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Journal of Approximation Theory 248 (2019) 105301

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

L^p approximation of completely monotone functions

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Received 10 September 2018; received in revised form 18 April 2019; accepted 10 September 2019
Available online 20 September 2019

Communicated by Tamás Erdélyi

Abstract

We show that any completely monotone L^p function on $[0, \infty)$ is the $\|\cdot\|_p$ limit of a sequence of Dirichlet series with non-negative coefficients. This answers a question of Liu (2001).

MSC: 41A30; 44A10

Keywords: Completely monotone functions; $\|\cdot\|_p$ approximation

1. Introduction

We begin with the basic

Definition 1.1. A function $f : (0, \infty) \to [0, \infty)$ is *completely monotone* if it is C^∞ and satisfies

$$(-1)^n f^{(n)}(t) \geqslant 0 \quad (t > 0\,,\ n = 0, 1, \ldots)\,. \tag{1}$$

In addition, following [7, Chapter IV], we will say that a function $f : [0, \infty) \to [0, \infty)$ is completely monotone if it is continuous at 0 and satisfies (1).

The class of completely monotone functions on $\mathbb{R}^+ = [0, \infty)$ will be denoted by $\mathcal{CM}(\mathbb{R}^+)$, and those on $(0, \infty)$ by $\mathcal{CM}$. For a general discussion of properties of completely monotone functions see [6].

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https://doi.org/10.1016/j.jat.2019.105301

For a set S of non-negative functions or measures on $[0,\infty)$ define

$$\mathrm{sp}^+(S) = \left\{ \sum_{k=0}^{n} \alpha_k f_k, \alpha_k \geq 0, f_k \in S, n = 1, 2, \ldots \right\}.$$

This is just the cone of non-negative linear combinations of elements of S.

By a (general) Dirichlet series we mean a finite series of the form

$$\sum_{k=1}^{n} a_k \exp(-\lambda_k s), \quad (s \in \mathbb{R}^+),$$

where $\{a_k\} \subset \mathbb{C}$, and $\{\lambda_k\} \subset \mathbb{R}^+$ is a strictly increasing sequence.

Let $1 \leq p < \infty$. In [4, Theorem 5] it was shown that the $\|\cdot\|_p$ limit of a sequence of Dirichlet series with nonnegative coefficients, which is continuous, lies in $\mathcal{CM}(\mathbb{R}^+)$. That is, $\overline{\mathrm{sp}^+\{\exp(-\lambda s), \lambda \geq 0\}}^{\|\cdot\|_p} \cap C(\mathbb{R}^+) \subset \mathcal{CM}(\mathbb{R}^+)$. The question as to whether or not every function in $\mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$ was necessarily such a $\|\cdot\|_p$ limit was left unresolved. This question is answered in the affirmative in Theorem 2.5 below.

Remark 1.2. The case of $p = \infty$ is fully dealt with in [4, Theorem 4].

The fundamental fact concerning the class $\mathcal{CM}$ is the following consequence of Bernstein [1]. See, for example, [7, Theorem IV.12b], [2, Theorem 5.2.5] or [6, Theorem 1.4]. The original is [1, page 56], where it is couched in terms of absolutely monotone functions.

Theorem 1.3 (*Bernstein*)**.** *A function $f : (0,\infty) \to \mathbb{R}^+$ is $\mathcal{CM}$ if and only if there is a positive locally finite Borel measure μ on $\mathbb{R}^+$ such that*

$$f(t) = \int_0^\infty \mathrm{e}^{-st} \mathrm{d}\mu(s) \quad (t > 0). \tag{2}$$

The measure μ is finite if and only if f extends continuously to a function in $\mathcal{CM}(\mathbb{R}^+)$.

Note that μ is necessarily unique, and is such that the integrals (2) are finite for each $t > 0$. We will refer to μ as the Bernstein measure for f.

Remark 1.4. The measure μ in Theorem 1.3 need not be finite, but its local finiteness is automatic. For taking $M > 0$, for $t > 0$ we have

$$\int_0^M \mathrm{d}\mu \leq \mathrm{e}^{tM} \int_0^M \mathrm{e}^{-st} \mathrm{d}\mu(s) \leq \mathrm{e}^{tM} \int_0^\infty \mathrm{e}^{-st} \mathrm{d}\mu(s) = \mathrm{e}^{tM} f(t) < \infty.$$

2. The result

We need the fact that a finite (indeed, σ-finite) measure μ on $\mathbb{R}^+$ has a unique decomposition $\mu = \mu_{\mathrm{c}} + \mu_{\mathrm{d}}$ where μ_c is continuous (that is, $\mu_{\mathrm{c}}(\{x\}) = 0$ for every $x \in \mathbb{R}^+$), and μ_{d} is discrete, [3, Theorem 19.57].

Lemma 2.1. *Take $1 \leq p < \infty$. Let $f \in \mathcal{CM}(\mathbb{R}^+)$ with Bernstein measure μ. Then $f \in L^p(\mathbb{R}^+)$ if and only if $\int_0^\infty s^{-1/p} \mathrm{d}\mu(s) < \infty$.*

Proof. For any nonnegative measurable function θ,

$$\int_0^\infty f(t)\theta(t)\mathrm{d}t = \int_0^\infty \left(\int_0^\infty e^{-st}\mathrm{d}\mu(s)\right)\theta(t)\mathrm{d}t = \int_0^\infty \left(\int_0^\infty e^{-st}\theta(t)\mathrm{d}t\right)\mathrm{d}\mu(s).$$

Let q be the conjugate index to p. Then for all such θ with $\|\theta\|_q = 1$, Cauchy–Schwarz gives

$$\int_0^\infty e^{-st}\theta(t)\mathrm{d}t \le \|e^{-st}\|_p = (sp)^{-1/p}.$$

Furthermore, there is a sequence (θ_n) with $\|\theta_n\|_q = 1$ and $\int_0^\infty e^{-st}\theta_n(t)\mathrm{d}t \nearrow (sp)^{-1/p}$. Monotone convergence then gives

$$\begin{aligned}\|f\|_p &= \sup_{\|\theta\|_q=1}\int_0^\infty f(t)\theta(t)\mathrm{d}t \ge \sup_n \int_0^\infty f(t)\theta_n(t)\mathrm{d}t = \int_0^\infty (sp)^{-1/p}\mathrm{d}\mu(s)\\ &\ge \sup_{\|\theta\|_q=1}\int_0^\infty \left(\int_0^\infty e^{-st}\theta(t)\mathrm{d}t\right)\mathrm{d}\mu(s) = \|f\|_p.\end{aligned}$$

Thus

$$\|f\|_p = \int_0^\infty (sp)^{-1/p}\mathrm{d}\mu(s). \quad \square$$

Our argument will follow the approach used in [5], with slightly modified notation to be consistent with that used here, and made a little more direct as our aim here is slightly different. Given $0 < \vartheta < \eta$, let $\pi = \{\vartheta = s_1 < s_2 < \cdots < s_{n+1} = \eta\}$ be a partition of $[\vartheta, \eta]$, and set $\mathrm{mesh}(\pi) = \max\{s_{i+1} - s_i : i = 1, \ldots, n\}$. Let π^* denote the pair $\{\pi, \{s_i, \ldots, s_n\}\}$. Given a positive locally finite measure μ on $\mathbb{R}^+$, set

$$a_i = \mu([s_i, s_{i+1})), i = 1, 2, \ldots, n,$$

and set

$$\sigma_{\pi^*,\varepsilon}(t) = \sum_{i=1}^n a_i \exp(-s_i t).$$

We note that this latter is a Dirichlet series. For suitable μ we first show that the $\sigma_{\pi^*,\varepsilon}$ give suitable approximations to the $\mathcal{CM}(\mathbb{R}^+)$ function which has Bernstein measure μ.

Theorem 2.2. *Take $1 \le p < \infty$. Let $f \in \mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$, with Bernstein measure μ. Suppose that μ is continuous. Then f lies in the $\|\cdot\|_p$-closure of* $\mathrm{sp}^+\{\exp(-\lambda s), \lambda \ge 0\}$.

Proof. By Lemma 2.1, $\int_0^\infty s^{-1/p}\mathrm{d}\mu(s) < \infty$. Take $\varepsilon > 0$. Choose $\eta > 1 > \vartheta > \xi > 0$ such that $\int_I (1 + s^{-1/p})\mathrm{d}\mu(s) < \varepsilon$ if the interval I has length less that ϑ, $\int_\eta^\infty (1 + s^{-1/p})\mathrm{d}\mu(s) < \varepsilon$, and $\int_J \mathrm{d}\mu(s) < \varepsilon\vartheta$ if the interval J has length less that ξ. Finally, take π, π^* as above, such that $\mathrm{mesh}(\pi) < \xi$. Fix $t \ge 0$. Then

$$\begin{aligned}&\sigma_{\pi^*,\varepsilon}(t) - f(t)\\ &= \sum_{i=1}^n a_i \exp(-s_i t) - \sum_{i=1}^n \int_{s_i}^{s_{i+1}} \exp(-st)\mathrm{d}\mu(s) - \left(\int_0^\vartheta + \int_\eta^\infty\right)\exp(-st)\mathrm{d}\mu(s).\end{aligned}$$

Since $s \mapsto \exp(-st)$ is decreasing, $\int_{s_i}^{s_{i+1}} \exp(-st)\mathrm{d}\mu(s) \geq a_i \exp(-s_{i+1}t)$. Further, $\max_i a_i < \varepsilon\vartheta$, so that

$$
\begin{aligned}
&|\sigma_{\pi^*,\varepsilon}(t) - f(t)| \\
&\quad \leq \varepsilon\vartheta \sum_{i=1}^{n}\Big(\exp(-s_i t) - \exp(-s_{i+1}t)\Big) + \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \exp(-st)\mathrm{d}\mu(s) \\
&\quad \leq \varepsilon\vartheta \exp(-\vartheta t) + \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \exp(-st)\mathrm{d}\mu(s)\,,
\end{aligned}
$$

since the first sum telescopes.

Thus it suffices to estimate the p-norm of the function

$$
\Psi(t) = \varepsilon\vartheta \exp(-\vartheta t) + \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \exp(-st)\mathrm{d}\mu(s)\,. \tag{3}
$$

By duality this equals $\sup_{\|\theta\|_q=1} \int_0^\infty \Psi(t)\theta(t)\mathrm{d}t$. Applying this to the first term of (3) we have

$$
\varepsilon\vartheta \sup_{\|\theta\|_q=1} \int_0^{\infty} \exp(-\vartheta t)\theta(t)\mathrm{d}t < \varepsilon p^{-1/p}.
$$

For the second pair,

$$
\begin{aligned}
&\int_0^{\infty} \theta(t) \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \exp(-st)\mathrm{d}\mu(s)\mathrm{d}t \\
&\qquad = \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \int_0^{\infty} \theta(t) \exp(-st)\mathrm{d}t\mathrm{d}\mu(s) \\
&\qquad \leq \left(\int_0^{\vartheta} + \int_{\eta}^{\infty}\right) \left(\frac{1}{sp}\right)^{1/p} \mathrm{d}\mu(s) \leq 2\varepsilon p^{-1/p}\,.
\end{aligned}
$$

Thus we have $\|\sigma_{\pi^*,\varepsilon} - f\|_p < 3\varepsilon p^{-1/p}$. It follows that f lies in the $\|\cdot\|_p$-closure of $\mathrm{sp}^+\{\exp(-\lambda s), \lambda \geq 0\}$. $\square$

Remark 2.3. This argument immediately gives an extension of [5, Corollary 4.4] to the case of a continuous Bernstein measure. As such it gives details of the outline sketched in [5, §6] for the case $p = 1$.

Lemma 2.4. *Take $1 \leq p < \infty$. Let $f \in \mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$ with Bernstein measure μ which is discrete. Then f lies in the $\|\cdot\|_p$-closure of* $\mathrm{sp}^+\{\exp(-\lambda s), \lambda \geq 0\}$.

Proof. Write $\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}$, where $\alpha_j \geq 0$, $\sum \alpha_j < \infty$. Since

$$
0 \leq \sum_{j=1}^{n} \alpha_j e^{-x_j s} \uparrow \sum_{j=1}^{\infty} \alpha_j e^{-x_j s} = f(s) \quad (0 \leq s < \infty),
$$

the monotone convergence theorem shows that $\|\sum_{j=1}^{n} \alpha_j e^{-x_j s} - f\|_p \to 0$. $\square$

Putting these results together gives a complete answer to the question of [4].

Theorem 2.5. *Take $1 \leq p < \infty$.* $\mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+) = \overline{\mathrm{sp}^+\{\exp(-\lambda s), \lambda \geq 0\}}^{\|\cdot\|_p}$.

Proof. The inclusion $\supseteq$ is [4, Theorem 5]. Suppose that $f \in \mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$, with Bernstein measure μ. Let $\mu = \mu_\mathrm{d} + \mu_\mathrm{c}$ be the decomposition into discrete and continuous parts. Define f_d to have Bernstein measure μ_d, f_c to have Bernstein measure μ_c. Since $0 \le f_\mathrm{c} \le f$ and $0 \le f_\mathrm{d} \le f$, f_d and f_c belong to $\mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$. Now apply Lemma 2.4 to f_d, and Theorem 2.2 to f_c to see $f \in \overline{\mathrm{sp}^+\{\exp(-\lambda s), \lambda \ge 0\}}^{\|\cdot\|_p}$. $\square$

Note that, since any $f \in \mathcal{CM}(\mathbb{R}^+)$ is bounded, any $f \in \mathcal{CM}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ satisfies $\|f\|_p^p \le \|f\|_\infty^{p-1} \cdot \|f\|_1$ for any $p > 1$. Thus, $\mathcal{CM}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \subset \bigcap_{p>1}\big(\mathcal{CM}(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)\big)$.

Remark 2.6. Set $f(x) = (1+x)^{-1}$ for $x \ge 0$. Then $f \in \mathcal{CM}(\mathbb{R}^+) \cap L^p(0,\infty)$ for any $p > 1$, but f is not integrable. The Bernstein measure here is given by the L^1-function e^{-t}.

Acknowledgment

We would like to thank the referee for insightful comments which led to improvements in the exposition of the paper.

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