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Approximation properties of periodic multivariate quasi-interpolation operators

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Abstract

We study approximation properties of general multivariate periodic quasi-interpolation operators, which are generated by distributions/functions $\widetilde{\varphi}_j$ and trigonometric polynomials φ_j. The class of such operators includes classical interpolation polynomials ($\widetilde{\varphi}_j$ is the Dirac delta function), Kantorovich-type operators ($\widetilde{\varphi}_j$ is a characteristic function), scaling expansions associated with wavelet constructions, and others. Under different compatibility conditions on $\widetilde{\varphi}_j$ and φ_j, we obtain upper and lower bound estimates for the L_p-error of approximation by quasi-interpolation operators in terms of the best and best one-sided approximation, classical and fractional moduli of smoothness, K-functionals, and other terms.

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1. Introduction

Quasi-interpolation operators are among the most important mathematical tools in many branches of science and engineering. They play a crucial role as a connecting link between

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continuous-time and discrete-time signals. For proper application of quasi-interpolation operators, it is very important to know the quality of approximation of functions by such operators in various settings. Recall that in the non-periodic case, quasi-interpolation operators, which are also often called quasi-projection operators, can be defined by

$$\sum_{k\in\mathbb{Z}^d} m^j \langle f, \widetilde{\varphi}(M^j\cdot -k)\rangle \varphi(M^j\cdot -k), \tag{1.1}$$

where φ is a function and $\widetilde{\varphi}$ is a distribution or a function, $\langle f, \widetilde{\varphi}(M^j\cdot -k)\rangle$ is an appropriate functional, M is a dilation matrix, and $m = |\det M|$. The class of these operators is very large. For example, if $\widetilde{\varphi}$ is the Dirac delta-function, operators (1.1) represent classical sampling expansions (see, e.g., [2,6,12,17,20,41]); if $\widetilde{\varphi}$ is a characteristic function of a certain bounded set, we obtain the so-called Kantorovich-type operators and their generalization (see, e.g., [3,7,19,21,26,42]); under particular conditions on φ and $\widetilde{\varphi}$, the class of operators (1.1) includes scaling expansions associated with wavelet constructions (see, e.g., [4,13,14,22,34]) and other types of operators.

In this paper, we study a periodic counterpart of (1.1), which can be defined in the following way

$$Q_j(f,\varphi_j,\widetilde{\varphi}_j) = \frac{1}{m^j}\sum_k \widetilde{\varphi}_j * f(M^{-j}k)\varphi_j(\cdot - M^{-j}k), \tag{1.2}$$

where the sum over k is finite, φ_j is a trigonometric polynomial, and $\widetilde{\varphi}_j * f$ is a certain bounded function associated with the distribution/function $\widetilde{\varphi}_j$ (see Section 2 for details).

Similar to the non-periodic case, approximation properties of operators (1.2) have also been intensively studied by many mathematicians (see, e.g., [10,11,18,28,33,35,36] and the references therein). It turns out that in the periodic case, such operators have been considered mainly in the form of sampling or interpolating-type operators (i.e., $\widetilde{\varphi}_j$ is the periodic Delta function) given by

$$I_j(f,\varphi_j) = \frac{1}{m^j}\sum_k f(M^{-j}k)\varphi_j(\cdot - M^{-j}k), \tag{1.3}$$

where, usually, φ_j is a so-called fundamental interpolant, e.g., the Dirichlet or de la Vallée-Poussin kernels, or periodic B-splines. At the same time, general periodic quasi-interpolation operators of type (1.2) have been studied only in a few works. In particular, the general case of operators (1.2) with some particular class of linear functionals instead of $f(M^{-j}k)$ was studied in [11] and in the recent paper [18].

The estimation of the L_p-error of approximation by interpolation operators (1.3), in which φ_j is the Dirichlet kernel was studied in [10]. A more general case of Hermite-type interpolation was considered in [28]. In the above mentioned two papers, the estimates of the error were given in terms of the best one-sided approximation by trigonometric polynomials and in terms of the τ-modulus of smoothness of arbitrary integer order. Approximation properties of operators (1.3) for various trigonometric polynomials φ_j (the so-called methods of summation of the discrete Fourier series) were considered in [36] and [37], in which the error estimates were investigated in the uniform norm. In the papers [33] and [35], the introduction of the periodic Strang–Fix conditions as well as their different modifications enabled the development of a unified approach to error estimates of periodic interpolation for functions from the Sobolev spaces and other function spaces. Some estimates of the L_p-error of approximation by operators (1.3) for functions from Nikol'skij–Besov spaces were derived in [32].

The goal of this paper is to estimate the L_p-error of approximation of a given function f, from above and below, by quasi-interpolation operators $Q_j(f,\varphi_j,\widetilde{\varphi}_j)$ for a wide range of distributions/functions $\widetilde{\varphi}_j$ and trigonometric polynomials φ_j. Under different compatibility conditions on φ_j and $\widetilde{\varphi}_j$ related in some sense to the Strang–Fix conditions, we obtain estimates for the error of approximation in terms of the best and best one-sided approximation (see the definition in Section 2), classical and fractional moduli of smoothness, K-functionals, and other terms. We pay a special attention to the case $\widetilde{\varphi}_j\in L_q$, for example, $\widetilde{\varphi}_j$ is a normalized characteristic function, which provides Kantorovich-type operators. In particular, we show that if $\varphi_j=\mathscr{D}_{2^j}$ is the Dirichlet kernel and $f\in L_p[-\frac12,\frac12]$, $1<p<\infty$, $\sigma\in(0,1/2]$, then (see Example 4.4)

$$\left\| f-\sum_{k=-2^{j-1}}^{2^{j-1}-1}\frac{1}{2\sigma}\int_{-2^{-j}\sigma}^{2^{-j}\sigma} f\left(t+2^{-j}k\right){\rm d}t\,\mathscr{D}_{2^j}\left(\cdot-2^{-j}k\right)\right\|_p\asymp \omega_2(f,2^{-j})_p, \tag{1.4}$$

where $\omega_2(f,2^{-j})_p$ is the classical modulus of smoothness of second order. At the same time, if $\varphi_j(x)=\mathscr{D}^{\chi}_{2^j,\sigma}(x)=\sum_{\ell=-2^{j-1}}^{2^{j-1}-1}\frac{\pi\sigma2^{-j+1}\ell}{\sin\pi\sigma2^{-j+1}\ell}{\rm e}^{2\pi {\rm i}\ell x}$ and $1<p<\infty$, then (see Example 4.2)

$$\left\| f-\sum_{k=-2^{j-1}}^{2^{j-1}-1}\frac{1}{2\sigma}\int_{-2^{-j}\sigma}^{2^{-j}\sigma} f\left(t+2^{-j}k\right){\rm d}t\,\mathscr{D}^{\chi}_{2^j,\sigma}\left(\cdot-2^{-j}k\right)\right\|_p\asymp E_{2^j}(f)_p, \tag{1.5}$$

where $E_{2^j}(f)_p$ is the L_p-error of the best approximation of f by trigonometric polynomials with frequencies in $[-2^{j-1},2^{j-1})$. In the above relations (1.4) and (1.5), the notation $\asymp$ denotes the two-sided inequality with positive constants that do not dependent on f and j.

The paper is organized as follows: in Section 2 we introduce basic notations, provide essential facts, and define the quasi-interpolation operator $Q_j(f,\varphi_j,\widetilde{\varphi}_j)$. Section 3 is devoted to auxiliary results. In this section, we obtain general upper estimates of the L_p-error for $Q_j(f,\varphi_j,\widetilde{\varphi}_j)$ and give auxiliary lemmas. In Section 4 we prove the main results. In Section 4.1, under strong compatibility conditions on φ_j and $\widetilde{\varphi}_j$, we estimate the L_p-error for operators (1.2) in terms of best approximation by trigonometric polynomials. In Section 4.2 we give two-sided estimates of the approximation error $\|f-Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|_p$ in terms of classical and fractional moduli of smoothness and K-functionals. In Section 4.3 we specify some error estimates from the previous section for functions f belonging to Besov-type spaces.

2. Basic notation

We use the standard multi-index notations. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}^d$ be the d-dimensional Euclidean space, $\mathbb{Z}^d$ be the integer lattice in $\mathbb{R}^d$, $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus. Further, let $x=(x_1,\dots,x_d)^{\rm T}$ and $y=(y_1,\dots,y_d)^{\rm T}$ be column vectors in $\mathbb{R}^d$. Then $(x,y):=x_1y_1+\dots+x_dy_d$, $|x|:=\sqrt{(x,x)}$; $\mathbf{0}=(0,\dots,0)^T\in\mathbb{R}^d$; $\mathbb{Z}^d_+:=\{x\in\mathbb{Z}^d:\ x_k\ge 0,k=1,\dots,d\}$. If $\alpha\in\mathbb{Z}^d_+$, we set $[\alpha]=\sum_{k=1}^d\alpha_k$, $D^\alpha f=\frac{\partial^{[\alpha]}f}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_d}x_d}$.

We denote by c and C some positive constants depending on the indicated parameters. By these letters we also denote some positive constants that are independent of the function f and the parameter j.

We use the notation L_p for the space $L_p(\mathbb{T}^d)$ with the usual norm

$$\|f\|_p=\left(\int_{\mathbb{T}^d}|f(x)|^p{\rm d}x\right)^{1/p}\quad\text{for}\quad 1\le p<\infty$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{T}^d}|f(x)| \quad \text{for} \quad p=\infty.$$

When $p=\infty$, we replace L_∞ with $C(\mathbb{T}^d)$. By $B=B(\mathbb{T}^d)$ we denote the space of all bounded measurable functions on $\mathbb{T}^d$.

If $f\in L_1(\mathbb{T}^d)$, then

$$\widehat{f}(k)=\int_{\mathbb{T}^d} f(x)\mathrm{e}^{-2\pi \mathrm{i}(k,x)}\mathrm{d}x, \quad k\in\mathbb{Z}^d,$$

denotes the kth Fourier coefficient of f. The Fourier transform of $f\in L_1(\mathbb{R}^d)$ is defined by $\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i(x,\xi)}\mathrm{d}x$.

Let $\mathcal{D}=C^\infty(\mathbb{T}^d)$ be the space of infinitely differentiable functions on $\mathbb{R}^d$ that are periodic with period 1. The linear space of periodic distributions (continuous linear functionals on $\mathcal{D}$) is denoted by $\mathcal{D}'$. It is known (see, e.g., [30, p. 144]) that any periodic distribution $\widetilde{\varphi}$ can be expanded in a weakly convergent (in $\mathcal{D}'$) Fourier series

$$\widetilde{\varphi}(x)=\sum_{k\in\mathbb{Z}^d}\widehat{\widetilde{\varphi}}(k)\mathrm{e}^{2\pi\mathrm{i}(k,x)}, \tag{2.1}$$

where the sequence $\{\widehat{\widetilde{\varphi}}(k)\}_k$ has at most polynomial growth. Also, conversely, for any sequence $\{\widehat{\widetilde{\varphi}}(k)\}_k$ of at most polynomial growth the series on the right-hand side of (2.1) converges weakly to a periodic distribution. The numbers $\widehat{\widetilde{\varphi}}(k)$ are called the Fourier coefficients of a periodic distribution $\widetilde{\varphi}$ and $\widehat{\widetilde{\varphi}}(k)=\langle \mathrm{e}^{-2\pi\mathrm{i}(k,\cdot)},\widetilde{\varphi}\rangle$.

In what follows, $M=\operatorname{diag}(m_1,m_2,\dots,m_d)$ is a diagonal dilation matrix, m_j is an integer with $|m_j|>1$, $m:=|\det M|$, $D(M):=(M[-1/2,1/2)^d)\cap\mathbb{Z}^d$.

For a given matrix M, we will use the following set of trigonometric polynomials:

$$\mathcal{T}_M:=\{T\ :\ \operatorname{spec} T\subset D(M)\}.$$

The L_p-error of the best approximation of $f\in L_p$ by trigonometric polynomials $T\in\mathcal{T}_M$ is denoted by

$$E_M(f)_p:=\inf\left\{\|f-T\|_p\ :\ T\in\mathcal{T}_M\right\}.$$

The L_p-error of the best one-sided approximation of $f\in B$ is given by

$$\widetilde{E}_M(f)_p:=\inf\left\{\|t-T\|_p\ :\ t,T\in\mathcal{T}_M,\quad t(x)\le f(x)\le T(x)\quad\text{for all}\quad x\in\mathbb{T}^d\right\}.$$

Note that for $p=\infty$ the error of the best one-sided approximation coincides up to a constant with the error of the unrestricted best approximation $E_M(f)_p$, see, e.g., [31, p. 163].

For a sequence $\{a_k\}_{k\in D(M)}\in\mathbb{C}$, we denote

$$\|\{a_k\}_k\|_{\ell_{p,M}}:=\begin{cases}\left(\dfrac{1}{m}\displaystyle\sum_{k\in D(M)}|a_k|^p\right)^{\frac{1}{p}}, & \text{if } 1\le p<\infty,\\ \displaystyle\sup_{k\in D(M)}|a_k|, & \text{if } p=\infty.\end{cases}$$

In this paper, we will use the following notation for the rectangular partial sums of the Fourier series and the de la Vallée Poussin means of f:

$$S_Mf(x):=\sum_{k\in D(M)}\widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}(k,x)},$$

$$V_M f(x) := \sum_{k\in D(M)} v(M^{-1}k)\widehat{f}(k)\mathrm{e}^{2\pi \mathrm{i}(k,x)},$$

where $v \in C^\infty(\mathbb{R}^d)$, $v(\xi) = 1$ for $\xi \in [-1/4, 1/4)^d$ and $v(\xi) = 0$ for $\xi \notin [-3/8, 3/8)^d$. Recall the following well-known inequalities (see, e.g., [40, 2.1.6, 2.4.5, and 4.1.1]):

$$\|f - S_M f\|_p \le c(p,d)E_M(f)_p, \quad 1 < p < \infty, \tag{2.2}$$

$$\|f - V_M f\|_p \le \left(1 + \|v\|_{L_1(\mathbb{R}^d)}\right) E_{\frac12 M}(f)_p \le c(d)E_{\frac12 M}(f)_p, \quad 1 \le p \le \infty. \tag{2.3}$$

The Dirichlet kernel with respect to the matrix M is defined by

$$\mathscr{D}_M(x) = \sum_{k\in D(M)} \mathrm{e}^{2\pi \mathrm{i}(k,x)}.$$

Let φ be a trigonometric polynomial and $f \in L_p$, $1 \le p \le \infty$. Denote

$$K_{\varphi,p} := \sup_{\|f\|_p\le 1} \|\varphi * f\|_p.$$

Note that (see, e.g., [40, Ch. 8]) if $\widehat{\varphi_j}(\xi) = \chi_{M^j[-\frac12,\frac12)^d}(\xi)$, where χ_G denote the characteristic function of the set G, then $\varphi_j * f = S_{M^j} f$ and

$$K_{\varphi_j,p} \asymp \begin{cases} 1, & 1 < p < \infty, \\ j^d, & p = 1 \text{ or } \infty. \end{cases}$$

The averaging operator with respect to the matrix M is defined by

$$\mathrm{Avg}_M f(x) = m^{-1} \int_{M[-\frac12,\frac12)^d} f(t + x)\mathrm{d}t.$$

Definition 2.1. Let $\widetilde{\varphi} \in \mathcal{D}'$ and $1 \le p \le \infty$. We will say that a function f belongs to the class $B_{\widetilde{\varphi},p}$ if $f \in L_p$ and

$$\sum_{\ell\in\mathbb{Z}^d} \widehat{\widetilde{\varphi}}(\ell)\widehat{f}(\ell)\mathrm{e}^{2\pi \mathrm{i}(\ell,x)}$$

is a Fourier series of a certain bounded function, which we denote by $\widetilde{\varphi} * f$.

Typical examples of $B_{\widetilde{\varphi},p}$ are the following: (1) if $\widetilde{\varphi}$ is a finite complex-valued Borel measure on $\mathbb{T}^d$ and $p = \infty$, then $B_{\widetilde{\varphi},p} = B$, see, e.g., [40, 7.1.4]; (2) if $\widetilde{\varphi} \in L_q$, $1/p + 1/q = 1$, then by Young's convolution inequality, we have that $B_{\widetilde{\varphi},p} = L_p$.

Now, let us introduce the main object of this paper. Let $j \in \mathbb{N}$, $\widetilde{\varphi}_j \in \mathcal{D}'$, $\varphi_j \in L_p$, and $f \in B_{\widetilde{\varphi}_j,p}$ be given. We define the general multivariate periodic quasi-interpolation operator by

$$Q_j(f, \varphi_j, \widetilde{\varphi}_j)(x) = \frac{1}{m^j} \sum_{k\in D(M^j)} \widetilde{\varphi}_j * f(M^{-j}k)\varphi_j(x - M^{-j}k). \tag{2.4}$$

Note that for functions f from some special Wiener and Besov classes, similar quasi-interpolation operators have been recently studied in [18]. Particularly, in terms of decay of the Fourier coefficients of f, there were obtained several types of estimates of approximation by operators (2.4) in the Wiener-type spaces and the spaces $L_p(\mathbb{T}^d)$ with $2 \le p \le \infty$. In the present paper, we essentially extend and improve the results given in [18] in several directions. First of all, using an approach based on the best one-sided approximations and Fourier multipliers, we obtain error estimates in $L_p(\mathbb{T}^d)$ for all $1 \le p \le \infty$. Second, using new type of compatibility conditions for φ_j and $\widetilde{\varphi}_j$, we give the corresponding error estimates in terms of classical and fractional moduli of smoothness and K-functionals, which are commonly used in approximation theory and in most cases provide sharper estimates than those given in [18] in terms of the Fourier coefficients of f. Third, together with estimates from above of the L_p-error of approximation, we obtain also the estimates from below, which show the sharpness of our results for particular classes of quasi-interpolation operators.

3. Auxiliary results

The next lemma is one of the main auxiliary results in this paper.

Lemma 3.1. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $\delta \in (0, 1]$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$. Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C \Big(\|\psi_j * T_j\|_p + E_{\delta M^j}(f)_p + K_{\varphi_j,q}\big(\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p + \|\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * T_j\|_p\big) \Big),$$

where

$$\psi_j(x) = \sum_{\ell \in D(M^j)} \big(1 - \widehat{\varphi}_j(\ell)\widehat{\widetilde{\varphi}}_j(\ell)\big)\, \mathrm{e}^{2\pi \mathrm{i}(\ell,x)}, \tag{3.1}$$

the polynomial $T_j \in \mathcal{T}_{M^j}$ is such that $\|f - T_j\|_p \le c(d, p, \delta)E_{\delta M^j}(f)_p$, and the constant C does not depend on f and j.

Before proving Lemma 3.1, we give one simple corollary of Lemma 3.1 for the partial sums of the Fourier series $S_{M^j} f$ and the de la Vallée Poussin means $V_{M^j} f$.

Corollary 3.1. *Under the conditions of Corollary 3.1, we have:*

(a) if $1 < p < \infty$, then

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\|\psi_j * S_{M^j} f\|_p + E_{M^j}(f)_p + K_{\varphi_j,q}\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p\Big), \tag{3.2}$$

(b) if $1 \le p \le \infty$, then

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\|\psi_j * V_{M^j} f\|_p + E_{\frac12 M^j}(f)_p + K_{\varphi_j,q}\widetilde{E}_{\frac12 M^j}(\widetilde{\varphi}_j * f)_p\Big), \tag{3.3}$$

where the constant C does not depend on f and j and the function ψ_j is given by (3.1).

Proof. The inequalities (3.2) and (3.3) can be obtained repeating the proof of Lemma 3.1 presented below by taking $T_j = S_{M^j} f$ in the case $1 < p < \infty$ and $T_j = V_{M^j} f$ in the case $1 \le p \le \infty$. We need also to use (2.2), (2.3), and the following simple inequalities

$$\begin{aligned}\|\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * V_{M^j} f\|_p &= \|\widetilde{\varphi}_j * f - V_{M^j}(\widetilde{\varphi}_j * f)\|_p \\ &\le CE_{\frac12 M^j}(\widetilde{\varphi}_j * f)_p \le C\widetilde{E}_{\frac12 M^j}(\widetilde{\varphi}_j * f)_p. \qquad \square\end{aligned} \tag{3.4}$$

To prove Lemma 3.1, we will use a standard Marcinkiewicz–Zygmund inequality for multivariate trigonometric polynomials given in the following lemma. Its proof follows easily from the corresponding one-dimensional result, see, e.g., [24].

Lemma 3.2. *Let* $1 \le p \le \infty$, $j \in \mathbb{N}$, *and* $T_j \in \mathcal{T}_{M^j}$. *Then*

$$\left\| \{T_j(M^{-j}k)\}_k \right\|_{\ell_{p,M^j}} \le c(d,p)\|T_j\|_p.$$

The next lemma was proved in [18, Lemma16].

Lemma 3.3. *Let* $1 \le p \le \infty$, $1/p + 1/q = 1$, $j \in \mathbb{N}$, $\{a_k\}_k \in \mathbb{C}$, *and* $\varphi_j \in \mathcal{T}_{M^j}$. *Then*

$$\left\| \frac{1}{m^j} \sum_{k\in D(M^j)} a_k \varphi_j(\cdot - M^{-j}k) \right\|_p \le C K_{\varphi_j,q} \, \|\{a_k\}_k\|_{\ell_{p,M^j}},$$

where the constant C *does not depend on* j *and* $\{a_k\}$.

Proof of Lemma 3.1. We consider only the case $1 \le p < \infty$. The case $p = \infty$ can be treated similarly. We have

$$\begin{aligned}
&\left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} \widetilde{\varphi}_j * f(M^{-j}k)\varphi_j(\cdot - M^{-j}k) \right\|_p \\
&\le \|f - T_j\|_p + \left\| T_j - \frac{1}{m^j} \sum_{k\in D(M^j)} \widetilde{\varphi}_j * T_j(M^{-j}k)\varphi_j(\cdot - M^{-j}k) \right\|_p \\
&\quad + \left\| \frac{1}{m^j} \sum_{k\in D(M^j)} \left(\widetilde{\varphi}_j * f(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)\right) \varphi_j(\cdot - M^{-j}k) \right\|_p := I_1 + I_2 + I_3.
\end{aligned} \tag{3.5}$$

First, we consider I_2. We have

$$\begin{aligned}
&T_j(x) - \frac{1}{m^j} \sum_{k\in D(M^j)} \widetilde{\varphi}_j * T_j(M^{-j}k)\varphi_j(x - M^{-j}k) \\
&= \sum_{\ell\in D(M^j)} \left(\widehat{T}_j(\ell) - \frac{\widehat{\varphi}_j(\ell)}{m^j} \sum_{k\in D(M^j)} \widetilde{\varphi}_j * T_j(M^{-j}k) \mathrm{e}^{-2\pi \mathrm{i}(\ell, M^{-j}k)} \right) \mathrm{e}^{2\pi \mathrm{i}(\ell,x)} \\
&= \sum_{\ell\in D(M^j)} \left(\widehat{T}_j(\ell) - \widehat{\varphi}_j(\ell) \sum_{\nu\in D(M^j)} \widehat{\widetilde{\varphi}}_j(\nu)\widehat{T}_j(\nu) \frac{1}{m^j} \sum_{k\in D(M^j)} \mathrm{e}^{2\pi \mathrm{i}(\nu-\ell, M^{-j}k)} \right) \mathrm{e}^{2\pi \mathrm{i}(\ell,x)} \\
&= \sum_{\ell\in D(M^j)} \left(\widehat{T}_j(\ell) - \widehat{\varphi}_j(\ell)\widehat{\widetilde{\varphi}}_j(\ell)\widehat{T}_j(\ell) \right) \mathrm{e}^{2\pi \mathrm{i}(\ell,x)} = \psi_j * T_j(x),
\end{aligned} \tag{3.6}$$

which implies that

$$I_2 = \|\psi_j * T_j\|_p. \tag{3.7}$$

Consider I_3. Let $u_j, U_j \in \mathcal{T}_{M^j}$ be such that $u_j(x) \le \widetilde{\varphi}_j * f(x) \le U_j(x)$ for all $x \in \mathbb{T}^d$ and $\|u_j - U_j\|_p \le 2\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p$. Then, using Lemmas 3.2 and 3.3, we derive

$$\begin{aligned}
I_3 &\le CK_{\varphi_j,q}\Big(\frac{1}{m^j}\sum_{k\in D(M^j)}|\widetilde{\varphi}_j * f(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)|^p\Big)^{\frac1p}\\
&\le CK_{\varphi_j,q}\Bigg(\Big(\frac{1}{m^j}\sum_{k\in D(M^j)}|U_j(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)|^p\Big)^{\frac1p}\\
&\qquad\qquad + \Big(\frac{1}{m^j}\sum_{k\in D(M^j)}|U_j(M^{-j}k) - \widetilde{\varphi}_j * f(M^{-j}k)|^p\Big)^{\frac1p}\Bigg)\\
&\le CK_{\varphi_j,q}\Bigg(\|U_j - \widetilde{\varphi}_j * T_j\|_p + \Big(\frac{1}{m^j}\sum_{k\in D(M^j)}|U_j(M^{-j}k) - u_j(M^{-j}k)|^p\Big)^{\frac1p}\Bigg)\\
&\le CK_{\varphi_j,q}\left(\|U_j - \widetilde{\varphi}_j * T_j\|_p + \|U_j - u_j\|_p\right)\\
&\le CK_{\varphi_j,q}\left(\|U_j - \widetilde{\varphi}_j * f\|_p + \|U_j - u_j\|_p + \|\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * T_j\|_p\right)\\
&\le CK_{\varphi_j,q}\left(\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p + \|\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * T_j\|_p\right).
\end{aligned} \tag{3.8}$$

Finally, combining (3.5), (3.7), and (3.8), we prove the lemma. □

In Lemma 3.1, the error estimate was given in terms of the best one-sided approximation $\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p$ for the function $f \in B_{\widetilde{\varphi}_j,p}$. Under more restrictive conditions on the function $\widetilde{\varphi}_j$, we can take $B_{\widetilde{\varphi}_j,p} = L_p$ and replace the best one-sided approximation with the unrestricted best approximation. For this, we will use the following special norms for a function $\widetilde{\varphi}_j \in L_q$, $j \in \mathbb{N}$:

$$\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}} := \left(m^j\int_{M^{-j}\mathbb{T}^d}\Big(\frac{1}{m^j}\sum_{k\in D(M^j)}|\widetilde{\varphi}_j(x - M^{-j}k)|\Big)^q \mathrm{d}x\right)^{\frac1q} \quad \text{if} \quad 1 \le q < \infty$$

and

$$\|\widetilde{\varphi}_j\|_{\mathcal{L}_{\infty,j}} := \frac{1}{m^j}\sup_{x\in\mathbb{R}^d}\sum_{k\in D(M^j)}|\widetilde{\varphi}_j(x - M^{-j}k)| \quad \text{if} \quad q = \infty.$$

We have the following improvement of Lemma 3.1 for $\widetilde{\varphi}_j \in L_q$:

Lemma 3.4. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $\delta \in (0, 1]$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in L_q$ and $\varphi_j \in \mathcal{T}_{M^j}$. Then, for any $f \in L_p$, we have*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\|\psi_j * T_j\|_p + (1 + K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}})E_{\delta M^j}(f)_p\Big),$$

where ψ_j is given by (3.1), the polynomial $T_j \in \mathcal{T}_{M^j}$ is such that $\|f - T_j\|_p \le c(d, p, \delta)E_{\delta M^j}(f)_p$, and the constant C does not depend on f and j.

The proof of Lemma 3.4 is based on the following result (see Lemma 17 in [18]):

Lemma 3.5. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $j \in \mathbb{N}$, and $\widetilde{\varphi}_j \in L_q$. Then, for any $f \in L_p$, we have*

$$\left\|\{\widetilde{\varphi}_j * f(M^{-j}k)\}_k\right\|_{\ell_{p,M^j}} \le \|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}}\|f\|_p.$$

Proof of Lemma 3.4. The proof is similar to the proof of Lemma 3.1. It is sufficient to use inequalities (3.5) and (3.7) as well as the following estimate

$$\begin{aligned} I_3 &\le C K_{\varphi_j,q}\left(\frac{1}{m^j}\sum_{k\in D(M^j)}|\widetilde{\varphi}_j*(f-T_j)(M^{-j}k)|^p\right)^{\frac1p} \\ &\le C K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}}\|f-T_j\|_p \end{aligned} \tag{3.9}$$

instead of inequality (3.8). The above estimate easily follows from Lemmas 3.3 and 3.5. □

4. Main results

4.1. Estimates of approximation in terms of best approximation

In this subsection, we give an explicit form of the error estimates from Lemmas 3.1 and 3.4 in the case of the so-called strictly compatible functions/distributions φ_j and $\widetilde{\varphi}_j$.

Theorem 4.1. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $0 < \delta \le \rho \le 1$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$ are such that*

$$\widehat{\varphi_j}(k)\widehat{\widetilde{\varphi}_j}(k) = 1 \quad \text{for all} \quad k \in D(\rho M^j). \tag{4.1}$$

Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have

$$\|f - Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|_p \le C\left(E_{\delta M^j}(f)_p + K_{\varphi_j,q}\left(\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p + \|\widetilde{\varphi}_j*(f-T_j)\|_p\right)\right), \tag{4.2}$$

where $T_j \in \mathcal{T}_{\rho M^j}$ is such that $\|f-T_j\|_p \le c(d,p,\delta)E_{\delta M^j}(f)_p$; if, additionally, $\widetilde{\varphi}_j \in L_q$, then, for any $f \in L_p$, we have

$$\|f - Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|_p \le C(1 + K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}})E_{\delta M^j}(f)_p, \tag{4.3}$$

where the constant C does not depend on f and j.

Note that inequality (4.3) was earlier obtained in [18].

Proof. To prove the theorem, it is enough to use Lemmas 3.1, 3.4 and to take into account that $\|\psi_j * T_j\|_p = 0$ and all estimates in the proof of Lemma 3.1 remain the same for $T_j \in \mathcal{T}_{\rho M^j}$. □

Similarly to Corollary 3.1, we derive the following result:

Corollary 4.1. *Under the conditions of Theorem 4.1, we have that inequality (4.2) can be replaced by*

$$\|f - Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|_p \le C\Big(E_{\delta M^j}(f)_p + K_{\varphi_j,q}\widetilde{E}_{\delta M^j}(\widetilde{\varphi}_j * f)_p\Big),$$

where $\delta < \rho$ if $p = 1, \infty$ and the constant C does not depend on f and j.

Example 4.1. If $\widetilde{\varphi}_j$ is the periodic Dirac delta function for all $j \in \mathbb{N}$ and $\varphi_j = \mathscr{D}_{M^j}$ is the Dirichlet kernel, then equality (4.1) obviously holds with $\rho = \delta = 1$ and inequality (4.2) implies the following well-known error estimate for the corresponding interpolation operator (cf. [10, Corollary 3]):

$$\left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} f(M^{-j}k)\mathscr{D}_{M^j}(\cdot - M^{-j}k) \right\|_p \le C\kappa_{j,p}\widetilde{E}_{M^j}(f)_p,$$

where $f \in B$, $1 \le p \le \infty$,

$$\kappa_{j,p} := \begin{cases} 1, & 1 < p < \infty, \\ j^d, & p = 1,\ \infty \end{cases} \tag{4.4}$$

and the constant C does not depend on f and j.

In the next example, we deal with a periodic Kantorovich-type quasi-interpolation operator generated by the samples $\{\mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\}_k$.

Example 4.2. Let $f \in L_p$, $1 \le p \le \infty$, $\sigma \in (0, 1]$, and $j \in \mathbb{N}$. Then

$$\left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} \mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\mathscr{D}^{\chi}_{M^j,\sigma}(\cdot - M^{-j}k) \right\|_p \le C\kappa_{j,p}E_{M^j}(f)_p, \tag{4.5}$$

where

$$\mathscr{D}^{\chi}_{M^j,\sigma}(x) = \sum_{\ell\in D(M^j)} \prod_{i=1}^{d} \frac{\pi\sigma m_i^{-j}\ell_i}{\sin \pi\sigma m_i^{-j}\ell_i} \mathrm{e}^{2\pi \mathrm{i}(\ell,x)},$$

the constant $\kappa_{j,p}$ is given in (4.4) and C does not depend on f and j.

The proof of estimate (4.5) easily follows from inequality (4.3) with $\varphi_j = \mathscr{D}^{\chi}_{M^j,\sigma}$ and $\widetilde{\varphi}_j = \sigma^{-d}m^j\chi_{M^{-j}[-\frac{\sigma}{2},\frac{\sigma}{2})^d}$. One only needs to take into account that (4.1) holds with $\rho = \delta = 1$,

$$\mathrm{Avg}_{\sigma M^{-j}} f(x) = f * \widetilde{\varphi}_j(x) \sim \sum_{\ell\in\mathbb{Z}^d} \prod_{i=1}^{d} \frac{\sin \pi\sigma m_i^{-j}\ell_i}{\pi\sigma m_i^{-j}\ell_i} \widehat{f}(\ell)\mathrm{e}^{2\pi \mathrm{i}(\ell,x)},$$

$\sup_j \|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}} < \infty$, and $\sup_{\|f\|_p\le 1} \|f * \mathscr{D}^{\chi}_{M^j,\sigma}\|_p \le C \sup_{\|f\|_p\le 1} \|f * \mathscr{D}_{M^j}\|_p \le C\kappa_{j,p}$. The last estimate follows from the fact that the function

$$\eta^{\chi}(\xi) = \eta(\xi) \prod_{i=1}^{d} \frac{\pi\sigma\xi_i}{\sin \pi\sigma\xi_i},$$

where $\eta \in C^\infty(\mathbb{R}^d)$, $\eta(\xi) = 1$ for $\xi \in [-1/2, 1/2)^d$ and $\eta(x) = 0$ for $\xi \notin [-1, 1)^d$, is a Fourier multiplier in $L_p(\mathbb{R}^d)$ for all $1 \le p \le \infty$ (see Lemma 4.3).

4.2. *Estimates of approximation in terms of moduli of smoothness and K-functionals*

We need to introduce some additional notation. For a given matrix M, $s \in \mathbb{N}$, and a function $f \in L_p$, we set

$$\Omega_s(f, M^{-1})_p := \sup_{|M\delta|<1,\delta\in\mathbb{R}^d} \|\Delta^s_\delta f\|_p,$$

where

$$\Delta^s_\delta f(x) := \sum_{\nu=0}^{s} (-1)^\nu \binom{s}{\nu} f(x + \delta\nu)$$

and $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$, $\binom{\alpha}{0} = 1$, for any $\alpha > 0$. This is the so-called (total) anisotropic modulus of smoothness. Together with this modulus of smoothness, we will also use the classical mixed modulus of smoothness, which for a given vector $\beta \in \mathbb{Z}_+^d$ and a diagonal matrix $M = \mathrm{diag}(m_1, \dots, m_d)$ is defined by

$$\omega_\beta(f, M^{-1})_p := \sup_{|\delta_i|<m_i^{-1},\, i=1,\dots,d} \|\Delta_{\delta_1 e_1}^{\beta_1} \dots \Delta_{\delta_d e_d}^{\beta_d} f\|_p.$$

The following relations for the moduli of smoothness defined above were proved in [39]:

$$\Omega_s(f, M^{-1})_p \asymp \sum_{i=1}^d \omega_{se_i}(f, M^{-1})_p, \quad f \in L_p, \quad 1 < p < \infty, \tag{4.6}$$

and

$$\Omega_s(f, M^{-1})_p \asymp \sum_{[\beta]=s,\, \beta\in\mathbb{Z}_+^d} \omega_\beta(f, M^{-1})_p, \quad f \in L_p, \quad 1 \le p \le \infty, \tag{4.7}$$

where $\asymp$ is a two-sided inequality with constants that do not depend on f and j.

Let us recall several basic properties of moduli of smoothness (see, e.g., [25, Ch. 4]). For $f, g \in L_p$, $1 \le p \le \infty$, and $s \in \mathbb{N}$, we have

(a) $\Omega_s(f + g, M^{-1})_p \le \Omega_s(f, M^{-1})_p + \Omega_s(g, M^{-1})_p$;
(b) $\Omega_s(f, M^{-1})_p \le 2^s \|f\|_p$;
(c) for $\lambda > 0$,

$$\Omega_s(f, \lambda M^{-1})_p \le (1 + \lambda)^s \Omega_s(f, M^{-1})_p.$$

We will also use the following Jackson-type theorem in L_p (see, e.g., [25, Theorem 5.2.1 (7)] or [38, 5.3.2]):

Lemma 4.1. *Let $f \in L_p$, $1 \le p \le \infty$, and $s \in \mathbb{N}$. Then, there exists $T_j \in \mathcal{T}_{M^j}$ such that*

$$\|f - T_j\|_p \le C \sum_{i=1}^d \omega_{se_i}(f, M^{-j})_p,$$

where C does not depend on f and T_j.

The next lemma provides the Nikol'skii–Stechkin–Riesz type inequality (see, e.g. [38, p. 215]).

Lemma 4.2. *Let $1 \le p \le \infty$, $s \in \mathbb{N}$, and $n \in \mathbb{N}$. Then, for any trigonometric polynomial $T_n(x) = \sum_{|k|\le n} c_k e^{2\pi i k x}$, $x \in \mathbb{T}$, we have*

$$\|T_n^{(s)}\|_{L_p(\mathbb{T})} \le \left(\frac{n}{2\sin\frac{n\delta}{2}}\right)^s \|\Delta_\delta^s T_n\|_{L_p(\mathbb{T})}, \quad 0 < \delta \le 1/n.$$

Recall that the sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}^d}$ is called a Fourier multiplier in L_p, $1 \le p \le \infty$, if for every function $f \in L_p$,

$$\sum_{k\in\mathbb{Z}^d} \lambda_k \widehat{f}(k) e^{2\pi i(k,x)}$$

is the Fourier series of a certain function $\Lambda f \in L_p$ and

$$\|\{\lambda_k\}_k\|_{\mathcal{M}_p} = \sup_{\|f\|_p \le 1} \|\Lambda f\|_p.$$

In the next theorem and below, we denote $v_\delta(\xi) = v(\delta^{-1}\xi)$, where $v \in C^\infty(\mathbb{R}^d)$, $v(\xi) = 1$ for $\xi \in [-1/4, 1/4)^d$ and $v(\xi) = 0$ for $\xi \notin [-3/8, 3/8)^d$.

Theorem 4.2. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $s \in \mathbb{N}$, $\delta \in (0, 1/2)$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$ are such that*

$$\widehat{\varphi_j}(k)\widehat{\widetilde{\varphi}_j}(k) = 1 + \sum_{[\beta]=s} (M^{-j}k)^\beta \Gamma_{j,s}(k) \quad \text{for all} \quad k \in D(\delta M^j), \tag{4.8}$$

where

$$\sup_j \|\{\Gamma_{j,s}(k) v_\delta(M^{-j}k)\}_k\|_{\mathcal{M}_p} < \infty. \tag{4.9}$$

Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\Omega_s(f, M^{-j})_p + K_{\varphi_j,q} \widetilde{E}_{\frac{\delta}{2}M^j}(\widetilde{\varphi}_j * f)_p\Big); \tag{4.10}$$

if, additionally, $\widetilde{\varphi}_j \in L_q$, then for any $f \in L_p$, we have

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C(1 + K_{\varphi_j,q} \|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}}) \Omega_s(f, M^{-j})_p, \tag{4.11}$$

where the constant C does not depend on f and j.

Proof. To prove estimate (4.10), we will use the following slightly modified version of inequality (3.3):

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\|\psi_j * V_{\delta M^j} f\|_p + E_{\frac{\delta}{2}M^j}(f)_p + K_{\varphi_j,q} \widetilde{E}_{\frac{\delta}{2}M^j}(\widetilde{\varphi}_j * f)_p\Big).$$

Thus, taking into account Lemma 4.1 and relations (4.7), we see that it is enough to show that

$$\|\psi_j * V_{\delta M^j} f\|_p \le C \Omega_s(f, M^{-j})_p. \tag{4.12}$$

Using (4.8), (4.9), and Lemma 4.2, we derive

$$\begin{aligned} \|\psi_j * V_{\delta M^j} f\|_p &\le \sum_{[\beta]=s} \Big\| \sum_k (M^{-j}k)^\beta \Gamma_{j,s}(k) v_\delta(M^{-j}k) \widehat{f}(k) \mathrm{e}^{2\pi \mathrm{i}(k,x)} \Big\|_p \\ &\le C \sum_{[\beta]=s} \Big\| \sum_k (M^{-j}k)^\beta v(M^{-j}k) \widehat{f}(k) \mathrm{e}^{2\pi \mathrm{i}(k,x)} \Big\|_p \\ &\le C \sum_{[\beta]=s} \Big\| \Delta^{\beta_1}_{\pi m_1^{-j}} \dots \Delta^{\beta_d}_{\pi m_d^{-j}} V_{M^j} f \Big\|_p \\ &\le C \Omega_s \left(V_{M^j} f, M^{-j}\right)_p. \end{aligned} \tag{4.13}$$

Next, applying the properties of moduli of smoothness (a)–(c), inequality (2.3), and Lemma 4.1 along with (4.7), we obtain

$$\begin{aligned}\Omega_s\left(V_{M^j} f, M^{-j}\right)_p &\le C\left(2^s\|f - V_{M^j} f\|_p + \Omega_s(f, M^{-j})_p\right)\\ &\le C\Omega_s(f, M^{-j})_p.\end{aligned} \tag{4.14}$$

Finally, combining (4.13) and (4.14), we get (4.12).

The proof of estimate (4.11) easily follows from Lemma 3.4, Lemma 4.1, and inequality (4.12). □

4.2.1. Two-sided estimates of approximation and fractional smoothness

Below, we will present some two-sided estimates of approximation by quasi-interpolation operators using fractional K-functionals and moduli of smoothness.

For our purposes, we will use the K-functional corresponding to the fractional Laplacian:

$$\mathcal{K}_s^{\Delta}(f, M^{-1})_p := \inf_g\{\|f - g\|_p + \|(-\Delta_{M^{-1}})^{s/2} g\|_p\},$$

where

$$(-\Delta_{M^{-1}})^{s/2} g(x) \sim \sum_{k\in\mathbb{Z}^d} |M^{-1}k|^s \widehat{g}(k) \mathrm{e}^{2\pi \mathrm{i}(k,x)}.$$

Recall that if $1 < p < \infty$, $s > 0$, and $M = \lambda I_d$, where $\lambda > 1$ is integer, then the K-functional $\mathcal{K}_s^{\Delta}(f, M^{-1})_p$ is equivalent to the following fractional modulus of smoothness (see, e.g., [43])

$$\omega_s(f, \lambda^{-1})_p := \sup_{|h|\le\lambda^{-1}} \left\| \sum_{l=0}^{\infty} (-1)^l \binom{s}{l} f(\cdot + hl)\right\|_p,$$

i.e.,

$$\mathcal{K}_s^{\Delta}(f, M^{-1})_p \asymp \omega_s(f, \lambda^{-1})_p, \tag{4.15}$$

where $\asymp$ is a two-sided inequality with positive constants that do not depend on f and λ.

Theorem 4.3. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $s \in \mathbb{N}$, $\delta \in (0, 1/2)$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$ are such that*

$$\sup_j \left\| \left\{ \frac{1 - \widehat{\varphi}_j(k)\widehat{\widetilde{\varphi}}_j(k)}{|M^{-j}k|^s} v_\delta(M^{-j}k)\right\}_k \right\|_{\mathcal{M}_p} < \infty. \tag{4.16}$$

Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\left(\mathcal{K}_s^{\Delta}(f, M^{-j})_p + K_{\varphi_j,q} \widetilde{E}_{\frac{\delta}{2}M^j}(\widetilde{\varphi}_j * f)_p\right); \tag{4.17}$$

if, additionally, $\widetilde{\varphi}_j \in L_q$, then

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C(1 + K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}})\mathcal{K}_s^{\Delta}(f, M^{-j})_p, \tag{4.18}$$

where the constant C does not depend on f and j.

Proof. As in the proof of Theorems 4.2, it is sufficient to show that

$$\|\psi_j * V_{\delta M^j} f\|_p \le C\mathcal{K}_s^{\Delta}(f, M^{-j})_p. \tag{4.19}$$

Using condition (4.16), we derive

$$\begin{aligned}\|\psi_j * V_{\delta M^j} f\|_p &= \left\| \sum_k \frac{1-\widehat{\varphi}_j(k)\widehat{\widetilde{\varphi}}_j(k)}{|M^{-j}k|^s} v_\delta(M^{-j}k)v(M^{-j}k)|M^{-j}k|^s \widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}(k,x)} \right\|_p \\ &\le C \left\| \sum_k v(M^{-j}k)|M^{-j}k|^s \widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}(k,x)} \right\|_p \\ &= C\|(-\Delta_{M^{-j}})^{s/2} V_{M^j} f\|_p.\end{aligned} \tag{4.20}$$

Next, taking into account the fact that

$$\sup_j \|\{v(M^{-j}k)|M^{-j}k|^s\}_k\|_{\mathcal{M}_p} < \infty \quad \text{for every} \quad s \ge 0 \tag{4.21}$$

(see Lemma 4.3) and choosing a function g such that

$$\|f-g\|_p + \|(-\Delta_{M^{-j}})^{s/2} g\|_p \le 2\mathcal{K}_s^\Delta(f, M^{-j})_p,$$

we obtain

$$\begin{aligned}\|(-\Delta_{M^{-j}})^{s/2} V_{M^j} f\|_p &\le \|(-\Delta_{M^{-j}})^{s/2} V_{M^j}(f-g)\|_p + \|(-\Delta_{M^{-j}})^{s/2} V_{M^j} g\|_p \\ &\le C\|f-g\|_p + \left\| V_{M^j}\left((-\Delta_{M^{-j}})^{s/2} g\right) \right\|_p \\ &\le C\left(\|f-g\|_p + \|(-\Delta_{M^{-j}})^{s/2} g\|_p\right) \le C\mathcal{K}_s^\Delta(f, M^{-j})_p.\end{aligned} \tag{4.22}$$

Thus, combining (4.20) and (4.22), we get (4.19). This implies that inequalities (4.17) and (4.18) are valid. □

Now we consider the estimates from below.

Theorem 4.4. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $s > 0$, $\delta \in (0, 1/2)$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$ are such that*

$$\sup_j \left\| \left\{ \frac{|M^{-j}k|^s}{1-\widehat{\varphi}_j(k)\widehat{\widetilde{\varphi}}_j(k)} v_{1/\delta}(M^{-j}k) \right\}_k \right\|_{\mathcal{M}_p} < \infty. \tag{4.23}$$

Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have

$$\mathcal{K}_s^\Delta(f, M^{-j})_p \le C\left(\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + E_{\frac12 M^j}(f)_p + K_{\varphi_j,q} \widetilde{E}_{\frac12 M^j}(\widetilde{\varphi}_j * f)_p \right); \tag{4.24}$$

if, additionally, $\widetilde{\varphi}_j \in L_q$, then for any $f \in L_p$, we have

$$\mathcal{K}_s^\Delta(f, M^{-j})_p \le C(1 + K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}})\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p, \tag{4.25}$$

where the constant C does not depend on f and j.

Remark 4.1. If in Theorem 4.4 instead of (4.23), we suppose that

$$\sup_j \left\| \left\{ \frac{|M^{-j}k|^s}{1-\widehat{\varphi}_j(k)\widehat{\widetilde{\varphi}}_j(k)} \chi_{D(M^j)}(k) \right\}_k \right\|_{\mathcal{M}_p} < \infty,$$

then, for any $f \in B_{\widetilde{\varphi}_j,p}$, $1 < p < \infty$, we have

$$\mathcal{K}_s^{\Delta}(f, M^{-j})_p \le C \left(\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + K_{\varphi_j,q} \widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p\right).$$

This follows from the proof of Theorem 4.4 presented below and Corollary 3.1(a).

Remark 4.2. If $d = 1$ and in conditions (4.16) or (4.23) we replace $|M^{-j}k|^s$ with $(\mathrm{i}M^{-j}k)^s$, $M > 1$, then for any $f \in L_p$, $1 \le p \le \infty$, and $s > 0$, the K-functional $\mathcal{K}_s^{\Delta}(f, M^{-j})_p$ can be replaced with the fractional modulus of smoothness $\omega_s(f, M^{-j})_p$. This easily follows from the proofs of Theorems 4.3 and 4.4 and the fact that for any $f \in L_p(\mathbb{T})$ and $s > 0$ (see, e.g., [5])

$$\omega_s(f, t)_p \asymp \inf_g \left(\|f - g\|_p + t^s \|g^{(s)}\|_p\right),$$

where $\asymp$ is a two-sided inequality with positive constants that do not depend on f and t.

Proof of Theorem 4.4. By the definition of the K-functional, we derive

$$\mathcal{K}_s^{\Delta}(f, M^{-j})_p \le \|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + \|(-\Delta_{M^{-j}})^{s/2} Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p. \tag{4.26}$$

Let $T_j \in \mathcal{T}_{M^j}$ be some trigonometric polynomial that will be chosen later. Taking into account condition (4.23) and using (4.21) and equality (3.6), we obtain

$$\begin{aligned}
&\|(-\Delta_{M^{-j}})^{s/2} Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \\
&\quad\le \|(-\Delta_{M^{-j}})^{s/2} \left(Q_j(f, \varphi_j, \widetilde{\varphi}_j) - T_j\right)\|_p + \|(-\Delta_{M^{-j}})^{s/2} T_j\|_p \\
&\quad\le C \left(\|Q_j(f, \varphi_j, \widetilde{\varphi}_j) - T_j\|_p + \|\psi_j * T_j\|_p\right) \\
&\quad= C \left(\|Q_j(f, \varphi_j, \widetilde{\varphi}_j) - T_j\|_p + \|Q_j(T_j, \varphi_j, \widetilde{\varphi}_j) - T_j\|_p\right) \\
&\quad\le C \left(\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + \|f - T_j\|_p + \|Q_j(f - T_j, \varphi_j, \widetilde{\varphi}_j)\|_p\right).
\end{aligned} \tag{4.27}$$

Now, to prove inequality (4.24), we choose $T_j = V_{M^j} f$. Then, applying estimates (3.4) and (3.8), we derive

$$\begin{aligned}
\|Q_j(f - T_j, \varphi_j, \widetilde{\varphi}_j)\|_p &\le C K_{\varphi_j,q} \left(\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p + \|\widetilde{\varphi}_j * (f - T_j)\|_p\right) \\
&\le C K_{\varphi_j,q} \left(\widetilde{E}_{M^j}(\widetilde{\varphi}_j * f)_p + \widetilde{E}_{\frac{1}{2}M^j}(\widetilde{\varphi}_j * f)_p\right) \\
&\le C K_{\varphi_j,q} \widetilde{E}_{\frac{1}{2}M^j}(\widetilde{\varphi}_j * f)_p.
\end{aligned} \tag{4.28}$$

Using also estimate (2.3), we see that inequalities (4.27) and (4.28) imply that

$$\begin{aligned}
&\|(-\Delta_{M^{-j}})^{s/2} Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \\
&\quad\le C \left(\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + E_{\frac{1}{2}M^j}(f)_p + K_{\varphi_j,q} \widetilde{E}_{\frac{1}{2}M^j}(\widetilde{\varphi}_j * f)_p\right).
\end{aligned} \tag{4.29}$$

Combining (4.26) and (4.29), we get (4.24).

To prove inequality (4.25), it is enough to set $T_j = Q_j(f, \varphi_j, \widetilde{\varphi}_j)$ and take into account that by (3.9) and (4.27), we have

$$\|(-\Delta_{M^{-j}})^{s/2} Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C(1 + K_{\varphi_j,q} \|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}}) \|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p,$$

which together with (4.26) implies (4.25). □

In the next results, we deal with functions/distributions φ_j and $\widetilde{\varphi}_j$ having the following special form:

$$\varphi_j(x) \sim \sum_{k \in \mathbb{Z}^d} \Phi(M^{-j}k) \mathrm{e}^{2\pi \mathrm{i}(k,x)}, \quad \widetilde{\varphi}_j(x) \sim \sum_{k \in \mathbb{Z}^d} \widetilde{\Phi}(M^{-j}k) \mathrm{e}^{2\pi \mathrm{i}(k,x)}, \tag{4.30}$$

where Φ, $\widetilde{\Phi} : \mathbb{R}^d \to \mathbb{C}$ are appropriate functions, which will be specified below. Actually, most of the quasi-interpolation operators (2.4) are defined by means of functions/distributions φ_j and $\widetilde{\varphi}_j$ given by (4.30). Below, we would like to give a version of Theorem 4.2, in which the conditions on φ_j and $\widetilde{\varphi}_j$ are given only in terms of some simple smoothness properties of the functions Φ and $\widetilde{\Phi}$.

For our purposes, we need to recall some facts about Fourier multipliers on $L_p(\mathbb{R}^d)$. First, we recall that a bounded function $\mu : \mathbb{R}^d \to \mathbb{C}$ is called a Fourier multiplier on $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$ (we will write $\mu \in \mathcal{M}_p(\mathbb{R}^d)$), if the operator T_μ defined by

$$\mathcal{F}(T_\mu f) = \mu \mathcal{F}(f), \quad f \in L_p(\mathbb{R}^d) \cap L_2(\mathbb{R}^d),$$

is bounded on $L_p(\mathbb{R}^d)$, i.e., there exists a constant C such that $\|T_\mu f\|_{L_p(\mathbb{R}^d)} \le C\|f\|_{L_p(\mathbb{R}^d)}$. The norm of the Fourier multiplier μ is given by

$$\|\mu\|_{\mathcal{M}_p(\mathbb{R}^d)} = \sup_{\|f\|_{L_p(\mathbb{R}^d)} \le 1} \|T_\mu f\|_{L_p(\mathbb{R}^d)}.$$

We will use the following basic properties of Fourier multipliers on $L_p(\mathbb{R}^d)$:

Lemma 4.3. *(a) If $\mu \in \mathcal{M}_p(\mathbb{R}^d)$, $1 \le p \le \infty$, and $\mu(t)$ is continuous at the points $t \in \mathbb{Z}^d$, then, for any dilation matrix M and $j \in \mathbb{N}$, the sequence $\{\mu(M^{-j}k)\}_{k\in\mathbb{Z}^d}$ is a bounded Fourier multiplier in the space $L_p(\mathbb{T}^d)$ and*

$$\sup_j \|\{\mu(M^{-j}k)\}_k\|_{\mathcal{M}_p} \le c(p,d)\|\mu\|_{\mathcal{M}_p(\mathbb{R}^d)}.$$

(b) Suppose that the function μ belongs to $C(\mathbb{R}^d)$ and has a compact support. If $\mu \in W_s^d(\mathbb{R}^d)$ for some $s > 1$, or more generally $\mathcal{F}(\mu) \in L_1(\mathbb{R}^d)$, then $\mu \in \mathcal{M}_p(\mathbb{R}^d)$ for all $1 \le p \le \infty$.

Proof. (a) This assertion follows from the well-known de Leeuw theorem (see [8]) and the fact that for every affine transformation $l : \mathbb{R}^d \to \mathbb{R}^d$, we have $\|\mu \circ l\|_{\mathcal{M}_p(\mathbb{R}^d)} = \|\mu\|_{\mathcal{M}_p(\mathbb{R}^d)}$ (see, e.g., [9, p. 147]).

(b) The assertion can be found, e.g., in [23]. □

Remark 4.3. The sufficient condition for Fourier multipliers given in assertion (b) is one of the simplest and is rather rough. For more advanced sufficient conditions for Fourier multipliers see, e.g., [9, Ch. 5], [16,23].

Now, we are ready to present an analogue of Theorem 4.2.

Theorem 4.5. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, $s \in \mathbb{N}$, $\delta \in (0, 1/2)$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'$ and $\varphi_j \in \mathcal{T}_{M^j}$, φ_j and $\widetilde{\varphi}_j$ are given by (4.30), Φ, $\widetilde{\Phi} \in C^{s+d}(2\delta\mathbb{T}^d)$ and $D^\alpha(1 - \widetilde{\Phi}\Phi)(0) = 0$ for all $|\alpha| < s$. Then, for any $f \in B_{\widetilde{\varphi}_j,p}$, we have*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\Big(\Omega_s(f, M^{-j})_p + K_{\varphi_j,q}\widetilde{E}_{\frac{\delta}{2}M^j}(\widetilde{\varphi}_j * f)_p\Big),$$

if, additionally, $\widetilde{\varphi}_j \in L_q$, then for any $f \in L_p$, we have

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C(1 + K_{\varphi_j,q}\|\widetilde{\varphi}_j\|_{\mathcal{L}_{q,j}})\Omega_s(f, M^{-j})_p,$$

where the constant C does not depend on f and j.

Proof. The proof easily follows from Theorem 4.2 and Lemma 4.3. One only needs to take into account that using Taylor's formula near zero, we have

$$\Phi(\xi)\widetilde{\Phi}(\xi) = 1 + \sum_{[\beta]=s} \frac{s}{\beta!} r^\beta \int_0^1 (1-t)^{s-1} D^\beta \Phi\widetilde{\Phi}(t\xi)\mathrm{d}t, \quad \beta \in \mathbb{Z}_+^d,\ [\beta] = s.$$

Then, denoting

$$G_\beta(\xi) = \rho(\xi) \int_0^1 (1-t)^{s-1} D^\beta \Phi\widetilde{\Phi}(t\xi)\mathrm{d}t,$$

where $\rho(\xi) \in C^\infty(\mathbb{R}^d)$, $\rho(\xi) = 1$ for $\xi \in \delta\mathbb{T}^d$ and $\rho(\xi) = 0$ for $\xi \notin 2\delta\mathbb{T}^d$, and taking into account that $G_\beta \in C^d(\mathbb{R}^d)$, we have that by Lemma 4.3, conditions (4.8) and (4.9) hold with $\Gamma_{j,\beta}(k) = G_\beta(M^{-j}k)$. $\square$

Example 4.3. Taking $\widetilde{\varphi}_j = m^j \chi_{M^{-j}[-\frac{1}{2},\frac{1}{2})^d}$ and $\varphi_j = \mathscr{D}_{M^j}$, it is not difficult to see that Theorem 4.5 provides the following error estimate for the corresponding Kantorovich-type operator (cf. [19, Proposition 19]):

$$\left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} \mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\mathscr{D}_{M^j}(\cdot - M^{-j}k) \right\|_p \le C\kappa_{j,p}\Omega_2(f, M^{-j})_p, \qquad (4.31)$$

where $f \in L_p$, $1 \le p \le \infty$, $\sigma \in (0, 1]$, the constant $\kappa_{j,p}$ is given in (4.4), and C does not depend on f and j.

We omit the formulations of the corresponding analogues of Theorems 4.3 and 4.4 in terms of the smoothness properties of Φ and $\widetilde{\Phi}$. Using Lemma 4.3 and Remark 4.3, one can directly and easily obtain appropriate statements. Instead of this, we give several examples of applications of Theorems 4.3 and 4.4 for some special quasi-interpolation operators.

First, we consider an estimate from below for the L_p-error of approximation by the quasi-interpolation operator from Example 4.3.

Example 4.4. Using Remark 4.1 and Lemma 4.3, we obtain that for any $f \in L_p$, $1 < p < \infty$, $\sigma \in (0, 1]$, and $j \in \mathbb{N}$

$$C\mathcal{K}_2^\Delta(f, M^{-j})_p \le \left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} \mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\mathscr{D}_{M^j}(\cdot - M^{-j}k) \right\|_p,$$

where C does not depend on f and j. Combining this estimate and inequality (4.31), we derive that

$$\left\| f - \frac{1}{m^j} \sum_{k\in D(M^j)} \mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\mathscr{D}_{M^j}(\cdot - M^{-j}k) \right\|_p \asymp \Omega_2(f, M^{-j})_p.$$

In the last estimate, we took into account the fact that $\Omega_2(f, M^{-j})_p \le C\mathcal{K}_2^\Delta(f, M^{-j})_p$, which easily follows from relation (4.6) and inequality $\|\Delta_h^2 g\|_{L_p(\mathbb{T})} \le \|g''\|_{L_p(\mathbb{T})}$.

Our next example concerns quasi-projection operators that are generated by an average sampling instead of the exact samples of f. Note that in the non-periodic case such operators are useful to reduce noise (see, e.g., [44]). However, we will show that some of these operators cannot provide as "good" an approximation order as in the case of the classical interpolation operator, cf. Example 4.1.

Example 4.5. Let $d = 1$ and $M \in \mathbb{N}$, $M \ge 2$. For $f \in B$, we denote

$$\lambda_j f(x) = \frac{1}{4} f(x - M^{-j-1}) + \frac{1}{2} f(x) + \frac{1}{4} f(x + M^{-j-1}) \sim \sum_{\ell\in\mathbb{Z}} \widehat{\widetilde{\varphi}_j}(\ell)\widehat{f}(\ell) \mathrm{e}^{2\pi \mathrm{i}\ell x},$$

where $\widehat{\widetilde{\varphi}_j}(\ell) = \cos^2(2\pi M^{-j-1}\ell)$. Using Theorems 4.3 and 4.4 and Lemma 4.3 for $\widetilde{\varphi}_j$ and $\varphi_j = \mathcal{D}_{M^j}$, taking also into account Remark 4.2, we derive

$$\begin{aligned} C_1\omega_2(f, M^{-j})_p &\le \Big\| f - \frac{1}{M^j}\sum_{k\in D(M^j)} \lambda_j f(M^{-j}k)\mathscr{D}_{M^j}(\cdot - M^{-j}k)\Big\|_p \\ &\le C_2\left(\omega_2(f, M^{-j})_p + \widetilde{E}_{M^j}(\lambda_j f)_p\right), \end{aligned}$$

where $1 < p < \infty$ and C_1, C_2 are some positive constants that do not depend on f and j.

Finally, we present two examples of the error estimates, in which we essentially use the fractional smoothness of a function f. For our purposes, we consider the following Riesz kernel

$$\mathscr{R}^\gamma_{s,M^j}(x) = \sum_k (1 - |c_d M^{-j}k|^s)^\gamma_+ \mathrm{e}^{2\pi\mathrm{i}(k,x)}, \quad s, \gamma > 0 \quad \text{and} \quad c_d = 4d^{1/2}.$$

Example 4.6. Let $1 \le p \le \infty$, $s > 0$, $\gamma > \frac{d-1}{2}$, and $j \in \mathbb{N}$.

(1) For any $f \in B$ ($f \in C(\mathbb{T}^d)$ in the case $p = \infty$), we have

$$\begin{aligned} C_1\mathcal{K}^\Delta_s(f, M^{-j})_p &\le \Big\| f - \frac{1}{m^j}\sum_{k\in D(M^j)} f(M^{-j}k)\mathscr{R}^\gamma_{s,M^j}(\cdot - M^{-j}k)\Big\|_p \\ &\le C_2\Big(\mathcal{K}^\Delta_s(f, M^{-j})_p + \widetilde{E}_{cM^j}(f)_p\Big), \end{aligned} \tag{4.32}$$

where c, C_1 and C_2 are some positive constants that do not depend on f and j

(2) For any $f \in L_p$, $s \in (0, 2]$, and $\sigma \in (0, 1]$, we have

$$\Big\| f - \frac{1}{m^j}\sum_{k\in D(M^j)} \mathrm{Avg}_{\sigma M^{-j}} f(M^{-j}k)\mathscr{R}^\gamma_{s,M^j}(\cdot - M^{-j}k)\Big\|_p \asymp \mathcal{K}^\Delta_s(f, M^{-j})_p, \tag{4.33}$$

where $\asymp$ is a two-sided inequality with positive constants that do not depend on f and j.

The proof of inequalities in (4.32) follows from Theorems 4.3 and 4.4, Lemma 4.3, and the fact that with an appropriate parameter $\delta \in (0, 1/2)$, the Fourier transforms of the functions

$$g_1(\xi) = \frac{|\xi|^s v_{1/\delta}(\xi)}{1 - (1 - |c_d\xi|^s)^\gamma_+} \quad \text{and} \quad g_2(\xi) = \frac{1 - (1 - |c_d\xi|^s)^\gamma_+ v_\delta(\xi)}{|\xi|^s}$$

belong to $L_1(\mathbb{R}^d)$ (see, e.g., [29], see also the proof of Theorem 2 in [15]).

The proof of (4.33) is similar. In this case, one only needs to investigate, by analogy with the previous case, the following two functions

$$g_2(\xi) = \frac{|\xi|^s v_{1/\delta}(\xi)}{1 - \widetilde{\Phi}(\xi)(1 - |c_d\xi|^s)^\gamma_+} \quad \text{and} \quad g_3(\xi) = \frac{1 - \widetilde{\Phi}(\xi)(1 - |c_d\xi|^s)^\gamma_+ v_\delta(\xi)}{|\xi|^s},$$

where $\widetilde{\Phi}(\xi) = \prod_{\ell=1}^d \frac{\sin \pi\sigma\xi_\ell}{\pi\sigma\xi_\ell}$.

4.3. Error estimates for functions from Besov-type spaces

In the previous sections, we obtained error estimates for the quasi-interpolation operators $Q_j(f,\varphi_j,\widetilde{\varphi}_j)$ under very general conditions on the distribution $\widetilde{\varphi}_j$. These estimates were given in terms of the best one-sided approximation $\widetilde{E}_{\delta M^j}(\widetilde{\varphi}_j * f)_p$ and appropriate moduli of smoothness and K-functionals. At the same time, we proved that in the case $\widetilde{\varphi}_j \in L_q$, the best one-sided approximation can be replaced by the classical best approximation $E_{\delta M^j}(f)_p$. In this section, we will present other possibilities (not so restrictive as the assumption $\widetilde{\varphi}_j \in L_q$) to avoid exploitation of a quite specific quantity $\widetilde{E}_{\delta M^j}(\widetilde{\varphi}_j * f)_p$.

First of all, we note that the best one-sided approximation can be estimated from above by means of the so-called τ-modulus of smoothness, which is defined by

$$\tau_s(g,u)_p := \|\omega(g,\cdot,u)\|_p, \quad s\in\mathbb{N}, \quad u>0,$$

where

$$\omega(g,x,u)=\sup\{|\Delta_h^s g(t)| \,:\, t,t+sh\in D(su,x)\}, \quad x\in\mathbb{R}^d,$$

$$D(u,x)=\{y\in\mathbb{R}^d \,:\, |x-y|\le u/2\}.$$

Recall (see [1]) that for any $g\in B$, $s\in\mathbb{N}$, and the isotropic matrix $M=\lambda I_d$, $\lambda>1$ we have

$$\widetilde{E}_{M^j}(g)_p \le C_{s,d}\,\tau_s(g,\lambda^{-j})_p, \tag{4.34}$$

where the constant C does not depend on g and j.

For smooth functions, one can estimate one-sided best approximation as follows (see [27]): if $f\in W_p^d\cap B$, then

$$\widetilde{E}_{M^j}(g)_p \le C_d \sum_{\alpha_j\in\{0,1\},\,[\alpha]>0} \lambda^{-j[\alpha]} E_{M^j}(D^\alpha g)_p. \tag{4.35}$$

Thus, using (4.34) or (4.35) with $g=\widetilde{\varphi}_j * f$, we can replace $\widetilde{E}_{\delta M^j}(\widetilde{\varphi}_j * f)_p$ in Theorems 4.1–4.5 by the corresponding approximation quantity from the right-hand sides of (4.34) or (4.35).

Below, using a special Besov space, we present another approach to replace $\widetilde{E}_{\delta M^j}(\widetilde{\varphi}_j * f)_p$ in the corresponding results. Note that this approach is based on some ideas from [10] and [18]. In contrast to formulas (4.34) and (4.35), we avoid calculations of special τ-moduli of smoothness and the consideration of functions from the Sobolev spaces.

We use the following anisotropic Besov spaces with respect to the matrix M. We say that $f\in\mathbb{B}_{p,q}^s(M)$, $1\le p\le\infty$, $0<q\le\infty$, and $s>0$, if $f\in L_p$ and

$$\|f\|_{\mathbb{B}_{p,q}^s(M)} := \|f\|_p + \left(\sum_{\nu=1}^\infty m^{\frac{s}{d}q\nu} E_{M^\nu}(f)_p^q\right)^{\frac1q} < \infty.$$

For our purposes, we need to specify the class of tempered distributions $\widetilde{\varphi}_j$. We say that a sequence of tempered distribution $\widetilde{\varphi}_j$ belongs to the class $\mathcal{D}'_{N,j,p}$ for some $N\ge 0$ and $1\le p\le\infty$ if there exists a positive constant C, which does not depend on j, such that for any trigonometric polynomial $T_\nu\in\mathcal{T}_{M^\nu}$, one has

$$\|\widetilde{\varphi}_j * T_\nu\|_p \le C m^{\frac{N}{d}(\nu-j)}\|T_\nu\|_p \quad \text{for all} \quad \nu\ge j, \quad j,\nu\in\mathbb{N}. \tag{4.36}$$

As a simple example of $\widetilde{\varphi}_j \in \mathcal{D}'_{N,j,p}$, we can take the distribution corresponding to some differential operator. Namely, if we set

$$\widehat{\widetilde{\varphi}}_j(\ell) = \sum_{[\beta]\le N} c_\beta (2\pi \mathrm{i} M^{-j}\ell)^\beta, \quad N \in \mathbb{Z}_+,$$

where the numbers c_β do not depend on j, then by the well-known Bernstein inequality for trigonometric polynomials (see, e.g., [38, p. 215])

$$\left\| \sum_{k=-n}^{n} (\mathrm{i}k)^r a_k \mathrm{e}^{2\pi \mathrm{i}kx} \right\|_{L_p(\mathbb{T})} \le n^r \left\| \sum_{k=-n}^{n} a_k \mathrm{e}^{2\pi \mathrm{i}kx} \right\|_{L_p(\mathbb{T})},$$

we can easily derive that $\widetilde{\varphi}_j \in \mathcal{D}'_{N,j,p}$.

Lemma 4.4. *Let $1 \le p \le \infty$, $M \ge 0$, $\delta \in (0,1]$, $j \in \mathbb{N}$, and $\widetilde{\varphi}_j \in \mathcal{D}'_{N,j,p}$. Then, for any $f \in \mathbb{B}^{N+d/p}_{p,1}(M)$,*

$$\sum_{\ell\in\mathbb{Z}^d} \widehat{\widetilde{\varphi}}_j(\ell) \widehat{f}(\ell) \mathrm{e}^{2\pi \mathrm{i}(\ell,x)} \tag{4.37}$$

is a Fourier series of a continuous function $\widetilde{\varphi}_j * f$ on $\mathbb{T}^d$, i.e., $\mathbb{B}^{N+d/p}_{p,1}(M) \subset B_{\widetilde{\varphi}_j,p}$, and

$$\|\{\widetilde{\varphi}_j * f(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)\}_k\|_{\ell_{p,M^j}} \le C m^{-(\frac{1}{p}+\frac{N}{d})j} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p, \tag{4.38}$$

where $T_j \in \mathcal{T}_{M^j}$ is such that $\|f - T_j\| \le c(d,p,\delta) E_{\delta M^j}(f)_p$ and the constant C does not depend on f and j.

Proof. First, we show that the series in (4.37) is a Fourier series of a certain continuous function, which we will denote by $\widetilde{\varphi}_j * f$.

Using Nikolskii's inequality of different metrics (see, e.g., [25, p. 133])

$$\|T_\nu\|_\infty \le C_p m^{\frac{\nu}{p}} \|T_\nu\|_p$$

and inequality (4.36), we derive

$$\begin{aligned} \sum_{\nu=1}^{\infty} \|\widetilde{\varphi}_j * T_{\nu+1} - \widetilde{\varphi}_j * T_\nu\|_\infty &\le C \sum_{\nu=1}^{\infty} m^{\frac{\nu}{p}} \|\widetilde{\varphi}_j * (T_{\nu+1} - T_\nu)\|_p \\ &\le C m^{-\frac{N}{d}j} \sum_{\nu=1}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} \|T_{\nu+1} - T_\nu\|_p \\ &\le C m^{-\frac{N}{d}j} \sum_{\nu=1}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p. \end{aligned} \tag{4.39}$$

The estimates (4.39) imply that the sequence $\{\widetilde{\varphi}_j * T_\nu\}_{\nu\in\mathbb{N}}$ is fundamental in $C(\mathbb{T}^d)$. We denote its limit by $\widetilde{\varphi}_j * f$. It is clear that this limit does not depend on the choice of polynomials T_ν. Thus, if T_ν is defined using the de la Vallée Poussin means $V_\nu f$, we derive that $\{\widehat{\widetilde{\varphi}}_j(\ell)\widehat{f}(\ell)\}_\ell$ are the Fourier coefficients of the function $\widetilde{\varphi}_j * f$ since for a fixed ℓ and a sufficiently large ν

$$\begin{aligned} |\widehat{\widetilde{\varphi}_j * f}(\ell) - \widehat{\widetilde{\varphi}}_j(\ell)\widehat{f}(\ell)| &= \left| \int_{\mathbb{T}^d} (\widetilde{\varphi}_j * f(x) - \widetilde{\varphi}_j * V_\nu f(x)) \mathrm{e}^{2\pi \mathrm{i}(\ell,x)} \mathrm{d}x \right| \\ &\le \|\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * V_\nu f\|_\infty \to 0 \quad \text{as} \quad \nu \to \infty. \end{aligned}$$

Now, we prove inequality (4.38). Using the representation

$$\widetilde{\varphi}_j * f - \widetilde{\varphi}_j * T_j = \sum_{\nu=j}^{\infty} \widetilde{\varphi}_j * (T_{\nu+1} - T_\nu) \quad \text{in} \quad C(\mathbb{T}^d),$$

Lemma 3.2, and (4.39), we obtain

$$\|\{\widetilde{\varphi}_j * f(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)\}_k\|_{\ell_{p,M^j}} \le \sum_{\nu=j}^{\infty} \|\{\widetilde{\varphi}_j * (T_{\nu+1} - T_\nu)(M^{-j}k)\}_k\|_{\ell_{p,M^j}}$$

$$\le m^{-\frac{j}{p}} \sum_{\nu=j}^{\infty} m^{\frac{\nu}{p}} \|\{\widetilde{\varphi}_j * (T_{\nu+1} - T_\nu)(M^{-\nu}k)\}_k\|_{\ell_{p,M^\nu}}$$

$$\le Cm^{-\frac{j}{p}} \sum_{\nu=j}^{\infty} m^{\frac{\nu}{p}} \|\widetilde{\varphi}_j * (T_{\nu+1} - T_\nu)\|_p$$

$$\le Cm^{-\frac{j}{p}} \sum_{\nu=j}^{\infty} m^{\frac{N}{d}(\nu+1-j)\frac{\nu}{p}} \|T_{\nu+1} - T_\nu\|_p$$

$$\le Cm^{-(\frac{1}{p}+\frac{N}{d})j} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})} E_{\delta M^\nu}(f)_p,$$

which proves the lemma. □

We have the following counterpart of Lemma 3.1:

Lemma 4.5. *Let $1 \le p \le \infty$, $1/p+1/q = 1$, $\delta \in (0, 1]$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'_{N,j,p}$ and $\varphi_j \in \mathcal{T}_{M^j}$. Then, for any $f \in \mathbb{B}_{p,1}^{d/p+N}(M)$, we have*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\|_p \le C \left(\|\psi_j * T_j\|_p + m^{-j(\frac{1}{p}+\frac{N}{d})} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p \right), \tag{4.40}$$

where ψ_j is given in (3.1), $T_j \in \mathcal{T}_{M^j}$ is such that $\|f - T_j\|_p \le c(d, p, \delta)E_{\delta M^j}(f)_p$, and the constant C does not depend on f and j.

Proof. The proof is similar to the one of Lemma 3.1. The only difference consists in the estimate of the norm I_3 in inequality (3.8). In particular, using Lemma 4.4 and the first inequality in (3.8), we derive that

$$\begin{aligned} I_3 &\le CK_{\varphi_j,q} \left(\frac{1}{m^j} \sum_{k\in D(M^j)} |\widetilde{\varphi}_j * f(M^{-j}k) - \widetilde{\varphi}_j * T_j(M^{-j}k)|^p \right)^{\frac{1}{p}} \\ &\le CK_{\varphi_j,q} m^{-(\frac{1}{p}+\frac{N}{d})j} \sum_{\nu=1}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p. \end{aligned} \tag{4.41}$$

Thus, combining (3.5), (3.7), and (4.41), we prove the lemma. □

Remark 4.4. If in Lemma 4.5 we replace the condition $\widetilde{\varphi}_j \in \mathcal{D}'_{0,j,\infty}$ by

$$\|\widetilde{\varphi}_j * f\|_\infty \le C\|f\|_\infty, \quad \text{for all} \quad f \in B, \quad j \in \mathbb{N}, \tag{4.42}$$

then, for any $f \in C(\mathbb{T}^d)$, the error estimate (4.40) can be improved in the following way

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_\infty \le C \left(\|\psi_j * T_j\|_\infty + K_{\varphi_j,1} E_{\delta M^j}(f)_\infty\right).$$

This estimate can be proved using the same argument as in the proof of Lemma 3.4.

Note also that condition (4.42) holds if, for example, $\widetilde{\varphi}_j$ is the periodic Dirac-delta function for all $j \in \mathbb{N}$.

Finally, we note that combining Lemma 4.5 with Theorems 4.1–4.4, we easily obtain the following error estimates given in terms of the unrestricted best approximation. Note also that inequality (4.43) was earlier obtained in [18].

Proposition 4.1. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$, and $j \in \mathbb{N}$. Suppose that $\widetilde{\varphi}_j \in \mathcal{D}'_{N,j,p}$, $\varphi_j \in \mathcal{T}_{M^j}$, and $f \in \mathbb{B}_{p,1}^{d/p+N}(M)$.*

(1) *If condition (4.1) holds for some $\delta \in (0, 1]$, then*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C K_{\varphi_j,q} m^{-j(\frac{1}{p}+\frac{N}{d})} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p. \tag{4.43}$$

(2) *If conditions (4.8)* and *(4.9) hold for some $\delta \in (0, 1/2)$ and $s \in \mathbb{N}$, then*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\left(\Omega_s(f, M^{-j})_p + K_{\varphi_j,q} m^{-j(\frac{1}{p}+\frac{N}{d})} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p\right).$$

(3) *If condition (4.16) holds for some $\delta \in (0, 1/2)$ and $s > 0$, then*

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p \le C\left(\mathcal{K}_s^{\Delta}(f, M^{-j})_p + K_{\varphi_j,q} m^{-j(\frac{1}{p}+\frac{N}{d})} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\delta M^\nu}(f)_p\right).$$

(4) *If condition (4.23) holds for some $\delta \in (0, 1/2)$ and $s > 0$, then*

$$\mathcal{K}_s^{\Delta}(f, M^{-j})_p \le C \Bigg(\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_p + K_{\varphi_j,q} m^{-j(\frac{1}{p}+\frac{N}{d})} \sum_{\nu=j}^{\infty} m^{(\frac{1}{p}+\frac{N}{d})\nu} E_{\frac{1}{2} M^\nu}(f)_p \Bigg).$$

In the above four inequalities, the constant C does not depend on f and j.

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