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Notes

Minimal degree univariate piecewise polynomials with prescribed Sobolev regularity

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Abstract

For $k \in \{1, 2, 3, \ldots\}$, we construct an even compactly supported piecewise polynomial ψ_k whose Fourier transform satisfies $A_k(1+\omega^2)^{-k} \le \widehat{\psi}_k(\omega) \le B_k(1+\omega^2)^{-k}$, $\omega \in \mathbb{R}$, for some constants $B_k \ge A_k > 0$. The degree of ψ_k is shown to be minimal, and is strictly less than that of Wendland's function $\phi_{1,k-1}$ when $k > 2$. This shows that, for $k > 2$, Wendland's piecewise polynomial $\phi_{1,k-1}$ is not of minimal degree if one places no restrictions on the number of pieces.

Keywords: Positive definite function; Compactly supported; B-spline

1. Introduction

A function $\Phi \in L_1(\mathbb{R}^d)$ is said to have *Sobolev regularity* $k > 0$ if its Fourier transform $\widehat{\Phi}(\omega) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(x) e^{-\imath x \cdot \omega}\, dx$ satisfies

$$A(1+\|\omega\|^2)^{-k} \le \widehat{\Phi}(\omega) \le B(1+\|\omega\|^2)^{-k}, \quad \omega \in \mathbb{R}^d,$$

for some constants $B \ge A > 0$. Such functions are useful in radial basis function methods since the generated native space will equal the Sobolev space $W_2^k(\mathbb{R}^d)$. The reader is referred to Schaback [3] for a description of the current state of the art in the construction of compactly supported functions Φ having prescribed Sobolev regularity. Wendland (see [4,5]) has constructed radial functions $\Phi_{d,\ell}(x) = \phi_{d,\ell}(\|x\|)$, where $\phi_{d,\ell}$ is a piecewise polynomial of the form $\phi_{d,\ell}(t) = \begin{cases} p(|t|), & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$, p being a polynomial. For $d \in \{1, 2, 3, \ldots\}$ and $\ell \in \{0, 1, 2, \ldots\}$,

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with the case $d = 1, \ell = 0$ excluded, $\Phi_{d,\ell}$ has Sobolev regularity $k = \ell + (d+1)/2$ and the degree of the piecewise polynomial $\phi_{d,\ell}$, namely $\lfloor d/2 \rfloor + 3\ell + 1$, is minimal with respect to this property. A natural question to ask is that of whether the degree of $\phi_{d,\ell}$ would still be minimal if we considered functions of the form $\Phi(x) = \phi(\|x\|)$ where ϕ is a piecewise polynomial having bounded support. In this note, we answer this question for the univariate case $d = 1$. Specifically, we construct a compactly supported even piecewise polynomial ψ_k, with Sobolev regularity k (see Theorem 2.8), and we show that the degree of ψ_k, namely $2k$, is minimal (see Theorem 2.10). From a comparison with Wendland's function $\Phi_{1,k-1}$ (which has Sobolev regularity k when $k > 1$), we see that $\deg \psi_k = \deg \phi_{1,k-1}$ if $k = 2$, while $\deg \psi_k = 2k < 3k - 2 = \deg \phi_{1,k-1}$ when $k > 2$.

2. Results

Wendland's piecewise polynomial $\phi_{d,\ell}$ can be identified as a constant multiple of the B-spline having $\ell + 1$ knots at the nodes -1 and 1 and $\lfloor d/2 \rfloor + \ell + 1$ knots at 0. This can be verified simply by observing that $\phi_{d,\ell}$ and the above-mentioned B-spline have the same degree, $\lfloor d/2 \rfloor + 3\ell + 1$, and satisfy the same number of continuity conditions across each of the nodes -1,0,1, namely $\lfloor d/2 \rfloor + 2\ell + 1$ at -1,1 and $2\ell + 1$ at 0. It is well understood in the theory of B-splines that multiple knots are to be avoided if one wishes to keep the degree low, and with this in mind, we define ψ_k as follows. For $k = 1, 2, 3, \ldots$, let ψ_k be the B-spline having knots $-k, \ldots, -2, -1, 0; 0, 1, 2, \ldots, k$ (note that 0 is the only double knot). For easy reference, we display $\psi_k(t)$ (normalized) for $t \in [0, k]$ and $k = 1, 2, 3$:

$$\psi_1(t) = (1-t)^2, \qquad \psi_2(t) = \begin{cases} 8 - 24t^2 + 24t^3 - 7t^4, & t \in [0,1] \\ (2-t)^4, & t \in (1,2] \end{cases}$$

$$\psi_3(t) = \begin{cases} 198 - 270t^2 + 270t^4 - 180t^5 + 37t^6, & t \in [0,1] \\ 153 + 270t - 945t^2 + 900t^3 - 405t^4 + 90t^5 - 8t^6, & t \in (1,2] \\ (3-t)^6, & t \in (2,3]. \end{cases}$$

We begin by mentioning several salient facts about the B-spline ψ_k which can be found in [1, pp. 108–131]. The piecewise polynomial ψ_k is supported on $[-k, k]$, positive on $(-k, k)$, even and of degree $2k$. Furthermore, it is $2k - 1$ times continuously differentiable on $\mathbb{R} \setminus \{0\}$ and $2k - 2$ times continuously differentiable on all of $\mathbb{R}$. It follows from this that the $2k - 1$-order derivative, $D^{2k-1}\psi_k$, is a piecewise linear function which is supported on $[-k, k]$ and is continuous except at the origin where it has a jump discontinuity. Consequently, the $2k$-order derivative has the form

$$D^{2k}\psi_k = \sqrt{2\pi} a_0 \delta_0 + \sum_{j=1}^{k} \sqrt{2\pi} a_j (\chi_{[j-1,j)} + \chi_{[-j,1-j)}),$$

for some constants $a_0, a_1, a_2, \ldots, a_k$ and where δ_0 is the Dirac δ-distribution defined by $\delta_0(f) = f(0)$. We can thus express the Fourier transform of $D^{2k}\psi_k$ as

$$\begin{aligned} \left(D^{2k}\psi_k\right)\hat{}(\omega) &= a_0 + 2\sum_{j=1}^{k} a_j \frac{\sin(j\omega) - \sin((j-1)\omega)}{\omega} \\ &= a_0 + \sum_{j=1}^{k} 2(a_j - a_{j+1}) \frac{\sin(j\omega)}{\omega}, \end{aligned}$$

with $a_{k+1} := 0$, whence it follows that

$$\widehat{\psi}_k(\omega) = (\imath\omega)^{-2k}\left(D^{2k}\psi_k\right)\widehat{\ }(\omega) = \frac{(-1)^k}{\omega^{2k+1}}\left(a_0\omega + \sum_{j=1}^{k} 2(a_j - a_{j+1})\sin(j\omega)\right). \tag{2.1}$$

Lemma 2.2. *Let $\beta \in \mathbb{R}$. Then there exist unique scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that*

$$\left|\beta + \sum_{j=1}^{k} c_j \cos(j\omega)\right| = O(|\omega|^{2k}) \quad \text{as } \omega \to 0. \tag{2.3}$$

Proof. Define $g(w) = \beta + \sum_{i=1}^{k} c_i \cos(i\omega)$. Since $g \in C^\infty(\mathbb{R})$ is even, (2.3) holds if and only if $D^{2\ell}g(0) = 0$ for $\ell = 0, 1, 2, \ldots, k-1$. These conditions form the system of linear equations $[c_1, c_2, \ldots, c_k]A = [-\beta, 0, 0, \ldots, 0]$, where A is the $k \times k$ matrix given by $A(i, j) = (-1)^{j-1} i^{2j-2}$. Writing $A(i, j) = (-i^2)^{j-1}$, we recognize A as a nonsingular Vandermonde matrix, and therefore, (2.3) holds if and only if $[c_1, c_2, \ldots, c_k] = [-\beta, 0, 0, \ldots, 0]A^{-1}$. $\square$

Theorem 2.4. *Let $\beta, c_1, c_2, \ldots, c_k \in \mathbb{R}$ be such that (2.3) holds. Then*

$$\beta + \sum_{j=1}^{k} c_j \cos(j\omega) = \beta\alpha_k(1 - \cos\omega)^k, \quad \omega \in \mathbb{R}, \tag{2.5}$$

where $\alpha_k > 0$ is defined by $\frac{1}{\alpha_k} = \frac{1}{\pi}\int_0^\pi (1 - \cos\omega)^k \, d\omega$.

Proof. Since $\cos^j \omega \in \operatorname{span}\{1, \cos\omega, \cos 2\omega, \ldots, \cos k\omega\}$ for $j = 0, 1, \ldots, k$, it follows that there exist $b_j \in \mathbb{R}$ such that $(1 - \cos\omega)^k = b_0 + \sum_{j=1}^{k} b_j \cos(j\omega)$. Note that

$$0 < \frac{1}{\alpha_k} = \frac{1}{\pi}\int_0^\pi (1 - \cos\omega)^k \, d\omega = \frac{1}{\pi}\int_0^\pi b_0 \, d\omega + \sum_{j=1}^{k} b_j \frac{1}{\pi}\int_0^\pi \cos(j\omega)\, d\omega = b_0,$$

and hence $\beta\alpha_k(1 - \cos\omega)^k = \beta + \sum_{j=1}^{k} \beta\alpha_k b_j \cos(j\omega)$. Since $\left|\beta\alpha_k(1 - \cos\omega)^k\right| = O(|\omega|^{2k})$ as $\omega \to 0$, it follows from the lemma that $c_j = \beta\alpha_k b_j$ for $j = 1, 2, \ldots, k$, and therefore (2.5) holds. $\square$

Corollary 2.6. *Let a_0 be as in (2.1). Then $(-1)^k a_0 > 0$ and*

$$\widehat{\psi}_k(\omega) = \frac{(-1)^k a_0 \alpha_k}{\omega^{2k+1}} \int_0^\omega (1 - \cos t)^k \, dt, \quad \omega \neq 0. \tag{2.7}$$

Proof. It follows from (2.1) that $\widehat{\psi}_k(\omega) = \frac{(-1)^k}{\omega^{2k+1}} f(\omega)$, where $f(\omega) := a_0\omega + \sum_{j=1}^{k} 2(a_j - a_{j+1})\sin(j\omega)$. Since ψ_k is supported on $[-k, k]$ and positive on $(-k, k)$, it follows that $\widehat{\psi}_k$ is continuous (in fact entire) and $\widehat{\psi}_k(0) > 0$. Consequently, $|f(\omega)| = O(|\omega|^{2k+1})$ as $\omega \to 0$. Since f is infinitely differentiable, it follows that $\left|f'(\omega)\right| = \left|a_0 + \sum_{j=1}^{k} 2j(a_j - a_{j+1})\cos(j\omega)\right| = O(|\omega|^{2k})$ as $\omega \to 0$, and so by Theorem 2.4, $f'(\omega) = a_0\alpha_k(1 - \cos\omega)^k$. Since $f(0) = 0$, we can write $f(\omega) = \int_0^\omega f'(t)\, dt = a_0\alpha_k \int_0^\omega (1 - \cos t)^k \, dt$, and hence obtain (2.7). That $(-1)^k a_0 > 0$ is now evident since $0 < \widehat{\psi}_k(0) = \lim_{\omega \to 0^+} \widehat{\psi}_k(w)$. $\square$

Remark. At this point, it is also easy to show that

$$\widehat{\psi}_k(\omega) = \frac{(-1)^k a_0}{\omega^{2k+1}} \left(\omega + \sum_{j=1}^{k} b_j \sin(j\omega) \right), \quad \omega \neq 0,$$

where the scalars $\{b_j\}$ are determined by the fact that $\widehat{\psi}_k$ is continuous at 0.

Theorem 2.8. *For $k \in \{1, 2, 3, \ldots\}$, ψ_k has Sobolev regularity k; that is, there exist constants $B_k \geq A_k > 0$ such that*

$$A_k(1 + |\omega|^2)^{-k} \leq \widehat{\psi}_k(\omega) \leq B_k(1 + |\omega|^2)^{-k}, \quad \omega \in \mathbb{R}. \tag{2.9}$$

Proof. As in the proof of Corollary 2.6, let us write $\widehat{\psi}_k(\omega) = \frac{(-1)^k}{\omega^{2k+1}} f(\omega)$, where $f(\omega) := a_0\omega + \sum_{j=1}^{k} 2(a_j - a_{j+1}) \sin(j\omega)$. Since $\lim_{\omega\to\infty} \frac{f(\omega)}{\omega} = a_0$, it follows that $\lim_{w\to\infty} w^{2k} \psi_k(w) = (-1)^k a_0$. Since $(-1)^k a_0 > 0$ (by Corollary 2.6), it follows that there exists $N > 0$ such that (2.9) holds for $\omega \geq N$. That $\widehat{\psi}_k(\omega) > 0$ for all $\omega > 0$ follows easily from Corollary 2.6, and since $\widehat{\psi}_k$ is continuous and $\widehat{\psi}_k(0) > 0$, we see that (2.9) holds for $0 \leq \omega \leq N$. We finally conclude that (2.9) holds for all $\omega \in \mathbb{R}$ since $\widehat{\psi}_k$ is an even function. □

We now show that the degree of ψ_k is minimal.

Theorem 2.10. *If ψ is an even, compactly supported, piecewise polynomial satisfying (2.9), then the degree of ψ is at least $2k$.*

Proof. Let ψ be an even, compactly supported piecewise polynomial satisfying (2.9) and let the ℓth derivative of ψ be the first discontinuous derivative of ψ (if ψ is itself discontinuous then $\ell = 0$). Then $D^{\ell+1}\psi$ can be written as

$$D^{\ell+1}\psi = g + \sum_{j=1}^{n} \sqrt{2\pi} c_j \delta_{x_j}, \tag{2.11}$$

where $g \in L_1(\mathbb{R})$ and c_j is the height (possibly 0) of the jump discontinuity at x_j. We can then express the Fourier transform of ψ as

$$\widehat{\psi}(\omega) = (\imath\omega)^{-\ell-1} \left(D^{\ell+1}\psi \right)\widehat{\ }(\omega) = (\imath\omega)^{-\ell-1} (\widehat{g}(\omega) + \Theta(\omega)),$$

where $\Theta(\omega) = \sum_{j=1}^{n} c_j e^{-\imath x_j \omega}$. Since Θ is bounded and $|\widehat{g}(\omega)|$ has limit 0 as $|\omega| \to \infty$ (by the Riemann–Lebesgue lemma), it follows that $\left|\widehat{\psi}(\omega)\right| = O(|\omega|^{-\ell-1})$ as $|\omega| \to \infty$. In view of the left side of (2.9), we conclude that $\ell + 1 \leq 2k$. Since Θ is a non-trivial almost periodic function (see [2, pp. 9–14]), it follows that $|\Theta(\omega)| \neq o(1)$ as $|\omega| \to \infty$, and in view of the right side of (2.9), we see that $\ell + 1 \geq 2k$. Therefore, $\ell + 1 = 2k$ and we conclude that ψ_k is $2k - 2$ times continuously differentiable. Since ψ_k is compactly supported (i.e. ψ_k is not a polynomial), it follows that ψ_k has degree at least $2k - 1$ (see [1, pp. 96–120]). In order to show that the degree of ψ_k is at least $2k$, let us assume to the contrary that the degree equals $2k - 1$. In this case the $\ell = 2k - 1$ derivative of ψ_k is piecewise constant and hence $g = 0$ and $\widehat{\psi}(\omega) = (-1)^k \omega^{-2k} \Theta(\omega)$. Since $\widehat{\psi}$ is continuous at 0, it follows that $\Theta(0) = 0$. Since Θ is an almost periodic function, there exist values $\omega_1 < \omega_2 < \cdots$ such that $\lim_{n\to\infty} \omega_n = \infty$ and $\lim_{n\to\infty} \Theta(\omega_n) = 0$; but this contradicts the left side of (2.9). Therefore, ψ has degree at least $2k$. □

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