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Phase space localization of orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$

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Abstract

The aim of this paper is to prove a quantitative extension of Shapiro's result on the time–frequency concentration of orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$. More precisely, we prove that, if $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$, then for all $N \geq 0$

$$\sum_{n=0}^{N}\left(\left\|x\varphi_n\right\|^2_{L^2_\alpha}+\left\|\xi\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right)\geq 2(N+1)(N+1+\alpha),$$

and the equality is attained for the sequence of Laguerre functions. Particularly if the elements of an orthonormal sequence and their Fourier–Bessel transforms (or Hankel transforms) have uniformly bounded dispersions then the sequence is finite.

Moreover we prove the following strong uncertainty principle for bases for $L^2_\alpha(\mathbb{R}_+)$, that is if $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$ and $s>0$, then

$$\sup_n\left(\left\|x^s\varphi_n\right\|^2_{L^2_\alpha}\left\|\xi^s\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right)=+\infty.$$

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1. Introduction

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function and its Fourier transform. The most familiar form is the Heisenberg–Pauli–Weil inequality. To be more precise, let $d \geq 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. Then the Heisenberg–Pauli–Weil inequality (see *e.g.* [9,20]) leads to the following classical formulation of the uncertainty principle in form of the lower bound of the product of the dispersions of a unit-norm function in $L^2(\mathbb{R}^d)$ and its Fourier transform:

$$\| |x| f \|_{L^2(\mathbb{R}^d)} \| |\xi| \mathcal{F}(f) \|_{L^2(\mathbb{R}^d)} \geq \frac{d}{2}, \tag{1.1}$$

with equality if and only if f is a multiple of a suitable Gaussian. Heisenberg's inequality (1.1) may be also written in the form

$$\| |x| f \|^2_{L^2(\mathbb{R}^d)} + \| |\xi| \mathcal{F}(f) \|^2_{L^2(\mathbb{R}^d)} \geq d, \tag{1.2}$$

where the Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by:

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} \, \mathrm{d}x,$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way. With this normalization, if $f(x) = \tilde{f}(|x|)$ is a radial function on $\mathbb{R}^d$, then $\mathcal{F}(f)(\xi) = \mathcal{H}_{d/2-1}(\tilde{f})(|\xi|)$, where for $\alpha > -1/2$, $\mathcal{H}_\alpha$ is the Fourier–Bessel transform (also known as the Hankel transform) defined by (see *e.g.* [23]):

$$\mathcal{H}_\alpha(\xi) = \int_{\mathbb{R}_+} f(x) j_\alpha(x\xi) \, \mathrm{d}\mu_\alpha(x), \quad \xi \in \mathbb{R}_+ = [0, +\infty).$$

Here $\mathrm{d}\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} \, \mathrm{d}x$ and j_α (see *e.g.* [23,25]) is the spherical Bessel function given by:

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(x)}{x^\alpha} := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} \left(\frac{x}{2} \right)^{2n}.$$

Note that J_α is the Bessel function of the first kind and Γ is the gamma function.

For $\alpha > -1/2$, let us recall the *Poisson representation formula* (see *e.g.* [24, (1.71.6), p. 15]):

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1-s^2)^{\alpha-1/2} \cos(sx) \, \mathrm{d}s.$$

Therefore, j_α is bounded with $|j_\alpha(x)| \leq j_\alpha(0) = 1$. As a consequence,

$$\|\mathcal{H}_\alpha(f)\|_\infty \leq \|f\|_{L^1_\alpha}, \tag{1.3}$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm and for $1 \leq p < +\infty$, we denote by $L^p_\alpha(\mathbb{R}_+)$ the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms:

$$\|f\|_{L^p_\alpha} = \left(\int_{\mathbb{R}_+} |f(x)|^p \, \mathrm{d}\mu_\alpha(x) \right)^{1/p}.$$

It is also well-known (see [23,26]) that the Fourier–Bessel transform extends to an isometry on $L^2_\alpha(\mathbb{R}_+)$:

$$\|\mathcal{H}_\alpha(f)\|_{L^2_\alpha} = \|f\|_{L^2_\alpha}. \tag{1.4}$$

Bowie [6] and then Rösler and Voit [21] proved the analogue of Heisenberg's uncertainty inequality for the Fourier–Bessel transform which can be written for unit-norm functions in $L^2_\alpha(\mathbb{R}_+)$ of the form:

$$\|xf\|_{L^2_\alpha}\|\xi\,\mathcal{H}_\alpha(f)\|_{L^2_\alpha} \geq (\alpha+1) \quad \text{or} \quad \|xf\|^2_{L^2_\alpha} + \|\xi\,\mathcal{H}_\alpha(f)\|^2_{L^2_\alpha} \geq 2(\alpha+1), \tag{1.5}$$

with equality, if and only if f is a multiple of a suitable Gaussian function.

Considerable attention has been devoted recently to discovering new mathematical formulations and new contexts for the uncertainty principle (see the surveys [4,9,20] and the book [13] for other forms of the uncertainty principle). This paper will adopt the broader view that the uncertainty principle can be seen not only as a statement about the phase space (or time–frequency) localization of a single function but also as a statement on the degradation of localization when one considers successive elements of an orthonormal basis. In particular, Heisenberg's inequality (1.5) states that a unit-norm function in $L^2_\alpha(\mathbb{R}_+)$ cannot occupy an arbitrarily small region in the phase space plane and the results that we consider show that the elements of an orthonormal basis as well as their Fourier–Bessel transforms cannot be uniformly concentrated in the time–frequency plane.

For some of the well-known results related to uncertainty principles for orthonormal sequences, Shapiro proved a number of uncertainty inequalities that are stronger than corresponding inequalities for a single function. For example, using compactness argument, see [22], one can conclude that for any orthonormal sequence $\{f_n\}_{n=0}^{+\infty}$ in $L^2(\mathbb{R})$:

$$\sup_n \left(\|xf_n\|^2_{L^2(\mathbb{R})} + \|\xi\mathcal{F}(f_n)\|^2_{L^2(\mathbb{R})}\right) = +\infty. \tag{1.6}$$

Some other results on time–frequency localization of orthonormal sequences and bases have been obtained by Benedetto [2] and Powell [18] and the quantitative version of Shapiro's result has been proved by Jaming and Powell [14] which states that, if $\{f_n\}_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2(\mathbb{R})$ then for all $N \geq 0$:

$$\sum_{n=0}^{N} \left(\|xf_n\|^2_{L^2(\mathbb{R})} + \|\xi\mathcal{F}(f_n)\|^2_{L^2(\mathbb{R})}\right) \geq (N+1)^2. \tag{1.7}$$

The equality in (1.7) is attained for the sequence of Hermite functions and the higher dimensional version of this result that involving generalized dispersions $\|\,|x|^s f_n\|^2_{L^2(\mathbb{R}^d)}$ and $\|\,|\xi|^s\mathcal{F}(f_n)\|^2_{L^2(\mathbb{R}^d)}$, $s > 0$, for orthonormal sequences in $L^2(\mathbb{R}^d)$ was obtained by Malinnikova [16].

The goal of this paper is to provide new uncertainty principles for the Fourier–Bessel transform which are all known for the usual Fourier transform (see *e.g.* [14,16]). The first of these results is an extension of Shapiro's result for orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$. Especially by using the Laguerre expansions instead of Hermite expansions in [14] and the Rayleigh–Ritz technique [19, Theorem XIII.3, p. 82] for eigenvalues of operators we prove the following quantitative uncertainty inequality for orthonormal sequences:

Theorem A. *If $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$, then for all $N \geq 0$*

$$\sum_{n=0}^{N} \left(\|x\varphi_n\|^2_{L^2_\alpha} + \|\xi \mathcal{H}_\alpha(\varphi_n)\|^2_{L^2_\alpha} \right) \geq 2(N+1)(N+1+\alpha). \tag{1.8}$$

This theorem implies in particular that, if the elements of an orthonormal sequence and their Fourier–Bessel transforms have uniformly bounded dispersions then the sequence is finite.

The last result is sharp and the equality in (1.8) is attained for the sequence of Laguerre functions, but the method of proof using Laguerre expansions is not applicable to generalized dispersions. Nevertheless based on an idea of Malinnikova [16] we adapt the proof of [16, Theorem 2] to establish the localization inequality (3.43) which implies the following strong uncertainty principle for orthonormal bases:

Theorem B. *If $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$ and $s > 0$, then*

$$\sup_n \left(\|x^s\varphi_n\|^2_{L^2_\alpha} \|\xi^s \mathcal{H}_\alpha(\varphi_n)\|^2_{L^2_\alpha} \right) = +\infty.$$

A related result for Riesz bases in $L^2(\mathbb{R}^d)$ appeared in a recent article by Gröchenig and Malinnikova [12] *which asserts in particular that the Bourgain basis* [5] *possesses the best possible phase space localization.*

The remainder of the paper is organized as follows. Section 2 is devoted to proving Theorem A and as a side result we give another proof of a Heisenberg-type uncertainty inequality for the Fourier–Bessel transform. In Section 3, we prove Theorem B.

2. Quantitative version of Shapiro's result in the Fourier–Bessel setting

2.1. Heisenberg-type uncertainty inequality for the Fourier–Bessel transform revisited

In this section we will revisit the Heisenberg uncertainty inequality for the Fourier–Bessel transform which was first proved by Bowie [6] and then by Rösler and Voit [21]. We give here a simple proof based on Laguerre expansions and prove a not so well-known extension that shows that Laguerre functions are successive optimizers of Heisenberg-type uncertainty principle for the Fourier–Bessel transform.

It is well-known (see [15, (4.17.1), p. 76]) that the Laguerre polynomials L_n^α are defined by the following Rodriguez formula

$$L_n^\alpha(x) = x^{-\alpha} \frac{e^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}x^{n+\alpha}\right), \quad n \in \mathbb{N},\ x > 0 \tag{2.9}$$

and they are a particular solution of the following second order linear differential equation (see [15, (4.18.8), p. 80]),

$$xu'' + (\alpha + 1 - x)u' + nu = 0. \tag{2.10}$$

Moreover the Laguerre polynomials satisfy the following recurrence formula (see [24, (5.1.14), p. 102] or [15, (4.18.4), p. 76])

$$xL_n^{\alpha+1}(x) = -(n+1)L_{n+1}^\alpha(x) + (\alpha + n + 1)L_n^\alpha(x), \quad n \in \mathbb{N}. \tag{2.11}$$

Therefore if we define ϕ_n^α by

$$\phi_n^\alpha(x) = \left(\frac{2^{\alpha+1}\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-x^2/2} L_n^\alpha(x^2), \quad n \in \mathbb{N},\ x > 0, \tag{2.12}$$

then a straightforward manipulation using (2.11) leads to the following new recurrence formula

$$x^2\phi_n^{\alpha+1}(x) = \sqrt{2(\alpha+1)}\left[-(n+1)^{1/2}\phi_{n+1}^\alpha(x) + (\alpha+n+1)^{1/2}\phi_n^\alpha(x)\right], \quad x > 0. \tag{2.13}$$

It is also well-known (see [15, p. 84]) that the sequence $\left\{\phi_n^\alpha\right\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $L_\alpha^2(\mathbb{R}_+)$ and each ϕ_n^α is an eigenfunction for the Fourier–Bessel transform associated to the eigenvalue $(-1)^n$ (see *e.g.* [8, 8.9 (3), p. 42]), that is,

$$\mathcal{H}_\alpha(\phi_n^\alpha) = (-1)^n\phi_n^\alpha, \quad n \in \mathbb{N}. \tag{2.14}$$

Now if we denote by $\ell_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha+1}{x}\frac{\mathrm{d}}{\mathrm{d}x}$ the differential Bessel operator, then by (2.10) it is easy to show that the ϕ_n^α's form the family of eigenfunctions of the differential operator $\mathcal{L}_\alpha = -\ell_\alpha + x^2$ with corresponding eigenvalues $(4n+2\alpha+2)$, that is,

$$\mathcal{L}_\alpha\phi_n^\alpha = (4n+2\alpha+2)\phi_n^\alpha, \quad n \in \mathbb{N}. \tag{2.15}$$

So that $\mathcal{L}_\alpha$ may also be seen as the densely defined, positive, self-adjoint, unbounded operator on $L_\alpha^2(\mathbb{R}_+)$ defined by

$$\mathcal{L}_\alpha f = \sum_{n=0}^{+\infty}(4n+2\alpha+2)\left\langle f, \phi_n^\alpha\right\rangle_\alpha \phi_n^\alpha, \tag{2.16}$$

where $\langle\cdot,\cdot\rangle_\alpha$ is the usual inner product in the Hilbert space $L_\alpha^2(\mathbb{R}_+)$ defined by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{R}_+} f(x)\overline{g(x)}\,\mathrm{d}\mu_\alpha(x). \tag{2.17}$$

The first part of the following theorem is the well-known Heisenberg-type uncertainty inequality for the Fourier–Bessel transform [6,21] and the second part is a stronger version that shows that Laguerre functions are successive optimal on Heisenberg's uncertainty principle.

Theorem 2.1. *For every $f \in L_\alpha^2(\mathbb{R}_+)$,*

$$\|xf\|_{L_\alpha^2}^2 + \|\xi\mathcal{H}_\alpha(f)\|_{L_\alpha^2}^2 \geq (2\alpha+2)\|f\|_{L_\alpha^2}^2, \tag{2.18}$$

with equality if and only if $f(x) = ce^{-x^2/2}$ for some $c \in \mathbb{C}$.

Moreover if f is orthogonal to the sequence $\{\phi_k^\alpha\}_{k=0}^{n-1}$, then

$$\|xf\|_{L_\alpha^2}^2 + \|\xi\mathcal{H}_\alpha(f)\|_{L_\alpha^2}^2 \geq (4n+2\alpha+2)\|f\|_{L_\alpha^2}^2, \tag{2.19}$$

with equality if and only if $f = c_n\phi_n^\alpha$ for some $c_n \in \mathbb{C}$.

Proof. By Parseval's equality for Laguerre expansions we can write

$$\|xf\|_{L_\alpha^2}^2 = 2(\alpha+1)\|f\|_{L_{\alpha+1}^2}^2 = 2(\alpha+1)\sum_{n=0}^{+\infty}\left|\left\langle f, \phi_n^{\alpha+1}\right\rangle_{\alpha+1}\right|^2$$

$$= \frac{1}{2(\alpha+1)} \sum_{n=0}^{+\infty} \left| \left\langle f, x^2 \phi_n^{\alpha+1} \right\rangle_\alpha \right|^2, \tag{2.20}$$

therefore by the recurrence formula (2.13) we obtain

$$\|xf\|_{L^2_\alpha}^2 = \sum_{n=0}^{+\infty} (n+1) \left| \langle f, \phi_{n+1}^\alpha \rangle_\alpha \right|^2 + \sum_{n=0}^{+\infty} (\alpha+n+1) \left| \langle f, \phi_n^\alpha \rangle_\alpha \right|^2. \tag{2.21}$$

In the same way and by taking into account Plancherel theorem (1.4) and (2.14) we obtain also

$$\|\xi \mathcal{H}_\alpha(f)\|_{L^2_\alpha}^2 = \sum_{n=0}^{+\infty} (n+1) \left| \langle f, \phi_{n+1}^\alpha \rangle_\alpha \right|^2 + \sum_{n=0}^{+\infty} (\alpha+n+1) \left| \langle f, \phi_n^\alpha \rangle_\alpha \right|^2. \tag{2.22}$$

Hence

$$\|xf\|_{L^2_\alpha}^2 + \|\xi \mathcal{H}_\alpha(f)\|_{L^2_\alpha}^2 = 2\sum_{n=0}^{+\infty} (\alpha+2n+1) \left| \langle f, \phi_n^\alpha \rangle_\alpha \right|^2 \tag{2.23}$$

$$\geq 2(\alpha+1) \sum_{n=0}^{+\infty} \left| \langle f, \phi_n^\alpha \rangle_\alpha \right|^2 = 2(\alpha+1) \|f\|_{L^2_\alpha}^2. \tag{2.24}$$

Moreover, equality in (2.24) can only occur if $\langle f, \phi_n^\alpha \rangle_\alpha = 0$ for all $n \neq 0$, that is, $f = c\phi_0^\alpha$.

Now, if f is orthogonal to the sequence $\{\phi_k^\alpha\}_{k=0}^{n-1}$, then (2.23) implies

$$\|xf\|_{L^2_\alpha}^2 + \|\xi \mathcal{H}_\alpha(f)\|_{L^2_\alpha}^2 \geq 2(\alpha+2n+1) \sum_{k=n}^{+\infty} \left| \langle f, \phi_k^\alpha \rangle_\alpha \right|^2$$

$$= 2(\alpha+2n+1) \|f\|_{L^2_\alpha}^2. \tag{2.25}$$

Moreover, equality in (2.25) can only occur if $\langle f, \phi_k^\alpha \rangle_\alpha = 0$ for all $k > n$ (*i.e.* $k \neq n$), that is, $f = c_n \phi_n^\alpha$. □

Our proof of Theorem 2.1 is inspired by de Bruijn [7] who proved a sharpened form of the one-dimensional Heisenberg-type inequality for the usual Fourier transform using the Hermite functions (see also [9, Section 3]). Since the sequence of Laguerre functions $\{\phi_n^\alpha\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$ that satisfies (2.14), we could prove the Heisenberg-type inequality (2.18) by using the recurrence relation (2.11) for Laguerre polynomials instead of that of Hermite polynomials in [7].

Now since the measure μ_α is absolutely continuous with respect to the Lebesgue measure and the kernel j_α is homogeneous, we can use a well-known dilation argument largely exploited in [11] (see also [7] and [9, Section 3]) to derive the following corollary.

Corollary 2.2. *For every* $f \in L^2_\alpha(\mathbb{R}_+)$,

$$\|xf\|_{L^2_\alpha} \|\xi \mathcal{H}_\alpha(f)\|_{L^2_\alpha} \geq (\alpha+1) \|f\|_{L^2_\alpha}^2, \tag{2.26}$$

with equality if and only if $f(x) = ce^{-\mu x^2/2}$ *for some* $c \in \mathbb{C}$ *and* $\mu > 0$.

Proof. For $\lambda > 0$ we define the dilation operator on $L^2_\alpha(\mathbb{R}_+)$ by

$$\delta_\lambda f(x) = \frac{1}{\lambda^{\alpha+1}} f\left(\frac{x}{\lambda}\right). \tag{2.27}$$

Then

$$\|\delta_\lambda f\|^2_{L^2_\alpha} = \|f\|^2_{L^2_\alpha} \quad \text{and} \quad \mathcal{H}_\alpha(\delta_\lambda f) = \delta_{\lambda^{-1}}\mathcal{H}_\alpha(f). \tag{2.28}$$

Replacing f by $\delta_\lambda f$ in (2.18) we get

$$\lambda^2\|xf\|^2_{L^2_\alpha} + \lambda^{-2}\|\xi\mathcal{H}_\alpha(f)\|^2_{L^2_\alpha} \geq 2(\alpha+1)\|f\|^2_{L^2_\alpha}. \tag{2.29}$$

Thus (2.26) follows by minimizing the left hand side of that inequality over $\lambda > 0$. Further, equality in (2.26) holds exactly if

$$2\|xf\|_{L^2_\alpha}\|\xi\mathcal{H}_\alpha(f)\|_{L^2_\alpha} = \min_{\lambda>0}\left(\lambda^2\|xf\|^2_{L^2_\alpha} + \lambda^{-2}\|\xi\mathcal{H}_\alpha(f)\|^2_{L^2_\alpha}\right) = 2(\alpha+1)\|f\|^2_{L^2_\alpha}.$$

By the equality cases in (2.18), this condition is satisfied if and only if $f(x) = ce^{-(\lambda x)^2/2}$ for some $c \in \mathbb{C}$. $\square$

Remark 2.3. Notice that in case $\alpha = -1/2$, $\mu_{-1/2}$ is the Lebesgue measure and $\mathcal{H}_{-1/2}$ is the Fourier-cosine transform defined for any even function $f \in L^2(\mathbb{R}_+)$ by

$$\mathcal{H}_{-1/2}(f)(\xi) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} f(x)\cos(x\xi)\,\mathrm{d}x.$$

In other words, $\mathcal{H}_{-1/2}$ is the Fourier transform $\mathcal{F}$ restricted to even functions in the sense that, if $\psi \in L^2(\mathbb{R})$ is even and $f = \psi|_{\mathbb{R}_+}$ the restriction of ψ to $\mathbb{R}_+$, then $\mathcal{F}(\psi)(\xi) = \mathcal{H}_{-1/2}(f)(\xi)$ for $\xi \geq 0$. It follows that Heisenberg's inequalities (1.1) and (2.26) coincide for $\alpha = -1/2$ and $d = 1$.

2.2. *Main dispersion result*

In this section, we use the classical Rayleigh–Ritz technique for estimating eigenvalues of operators to give a quantitative version of Shapiro's theorem. For sake of completeness let us recall the proofs of the following theorem and its corollary which can be found respectively in [19, Theorem XIII.3, p. 82] and [14, Corollary 2.2].

Theorem 2.4 (*The Rayleigh–Ritz Technique*)**.** *Let H be a positive self-adjoint operator and define*

$$\lambda_n(H) = \sup_{e_0,\dots,e_{n-1}} \inf_{\psi\in[e_0,\dots,e_{n-1}]^\perp, \|\psi\|_2=1, \psi\in D(H)} \langle H\psi, \psi\rangle,$$

where $D(H)$ is the domain of H. Let V be a $N+1$ dimensional subspace, $V \subset D(H)$, and let P be the orthogonal projection onto V. Let $H_V = PHP$ and let $\widetilde{H_V}$ denote the restriction of H_V to V. Let $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_N$ be the eigenvalues of $\widetilde{H_V}$. Then

$$\lambda_n(H) \leq \mu_n, \quad n = 0,\dots,N.$$

Proof. By the min–max principle (see [19, Theorem XIII.2, p. 78]), $\widetilde{H_V}$ has eigenvalues given by

$$\begin{aligned}\mu_n &= \sup_{e_0,\dots,e_{n-1}\in V} \inf_{\psi\in[e_0,\dots,e_{n-1}]^\perp, \|\psi\|_2=1, \psi\in V} \langle H\psi, \psi\rangle \\ &= \sup_{e_0,\dots,e_{n-1}} \inf_{\psi\in[Pe_0,\dots,Pe_{n-1}]^\perp, \|\psi\|_2=1, \psi\in V} \langle H\psi, \psi\rangle.\end{aligned}$$

Since $\psi \in V \cap [Pe_0, \dots, Pe_{n-1}]^\perp$, then $\langle \psi, e_i \rangle = \langle \psi, Pe_i \rangle = 0, i = 0, \dots, n-1$. Therefore

$$\begin{aligned}\mu_n &= \sup_{e_0,\dots,e_{n-1}} \inf_{\psi \in [e_0,\dots,e_{n-1}]^\perp, \|\psi\|_2=1, \psi \in V} \langle H\psi, \psi \rangle \\ &\geq \sup_{e_0,\dots,e_{n-1}} \inf_{\psi \in [e_0,\dots,e_{n-1}]^\perp, \|\psi\|_2=1, \psi \in D(H)} \langle H\psi, \psi \rangle ,\end{aligned}$$

which completes the proof of the theorem. □

Corollary 2.5. *Let H be a positive self-adjoint operator, and let $\varphi_0, \dots, \varphi_N$ be an orthonormal set of functions. Then*

$$\sum_{n=0}^{N} \lambda_n(H) \leq \sum_{n=0}^{N} \langle H\varphi_n, \varphi_n \rangle. \tag{2.30}$$

Proof. We may assume that $\varphi_0, \dots, \varphi_N \in D(H)$, since if some $\varphi_n \notin D(H)$ then (2.30) is trivial.

Define the $N+1$ dimensional subspace $V = \operatorname{span} \{\varphi_n\}_{n=0}^N$ and note that the operator $\widetilde{H_V}$ is given by the matrix $M = \left[\langle H\varphi_j, \varphi_k \rangle\right]_{0 \leq j,k \leq N}$. Let $\mu_0, \dots, \mu_N$ be the eigenvalues of $\widetilde{H_V}$, *i.e.* of the matrix M. By Theorem 2.4,

$$\sum_{n=0}^{N} \lambda_n(H) \leq \sum_{n=0}^{N} \mu_n = \operatorname{tr}(M) = \sum_{n=0}^{N} \langle H\varphi_n, \varphi_n \rangle$$

which completes the proof of the corollary. □

Now we will state the main result of this section. The proof of the next theorem is an adaptation of the proof for the usual Fourier transform in [14] by using Laguerre expansions instead of Hermite expansions.

Theorem 2.6. *Let $\{\varphi_n\}_{n=0}^{+\infty}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. Then for all $N \geq 0$*

$$\sum_{n=0}^{N} \left(\left\| x\varphi_n \right\|_{L^2_\alpha}^2 + \left\| \xi \mathcal{H}_\alpha(\varphi_n) \right\|_{L^2_\alpha}^2 \right) \geq 2(N+1)(N+1+\alpha). \tag{2.31}$$

Moreover, if equality holds for all $0 \leq N \leq N_0$, then there exists $\{c_n\}_{n=0}^{N_0} \subset \mathbb{C}$ such that $|c_n| = 1$ and $\varphi_n = c_n \phi_n^\alpha$ for each $0 \leq n \leq N_0$.

Proof. From (2.16) and (2.23) we have

$$\langle \mathcal{L}_\alpha f, f \rangle_\alpha = \sum_{n=0}^{+\infty} (4n + 2\alpha + 2) \left| \langle f, \phi_n^\alpha \rangle_\alpha \right|^2 = \|xf\|_{L^2_\alpha}^2 + \|\xi \mathcal{H}_\alpha(f)\|_{L^2_\alpha}^2. \tag{2.32}$$

In particular

$$\langle \mathcal{L}_\alpha \varphi_n, \varphi_n \rangle_\alpha = \left\| x\varphi_n \right\|_{L^2_\alpha}^2 + \left\| \xi \mathcal{H}_\alpha(\varphi_n) \right\|_{L^2_\alpha}^2.$$

It follows then from Corollary 2.5 that for each $N \geq 0$ one has

$$\sum_{n=0}^{N} (4n + 2\alpha + 2) \leq \sum_{n=0}^{N} \langle \mathcal{L}_\alpha \varphi_n, \varphi_n \rangle_\alpha = \sum_{n=0}^{N} \left(\left\| x\varphi_n \right\|_{L^2_\alpha}^2 + \left\| \xi \mathcal{H}_\alpha(\varphi_n) \right\|_{L^2_\alpha}^2 \right). \tag{2.33}$$

This completes the proof of (2.31).

Now assume equality holds in (2.33) for all $N = 0, \dots, N_0$, then for each $N = 0, \dots, N_0$,

$$\langle \mathcal{L}_\alpha \varphi_N, \varphi_N \rangle_\alpha = \left\| x\varphi_N \right\|^2_{L^2_\alpha} + \left\| \xi \mathcal{H}_\alpha(\varphi_N) \right\|^2_{L^2_\alpha} = (4N + 2\alpha + 2). \tag{2.34}$$

Let us first apply (2.32) for $f = \varphi_0$. Then from (2.34) and the fact that $\|\varphi_0\|^2_{L^2_\alpha} = \sum_{n=0}^{+\infty} \left|\left\langle \varphi_0, \phi_n^\alpha \right\rangle_\alpha\right|^2 = 1$,

$$\sum_{n=0}^{+\infty} (4n + 2\alpha + 2) \left|\left\langle \varphi_0, \phi_n^\alpha \right\rangle_\alpha\right|^2 = \langle \mathcal{L}_\alpha \varphi_0, \varphi_0 \rangle_\alpha = (2\alpha + 2) = \sum_{n=0}^{+\infty} (2\alpha + 2) \left|\left\langle \varphi_0, \phi_n^\alpha \right\rangle_\alpha\right|^2.$$

Thus, for all $n \neq 0$, one has $\left\langle \varphi_0, \phi_n^\alpha \right\rangle_\alpha = 0$ and hence $\varphi_0 = c_0 \phi_0^\alpha$, with $|c_0| = 1$, since $\|\varphi_0\|_{L^2_\alpha} = 1$.

Next, assume that we have proved $\varphi_n = c_n \phi_n^\alpha$ for $n = 0, \dots, N-1$. Since φ_N is orthogonal to φ_n for $n < N$, one has $\left\langle \varphi_N, \phi_n^\alpha \right\rangle = 0$. Applying now (2.32) for $f = \varphi_N$ we obtain with the same way that $\left\langle \varphi_N, \phi_n^\alpha \right\rangle_\alpha = 0$ for all $n \neq N$. Then $\varphi_N = c_N \phi_N^\alpha$ and $|c_N| = 1$, since $\|\varphi_N\|_{L^2_\alpha} = 1$. □

Remark 2.7. We will exploit the well-known relation between Hermite and Laguerre polynomials to show that Inequalities (2.31) and (1.7) coincide for the critical case $\alpha = -1/2$. To make this precise, recall, see [15, Chap. 4, p. 66], that the normalized Hermite functions $\{h_n\}_{n=0}^{+\infty}$ are defined on the real line by

$$h_n(x) = \left(2^{n-1/2} n!\right)^{-1/2} e^{-x^2/2} H_n(x), \tag{2.35}$$

form an orthonormal basis for the Hilbert space $L^2\left(\mathbb{R}, \frac{\mathrm{d}x}{\sqrt{2\pi}}\right)$. Here H_n is the Hermite polynomial of degree n defined by the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2}), \quad n \in \mathbb{N}. \tag{2.36}$$

Consequently any even function $f \in L^2\left(\mathbb{R}, \frac{\mathrm{d}x}{\sqrt{2\pi}}\right)$ admits the expansion

$$f = \sum_{n=0}^{+\infty} \langle f, h_{2n} \rangle\, h_{2n}, \tag{2.37}$$

since h_{2n} are even and h_{2n+1} are odd.

It is well-known that the normalized Hermite functions are eigenfunctions of the Fourier transform and of the Hermite operator (or harmonic oscillator) $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$, satisfy in particular, see [9, Section 3],

$$H h_{2n} = (4n+1) h_{2n} \quad \text{and} \quad \mathcal{F}(h_{2n}) = (-1)^n h_{2n}. \tag{2.38}$$

Now since, see [15, (4.19.5), p. 81],

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2), \tag{2.39}$$

then it immediately follows that

$$h_{2n} = \phi_n^{-1/2}, \quad n \in \mathbb{N}. \tag{2.40}$$

From this we deduce that for $\alpha = -1/2$, Inequalities (2.31) and (1.7) coincide for even functions with the optimal constant $\sum_{n=0}^N (4n+1) = (N+1)(2N+1)$.

The last theorem implies in particular that, there does not exist an infinite sequence $\{\varphi_n\}_{n=0}^{+\infty}$ in $L^2_\alpha(\mathbb{R}_+)$ such that the sequences of dispersions $\left\{\|x\varphi_n\|_{L^2_\alpha}\right\}_{n=0}^{+\infty}$ and $\left\{\|\xi\mathcal{H}_\alpha(\varphi_n)\|_{L^2_\alpha}\right\}_{n=0}^{+\infty}$ are all bounded.

Corollary 2.8. *Fix $A > 0$ and let $\{\varphi_n\}_{n=0}^{+\infty}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. If*

$$\left\|x\varphi_n\right\|_{L^2_\alpha},\ \left\|\xi\mathcal{H}_\alpha(\varphi_n)\right\|_{L^2_\alpha} \le A,$$

then the sequence has at most A^2 elements. In particular for every $N \ge 0$

$$\sup_{0\le n\le N}\left\{\left\|x\varphi_n\right\|^2_{L^2_\alpha},\ \left\|\xi\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right\} \ge N+\alpha+1. \tag{2.41}$$

Proof. From Theorem 2.6, we have for every $N \ge 0$,

$$2(N+1)A^2 \ge \sum_{n=0}^{N}\left(\left\|x\varphi_n\right\|^2_{L^2_\alpha} + \left\|\xi\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right) \ge 2(N+1)(N+\alpha+1),$$

then $A^2 \ge N+\alpha+1 > N$. $\square$

3. Strong uncertainty principle in terms of generalized dispersions

In this section we will prove a strong uncertainty principle (Theorem B) for orthonormal bases for $L^2_\alpha(\mathbb{R}_+)$ involving generalized dispersions with respect to t^s power weight dispersions, $s > 0$. Our proof here is inspired from similar results established in [16]. To do so we will use the time-limiting and the frequency-limiting operators on $L^2_\alpha(\mathbb{R}_+)$ defined by

$$E_S f = \chi_S f, \qquad F_\Sigma f = \mathcal{H}_\alpha\Big[\chi_\Sigma\mathcal{H}_\alpha(f)\Big],$$

where S and Σ are measurable subsets of $\mathbb{R}_+$ of finite measure $\mu_\alpha(S),\ \mu_\alpha(\Sigma) < +\infty$ and χ_A denotes the characteristic function of the set $A \subset \mathbb{R}_+$.

A straightforward computation shows $F_\Sigma E_S$ is an integral operator with kernel (see [10, Lemma 4.2])

$$\mathcal{N}(x,\xi) = \chi_S(x)\mathcal{H}_\alpha\big(\chi_\Sigma j_\alpha(x\,\cdot)\big).$$

Then $F_\Sigma E_S$ is a Hilbert–Schmidt operator with

$$\|F_\Sigma E_S\|^2_{HS} \le \mu_\alpha(S)\mu_\alpha(\Sigma). \tag{3.42}$$

The phase space restriction operator is defined by

$$L_{S,\Sigma} = (F_\Sigma E_S)^* F_\Sigma E_S = E_S F_\Sigma E_S,$$

where $(F_\Sigma E_S)^* = E_S F_\Sigma$.

Theorem 3.1. *Let $\{\varphi_n\}_{n=1}^{N}$ be an orthonormal system in $L^2_\alpha(\mathbb{R}_+)$. If*

$$\|E_{S^c}\varphi_n\|^2_{L^2_\alpha} \le a_n^2 \quad \textit{and} \quad \|F_{\Sigma^c}\varphi_n\|^2_{L^2_\alpha} \le b_n^2,$$

then

$$\sum_{n=1}^{N}\left(1-\frac{3}{2}a_n-\frac{3}{2}b_n\right) < \mu_\alpha(S)\mu_\alpha(\Sigma). \tag{3.43}$$

Proof. We will apply a standard estimate of the trace of the time–frequency restriction operator $L_{S,\Sigma}$ to conclude that

$$\operatorname{tr}\left(L_{S,\Sigma}\right) = \|F_\Sigma E_S\|_{HS}^2 \le \mu_\alpha(S)\mu_\alpha(\Sigma).$$

Then

$$\sum_{n=1}^{N} \left\langle L_{S,\Sigma}\varphi_n, \varphi_n\right\rangle_\alpha \le \operatorname{tr}\left(L_{S,\Sigma}\right) \le \mu_\alpha(S)\mu_\alpha(\Sigma).$$

On the other hand, as the identity operator $I = E_S + E_{S^c} = F_\Sigma + F_{\Sigma^c}$, then

$$\begin{aligned}\left\langle L_{S,\Sigma}\varphi_n, \varphi_n\right\rangle_\alpha &= \langle F_\Sigma E_S\varphi_n, E_S\varphi_n\rangle_\alpha \\ &= \langle \varphi_n, \varphi_n\rangle_\alpha - \langle E_{S^c}\varphi_n, \varphi_n\rangle_\alpha - \langle E_S\varphi_n, F_{\Sigma^c}\varphi_n\rangle_\alpha - \langle F_\Sigma E_S\varphi_n, E_{S^c}\varphi_n\rangle_\alpha .\end{aligned}$$

Therefore $\left\langle L_{S,\Sigma}\varphi_n, \varphi_n\right\rangle_\alpha > 1 - 2a_n - b_n$ and

$$\sum_{n=1}^{N} (1 - 2a_n - b_n) < \mu_\alpha(S)\mu_\alpha(\Sigma). \tag{3.44}$$

Now if we consider the operator $\tilde{L}_{S,\Sigma} = (E_S F_\Sigma)^* E_S F_\Sigma = F_\Sigma E_S F_\Sigma$, we obtain similarly

$$\sum_{n=1}^{N} (1 - a_n - 2b_n) < \mu_\alpha(S)\mu_\alpha(\Sigma). \tag{3.45}$$

Combining (3.44) and (3.45), we deduce the desired result. □

Definition 3.2. Let $0 < \varepsilon < 1$ and $f \in L^2_\alpha(\mathbb{R}_+)$. Then

(1) we say that f is ε-concentrated on S if $\|E_{S^c} f\|_{L^2_\alpha} \le \varepsilon\|f\|_{L^2_\alpha}$,
(2) we say that f is ε-bandlimited on Σ if $\|F_{\Sigma^c} f\|_{L^2_\alpha} \le \varepsilon\|f\|_{L^2_\alpha}$.

It is clear that, if f is ε-bandlimited on Σ then by Plancherel theorem (1.4), $\mathcal{H}_\alpha(f)$ is ε-concentrated on Σ.

From Theorem 3.1, we can obtain immediately the following corollary.

Corollary 3.3. *Let $a, b > 0$ and $0 < \varepsilon_1, \varepsilon_2 < 1$ such that $\varepsilon_1 + \varepsilon_2 < \frac{2}{3}$. Let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system in $L^2_\alpha(\mathbb{R}_+)$. If φ_n is ε_1-concentrated on $[0, a]$ and ε_2-bandlimited on $[0, b]$, then*

$$N \le \frac{1}{1 - 3/2(\varepsilon_1 + \varepsilon_2)} \left[\frac{(ab)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha + 2)}\right]^2.$$

Therefore if the generalized dispersions of the elements of an orthonormal sequence are uniformly bounded then this sequence is finite and we can give a bound on the number of elements in that sequence. More precisely:

Corollary 3.4. *Fix $A, B > 0$. Let $s > 0$ and $\{\varphi_n\}_{n=1}^N$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$ that satisfies $\left\|x^s\varphi_n\right\|_{L^2_\alpha}^{1/s} \le A$ and $\left\|\xi^s\,\mathcal{H}_\alpha(\varphi_n)\right\|_{L^2_\alpha}^{1/s} \le B$. Then there exists a positive constant $c(s, \alpha)$ such that*

$$N \le c(s, \alpha)(AB)^{2(\alpha+1)}.$$

Proof. As $\left\|x^s\varphi_n\right\|^2_{L^2_\alpha} \le A^{2s}$, then

$$\int_{x>4^{\frac{1}{s}}A} |\varphi_n(x)|^2 \,\mathrm{d}\mu_\alpha(x) = \int_{x>4^{\frac{1}{s}}A} x^{-2s}x^{2s}|\varphi_n(x)|^2 \,\mathrm{d}\mu_\alpha(x) \le \frac{1}{16A^{2s}}\left\|x^s\varphi_n\right\|^2_{L^2_\alpha} \le \frac{1}{16}.$$

In the same way we get

$$\int_{\xi>4^{\frac{1}{s}}B} |\mathcal{H}_\alpha(\varphi_n)(\xi)|^2 \,\mathrm{d}\mu_\alpha(\xi) \le \frac{1}{16}.$$

Thus φ_n is $\frac{1}{4}$-concentrated on $\left[0, 4^{\frac{1}{s}}A\right]$ and $\frac{1}{4}$-bandlimited on $\left[0, 4^{\frac{1}{s}}B\right]$. Therefore from Corollary 3.3 we obtain the desired result. □

Lemma 3.5. *Let S and Σ be measurable subsets of finite measure $\mu_\alpha(S)$, $\mu_\alpha(\Sigma) < +\infty$. Then there exists a nonzero function $f \in L^2_\alpha(\mathbb{R}_+)$ such that* supp $f \subset S^c$ *and* supp $\mathcal{H}_\alpha(f) \subset \Sigma^c$.

Proof. Let $PW_\alpha(\Sigma) = \left\{f \in L^2_\alpha(\mathbb{R}_+) : \operatorname{supp}\mathcal{H}_\alpha(f) \subset \Sigma\right\}$ be the Paley–Wiener space. As (S, Σ) form a strong annihilating pair, then from [10, Theorem A] there exists a positive constant $c = c(\alpha, S, \Sigma)$ such that for all $f \in PW_\alpha(\Sigma)$

$$\|f\|_{L^2_\alpha} \le c\|f\|_{L^2_\alpha(S^c)}. \tag{3.46}$$

The last inequality implies in particular that the restriction map $f \to f|_{S^c}$ is invertible on $PW_\alpha(\Sigma)$ and the trace space $PW_\alpha(\Sigma)|_{S^c} = \{f|_{S^c} : f \in PW_\alpha(\Sigma)\}$ form a closed subspace in $L^2_\alpha(S^c)$ which is obviously not the whole space. Thus there exists a nonzero function $f \in L^2_\alpha(S^c)$ such that $f \notin PW_\alpha(\Sigma)|_{S^c}$, *i.e.* f is supported in S^c and its Fourier–Bessel transform is supported in Σ^c. We extend f by zero on S in order to get the required function. □

Lemma 3.5 is a special form of uncertainty principle and it is well-known for the usual Fourier transform, which follows for example from [1, Proposition 3] (see also [17]). Moreover there are several examples of the uncertainty inequality of the form (3.46) for the Fourier transform, one of them is the Amrein–Berthier theorem [1] which is a quantitative version of a result due to Benedicks [3] showing that a pair of sets of finite Lebesgue measure is an annihilating pair.

Remark 3.6. Let S be a measurable subset of $\mathbb{R}_+$. Then the Lebesgue measure of S, $|S|$ satisfies

$$|S| = \int_0^1 \chi_S(x)\,\mathrm{d}x + \int_1^{+\infty} \chi_S(x)\,\mathrm{d}x \le 1 + c_\alpha\mu_\alpha(S), \tag{3.47}$$

which means that the condition $|S| < +\infty$ is weaker than $\mu_\alpha(S) < +\infty$. Therefore we can obtain slightly stronger results, namely by replacing in all this section the hypothesis $\mu_\alpha(S)$, $\mu_\alpha(\Sigma) < +\infty$ by $|S|$, $|\Sigma| < +\infty$ since in this case the Hilbert–Schmidt norm of $F_\Sigma E_S$ satisfies, see [10, Lemma 4.2],

$$\|F_\Sigma E_S\|^2_{HS} \le \kappa_\alpha\,|S|\,|\Sigma|\,, \tag{3.48}$$

and then the localization inequality (3.43) becomes

$$\sum_{n=1}^N \left(1 - \frac{3}{2}a_n - \frac{3}{2}b_n\right) < \kappa_\alpha\,|S|\,|\Sigma|\,. \tag{3.49}$$

Theorem 3.7. *Let $s > 0$ and let $\{\varphi_n\}_{n=1}^{+\infty}$ be an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$. Then*

$$\sup_n \left(\left\| x^s \varphi_n \right\|_{L^2_\alpha} \left\| \xi^s \mathcal{H}_\alpha(\varphi_n) \right\|_{L^2_\alpha} \right) = +\infty.$$

Proof. Assume that there exists an orthonormal basis $\{\varphi_n\}_{n=1}^{+\infty}$ such that

$$\left\| x^s \varphi_n \right\|^{1/s}_{L^2_\alpha} \left\| \xi^s \mathcal{H}_\alpha(\varphi_n) \right\|^{1/s}_{L^2_\alpha} \le C^2.$$

Let $k \in \mathbb{Z}$ and let

$$A_k = \left\{ \varphi_n : \left\| x^s \varphi_n \right\|^{1/s}_{L^2_\alpha} \in \left(2^{-k} C, 2^{-k+1} C \right] \right\}.$$

Clearly, $\{\varphi_n\}_{n=1}^{+\infty} = \bigcup_k A_k$, and for $\varphi_n \in A_k$, we have

$$\left\| x^s \varphi_n \right\|^{1/s}_{L^2_\alpha} \le 2^{-k+1} C \quad \text{and} \quad \left\| \xi^s \mathcal{H}_\alpha(\varphi_n) \right\|^{1/s}_{L^2_\alpha} \le C 2^k.$$

Then by Corollary 3.4, A_k is finite, and if N_k is the number of elements in A_k then N_k is bounded by a constant $c_{\alpha,s}$ that does not depend on k.

Let $R > 0$, then by using Lemma 3.5, we take a nonzero function $f \in L^2_\alpha(\mathbb{R}_+)$ with $\|f\|_{L^2_\alpha} = 1$, that vanishes on $[0, R]$ with its Fourier–Bessel transform. Then for $k \ge 0$ and $\varphi_n \in A_k$ we get by the Cauchy–Schwarz inequality that

$$\left| \langle f, \varphi_n \rangle_\alpha \right|^2 \le R^{-2s} \|f\|^2_{L^2_\alpha} \left\| x^s \varphi_n \right\|^2_{L^2_\alpha} \le (2CR^{-1})^{2s} 4^{-sk}. \tag{3.50}$$

Similarly, for $k < 0$ and $\varphi_n \in A_k$ we get by Plancherel theorem (1.4),

$$\begin{aligned} \left| \langle f, \varphi_n \rangle_\alpha \right|^2 &= \left| \langle \mathcal{H}_\alpha(f), \mathcal{H}_\alpha(\varphi_n) \rangle_\alpha \right|^2 \le R^{-2s} \|f\|^2_{L^2_\alpha} \left\| \xi^s \mathcal{H}_\alpha(\varphi_n) \right\|^2_{L^2_\alpha} \\ &\le (CR^{-1})^{2s} 4^{sk}. \end{aligned} \tag{3.51}$$

Now as $\{\varphi_n\}_{n=1}^{+\infty}$ is a basis for $L^2_\alpha(\mathbb{R}_+)$,

$$1 = \|f\|^2_{L^2_\alpha} = \sum_k \sum_{\varphi_n \in A_k} \left| \langle f, \varphi_n \rangle_\alpha \right|^2,$$

then by combining Inequalities (3.50) and (3.51) we obtain

$$\begin{aligned} 1 &\le (2CR^{-1})^{2s} \sum_{k=0}^{+\infty} 4^{-sk} N_k + c_{\alpha,s} (CR^{-1})^{2s} \sum_{k=1}^{+\infty} 4^{-sk} N_{-k} \\ &\le c_{\alpha,s} (2CR^{-1})^{2s} \sum_{k=0}^{+\infty} 4^{-sk} + c_{\alpha,s} (CR^{-1})^{2s} \sum_{k=1}^{+\infty} 4^{-sk} \\ &\le \frac{4 c_{\alpha,s} (2C)^{2s}}{3 R^{2s}}. \end{aligned}$$

Choosing R large enough, we get a contradiction. The theorem is proved. $\square$

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