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Journal of Approximation Theory 250 (2020) 105322

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

Two double-angle formulas of generalized trigonometric functions☆

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Received 22 April 2019; received in revised form 8 October 2019; accepted 25 October 2019
Available online 2 November 2019

Communicated by K. Groechenig

Dedicated to Professor Hiroyuki Usami on the occasion of his 60th birthday

Abstract

With respect to generalized trigonometric functions, since the discovery of double-angle formula for a special case by Edmunds, Gurka and Lang in 2012, no double-angle formulas have been found. In this paper, we will establish new double-angle formulas of generalized trigonometric functions in two special cases.

MSC: 33E05; 34L40

Keywords: Generalized trigonometric functions; Double-angle formulas; Lemniscate function; Jacobian elliptic functions; Dixon's elliptic functions; p-Laplacian

1. Introduction

Let $1 < p,\ q < \infty$ and

$$F_{p,q}(x) := \int_0^x \frac{dt}{(1-t^q)^{1/p}}, \qquad x \in [0, 1].$$

We will denote by $\sin_{p,q}$ the inverse function of $F_{p,q}$, i.e.,

$$\sin_{p,q} x := F_{p,q}^{-1}(x).$$

☆ The work of S. Takeuchi was supported by JSPS, Japan KAKENHI Grant Number 17K05336.
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https://doi.org/10.1016/j.jat.2019.105322

Clearly, $\sin_{p,q} x$ is an increasing function in $[0, \pi_{p,q}/2]$ to $[0, 1]$, where

$$\pi_{p,q} := 2F_{p,q}(1) = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}}.$$

We extend $\sin_{p,q} x$ to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q} - x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q} x \in C^1(\mathbb{R})$, we also define $\cos_{p,q} x$ by $\cos_{p,q} x := (d/dx)(\sin_{p,q} x)$. Then, it follows that

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.$$

In case $(p, q) = (2, 2)$, it is obvious that $\sin_{p,q} x,\ \cos_{p,q} x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x,\ \cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p, q)) and the *generalized* π, respectively.

The generalized trigonometric functions are well studied in the context of nonlinear differential equations (see [3] and the references given there). Suppose that u is a solution of the initial value problem of p-Laplacian

$$-(|u'|^{p-2}u')' = \frac{(p-1)q}{p}|u|^{q-2}u, \quad u(0) = 0,\ u'(0) = 1,$$

which is reduced to the equation $-u'' = u$ of simple harmonic motion for $u = \sin x$ in case $(p, q) = (2, 2)$. Then,

$$\frac{d}{dx}(|u'|^p + |u|^q) = \left(\frac{p}{p-1}(|u'|^{p-2}u')' + q|u|^{q-2}u\right)u' = 0.$$

Therefore, $|u'|^p + |u|^q = 1$, hence it is reasonable to define u as a generalized sine function and u' as a generalized cosine function. Indeed, it is possible to show that u coincides with $\sin_{p,q}$ defined as above. The generalized trigonometric functions are often applied to the eigenvalue problem of p-Laplacian.

Now, we are interested in finding double-angle formulas for generalized trigonometric functions. It is possible to discuss addition formulas for these functions, but for simplicity we will not develop this point here.

No one doubts that the most basic double-angle formula is

$$\sin_{2,2}(2x) = 2\sin_{2,2} x \cos_{2,2} x, \quad x \in \mathbb{R},$$

which is said to have been developed by Abu al-Wafa' (940–998), a Persian mathematician and astronomer. In case $(p, q) = (2, 4)$, it is easy to see that $\sin_{2,4} x$ coincides with the lemniscate function. Since this classical function has the double-angle formula (see, e.g. [4, p. 81]), which is due to Fagnano in 1718, we have

$$\sin_{2,4}(2x) = \frac{2\sin_{2,4} x \cos_{2,4} x}{1 + \sin_{2,4}^4 x}, \quad x \in \mathbb{R}.$$

The case $(p, q) = (3/2, 3)$ goes back to the work of Dixon [1] in 1890. It is simple matter to check that $\sin_{3/2,3} x$ is identical to his elliptic function for $x \in [0, \pi_{3/2,3}/2]$, so that the double-angle formula of his function yields

$$\sin_{3/2,3}(2x) = \frac{\sin_{3/2,3} x(1 + \cos_{3/2,3}^{3/2} x)}{\cos_{3/2,3}^{1/2} x(1 + \sin_{3/2,3}^3 x)}, \quad x \in [0, \pi_{3/2,3}/4]. \tag{1.1}$$

Recently, Edmunds, Gurka and Lang [2] give a remarkable formula for $(p, q) = (4/3, 4)$. Function $\sin_{4/3,4} x$ can be written in terms of Jacobian elliptic function, hence the double-angle

Table 1
The parameters for which the double-angle formulas have been obtained.

q	$(q^*, 2)$	$(2, q)$	(q^*, q)
2	$(2, 2)$ by Abu al-Wafa'	$(2, 2)$ by Abu al-Wafa'	$(2, 2)$ by Abu al-Wafa'
3	$(3/2, 2)$ open	$(2, 3)$ Theorem 1.1	$(3/2, 3)$ by Dixon
4	$(4/3, 2)$ Theorem 1.2	$(2, 4)$ by Fagnano	$(4/3, 4)$ by Edmunds et al.

formula of Jacobian elliptic function gives

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x \cos_{4/3,4}^{1/3} x}{(1+4\sin_{4/3,4}^4 x \cos_{4/3,4}^{4/3} x)^{1/2}}, \quad x \in [0, \pi_{4/3,4}/4].$$

To the best of our knowledge, as far as the generalized trigonometric functions are concerned, no other double-angle formulas have never been published.

In this paper, we will deal with the cases $(p, q) = (2, 3)$ and $(4/3, 2)$. The following double-angle formulas for the two special cases will be established.

Theorem 1.1. *Let $(p, q) = (2, 3)$. Then,*

$$\sin_{2,3}(2x) = \frac{4\sin_{2,3} x \cos_{2,3} x(3+\cos_{2,3} x)^3}{(1+\cos_{2,3} x)(8+\sin_{2,3}^3 x)^2}, \quad x \in [0, \pi_{2,3}/2].$$

Theorem 1.2. *Let $(p, q) = (4/3, 2)$. Then,*

$$\sin_{4/3,2}(2x) = \frac{4\sin_{4/3,2} x \cos_{4/3,2}^{1/3} x(1+\cos_{4/3,2}^{4/3} x)}{(2\cos_{4/3,2}^{2/3} x + \sin_{4/3,2}^2 x)^2}, \quad x \in [0, \pi_{4/3,2}/2].$$

The double-angle formulas for $\cos_{2,3} x$ and $\cos_{4/3,2} x$ are also obtained by differentiating both sides of those for $\sin_{2,3} x$ and $\sin_{4/3,2} x$, respectively.

Finally, we summarize the relationship between parameters for which double-angle formulas have been obtained (Table 1). Lemma 2.1 (resp. Lemma 2.2) connects $(2, q)$ to (q^*, q) (resp. $(q^*, 2)$), where $q^* := q/(q-1)$. Thus, there also exists an alternative proof of case $(4/3, 4)$ such that one uses Lemma 2.1 and the double-angle formula for $(2, 4)$ (see [5, Section 3.1]). Nevertheless, the case $(3/2, 2)$ is an open problem because of difficulty of the inverse problem corresponding to (2.8).

2. Proofs of theorems

To prove Theorem 1.1, we use the following multiple-angle formulas.

Lemma 2.1 ([5]). *Let $1 < q < \infty$ and $q^* := q/(q-1)$. If $x \in [0, \pi_{2,q}/(2^{2/q})] = [0, \pi_{q^*,q}/2]$, then*

$$\sin_{2,q}(2^{2/q} x) = 2^{2/q} \sin_{q^*,q} x \cos_{q^*,q}^{q^*-1} x, \tag{2.1}$$

$$\cos_{2,q}(2^{2/q} x) = \cos_{q^*,q}^{q^*} x - \sin_{q^*,q}^{q} x$$
$$= 1 - 2\sin_{q^*,q}^{q} x = 2\cos_{q^*,q}^{q^*} x - 1. \tag{2.2}$$

Proof of Theorem 1.1. From (2.2) in Lemma 2.1, we have

$$\sin_{q^*,q} x = \left(\frac{1-\cos_{2,q}(2^{2/q}x)}{2}\right)^{1/q}, \tag{2.3}$$

$$\cos_{q^*,q} x = \left(\frac{1+\cos_{2,q}(2^{2/q}x)}{2}\right)^{1/q^*}. \tag{2.4}$$

Let $x \in [0, \pi_{2,3}/2]$ and $y := x/(2^{2/3})$. It follows from (2.1) in Lemma 2.1 that since $2y \in [0, \pi_{2,3}/(2^{2/3})] = [0, \pi_{3/2,3}/2]$,

$$\begin{aligned}\sin_{2,3}(2x) &= \sin_{2,3}(2^{2/3}\cdot 2y)\\ &= 2^{2/3}\sin_{3/2,3}(2y)\cos_{3/2,3}^{1/2}(2y).\end{aligned} \tag{2.5}$$

Dixon's formula (1.1) with (2.3) and (2.4) yields

$$\sin_{3/2,3}(2y) = \frac{\sin_{3/2,3} y(1+\cos_{3/2,3}^{3/2} y)}{\cos_{3/2,3}^{1/2} y(1+\sin_{3/2,3}^{3} y)} = \frac{(1-C)^{1/3}(3+C)}{(1+C)^{1/3}(3-C)},$$

where $C = \cos_{2,3}(2^{2/3}y) = \cos_{2,3} x$. Moreover,

$$\cos_{3/2,3}^{1/2}(2y) = (1-\sin_{3/2,3}^3(2y))^{1/3} = \frac{2^{4/3}C}{(1+C)^{1/3}(3-C)}.$$

Therefore, from (2.5) we have

$$\begin{aligned}\sin_{2,3}(2x) &= 2^{2/3}\cdot\frac{(1-C)^{1/3}(3+C)}{(1+C)^{1/3}(3-C)}\cdot\frac{2^{4/3}C}{(1+C)^{1/3}(3-C)}\\ &= \frac{4(1-C^2)^{1/3}C(3+C)^3}{(1+C)(9-C^2)^2}.\end{aligned}$$

Since $1-C^2 = \sin_{2,3}^3 x$, the proof is complete. □

To show Theorem 1.2, the following lemma is useful.

Lemma 2.2 ([2,3]). *Let $1 < p,\ q < \infty$. For $x \in [0, 2]$,*

$$q\pi_{p,q} = p^*\pi_{q^*,p^*},$$

$$\sin_{p,q}\left(\frac{\pi_{p,q}}{2}x\right) = \cos_{q^*,p^*}^{q^*-1}\left(\frac{\pi_{q^*,p^*}}{2}(1-x)\right).$$

Proof of Theorem 1.2. Let $x \in [0, \pi_{4/3,2}/2]$. Then, since $4x/\pi_{4/3,2} \in [0, 2]$, it follows from Lemma 2.2 that

$$\sin_{4/3,2}(2x) = \cos_{2,4}\left(\frac{\pi_{2,4}}{2}\left(1-\frac{4x}{\pi_{4/3,2}}\right)\right) = \cos_{2,4}\left(\frac{\pi_{2,4}}{2}-x\right).$$

Thus,

$$\sin_{4/3,2}(2x) = \sqrt{1-\sin_{2,4}^4\left(\frac{\pi_{2,4}}{2}-x\right)}. \tag{2.6}$$

Since $\sin_{2,4}$ coincides with the lemniscate function, it has the addition formula: for any $u,\ v \in \mathbb{R}$,

$$\sin_{2,4}(u+v) = \frac{\sin_{2,4} u\cos_{2,4} v + \cos_{2,4} u\sin_{2,4} v}{1+\sin_{2,4}^2 u\sin_{2,4}^2 v}. \tag{2.7}$$

Applying (2.7) to the right-hand side of (2.6), we obtain

$$\sin_{4/3,2}(2x) = \frac{2\sin_{2,4} x}{1+\sin_{2,4}^2 x}.$$

We need only consider case $x \in (0, \pi_{4/3,2}/2) = (0, \pi_{2,4})$. Let $f(x) := \sin_{4/3,2} x$ and $g(x) := \sin_{2,4} x$. Then, $g(x) \neq 0$ and

$$f(2x) = \frac{2g(x)}{1+g(x)^2} = \frac{2}{1/g(x)+g(x)}. \tag{2.8}$$

Therefore, it is easy to see that

$$\frac{1}{g(x)} + g(x) = \frac{2}{f(2x)},$$

$$\frac{1}{g(x)^2} + g(x)^2 = \frac{4}{f(2x)^2} - 2, \tag{2.9}$$

$$\frac{1}{g(x)^2} - g(x)^2 = \frac{4}{f(2x)}\sqrt{\frac{1}{f(2x)^2} - 1}. \tag{2.10}$$

Moreover, letting $u = v = x/2$ in (2.7), we see that $g(x)$ satisfies

$$g(x) = \frac{2g(x/2)\sqrt{1-g(x/2)^4}}{1+g(x/2)^4}. \tag{2.11}$$

Thus, substituting (2.11) into (2.8), we obtain

$$f(2x) = \frac{4(1/g(x/2)^2 + g(x/2)^2)\sqrt{1/g(x/2)^2 - g(x/2)^2}}{(1/g(x/2)^2 + g(x/2)^2)^2 + 4(1/g(x/2)^2 - g(x/2)^2)}.$$

Since (2.9) and (2.10) hold true for x replaced with $x/2$, we can express $f(2x)$ in terms of $f(x)$, i.e.,

$$f(2x) = \frac{4f(x)(1-f(x)^2)^{1/4}(2-f(x)^2)}{(f(x)^2 + 2\sqrt{1-f(x)^2})^2}.$$

Since $1 - f(x)^2 = \cos_{4/3,2}^{4/3} x$, the proof is complete. □

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