{1-\left|\Phi_{n,n+k+1}(0)\right|^2}}, \quad n \geqslant -k.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\left|\frac{\varphi_{n,n+k+1}(z)}{z\,\varphi_{n,n+k}(z)} - 1\right| \leqslant \frac{1}{\sqrt{1-\widetilde{\mathcal{L}}_1(\widetilde{\delta})^2}}\widetilde{\mathcal{L}}_1(\widetilde{\delta}) + \frac{1-\sqrt{1-\widetilde{\mathcal{L}}_1(\widetilde{\delta})^2}}{\sqrt{1-\widetilde{\mathcal{L}}_1(\widetilde{\delta})^2}}.$$

Therefore

$$\limsup_{n\to\infty}\left|\frac{\varphi_{n,n+k+1}(z)}{z\,\varphi_{n,n+k}(z)} - 1\right| \leqslant \widetilde{\mathcal{L}}_5(\widetilde{\delta}),$$

where $\widetilde{\mathcal{L}}_5(\widetilde{\delta})$ tends to 0 as $\widetilde{\delta}$ tends to 0. Hence,

$$1 - \widetilde{\mathcal{L}}_5(\widetilde{\delta}) \leqslant \liminf_{n\to\infty}\left|\frac{\varphi_{n,n+k+1}(z)}{z\varphi_{n,n+k}(z)}\right| \leqslant \limsup_{n\to\infty}\left|\frac{\varphi_{n,n+k+1}(z)}{z\varphi_{n,n+k}(z)}\right| \leqslant 1 + \widetilde{\mathcal{L}}_5(\widetilde{\delta})$$

for $|z| \geqslant 1$. In particular, for $|z| = 1$, we have

$$\begin{aligned}0 \leqslant u(\widetilde{\delta}, m) &\leqslant \liminf_{n\to\infty}\left|\frac{\varphi_{n,n+k+m}(z)}{\varphi_{n,n+k}(z)}\right|^2\\ &\leqslant \limsup_{n\to\infty}\left|\frac{\varphi_{n,n+k+m}(z)}{\varphi_{n,n+k}(z)}\right|^2 \leqslant \left(1 + \widetilde{\mathcal{L}}_5(\widetilde{\delta})\right)^{2m}\end{aligned}$$

for each fixed $m \in \mathbb{Z}^+$, where $u(\widetilde{\delta}, m)$ tends to 1 as $\widetilde{\delta}$ tends to 0. Then

$$u(\widetilde{\delta}, m) - 1 \leqslant \liminf_{n\to\infty} \left(\left| \frac{\varphi_{n,n+k+m}(z)}{\varphi_{n,n+k}(z)} \right|^2 - 1 \right) \leqslant \limsup_{n\to\infty} \left(\left| \frac{\varphi_{n,n+k+m}(z)}{\varphi_{n,n+k}(z)} \right|^2 - 1 \right) \leqslant \left(1 + \widetilde{\mathcal{L}}_5(\widetilde{\delta})\right)^{2m} - 1$$

for each fixed $m \in \mathbb{Z}^+$. Therefore, we have proved (19) for each fixed $m \in \mathbb{Z}^+$. An analogous argument may be used for $m \in \mathbb{Z}^-$. $\square$

The next lemma plays a key role in the proof of Theorem 2. We use the fact that given an arbitrary Riemann integrable function $f(\theta)$ on $[0, 2\pi]$, for each $\eta > 0$ (see [25, Theorem 1.5.4]), there exist two trigonometrical polynomials $R_m(\theta)$ and $T_m(\theta)$ of the same degree m (m depends on η), such that

$$\inf_{\theta\in[0,2\pi]} f(\theta) - \eta \leqslant R_m(\theta) \leqslant f(\theta) \leqslant T_m(\theta) \leqslant \sup_{\theta\in[0,2\pi]} f(\theta) + \eta \tag{21}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} (T_m(\theta) - R_m(\theta))\, d\theta < \eta. \tag{22}$$

In parts (i) and (ii) of Lemma 4, $\{\psi_{n,m}\}_{m\in\mathbb{N}}$ stands for the sequence of orthonormal polynomials with respect to $\widetilde{h}_n\, d\sigma_n$.

Lemma 4. *Let $\{d\rho_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures supported on the compact set $\widetilde{S} \subset [0, 2\pi]$. Let $\widetilde{h}$ be a non-negative Borel measurable function on $\widetilde{S}$. Let $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions that converges to $\widetilde{g}(\theta) > 0$ uniformly on $\widetilde{S}$. Set $\widetilde{h}_n(\theta) = \widetilde{g}_n(\theta)\,\widetilde{h}(\theta)$, $\theta \in \widetilde{S}$, $n \in \mathbb{N}$.*

(i) *Suppose that $(\{\widetilde{h}\, d\rho_n\}, \{W_n\}, k)$ is strongly admissible on $[0, 2\pi]$ for each $k \in \mathbb{Z}$. Set $\widetilde{\mathcal{K}} = \{\theta \in \widetilde{S} : \rho'(\theta) > 0\}$ with $|\widetilde{\mathcal{K}}| \geqslant 2\pi - \widetilde{\delta}$. Assume that there exists an algebraic polynomial $\widetilde{Q}(z)$, such that $\widetilde{Q}(e^{i\theta})\widetilde{h}(\theta)^{-1} \in L^\infty(\widetilde{S})$. Then, for each $\eta > 0$, for any Riemann integrable function f on $[0, 2\pi]$, and each fixed $k \in \mathbb{Z}$,*

$$\limsup_{n\to\infty} \left| \frac{1}{2\pi}\int_0^{2\pi} f(\theta)|\widetilde{Q}(z)|^2 \left| \frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)} \right|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta \right| \leqslant \widetilde{\mathcal{L}}_7(\widetilde{\delta}, \eta, f), \quad z = e^{i\theta}. \tag{23}$$

(ii) *Suppose that $(\{d\rho_n\}, \{W_n\}, k)$ is strongly admissible on $[0, 2\pi]$ for each $k \in \mathbb{Z}$. Set $\widetilde{\mathcal{K}} = \{\theta \in \widetilde{S} : \rho'(\theta) > 0\}$ with $|\widetilde{\mathcal{K}}| \geqslant 2\pi - \widetilde{\delta}$. Assume that there exists an algebraic polynomial $\widetilde{Q}(z)$, such that $\widetilde{Q}(e^{i\theta})\widetilde{h}(\theta) \in L^\infty(\widetilde{S})$. Then, for each $\eta > 0$, for any Riemann integrable* *function f on* $[0, 2\pi]$, *and each fixed* $k \in \mathbb{Z}$,

$$\limsup_{n\to\infty} \left| \int_0^{2\pi} f(\theta)|\widetilde{Q}(z)|^2 \left|\frac{\varphi_{n,n+k}(z)}{\psi_{n,n+k}(z)}\right|^2 d\theta - \int_0^{2\pi} f(\theta)|\widetilde{Q}(z)|^2 \widetilde{h}(\theta)\,\widetilde{g}(\theta)\,d\theta \right| \leqslant 2\pi\,\widetilde{\mathcal{L}}_8(\widetilde{\delta}, \eta, f), \quad z = e^{i\theta}. \tag{24}$$

Each bound $\widetilde{\mathcal{L}}_i(\widetilde{\delta}, \eta, f)$ *tends to* $C_i\eta$ *as* $\widetilde{\delta}$ *tends to* 0, *where* $C_i \geqslant 0$ *is a constant*, $i = 7, 8$.

Proof. We will only prove part (i), since part (ii) is deduced analogously. Set $z = e^{i\theta}$. From hypothesis we know that $(\{\widetilde{h}_n\, d\rho_n\}, \{W_n\}, k)$ is strongly admissible on $[0, 2\pi]$ for each $k \in \mathbb{Z}$. We can assume that $\widetilde{h}_n(\theta) \geqslant 0$ for each $\theta \in \widetilde{S}$ and $\widetilde{Q}(e^{i\theta})\widetilde{h}_n(\theta)^{-1} \in L^\infty(\widetilde{S})$, $n \in \mathbb{N}$.

On one hand, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |\widetilde{Q}(z)|^2 \left|\psi_{n,n+k}(z)\right|^2 d\sigma_n(\theta)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}_n(\theta)} - \frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\right) \left|\psi_{n,n+k}(z)\right|^2 \widetilde{h}_n(\theta)\, d\sigma_n(\theta)\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)} \left|\psi_{n,n+k}(z)\right|^2 \widetilde{h}_n(\theta)\, d\sigma_n(\theta).
\end{aligned}$$

Take limit, as n tends to infinity, in the expression above. Using the orthonormality of the sequence $\{\psi_{n,m}\}_{m\in\mathbb{N}}$ and the convergence of the sequence $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ it follows that

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{Q}(z)|^2 \left|\psi_{n,n+k}(z)\right|^2 d\sigma_n(\theta) \leqslant \left\| \frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}} \right\|_{\widetilde{S}}. \tag{25}$$

On the other hand, fix $\eta > 0$ and use (21) to obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f(\theta)|\widetilde{Q}(z)|^2 \left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta\\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2 \left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} \frac{R_m(\theta)|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta.
\end{aligned} \tag{26}$$

Furthermore, for any $j \in \mathbb{Z}$

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2 \left|\psi_{n,n+j}(z)\right|^2 d\sigma_n(\theta)\\
&= \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)} |\psi_{n,n+j}(z)|^2 \widetilde{h}_n(\theta)\, d\sigma_n(\theta)\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\left(\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}_n(\theta)} - \frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\right) |\psi_{n,n+j}(z)|^2 \widetilde{h}_n(\theta)\, d\sigma_n(\theta)\\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)} |\psi_{n,n+j}(z)|^2 \widetilde{h}_n(\theta)\, d\sigma_n(\theta) + \left\| \frac{|\widetilde{Q}|^2}{\widetilde{h}\widetilde{g}_n} - \frac{|\widetilde{Q}|^2}{\widetilde{h}\widetilde{g}} \right\|_{\widetilde{S}} \|T_m\|_{\widetilde{S}}.
\end{aligned}$$

Therefore, on account of (9) and the convergence of the sequence $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$, we have

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2|\psi_{n,n+j}(z)|^2\,d\sigma_n(\theta) \leqslant \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\,d\theta + \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}} \|T_m\|_{[0,2\pi]}\widetilde{\mathcal{L}}_3(\widetilde{\delta}). \tag{27}$$

Now, let us consider the first term on the right-hand side of (26). Set $q=\deg(\widetilde{Q})$, then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \\
&\qquad+\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2\left(\left|\frac{\psi_{n,n+k}(z)}{\psi_{n,n+k-m-q}(z)}\right|^2-1\right) d\theta \\
&\quad\leqslant \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \\
&\qquad+\left\|\left|\frac{\psi_{n,n+k}}{\psi_{n,n+k-m-q}}\right|^2-1\right\|_{[0,2\pi]}\frac{1}{2\pi}\int_0^{2\pi}|T_m(\theta)|\,|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta.
\end{aligned}$$

Now, we take limit, as n tends to infinity, in the expression above and use (19) and (21) to obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \\
&\quad\leqslant \widetilde{\mathcal{L}}_6(m+q,\widetilde{\delta})\left(\|T_m\|_{[0,2\pi]}\right)\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \\
&\qquad+\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta.
\end{aligned}\tag{28}$$

Since (see [9, (1.20)])

$$\int_0^{2\pi}|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta = \int_0^{2\pi}|\widetilde{Q}(z)|^2\left|\psi_{n,n+k-m-q}(z)\right|^2 d\sigma_n(\theta),$$

from (25) it follows that

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \leqslant \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}}. \tag{29}$$

Analogously, using (27) instead of (25), we have

$$\limsup_{n\to\infty}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2\left|\frac{\psi_{n,n+k-m-q}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\,d\theta + \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}}\|T_m\|_{[0,2\pi]}\widetilde{\mathcal{L}}_3(\widetilde{\delta}). \tag{30}$$

From (28)–(30) it follows that

$$\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\widetilde{Q}(z)|^2 \left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta$$
$$\leqslant \widetilde{\mathcal{L}}_6(m+q,\widetilde{\delta})\|T_m\|_{[0,2\pi]} \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}}$$
$$+\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta + \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}} \|T_m\|_{[0,2\pi]}\widetilde{\mathcal{L}}_3(\widetilde{\delta})$$
$$= \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta + \widetilde{\mathcal{L}}'_6(m+q,\widetilde{\delta})\|T_m\|_{[0,2\pi]}, \tag{31}$$

where $\widetilde{\mathcal{L}}'_6(m+q,\widetilde{\delta})$ tends to 0 as $\widetilde{\delta}$ tends to 0. Finally, from (26), (31), and (22) we conclude that

$$\limsup_{n\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} f(\theta)|\widetilde{Q}(z)|^2 \left|\frac{\psi_{n,n+k}(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta\right)$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} (T_m(\theta)-R_m(\theta))\,\frac{|\widetilde{Q}(z)|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\, d\theta + \widetilde{\mathcal{L}}'_6(m+q,\widetilde{\delta})\|T_m\|_{[0,2\pi]}$$
$$\leqslant \left\|\frac{|\widetilde{Q}|^2}{\widetilde{h}\widetilde{g}}\right\|_{\widetilde{S}} \eta + \widetilde{\mathcal{L}}'_6(m+q,\widetilde{\delta})\left(\sup_{\theta\in[0,2\pi]} |f(\theta)|+\eta\right). \qquad \square$$

3. Proof of Theorem 1

As pointed out after the statement of Theorem 1, in proving this result we can limit ourselves to the proof of (3). Let us begin with some elementary facts. Fix $k \in \mathbb{Z}$. The $n+k$ simple zeros of the monic orthogonal polynomial $L_{n,n+k}$ lie in the smallest interval containing the support of the measure $d\mu_n/w_{2n}$ with respect to which it is orthogonal. Moreover, between two consecutive mass points of μ_n contained in $S \setminus [-1, 1]$ there may be at most one zero of $L_{n,n+k}$. These are well-known properties of polynomials orthogonal with respect to a fixed measure, and nothing changes here because the parameter n is fixed. Let $x_1^{(n)} < \cdots < x_{n+k}^{(n)}$ be the zeros of $L_{n,n+k}$.

Lemma 5. *We have*

$$\int \frac{p(x)}{w_{2n}(x)}\, d\mu_n(x) = \sum_{j=0}^{n+k} \lambda_{n,j}\frac{p(x_j^{(n)})}{w_{2n}(x_j^{(n)})}$$

for any polynomial p of degree $\leqslant 2n+2k-1$, where

$$\lambda_{n,j} = \int \left(\frac{L_{n,n+k}(x)}{L'_{n,n+k}(x_j^{(n)})(x-x_j^{(n)})}\right)^2 \frac{w_{2n}(x_j^{(n)})\, d\mu_n(x)}{w_{2n}(x)} > 0.$$

Proof. Since n is fixed, the Gauss–Jacobi quadrature formula gives

$$\int p(x)\frac{d\mu_n(x)}{w_{2n}(x)} = \sum_{j=0}^{n+k} \Lambda_{n,j}\, p(x_j^{(n)})$$

with

$$\Lambda_{n,j} = \int \left(\frac{L_{n,n+k}(x)}{L'_{n,n+k}(x_j^{(n)})(x - x_j^{(n)})} \right)^2 \frac{d\mu_n(x)}{w_{2n}(x)}.$$

We only have to take $\lambda_{n,j} = w_{2n}(x_j^{(n)})\Lambda_{n,j}$ and observe that for each j, $w_{2n}(x_j^{(n)})$ has the same sign as $w_{2n}(x)$, $x \in S$. $\square$

From this lemma, we obtain.

Lemma 6. *Suppose that, for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, k)$ is admissible on S. Then, for any function f, continuous on the convex hull $[a, b]$ of S*

$$\int f(x)\, d\mu(x) = \lim_{n\to\infty} \sum_{j=0}^{n+k} \lambda_{n,j} f(x_j^{(n)}).$$

Proof. Using the quadrature formula, it follows that

$$\begin{aligned} &\int f(x)\, d\mu(x) - \sum_{j=0}^{n+k} \lambda_{n,j} f(x_j^{(n)}) \\ &\quad = \int f(x)\, d\mu(x) - \int f(x)\, d\mu_n(x) + \int \left(f(x) - \frac{p(x)}{w_{2n}(x)} \right) d\mu_n(x) \\ &\qquad + \sum_{j=0}^{n+k} \left(\frac{p(x_j^{(n)})}{w_{2n}(x_j^{(n)})} - f(x_j^{(n)}) \right) \lambda_{n,j}, \end{aligned}$$

where $\deg p \leqslant 2n + 2k - 1$.

It is well known that the condition (iii) of admissibility implies that the rational functions of the form p/w_{2n} are dense in the space of continuous functions on $[a, b]$ (see, for example, [4, Corollary 1]). Let $\varepsilon > 0$ be arbitrary. Take p/w_{2n} so that $|f(x) - p(x)/w_{2n}(x)| < \varepsilon$, $x \in [a, b]$. From the previous equality we see that

$$\begin{aligned} &\left| \int f(x)\, d\mu(x) - \sum_{j=0}^{n+k} \lambda_{n,j} f(x_j^{(n)}) \right| \\ &\quad \leqslant \left| \int f(x)\, d\mu(x) - \int f(x)\, d\mu_n(x) \right| + \varepsilon \left(\int d\mu_n(x) + \sum_{j=0}^{n+k} \lambda_{n,j} \right). \end{aligned}$$

From the condition (i) of admissibility, it follows that

$$\limsup_{n\to\infty} \left| \int f(x)\, d\mu(x) - \sum_{j=0}^{n+k} \lambda_{n,j} f(x_j^{(n)}) \right| \leqslant \varepsilon \left(1 + \int d\mu(x) + \limsup_{n\to\infty} \sum_{j=0}^{n+k} \lambda_{n,j} \right).$$

If $k > 0$ one can take $p = w_{2n}$ in Lemma 5 and get $\sum_{j=0}^{n+k} \lambda_{n,j} = \int d\mu_n$. When $k \leqslant 0$, using the quadrature formula we can still eliminate $2n + 2k - 2$ factors of w_{2n} and from (ii) it follows that

$$\sum_{j=0}^{n+k} \lambda_{n,j} \leqslant M_{2k-2} \left(\sup\{1 + |x/y| : x \in [a, b],\, y \in \mathbb{R} \setminus [a, b]\} < \infty\right)^{2-2k} .$$

This and the inequality above complete the proof taking into consideration that $\varepsilon > 0$ is arbitrary. □

Remark 1. An immediate consequence of Lemma 6 is that each point in $\operatorname{supp}(\mu) \setminus [-1, 1]$ is a limit point of zeros of the sequence of orthogonal polynomials $\{l_{n,n+k}\}_{n\in\mathbb{N}}$. To prove this, take a small neighborhood of a mass point of μ in $\operatorname{supp}(\mu) \setminus [-1, 1]$ containing no other mass points of μ and assume that there exists a subsequence Λ of indices for which the polynomials $\{l_{n,n+k}\}_{n\in\Lambda}$ have no zeros in the prescribed neighborhood. Take a continuous function *f*, positive on the chosen neighborhood and equal to zero outside. Applying Lemma 6 to such an *f* we obtain a contradiction. This observation is used in the proof of Lemma 9 below.

Let us use the well-known connection between measures supported on $[-1, 1]$ and on $[0, 2\pi]$. Let α be a finite positive Borel measure on $[-1, 1]$ and let ρ be the measure supported on $[0, 2\pi]$ given by

$$\rho(\theta) = \begin{cases} -\alpha(\cos\theta), & \theta \in [0, \pi], \\ \alpha(\cos\theta), & \theta \in [\pi, 2\pi]. \end{cases} \tag{32}$$

Since w_{2n} is non-negative on $[-1, 1]$, there exists an algebraic polynomial (see [25, p. 3]) $W'_{2n}(z)$, $\deg(W'_{2n}) = i_n$, whose zeros lie in $\{|z| \leqslant 1\}$, such that

$$w_{2n}(\cos\theta) = |W'_{2n}(e^{i\theta})|^2, \quad \theta \in [0, 2\pi].$$

Take $W_{2n}(z) = z^{2n-i_n} W'_{2n}(z)$. Then, $\deg W_{2n} = 2n$ and

$$w_{2n}(\cos\theta) = |W_{2n}(e^{i\theta})|^2, \quad \theta \in [0, 2\pi].$$

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures supported on $[-1, 1]$ and $\{\rho_n\}_{n\in\mathbb{N}}$ the corresponding measures on $[0, 2\pi]$ given by (32). Set $d\tau_n = d\alpha_n / w_{2n}$ and $d\sigma_{2n} = d\rho_n / |W_{2n}|^2$. Then, τ_n and σ_{2n} are also connected by formulas similar to (32). Let us denote by $\{\Phi_{2n,m}\}_{m\in\mathbb{N}}$ and $\{\varphi_{2n,m}\}_{m\in\mathbb{N}}$, the sequences of monic orthogonal polynomials and orthonormal polynomials, respectively, with respect to $d\sigma_{2n}$. It is well known that (see [25, Theorem 11.5])

$$\left.\begin{aligned} a_m^{(\tau_n)} &= \frac{1}{2}\sqrt{\left(1 - \Phi_{2n,2m}(0)\right)\left(1 - \left(\Phi_{2n,2m-1}(0)\right)^2\right)\left(1 + \Phi_{2n,2m-2}(0)\right)}, \\ b_m^{(\tau_n)} &= \tfrac{1}{2}\Phi_{2n,2m-1}(0)\left(1 - \Phi_{2n,2m}(0)\right) - \tfrac{1}{2}\Phi_{2n,2m+1}(0)\left(1 + \Phi_{2n,2m}(0)\right), \end{aligned}\right\} \tag{33}$$

where $\{a_m^{(\tau_n)}\}_{m\in\mathbb{N}}$ and $\{b_m^{(\tau_n)}\}_{m\in\mathbb{N}}$ are the sequences of Jacobi parameters of the measure τ_n, $n \in \mathbb{N}$.

As a consequence of Lemma 2, we have

Lemma 7. *Suppose that* $(\{d\alpha_n\}, \{w_{2n}\}, k)$ *is strongly admissible on the set* $S \subset [-1, 1]$ *for all* $k \in \mathbb{Z}$. *Set* $\mathcal{K} = \{x \in [-1, 1] : \alpha'(x) > 0\}$, *where* α *is the weak** *limit of* $\{\alpha_n\}$. *Assume that* $|\mathcal{K}| \geqslant 2 - \delta$. *Then, for each fixed* $k \in \mathbb{Z}$

$$\limsup_{n\to\infty} \left(\left| a_{n+k}^{(\tau_n)} - \frac{1}{2} \right| + \left| b_{n+k}^{(\tau_n)} \right| \right) \leqslant \mathcal{L}(\delta), \tag{34}$$

where $\mathcal{L}(\delta)$ *tends to* 0 *as* δ *tends to* 0.

Proof. It is easy to see that the corresponding sequence of measures on the unit circle, $\{\rho_n\}_{n\in\mathbb{N}}$, verifies that $(\{\rho_n\}, \{W_{2n}\}, k)$ is strongly admissible on $[0, 2\pi]$ for all $k \in \mathbb{Z}$. Therefore, we can apply Lemma 2. Notice that $\mathcal{K} = \{x \in [-1, 1] : \arccos x \in \widetilde{\mathcal{K}}\}$, where $\widetilde{\mathcal{K}} = \{\theta \in [0, 2\pi] : \rho'(\theta) > 0\}$ and ρ is the weak* limit of $\{\rho_n\}$. Thus, $|\widetilde{\mathcal{K}}| = \pi|\mathcal{K}| \geqslant 2\pi - \pi\delta$. From Lemma 2 and formula (5), we have

$$\limsup_{n\to\infty} |\Phi_{2n,2n+2k-j}(0)| \leqslant \widetilde{\mathcal{L}}_1(\pi\delta), \quad j = 0, 1, 2, -1$$

for each fixed $k \in \mathbb{Z}$. Therefore, (34) follows using (33). □

Lemma 8. *Let* $\varepsilon > 0$ *be fixed. Suppose that* $(\{d\nu_n\}, \{w_{2n}\}, k)$ *is strongly admissible on* $[-1 - \varepsilon, 1 + \varepsilon]$ *for all* $k \in \mathbb{Z}$ *and* $\nu'(x) > 0$ *a.e. on* $[-1, +1]$, *where* ν *is the weak** *limit of* $\{\nu_n\}$. *Then, for each fixed* $k \in \mathbb{Z}$, *we have*

$$\limsup_{n\to\infty} \left(\left| a_{n+k}^{(\tau_n)} - \frac{1}{2} \right| + \left| b_{n+k}^{(\tau_n)} \right| \right) \leqslant \mathcal{L}^*(\varepsilon), \tag{35}$$

where $\mathcal{L}^*(\varepsilon)$ *tends to* 0 *as* ε *tends to* 0 *and* $d\tau_n = d\nu_n/w_{2n}$.

Proof. We define $d\widetilde{\nu}(x) = d\nu((1+\varepsilon)x)$, $d\widetilde{\nu}_n(x) = d\nu_n((1+\varepsilon)x)$, $n \in \mathbb{N}$. Since $\mathrm{supp}(\nu) = (1+\varepsilon)\mathrm{supp}(\widetilde{\nu})$ we have $\mathrm{supp}(\widetilde{\nu}) \subset [-1, +1]$ and $\mathrm{supp}(\widetilde{\nu}_n) \subset [-1, +1]$. Furthermore, $\widetilde{\nu}'(x) > 0$ a.e. on the interval $[-1/(1+\varepsilon), 1/(1+\varepsilon)]$. If we define $\mathcal{K}$ as in Lemma 7 (for $\widetilde{\nu}$), it follows that $|\mathcal{K}| \geqslant 2 - 2\varepsilon/(1+\varepsilon)$. Set $\widetilde{w}_{2n}(x) = w_{2n}((1+\varepsilon)x)$, then the polynomials $\widetilde{w}_{2n}$ have real coefficients and $\widetilde{w}_{2n}(x) \geqslant 0$ for $x \in [-1, 1]$. It is easy to see that $(\{d\widetilde{\nu}_n\}, \{\widetilde{w}_{2n}\}, k)$ is strongly admissible on $[-1, 1]$ for all $k \in \mathbb{Z}$ with $\widetilde{\nu}$ the weak* limit. Thus, from Lemma 7 it follows that

$$\limsup_{n\to\infty} \left(\left| a_{n+k}^{(\widetilde{\tau}_n)} - \frac{1}{2} \right| + \left| b_{n+k}^{(\widetilde{\tau}_n)} \right| \right) \leqslant \mathcal{L}(\delta), \tag{36}$$

where $\delta = 2\varepsilon/(1+\varepsilon)$ and $d\widetilde{\tau}_n = d\widetilde{\nu}_n/\widetilde{w}_{2n}$, $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, denote by $l_{n,m}^{(\tau_n)}$, respectively, $l_{n,m}^{(\widetilde{\tau}_n)}$, the mth orthonormal polynomial with respect to the measure $d\tau_n$, respectively, $d\widetilde{\tau}_n$. From the orthogonality relations satisfied by both sequences of polynomials it follows that

$$l_{n,m}^{(\tau_n)}(x) = l_{n,m}^{(\widetilde{\tau}_n)}\left(\frac{x}{1+\varepsilon}\right).$$

Using the recurrence relations (1) applied to both sequences of polynomials, we obtain

$$a_m^{(\widetilde{\tau}_n)} = \frac{a_m^{(\tau_n)}}{1+\varepsilon}, \quad b_m^{(\widetilde{\tau}_n)} = \frac{b_m^{(\tau_n)}}{1+\varepsilon}, \quad n, m \in \mathbb{N}.$$

In view of (36), we have

$$\limsup_{n\to\infty}\left(\left|a_{n+k}^{(\tau_n)}-\frac{1+\varepsilon}{2}\right|+\left|b_{n+k}^{(\tau_n)}\right|\right)\leqslant(1+\varepsilon)\mathcal{L}\left(\frac{2\varepsilon}{1+\varepsilon}\right).$$

Since

$$\left|a_{n+k}^{(\tau_n)}-\frac{1}{2}\right|\leqslant\left|a_{n+k}^{(\tau_n)}-\frac{1+\varepsilon}{2}\right|+\frac{\varepsilon}{2},$$

we obtain (35) with

$$\mathcal{L}^*(\varepsilon)=\frac{\varepsilon}{2}+(1+\varepsilon)\mathcal{L}\left(\frac{2\varepsilon}{1+\varepsilon}\right). \qquad \square$$

Let us return to the proof of Theorem 1. For each $N\in\mathbb{N}$, we can define a new measure $\mu^{(N)}$ obtained from μ removing N isolated points $x_1, x_2, \ldots, x_N$ from S, as follows

$$d\mu^{(N)}(x)=\left(\prod_{i=1}^{N}(x-x_i)\right)^2 d\mu(x).$$

We construct these measures in the following way. For each $\varepsilon>0$, choose $N=N(\varepsilon)\in\mathbb{N}$, such that, $x_1,\ldots,x_N\in\mathbb{R}\setminus[1-\varepsilon,\ 1+\varepsilon]$ and $\operatorname{supp}(\mu^{(N)})\subset[1-\varepsilon,\ 1+\varepsilon]$. Analogously, for each varying measure μ_n, we define

$$d\mu_n^{(N)}(x)=\left(\prod_{i=1}^{N}(x-x_i)\right)^2 d\mu_n(x). \tag{37}$$

For each $N\in\mathbb{N}$, it is easy to prove that $(\{d\mu_n^{(N)}\},\{w_{2n}\},k)$ is strongly admissible on $S\backslash\{x_1,\ldots,x_N\}$ for all $k\in\mathbb{Z}$. Also, $\{\mu_n^{(N)}\}$ is a Denisov-type sequence. By $\{a_{n,m}^{(N)}\}_{m\geqslant 0}$, $\{b_{n,m}^{(N)}\}_{m\geqslant 0}$ denote the Jacobi parameters of the measure $d\mu_n^{(N)}/w_{2n}$, $n\in\mathbb{N}$. We can apply Lemma 8 to $(\{d\mu_n^{(N)}\},\{w_{2n}\},k)$ and the following result completes the proof of Theorem 1. $\square$

Lemma 9. *For all $N\in\mathbb{N}$ and each fixed $k\in\mathbb{Z}$, we have*

$$\limsup_{n\to\infty} a_{n,n+k-N}^{(N)}=\limsup_{n\to\infty} a_{n,n+k}, \tag{38}$$

$$\liminf_{n\to\infty} a_{n,n+k-N}^{(N)}=\liminf_{n\to\infty} a_{n,n+k}, \tag{39}$$

$$\limsup_{n\to\infty}\left|b_{n,n+k-N}^{(N)}\right|=\limsup_{n\to\infty}\left|b_{n,n+k}\right|. \tag{40}$$

Proof. Without loss of generality, we can limit ourselves to the proof of (38)–(40) for $N=1$. For $n,m\in\mathbb{N}$, denote by $L_{n,m}^{(1)}$ the monic orthogonal polynomial with respect to $d\mu_n^{(1)}(x)/w_{2n}(x)$ and by $l_{n,m}^{(1)}(x)=\gamma_{n,m}^{(1)}L_{n,m}^{(1)}(x)$ the corresponding orthonormal polynomial. Set $l_{n,m}(x)=\gamma_{n,m}L_{n,m}(x)$. Therefore, (see (2))

$$a_{n,m}=\frac{\gamma_{n,m}}{\gamma_{n,m+1}},\quad a_{n,m}^{(1)}=\frac{\gamma_{n,m}^{(1)}}{\gamma_{n,m+1}^{(1)}}.$$

Let $k \in \mathbb{Z}$ be fixed. As a consequence of Lemma 6 we have that there exists a sequence $\{x_{n+k}^{(n)}\}$, $n \in \mathbb{N}, n \geqslant -k$, such that $l_{n,n+k}(x_{n+k}^{(n)}) = 0$ and $x_1 = \lim_{n\to\infty} x_{n+k}^{(n)}$. From this, we can deduce that there exists a sequence $\{\tau_{n,n+k}\}$, $n \in \mathbb{N}, n \geqslant -k$, of non-negative real numbers, such that

$$\left|\frac{x - x_1}{x - x_{n+k}^{(n)}}\right| \leqslant 1 + \tau_{n,n+k}, \tag{41}$$

uniformly on the compact set $S \setminus \{x_1\}$ and

$$\lim_{n\to\infty} \tau_{n,n+k} = 0. \tag{42}$$

Taking (37) and (41) into consideration, we have

$$\begin{aligned}
\frac{1}{\gamma_{n,n+k}^2} &= \min_{P(x)=x^{n+k}+\cdots} \int_{\mathbb{R}} P^2(x) \frac{d\mu_n(x)}{w_{2n}(x)} \\
&\leqslant \min_{P(x)=x^{n+k-1}+\cdots} \int_{\mathbb{R}} P^2(x)(x - x_1)^2 \frac{d\mu_n(x)}{w_{2n}(x)} = \frac{1}{\left(\gamma_{n,n+k-1}^{(1)}\right)^2} \\
&\leqslant \min_{P(x)=x^{n+k-1}+\cdots} \int_{\mathbb{R}} P^2(x)(x - x_{n+k}^{(n)})^2 (1 + \tau_{n,n+k})^2 \frac{d\mu_n(x)}{w_{2n}(x)} \\
&\leqslant (1 + \tau_{n,n+k})^2 \int_{\mathbb{R}} \left(\frac{l_{n,n+k}(x)}{\gamma_{n,n+k}(x - x_{n+k}^{(n)})}\right)^2 (x - x_{n+k}^{(n)})^2 \frac{d\mu_n(x)}{w_{2n}(x)} \\
&= \frac{\left(1 + \tau_{n,n+k}\right)^2}{\gamma_{n,n+k}^2}.
\end{aligned}$$

Then,

$$\gamma_{n,n+k-1}^{(1)} \leqslant \gamma_{n,n+k} \leqslant (1 + \tau_{n,n+k})\, \gamma_{n,n+k-1}^{(1)}. \tag{43}$$

Also consider (43) replacing k by $k + 1$. From both sets of inequalities, we deduce

$$\frac{a_{n,n+k-1}^{(1)}}{1 + \tau_{n,n+k+1}} \leqslant a_{n,n+k} \leqslant (1 + \tau_{n,n+k}) a_{n,n+k-1}^{(1)},$$

which proves (38) and (39), because of (42).

In order to prove (40), we use the following representation for the Jacobi parameters:

$$b_{n,m} = \int_{\mathbb{R}} x \left(l_{n,m}(x)\right)^2 \frac{d\mu_n(x)}{w_{2n}(x)}, \quad b_{n,m}^{(1)} = \int_{\mathbb{R}} x \left(l_{n,m}^{(1)}(x)\right)^2 \frac{d\mu_n^{(1)}(x)}{w_{2n}(x)}.$$

Then

$$\begin{aligned}
&\left| b_{n,n+k} - \frac{\gamma_{n,n+k}^2}{\gamma_{n,n+k-1}^{(1)2}} b_{n,n+k-1}^{(1)} \right| \\
&\quad = \left| \int_{\mathbb{R}} x \left(l_{n,n+k}^2(x) - \frac{\gamma_{n,n+k}^2}{\gamma_{n,n+k-1}^{(1)2}} l_{n,n+k-1}^{(1)2}(x)(x - x_1)^2 \right) \frac{d\mu_n(x)}{w_{2n}(x)} \right|
\end{aligned}$$

$$\leqslant B \int_{\mathbb{R}} \left| l_{n,n+k}(x) - \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} l^{(1)}_{n,n+k-1}(x)(x-x_1) \right|$$
$$\times \left| l_{n,n+k}(x) + \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} l^{(1)}_{n,n+k-1}(x)(x-x_1) \right| \frac{d\mu_n(x)}{w_{2n}(x)}$$
$$\leqslant B \left(\int_{\mathbb{R}} \left(l_{n,n+k}(x) - \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} l^{(1)}_{n,n+k-1}(x)(x-x_1) \right)^2 \frac{d\mu_n(x)}{w_{2n}(x)} \right)^{1/2}$$
$$\times \left(\int_{\mathbb{R}} \left(l_{n,n+k}(x) + \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} l^{(1)}_{n,n+k-1}(x)(x-x_1) \right)^2 \frac{d\mu_n(x)}{w_{2n}(x)} \right)^{1/2},$$

where B only depends on S. Since,

$$\int_{\mathbb{R}} \left(l_{n,n+k}(x) \pm \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} l^{(1)}_{n,n+k-1}(x)(x-x_1) \right)^2 \frac{d\mu_n(x)}{w_{2n}(x)} = 1 \pm 2 + \frac{\gamma^2_{n,n+k}}{\gamma^{(1)2}_{n,n+k-1}},$$

we have

$$\left| b_{n,n+k} - \frac{\gamma^2_{n,n+k}}{\gamma^{(1)2}_{n,n+k-1}} b^{(1)}_{n,n+k-1} \right| \leqslant B \left(\frac{\gamma^2_{n,n+k}}{\gamma^{(1)2}_{n,n+k-1}} - 1 \right)^{1/2} \left(3 + \frac{\gamma^2_{n,n+k}}{\gamma^{(1)2}_{n,n+k-1}} \right)^{1/2}.$$

Now, take limit, as n tends to infinity, in the expression above. This, together with (42) and (43), implies (40). □

Following the same scheme as in the proof of Theorem 9 in [13] (see also [5, Theorem 8]), from Theorem 1 one obtains

Corollary 3. *Suppose that, for each $k \in \mathbb{Z}$, $(\{d\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on S and $\{\mu_n\}$ is a Denisov-type sequence on S. Then, for each $k \in \mathbb{Z}$, and any function f continuous on S, we have*

$$\lim_{n\to\infty} \int f(x) \frac{l^2_{n,n+k}(x)\, d\mu_n(x)}{w_{2n}(x)} = \frac{1}{\pi} \int f(x) \frac{dx}{\sqrt{1-x^2}}.$$

4. Proof of Theorem 2

We can assume that each function $h_n = hg_n$ is non-negative on S and $Qh_n^{\pm 1} \in L^\infty(S)$, $n \in \mathbb{N}$. It is obvious that $(\{h_n\, d\mu_n\}, \{w_{2n}\}, 2k)$ is strongly admissible on S for all $k \in \mathbb{Z}$.

For each $\varepsilon > 0$ and $n \in \mathbb{N}$, we define $d\widetilde{\mu}_n^{(\varepsilon)} = d\mu_n^{(N)}$, where $d\mu_n^{(N)}$ is given by (37) and $S\backslash[-1-\varepsilon, 1+\varepsilon] = \{x_1, \ldots, x_N\}$. Let $\{\widetilde{l}^{(\varepsilon)}_{n,m}\}_{m\in\mathbb{N}}$ and $\{\widetilde{q}^{(\varepsilon)}_{n,m}\}_{m\in\mathbb{N}}$ be the sequences of orthonormal polynomials with respect to $d\widetilde{\mu}_n^{(\varepsilon)}/w_{2n}$ and $h_n\, d\widetilde{\mu}_n^{(\varepsilon)}/w_{2n}$, respectively.

Lemma 10. *For each fixed* $k \in \mathbb{Z}$, *for each* $\varepsilon > 0$, *and for all* x *in* $\mathbb{C} \setminus S$,

$$\frac{q_{n,n+k}(x)}{l_{n,n+k}(x)} = \frac{\widetilde{q}^{(\varepsilon)}_{n,n+k}(x)}{\widetilde{l}^{(\varepsilon)}_{n,n+k}(x)} u_{n,n+k}(x,\varepsilon), \quad n \geqslant -k,$$

where $\lim_{n\to\infty} u_{n,n+k}(x,\varepsilon) = 1$ *uniformly on compact subsets of* $\mathbb{C} \setminus S$.

Proof. As in the proof of Lemma 9, we can limit ourselves to the case when $N = 1$. As in that Lemma, we denote by $l^{(1)}_{n,m}(x) = \gamma^{(1)}_{n,m} x^m + \cdots$, $\gamma^{(1)}_{n,m} > 0$, the orthonormal polynomial of degree m with respect to $d\mu^{(1)}_n(x)/w_{2n}(x)$, where $(x - x_1)^2\, d\mu_n(x) = d\mu^{(1)}_n(x)$. Set $l_{n,m}(x) = \gamma_{n,m} x^m + \cdots$, $\gamma_{n,m} > 0$.

From the orthogonality conditions satisfied by $l_{n,n+k}$, we have that

$$l_{n,n+k} = \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k}} l^{(1)}_{n,n+k} + c_{n,1} l^{(1)}_{n,n+k-1} + c_{n,2} l^{(1)}_{n,n+k-2}, \tag{44}$$

where

$$c_{n,i} = \int l_{n,n+k}(x) l^{(1)}_{n,n+k-i}(x) \frac{d\mu^{(1)}_n(x)}{w_{2n}(x)}, \quad i = 1, 2.$$

Using the Cauchy–Schwarz inequality it follows that

$$\sup\{|c_{n,i}| : n \in \mathbb{N}, i = 1, 2\} \leqslant M < \infty.$$

On the other hand, using (3) (for the Denisov-type sequence $\{\mu^{(1)}_n\}$), (42), and (43)

$$\lim_{n\to\infty} \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k}} = \lim_{n\to\infty} \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k-1}} \frac{\gamma^{(1)}_{n,n+k-1}}{\gamma^{(1)}_{n,n+k}} = \frac{1}{2}.$$

Consequently, on account of (4) (for the Denisov-type sequence $\{\mu^{(1)}_n\}$), the sequence $\{l_{n,n+k}/l^{(1)}_{n,n+k}\}$, $n \in \mathbb{N}$, is uniformly bounded on each compact subset of $\mathbb{C} \setminus (S \setminus \{x_1\})$. Let us prove that in fact it is convergent.

Let $\Lambda \subset \mathbb{N}$ be such that the subsequence $\{l_{n,n+k}/l^{(1)}_{n,n+k}\}$, $n \in \Lambda$, is convergent on compact subsets of $\mathbb{C} \setminus (S \setminus \{x_1\})$. Passing to a subsequence, if necessary, we can assume that

$$\lim_{n\in\Lambda} c_{n,i} = c_i, \quad i = 1, 2.$$

Because of (4) and (44)

$$\lim_{n\in\Lambda} \frac{l_{n,n+k}}{l^{(1)}_{n,n+k}} = \frac{1}{2} + \frac{c_1}{\Psi} + \frac{c_2}{\Psi^2} = p_\Lambda(1/\Psi),$$

uniformly on compact subsets of $\mathbb{C} \setminus (S \setminus \{x_1\})$, where p_Λ is an algebraic polynomial of second degree whose independent term does not depend on Λ and is equal to $\frac{1}{2}$. In order to prove that the whole sequence converges it is sufficient to show that the two zeros of p_Λ are the same for any Λ.

One of the zeros of p_Λ must be equal to $1/\Psi(x_1)$. Indeed, we know that the sequence of polynomials $\{l_{n,n+k}\}$, $n \in \mathbb{N}$, has a sequence of zeros which converges to x_1 and the limit function is analytic in a neighborhood of that point. By Hurwitz' theorem either $p_\Lambda(1/\Psi(x_1)) = 0$ or the sequence of polynomials $\{l^{(1)}_{n,n+k}\}$, $n \in \mathbb{N}$, must have a sequence of zeros which converges to x_1, but we know that this last assertion is not possible since $d\mu = (x - x_1)^2 d\mu$ has no mass point at x_1 (see Lemma 6 and the remark following it).

Let us find the second zero of p_Λ. Using the orthogonality conditions of $l_{n,n+k}$, $l^{(1)}_{n,n+k}$, and (44), we obtain

$$
\begin{aligned}
0 &= \int \frac{l_{n,n+k}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} = \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k}} \int \frac{l^{(1)}_{n,n+k}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} \\
&\quad + c_{n,1} \int \frac{l^{(1)}_{n,n+k-1}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} + c_{n,2} \int \frac{l^{(1)}_{n,n+k-2}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} \\
&= \frac{\gamma_{n,n+k}}{\gamma^{(1)}_{n,n+k} l^{(1)}_{n,n+k}(x_1)} \int \frac{l^{(1)^2}_{n,n+k}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} + \frac{c_{n,1}}{l^{(1)}_{n,n+k-1}(x_1)} \int \frac{l^{(1)^2}_{n,n+k-1}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)} \\
&\quad + \frac{c_{n,2}}{l^{(1)}_{n,n+k-2}(x_1)} \int \frac{l^{(1)^2}_{n,n+k-2}(x)\, d\mu^{(1)}_n(x)}{(x_1 - x) w_{2n}(x)}.
\end{aligned}
$$

Multiplying this equality by $l^{(1)}_{n,n+k}(x_1)$, using (3), and Corollary 3 applied to the function $(x_1 - x)^{-1}$ which is continuous on $S \setminus x_1$, and taking limit on $n \in \Lambda$, it follows that

$$0 = \tfrac{1}{2} + c_1 \Psi(x_1) + c_2 \Psi^2(x_1) = p_\Lambda(\Psi(x_1)).$$

Consequently,

$$p_\Lambda(z) = \tfrac{1}{2}(1 - z\Psi(x_1))(1 - z\Psi^{-1}(x_1)),$$

independent of Λ and, thus,

$$\lim_{n\to\infty} \frac{l_{n,n+k}(z)}{l^{(1)}_{n,n+k}(z)} = \frac{1}{2}(1 - \Psi^{-1}(z)\Psi(x_1))(1 - \Psi^{-1}(z)\Psi^{-1}(x_1)),$$

uniformly on compact subsets of $\mathbb{C} \setminus (S \setminus \{x_1\})$. By the same token

$$\lim_{n\to\infty} \frac{q_{n,n+k}}{q^{(1)}_{n,n+k}} = \frac{1}{2}(1 - \Psi(z)^{-1}\Psi(x_1))(1 - \Psi^{-1}(z)\Psi^{-1}(x_1)),$$

uniformly on compact subsets of $\mathbb{C} \setminus (S \setminus \{x_1\})$ and the assertion of the lemma readily follows when $N = 1$. The general case is obtained in a finite number of steps. $\square$

As a consequence of Lemma 10, for any fixed $k \in \mathbb{Z}$ and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{q_{n,n+k}(x)}{l_{n,n+k}(x)} = \lim_{n\to\infty} \frac{\widetilde{q}^{(\varepsilon)}_{n,n+k}(x)}{\widetilde{l}^{(\varepsilon)}_{n,n+k}(x)}, \tag{45}$$

uniformly on compact subsets of $\mathbb{C} \setminus S$ if one of two limits exists.

For each $\varepsilon > 0$ we define a new sequence of measures given by

$$d\mu^{(\varepsilon)}_n(x) = d\widetilde{\mu}^{(\varepsilon)}_n\left((1+\varepsilon)x\right), \quad x \in [-1, 1], \quad n \in \mathbb{N}.$$

Then, $\mathrm{supp}(\mu_n^{(\varepsilon)}) \subset [-1, 1]$, $\mu_n^{(\varepsilon)'}(x) > 0$ a.e. in $[-1/(1+\varepsilon), 1/(1+\varepsilon)]$ and $\mathrm{supp}(\mu_n^{(\varepsilon)}) \setminus [-1/(1+\varepsilon), 1/(1+\varepsilon)]$ is, at most, a denumerable set whose only possible accumulation points are $\pm 1/(1+\varepsilon)$. For each $\varepsilon > 0$, define the functions $h^{(\varepsilon)}(x) = h((1+\varepsilon)x)$, $g^{(\varepsilon)}(x) = g((1+\varepsilon)x)$, $g_n^{(\varepsilon)}(x) = g_n((1+\varepsilon)x)$, $h_n^{(\varepsilon)} = h^{(\varepsilon)} g_n^{(\varepsilon)}$, $x \in S$. From the fact that the functions h, g, g_n, h_n are defined on S, it follows that the corresponding functions $h^{(\varepsilon)}, g^{(\varepsilon)}, g_n^{(\varepsilon)}, h_n^{(\varepsilon)}$ are defined on $S^{(\varepsilon)} = \{x/(1+\varepsilon) : x \in S\}$. Denote by $\{l_{n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$ and $\{q_{n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$ the sequences of orthonormal polynomials with respect to $d\mu_n^{(\varepsilon)}/w_{2n}^{(\varepsilon)}$ and $h_n^{(\varepsilon)}\, d\mu_n^{(\varepsilon)}/w_{2n}^{(\varepsilon)}$ respectively, where $w_{2n}^{(\varepsilon)}(x) = w_{2n}((1+\varepsilon)x)$. Then, as in the proof of Lemma 8, we have

$$\widetilde{q}_{n,m}^{(\varepsilon)}(x) = q_{n,m}^{(\varepsilon)}\left(\frac{x}{1+\varepsilon}\right), \ \widetilde{l}_{n,m}^{(\varepsilon)}(x) = l_{n,m}^{(\varepsilon)}\left(\frac{x}{1+\varepsilon}\right)$$

and, because of (45), it is sufficient to study the ratio

$$\frac{q_{n,n+k}^{(\varepsilon)}\left(\dfrac{x}{1+\varepsilon}\right)}{l_{n,n+k}^{(\varepsilon)}\left(\dfrac{x}{1+\varepsilon}\right)} = \frac{1}{u_{n,n+k}(x,\varepsilon)} \frac{q_{n,n+k}(x)}{l_{n,n+k}(x)}, \quad n \in \mathbb{N}, \quad x \in \mathbb{C} \setminus S \tag{46}$$

for each $k \in \mathbb{Z}$. In other words, the convergence of $\left\{q_{n,n+k}/l_{n,n+k}\right\}$ on $\mathbb{C} \setminus S$ is equivalent to the convergence of $\left\{q_{n,n+k}^{(\varepsilon)}/l_{n,n+k}^{(\varepsilon)}\right\}$ on $\mathbb{C} \setminus S^{(\varepsilon)}$.

Let us go to the unit circle again in order to apply Lemmas 3 and 4. Set $\widetilde{h}^{(\varepsilon)}(\theta) = h^{(\varepsilon)}(\cos\theta)$, $\widetilde{g}^{(\varepsilon)}(\theta) = g^{(\varepsilon)}(\cos\theta)$, $\widetilde{g}_n^{(\varepsilon)}(\theta) = g_n^{(\varepsilon)}(\cos\theta)$, $\widetilde{h}_n^{(\varepsilon)} = \widetilde{h}^{(\varepsilon)}\widetilde{g}_n^{(\varepsilon)}$, where $\theta \in \widetilde{S}^{(\varepsilon)} = \{\beta \in [0, 2\pi] : \cos\beta \in S^{(\varepsilon)}\}$. For each $n \in \mathbb{N}$, let $d\sigma_{2n}^{(\varepsilon)}$ be the measure supported on $[0, 2\pi]$ associated with $d\mu_n^{(\varepsilon)}/w_{2n}^{(\varepsilon)}$ according to (32). That is, $d\sigma_n^{(\varepsilon)}(\theta) = d\rho_n^{(\varepsilon)}(\theta)/|W_{2n}^{(\varepsilon)}(e^{i\theta})|^2$ where $d\rho_n^{(\varepsilon)}(\theta) = d\mu_n^{(\varepsilon)}(\cos\theta)$ and $|W_{2n}^{(\varepsilon)}(e^{i\theta})|^2 = w_{2n}^{(\varepsilon)}(\cos\theta)$. Then, the support of $\sigma_{2n}^{(\varepsilon)}$ is contained in $\widetilde{S}^{(\varepsilon)}$. From the hypothesis of Theorem 2, it follows that $\left(\{d\rho_n^{(\varepsilon)}\}, \{W_{2n}^{(\varepsilon)}\}, 2k\right)$ and $\left(\{\widetilde{h}_n^{(\varepsilon)}\, d\rho_n^{(\varepsilon)}\}, \{W_{2n}^{(\varepsilon)}\}, 2k\right)$ are strongly admissible on $[0, 2\pi]$ for all $k \in \mathbb{Z}$.

By $\{\varphi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$, $\{\psi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$ denote the sequences of orthonormal polynomials with respect to $d\sigma_{2n}^{(\varepsilon)}$ and $\widetilde{h}_n^{(\varepsilon)}\, d\sigma_{2n}^{(\varepsilon)}$, respectively. For each fixed $k \in \mathbb{Z}$, we have

$$\frac{q_{n,n+k}^{(\varepsilon)}(x)}{l_{n,n+k}(x)}^{(\varepsilon)} = \frac{\psi_{2n,2n+2k}^{(\varepsilon)}(w)}{\varphi_{2n,2n+2k}^{(\varepsilon)}(w)} V_{n,n+k}^{(\varepsilon)}(w), \quad w = x + \sqrt{x^2-1}, \quad n \in \mathbb{N}, \tag{47}$$

where (see [5, formula (29)])

$$V_{n,n+k}^{(\varepsilon)}(w) = \frac{1 + \dfrac{\psi_{2n,2n+2k}^{(\varepsilon)^*}(w)}{\psi_{2n,2n+2k}^{(\varepsilon)}(w)}}{1 + \dfrac{\varphi_{2n,2n+2k}^{(\varepsilon)^*}(w)}{\varphi_{2n,2n+2k}^{(\varepsilon)}(w)}} \sqrt{\frac{1 + \Phi_{2n,2n+2k}^{(\varepsilon)}(0)}{1 + \Psi_{2n,2n+2k}^{(\varepsilon)}(0)}}, \quad |w| > 1 \tag{48}$$

and $\{\Phi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$, $\{\Psi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$ are the sequences of monic orthogonal polynomials corresponding to $\{\phi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$ and $\{\psi_{2n,m}^{(\varepsilon)}\}_{m\in\mathbb{N}}$, respectively.

First, we will prove that $\left\{q_{n,n+k}/l_{n,n+k}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on each compact subset of $\mathbb{C}\setminus S$. Let K be such a compact set. Fix $\varepsilon > 0$ sufficiently small so that $K\cap[-1-\varepsilon, 1+\varepsilon] = \emptyset$. Set $K^{(\varepsilon)} = \{w \in \mathbb{C} : (1+\varepsilon)w \in K\}$, notice that $K^{(\varepsilon)}\cap[-1,1] = \emptyset$. It is sufficient to prove that $\left\{q^{(\varepsilon)}_{n,n+k}/l^{(\varepsilon)}_{n,n+k}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on $K^{(\varepsilon)}$. Obviously

$$\frac{1-\left|\dfrac{\psi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\psi^{(\varepsilon)}_{2n,2n+2k}(w)}\right|}{1+\left|\dfrac{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\varphi^{(\varepsilon)}_{2n,2n+2k}(w)}\right|}\sqrt{\frac{1-\left|\Phi^{(\varepsilon)}_{2n,2n+2k}(0)\right|}{1+\left|\Psi^{(\varepsilon)}_{2n,2n+2k}(0)\right|}} \leqslant \left|V^{(\varepsilon)}_{n,n+k}(w)\right|$$

$$\leqslant \frac{1+\left|\dfrac{\psi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\psi^{(\varepsilon)}_{2n,2n+2k}(w)}\right|}{1-\left|\dfrac{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\varphi^{(\varepsilon)}_{2n,2n+2k}(w)}\right|}\sqrt{\frac{1+\left|\Phi^{(\varepsilon)}_{2n,2n+2k}(0)\right|}{1-\left|\Psi^{(\varepsilon)}_{2n,2n+2k}(0)\right|}}.$$

Using (5), (7), and (18) (for both sequences of orthonormal polynomials) we deduce that $V^{(\varepsilon)}_{n,n+k}(w)$ is uniformly bounded on $\widetilde{K}^{(\varepsilon)} = \{x+\sqrt{x^2-1} : x \in K^{(\varepsilon)}\}$. Thus, we only need to prove that $\left\{\psi^{(\varepsilon)}_{2n,2n+2k}(w)/\varphi^{(\varepsilon)}_{2n,2n+2k}(w)\right\}_{n\in\mathbb{N}}$ is uniformly bounded on $\widetilde{K}^{(\varepsilon)}$. Set $\widetilde{K}^{(\varepsilon)}_* = \{w \in \mathbb{C} : 1/\overline{w} \in \widetilde{K}^{(\varepsilon)}\}$. We have

$$\left|\frac{\psi^{(\varepsilon)}_{2n,2n+2k}(1/\overline{w})}{\varphi^{(\varepsilon)}_{2n,2n+2k}(1/\overline{w})}\right| = \left|\frac{\psi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(w)}\right|, \quad w \in \widetilde{K}^{(\varepsilon)}_* \subset \{z : |z| < 1\}.$$

We can assume that the zeros of Q are in S, since other zeros do not have any influence on the condition $Qh^{\pm 1} \in L^\infty(S)$. Therefore, Q has real coefficients. Set $Q^{(\varepsilon)}(x) = Q((1+\varepsilon)x)$, then $Q^{(\varepsilon)}$ is an algebraic polynomial with real coefficients, such that $Q^{(\varepsilon)}h^{(\varepsilon)^{\pm 1}} \in L^\infty(S^{(\varepsilon)})$. If we take $\widehat{Q}^{(\varepsilon)}(\theta) = \left(Q^{(\varepsilon)}(\cos\theta)\right)^2$, then $\widehat{Q}^{(\varepsilon)}$ is a trigonometric polynomial with real coefficients, non-negative for all $\theta \in [0, 2\pi]$. From [25, Theorem 1.2.1] there exists an algebraic polynomial $\widetilde{Q}^{(\varepsilon)}$, such that $\widehat{Q}^{(\varepsilon)}(\theta) = \left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\right|^2$. Thus, $\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\widetilde{h}^{(\varepsilon)}(\theta)^{\pm 1} \in L^\infty(\widetilde{S}^{(\varepsilon)})$. Analogously, $(Q(\cos\theta))^2$ is a non-negative trigonometric polynomial with real coefficients. Thus, there also exists an algebraic polynomial $\widetilde{Q}$ such that

$$\left|\widetilde{Q}(e^{i\theta})\right|^2 = (Q(\cos\theta))^2 \quad \text{and} \quad \lim_{\varepsilon\to 0}\left|\widetilde{Q}^{(\varepsilon)}(z)\right|^2 = \left|\widetilde{Q}(z)\right|^2,$$

uniformly on the set $\{z \in \mathbb{C} : |z| = 1\}$. For each $n \in \mathbb{N}$, the function

$$\frac{\widetilde{Q}^{(\varepsilon)}\psi^{(\varepsilon)^*}_{2n,2n+2k}}{\varphi^{(\varepsilon)^*}_{2n,2n+2k}}$$

is analytic on an open neighborhood of $\{w \in \mathbb{C} : |w| \leqslant 1\}$. Then, for all $w \in \widetilde{K}^{(\varepsilon)}_*$, we have

$$\left|\frac{\widetilde{Q}^{(\varepsilon)}(w)\psi^{(\varepsilon)^*}_{2n,2n+2k}(w)}{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(w)}\right|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\psi^{(\varepsilon)}_{2n,2n+2k}(e^{i\theta})}{\varphi^{(\varepsilon)}_{2n,2n+2k}(e^{i\theta})}\right|^2 P(w,\theta)\, d\theta, \tag{49}$$

where $P(w, \theta)$ is the Poisson kernel. Let us use Lemma 4 to estimate the right-hand side of (49). For this purpose, fix $\eta > 0$. Since

$$P(w, \theta) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\alpha-\theta)}, \quad w = re^{i\alpha}, \tag{50}$$

the trigonometric polynomials T_m and R_m employed in the proof of Lemma 4 may be chosen so that they verify (21) and (22) (with $P(w, \theta)$ playing the role of f) independently of $w \in K_*^{(\varepsilon)}$. Therefore, using (23), for all $w \in \widetilde{K}_*^{(\varepsilon)}$ we obtain

$$\limsup_{n\to\infty} \left| \frac{\widetilde{Q}^{(\varepsilon)}(w)\psi_{2n,2n+2k}^{(\varepsilon)*}(w)}{\varphi_{2n,2n+2k}^{(\varepsilon)*}(w)} \right|^2 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\right|^2 P(w, \theta)}{\widetilde{h}^{(\varepsilon)}(\theta)\widetilde{g}^{(\varepsilon)}(\theta)} \, d\theta + \widetilde{\mathcal{L}}_7(\widetilde{\delta}, \eta, P).$$

Since the Poisson kernel is bounded on compact subsets of $\{w \in \mathbb{C} : |w| < 1\}$, we have proved that $\left\{(\widetilde{Q}^{(\varepsilon)}\psi_{2n,2n+2k}^{(\varepsilon)*})/\varphi_{2n,2n+2k}^{(\varepsilon)*}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on $\widetilde{K}_*^{(\varepsilon)}$, thus $\left\{\psi_{2n,2n+2k}^{(\varepsilon)}/\varphi_{2n,2n+2k}^{(\varepsilon)}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on $\widetilde{K}^{(\varepsilon)}$, as we wanted to prove. In fact, the sequence $\left\{\psi_{2n,2n+2k}^{(\varepsilon)}/\varphi_{2n,2n+2k}^{(\varepsilon)}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on compact subsets of $\{w \in \mathbb{C} : |w| > 1\}$.

Let $\Delta \subset \mathbb{N}$ be an infinite set such that

$$\lim_{n\in\Delta} \frac{q_{n,n+k}(x)}{l_{n,n+k}(x)} = T_\Delta(x), \quad x \in \mathbb{C} \setminus S. \tag{51}$$

To conclude the proof of Theorem 2, it is sufficient to prove that $T_\Delta(x) = D\left(1/(\widetilde{h}\widetilde{g}), 1/\Psi(x)\right)$, $x \in \mathbb{C} \setminus S$. In view of (46) and (51), we have

$$\lim_{n\in\Delta} \frac{q_{n,n+k}^{(\varepsilon)}(x)}{l_{n,n+k}^{(\varepsilon)}(x)} = T_\Delta((1+\varepsilon)x), \tag{52}$$

uniformly on compact subsets of $\mathbb{C} \setminus S^{(\varepsilon)}$. For each $\varepsilon > 0$, we can choose $\Delta_\varepsilon \subset \Delta$ such that

$$\lim_{n\in\Delta_\varepsilon} \frac{\psi_{2n,2n+2k}^{(\varepsilon)}(w)}{\varphi_{2n,2n+2k}^{(\varepsilon)}(w)} = T^{(\varepsilon)}(w) \quad \text{and} \quad \lim_{n\in\Delta_\varepsilon} V_{n,n+k}^{(\varepsilon)}(w) = V^{(\varepsilon)}(w), \tag{53}$$

uniformly on compact subsets of $\{w \in \mathbb{C} : |w| > 1\}$. Additionally,

$$\lim_{\varepsilon\to 0} V^{(\varepsilon)}(w) = 1,$$

uniformly on compact subsets of $\{w \in \mathbb{C} : |w| > 1\}$ follows from (48), (5), (7), (18), and the fact that $\widetilde{\delta}$ tends to 0 as ε tends to 0 ($\widetilde{\delta} = (2\pi\varepsilon)/(1+\varepsilon)$, cf. proofs of Lemmas 7 and 8). For all w with $|w| > 1$, set $\widetilde{T}_\Delta(w, \varepsilon) = T^{(\varepsilon)}(w)V^{(\varepsilon)}(w)$. Then, due to (47), (52), and (53), we have

$$\widetilde{T}_\Delta(w, \varepsilon) = T_\Delta((1+\varepsilon)x), \quad w = x + \sqrt{x^2 - 1},$$

on $\{w : |w| > 1\} \setminus \widetilde{\mathcal{F}}_\varepsilon$, where $\widetilde{\mathcal{F}}_\varepsilon = \{x + \sqrt{x^2 - 1} : x \in S^{(\varepsilon)} \setminus [-1, 1]\}$ is a finite set. Moreover, the accumulation points of $\widetilde{\mathcal{F}} = \bigcup_{\varepsilon>0} \widetilde{\mathcal{F}}_\varepsilon$ are the points $x_i + \sqrt{x_i^2 - 1}$ such that $x_i \in S \setminus [-1, 1]$.

Therefore

$$\lim_{\varepsilon\to 0} \widetilde{T}_\Delta(w,\varepsilon) = \lim_{\varepsilon\to 0} T^{(\varepsilon)}(w) = T_\Delta(x), \quad w = x+\sqrt{x^2-1}, \tag{54}$$

uniformly on compact subsets of $\{w : |w| > 1\}\backslash\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}} = \{x_i + \sqrt{x_i^2-1} : x_i \in S\backslash[-1,1]\}$. Since the functions $\widetilde{T}_\Delta(w,\varepsilon)$ are analytic on $\{w : |w| > 1\}$, we can extend the convergence in (54) to all $\{w : |w| > 1\}$. Set

$$\widetilde{T}_\Delta(w) = \lim_{\varepsilon\to 0} \widetilde{T}_\Delta(w,\varepsilon) = \lim_{\varepsilon\to 0} T^{(\varepsilon)}(w), \quad |w| > 1.$$

Thus, for all $w \in \{w : |w| > 1\} \setminus \widehat{\mathcal{F}}$,

$$\widetilde{T}_\Delta(w) = T_\Delta(x), \quad w = x+\sqrt{x^2-1}.$$

Define $\widetilde{T}^*_\Delta(w) = \overline{\widetilde{T}_\Delta(1/\overline{w})}$ and $T^{(\varepsilon)}_*(w) = \overline{T^{(\varepsilon)}(1/\overline{w}}$, then

$$\widetilde{T}^*_\Delta(w) = \lim_{\varepsilon\to 0} T^{(\varepsilon)}_*(w), \tag{55}$$

uniformly on compact subsets of $\{w : |w| < 1\}$, where

$$T^{(\varepsilon)}_*(z) = \lim_{n\in\Delta_\varepsilon} \frac{\psi^{(\varepsilon)^*}_{2n,2n+2k}(z)}{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(z)}.$$

We will prove that $\widetilde{T}^*_\Delta(w) = D\left(1/(\widetilde{h}\widetilde{g}), w\right)$ which, in turn, proves Theorem 2. First, we show that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}(re^{i\theta})\widetilde{T}^*_\Delta(re^{i\theta})\right|^2 d\theta$$

is bounded for any $r \in (0,1)$. That is, $\widetilde{Q}\widetilde{T}^*_\Delta \in \mathbb{H}^2$, where $\mathbb{H}^2$ stands for the usual Hardy space on the unit disk.

Using (55), for each $r < 1$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}(re^{i\theta})\widetilde{T}^*_\Delta(re^{i\theta})\right|^2 d\theta \\
&\quad = \lim_{\varepsilon\to 0} \frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}^{(\varepsilon)}(re^{i\theta})T^{(\varepsilon)}_*(re^{i\theta})\right|^2 d\theta \\
&\quad = \lim_{\varepsilon\to 0}\lim_{n\in\Delta_\varepsilon} \frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}^{(\varepsilon)}(re^{i\theta})\frac{\psi^{(\varepsilon)^*}_{2n,2n+2k}(re^{i\theta})}{\varphi^{(\varepsilon)^*}_{2n,2n+2k}(re^{i\theta})}\right|^2 d\theta \\
&\quad \leqslant \limsup_{\varepsilon\to 0}\left(\limsup_{n\in\Delta_\varepsilon} \frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\frac{\psi^{(\varepsilon)}{}_{2n,2n+2k}(e^{i\theta})}{\varphi^{(\varepsilon)}{}_{2n,2n+2k}(e^{i\theta})}\right|^2 d\theta\right).
\end{aligned}$$

For each $\varepsilon > 0$, let us apply formula (23) with $f(\theta) = 1$ and $\widetilde{S} = \widetilde{S}^{(\varepsilon)}$. Fix $\eta > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{Q}(re^{i\theta})\widetilde{T}_\Delta^*(re^{i\theta})\right|^2 d\theta \leqslant \limsup_{\varepsilon\to 0}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\right|^2}{\widetilde{h}^{(\varepsilon)}(\theta)\widetilde{g}^{(\varepsilon)}(\theta)}\,d\theta + \widetilde{\mathcal{L}}_7(\widetilde{\delta},\eta,1)\right), \tag{56}$$

where $\widetilde{\mathcal{L}}_7$ implicitly depends on ε. A careful study of the proof of Lemma 4 shows that this dependence is expressed in terms of $\|(\widetilde{Q}^{(\varepsilon)})^2/(\widetilde{h}^{(\varepsilon)}\widetilde{g}^{(\varepsilon)})\|_{\widetilde{S}^{(\varepsilon)}}$. Since

$$\lim_{\varepsilon\to 0}\left\|\frac{(\widetilde{Q}^{(\varepsilon)})^2}{\widetilde{h}^{(\varepsilon)}\widetilde{g}^{(\varepsilon)}}\right\|_{\widetilde{S}^{(\varepsilon)}} = \left\|\frac{\widetilde{Q}^2}{\widetilde{h}\,\widetilde{g}}\right\|_{[0,2\pi]} \quad \text{and} \quad \widetilde{\delta} = \frac{2\pi\varepsilon}{1+\varepsilon},$$

we have

$$\limsup_{\varepsilon\to 0}\ \widetilde{\mathcal{L}}_7(\widetilde{\delta},\eta,1) \leqslant C\eta,$$

where C is a constant. Since η is arbitrary, applying the Dominated Convergence theorem to (56), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{Q}(re^{i\theta})\widetilde{T}_\Delta^*(re^{i\theta})\right|^2 d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\widetilde{Q}(e^{i\theta})\right|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)}\,d\theta, \quad \forall r \in (0,1), \tag{57}$$

as we wanted to show. From this fact we can deduce that there exist radial limits

$$\lim_{r\to 1^-}\ \widetilde{Q}(re^{i\theta})\widetilde{T}_\Delta^*(re^{i\theta}) = \widetilde{Q}(e^{i\theta})\widetilde{T}_\Delta^*(e^{i\theta}) \quad \text{a.e. in } [0,2\pi]$$

and, obviously

$$\lim_{r\to 1^-}\ \widetilde{T}_\Delta^*(re^{i\theta}) = \widetilde{T}_\Delta^*(e^{i\theta}) \quad \text{a.e. in } [0,2\pi].$$

For $r \in (0,1)$ and z such that $|z| = 1$, using (49) and (55), we have

$$\begin{aligned}\left|\widetilde{Q}(rz)\widetilde{T}_\Delta^*(rz)\right|^2 &= \lim_{\varepsilon\to 0}\left|\widetilde{Q}^{(\varepsilon)}(rz)T_*^{(\varepsilon)}(rz)\right|^2\\ &= \lim_{\varepsilon\to 0}\lim_{n\in\Delta_\varepsilon}\left|\widetilde{Q}^{(\varepsilon)}(rz)\frac{{\psi^{(\varepsilon)}}^*_{2n,2n+2k}(rz)}{{\varphi^{(\varepsilon)}}^*_{2n,2n+2k}(rz)}\right|^2\\ &\leqslant \limsup_{\varepsilon\to 0}\left(\limsup_{n\in\Delta_\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\frac{{\psi^{(\varepsilon)}}_{2n,2n+2k}(e^{i\theta})}{{\varphi^{(\varepsilon)}}_{2n,2n+2k}(e^{i\theta})}\right|^2 P(rz,\theta)\,d\theta\right).\end{aligned}$$

For each $\varepsilon > 0$, apply again formula (23) with $f(\theta) = P(rz,\theta)$ and $\widetilde{S} = \widetilde{S}^{(\varepsilon)}$. Fix $\eta > 0$ and consider $r \leqslant R < 1$. Then

$$\begin{aligned}\left|\widetilde{Q}(rz)\widetilde{T}_\Delta^*(rz)\right|^2 \leqslant{}& \limsup_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\widetilde{Q}^{(\varepsilon)}(e^{i\theta})\right|^2}{\widetilde{h}^{(\varepsilon)}(\theta)\widetilde{g}^{(\varepsilon)}(\theta)}P(rz,\theta)\,d\theta\\ &+ \limsup_{\varepsilon\to 0}\ \widetilde{\mathcal{L}}_7(\widetilde{\delta},\eta,P(rz,\cdot)).\end{aligned}$$

The same considerations used to prove (57) equally work here taking into account (50) and the fact that $r \leqslant R < 1$. Since R is arbitrary,

$$\left|\widetilde{Q}(rz)\widetilde{T}_{\Delta}^{*}(rz)\right|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|\widetilde{Q}(e^{i\theta})\right|^2}{\widetilde{h}(\theta)\widetilde{g}(\theta)} P(rz,\theta)\,d\theta,$$

for all $r \in (0, 1)$. Taking limit as r tends to 1, we obtain (see [23, Theorem 11.8])

$$\left|\widetilde{Q}(e^{it})\widetilde{T}_{\Delta}^{*}(e^{it})\right|^2 \leqslant \frac{\left|\widetilde{Q}(e^{it})\right|^2}{\widetilde{h}(t)\widetilde{g}(t)} \quad \text{a.e. in } [0, 2\pi].$$

Therefore,

$$\left|\widetilde{T}_{\Delta}^{*}(e^{it})\right|^2 \leqslant \frac{1}{\widetilde{h}(t)\widetilde{g}(t)} \quad \text{a.e. in } [0, 2\pi], \tag{58}$$

which, in particular, implies $\widetilde{T}_{\Delta}^{*} \in \mathbb{H}^2$.

Similarly, it is possible to prove that $\left\{l_{n,n+k}/q_{n,n+k}\right\}$ is also uniformly bounded on compact subsets of $\mathbb{C} \setminus S$. Therefore, we can assume that $\Delta \subset \mathbb{N}$ was chosen so as to additionally fulfill

$$\lim_{n\in\Delta} \frac{l_{n,n+k}(x)}{q_{n,n+k}(x)} = \frac{1}{T_{\Delta}(x)}, \quad x \in \mathbb{C} \setminus S.$$

An analogous statement is valid for $\Delta_{\varepsilon} \subset \Delta$. We can then repeat the above calculations, this time with $1/\widetilde{T}_{\Delta}^{*}$, replacing the use of (23) with that of (24). We conclude that

$$\frac{1}{\left|\widetilde{T}_{\Delta}^{*}(e^{it})\right|^2} \leqslant \widetilde{h}(t)\,\widetilde{g}(t) \quad \text{a.e. in } [0, 2\pi]. \tag{59}$$

Formulas (58) and (59) imply

$$\frac{1}{\left|\widetilde{T}_{\Delta}^{*}(e^{it})\right|^2} = \widetilde{h}(t)\widetilde{g}(t) \quad \text{a.e. in } [0, 2\pi].$$

Furthermore, $\widetilde{T}_{\Delta}^{*}$ and $\left(\widetilde{T}_{\Delta}^{*}\right)^{-1}$ belong to $\mathbb{H}^2$. Therefore, $\log \widetilde{T}_{\Delta}^{*} \in \mathbb{H}^1$ and we have

$$\log\left|\widetilde{T}_{\Delta}^{*}(z)\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\widetilde{T}_{\Delta}^{*}(e^{i\theta})\right| P(z,\theta)\,d\theta, \quad |z| < 1.$$

In particular,

$$\log\left|\widetilde{T}_{\Delta}^{*}(0)\right| = \log\left|D\left(1/(\widetilde{h}\,\widetilde{g}), 0\right)\right| = \log D\left(1/(\widetilde{h}\widetilde{g}), 0\right).$$

From (55) we know that $\widetilde{T}_{\Delta}^{*}(0) \geqslant 0$. Then

$$\log \widetilde{T}_{\Delta}^{*}(0) = \log D(1/(\widetilde{h}\widetilde{g}), 0). \tag{60}$$

From this fact (see [23, Theorem 17.17]), it follows that

$$\widetilde{T}_{\Delta}^{*}(z) = \lambda D\left(1/(\widetilde{h}\widetilde{g}), z\right), \quad |z| < 1,$$

where λ is a constant. But (60) implies that $\lambda = 1$; therefore,

$$\widetilde{T}^*_\Delta(z) = D\left(1/(\widetilde{h}\widetilde{g}), z\right), \quad |z| < 1,$$

with which we conclude the proof of Theorem 2. □

5. Nikishin orthogonal polynomials

Let σ_1, σ_2 be two finite Borel measures with constant sign, whose supports $\mathrm{supp}(\sigma_1)$, $\mathrm{supp}(\sigma_2)$ are contained in non-intersecting intervals Δ_1, Δ_2, respectively, of the real line $\mathbb{R}$. Set

$$d\langle\sigma_1, \sigma_2\rangle(x) = \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x).$$

This expression defines a new measure with constant sign whose support coincides with that of σ_1. Whenever we find it convenient we use the differential notation of a measure.

Let $\Sigma = (\sigma_1, \ldots, \sigma_m)$ be a system of finite Borel measures on the real line with constant sign and compact support. Let Δ_k denote the smallest interval which contains the support of σ_k. Assume that $\Delta_k \cap \Delta_{k+1} = \emptyset,\ k = 1, \ldots, m-1$. By definition, $S = (s_1, \ldots, s_m) = \mathcal{N}(\Sigma)$ is called the *Nikishin system* generated by Σ if

$$s_1 = \sigma_1, \quad s_2 = \langle\sigma_1, \sigma_2\rangle, \ldots, \quad s_m = \langle\sigma_1, \langle\sigma_2, \ldots, \sigma_m\rangle\rangle.$$

Fix a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m_+$. The polynomial $Q_{\mathbf{n}}(x)$ is called an $\mathbf{n}$th *multi-orthogonal polynomial* with respect to S if it is not identically equal to zero, $\deg Q_{\mathbf{n}} \leqslant |\mathbf{n}| = n_1 + \cdots + n_m$, and satisfies the orthogonality relations

$$\int Q_{\mathbf{n}}(x) x^\nu\, ds_k(x) = 0, \quad \nu = 0, \ldots, n_k - 1, \quad k = 1, \ldots, m.$$

In the sequel, we assume that $Q_{\mathbf{n}}$ is monic; that is, has leading coefficient equal to 1.

Let

$$\mathbb{Z}^m_+(\circledast) = \{\mathbf{n} \in \mathbb{Z}^m_+ : 1 \leqslant i < j \leqslant m \Rightarrow n_j \leqslant n_i + 1\}.$$

In [8] it was proved that, for all $\mathbf{n} \in \mathbb{Z}^m_+(\circledast)$, the zeros of $Q_{\mathbf{n}}$ are simple and lie in the interior of Δ_1. For each $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m_+(\circledast)$, define recursively the following functions

$$\Psi_{\mathbf{n},0}(x) = Q_{\mathbf{n}}(x), \quad \Psi_{\mathbf{n},k}(x) = \int \frac{\Psi_{\mathbf{n},k-1}(t)}{x-t}\, d\sigma_k(t), \quad k = 1, \ldots, m.$$

In Proposition 1 of Gonchar et al. [10] it was proved that for each $k = 1, \ldots, m$

$$\int \Psi_{\mathbf{n},k-1}(t) t^\nu\, d\langle\sigma_k, \ldots, \sigma_{k+r}\rangle(t) = 0, \quad \nu = 0, \ldots, n_{k+r} - 1, \quad k \leqslant k + r \leqslant m.$$

From here, the authors deduce that $\Psi_{\mathbf{n},k-1}$, $k = 1, \ldots, m$, has exactly $N_{\mathbf{n},k} = n_k + \cdots + n_m$ zeros in $\mathbb{C} \setminus \Delta_{k-1}$, that they are all simple, and lie in the interior of Δ_k. Let $Q_{\mathbf{n},k}$ be the monic polynomial of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\Psi_{\mathbf{n},k-1}$ vanishes in Δ_k and let $Q_{\mathbf{n},m+1} \equiv 1$. In Proposition 2 (see also Proposition 3) of [10] the authors prove that

$$\int x^\nu \Psi_{\mathbf{n},k-1}(x) \frac{d\sigma_k(x)}{Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m. \tag{61}$$

Set

$$H_{\mathbf{n},k} := \frac{Q_{\mathbf{n},k-1}\Psi_{\mathbf{n},k-1}}{Q_{\mathbf{n},k}}.$$

From (61), we have that for each multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$ there exists a system of monic polynomials

$$\{Q_{\mathbf{n},k}\}_{k=1}^m, \quad \deg Q_{\mathbf{n},k} = \sum_{\alpha=k}^m n_\alpha := N_{\mathbf{n},k}, \quad Q_{\mathbf{n},0} = Q_{\mathbf{n},m+1} \equiv 1, \tag{62}$$

satisfying the system of full orthogonality relations

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{|H_{\mathbf{n},k}(x)|\, d\sigma_k(x)}{|Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)|} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \quad k = 1, \ldots, m,$$

with respect to a varying measure. (Notice that $H_{\mathbf{n},k}$ and $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ have constant sign on Δ_k, thus we can take absolute value of these functions under the integral sign without affecting the value of the integral.)

Our goal is to state a result on ratio asymptotics for the polynomials $\{Q_{\mathbf{n},k}\}_{k=1}^m$ when the measures σ_k, $k = 1, \ldots, m$, are of Denisov type. In particular, for $Q_{\mathbf{n}} = Q_{\mathbf{n},1}$. It extends to this class of measures Theorem 1.2 of [1]. The proof is basically the same as in that paper. The answer is given in terms of certain algebraic functions of order $m + 1$ (as in the Denisov–Rakhmanov theorem for $m = 1$). In the sequel we will assume that for each $k = 1, \ldots, m$, $\operatorname{supp}(\sigma_k) = \widetilde{\Delta}_k \cup E_k$, where $\sigma_k' > 0$ a.e. on $\widetilde{\Delta}_k$ and E_k is a set of isolated points in $\mathbb{R} \setminus \widetilde{\Delta}_k$.

To introduce these functions, we consider the $(m + 1)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^m \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \widetilde{\Delta}_1, \quad \mathcal{R}_k := \overline{\mathbb{C}} \setminus \{\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}\}, \quad k = 1, \ldots, m-1, \quad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \widetilde{\Delta}_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \ldots, m\}$. Let $\psi^{(l)}$, $l = 1, \ldots, m$, be a single-valued rational function on $\mathcal{R}$ whose divisor consists of one simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and one simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. Therefore,

$$\psi^{(l)}(z) = C_1/z + \mathcal{O}(1/z^2), \quad z \to \infty^{(0)}, \quad \psi^{(l)}(z) = C_2 z + \mathcal{O}(1), \quad z \to \infty^{(l)}, \tag{63}$$

where C_1 and C_2 are constants different from zero. Since the genus of $\mathcal{R}$ equals zero, such a single-valued function on $\mathcal{R}$ exists and is uniquely determined up to a multiplicative constant. We denote the branches of the algebraic function $\psi^{(l)}$, corresponding to the different sheets $k = 0, \ldots, m$ of $\mathcal{R}$ by

$$\psi^{(l)} := \{\psi_k^{(l)}\}_{k=0}^m.$$

In the sequel, we fix the multiplicative constant so that

$$\prod_{k=0}^m \psi_k^{(l)}(\infty) = 1. \tag{64}$$

For any fixed multi-index $\mathbf{n} = (n_1, \dots, n_m)$, set

$$\mathbf{n}^l := (n_1, \dots, n_{l-1}, n_l + 1, n_{l+1}, \dots, n_m).$$

Given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1})$, $C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := \frac{F}{C}.$$

Now, we can state the general theorem on ratio asymptotics for multiple orthogonal polynomials of a Nikishin system.

Theorem 3. *Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system with* $\operatorname{supp}(\sigma_k) = \widetilde{\Delta}_k \cup E_k$ *and $\sigma_k' > 0$ almost everywhere on $\widetilde{\Delta}_k$, $k = 1, \dots, m$. Let $\Lambda \subset \mathbb{Z}_+(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \dots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+(\circledast)$ and $n_1 - n_m \leqslant d$, where d is a constant. Let $\{Q_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, be the corresponding system of monic polynomials* (62). *Then for each fixed $k \in \{1, \dots, m\}$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}^l,k}(z)}{Q_{\mathbf{n},k}(z)} = \widetilde{F_k^{(l)}}(z), \quad z \in K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k)$$

where

$$F_k^{(l)} := \prod_{\nu=k}^{m} \psi_\nu^{(l)}$$

and the algebraic functions $\psi_\nu^{(l)}$ are defined by (63)–(64).

The proof of Theorem 3 is based on three steps. First, you show that the zeros of the multiple orthogonal polynomials $Q_{\mathbf{n},k}$, $k = 1, \dots, m$, interlace. To prove this, follow Section 2 in [1]. On the other hand, the zeros of the polynomials $Q_{\mathbf{n},k}$ which lie in $\Delta_k \setminus \widetilde{\Delta}_k$ are attracted to the mass points of σ_k as we saw follows from Lemma 6. Therefore, given l, for each fixed $k = 1, \dots, m$, the ratios $Q_{\mathbf{n}^l,k}/Q_{\mathbf{n},k}$, $\mathbf{n} \in \Lambda$, form normal families of analytic functions in $\mathbb{C} \setminus \operatorname{supp}(\sigma_k)$, respectively. Secondly, using Theorems 1, 2, and Corollary 3 one proves that the limit functions of any convergent subsequence satisfy a system of boundary-value problems on the intervals $\widetilde{\Delta}_k$. This is done as in Section 3 of [1]. The varying measures to be considered are of the form

$$\frac{C_{\mathbf{n},k}|H_{\mathbf{n},k}(x)|\, d\sigma_k(x)}{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|},$$

where the $C_{\mathbf{n},k}$ are normalizing constants such that for each $k = 1, \dots, m-1$,

$$\lim_{n\in\Lambda} C_{\mathbf{n},k+1}|H_{\mathbf{n},k+1}(z)| = \frac{1}{|\sqrt{(z-b_k)(z-a_k)}|},$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_k$. The existence of such normalizing constants is clearly indicated in [1] and is based in the present situation on Corollary 3. In [1], instead of Theorems 1, 2 and Corollary 3, the authors make use of similar results for orthogonal polynomials with respect to varying measures without mass points outside of $\widetilde{\Delta}_k$ developed earlier by B. de la Calle and

G. López contained in [5,6]. To conclude, you show that the system of boundary-value problems has a unique solution which may be expressed by means of the algebraic functions defined above. The proof is exactly as in Section 4 of [1].

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