

# On the optimal approximation rate of certain stochastic integrals

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## Abstract

Given an increasing function  $H : [0, 1) \rightarrow [0, \infty)$  and

$$A_n(H) := \inf_{\tau \in \mathcal{T}_n} \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}},$$

where  $\mathcal{T}_n := \{\tau = (t_i)_{i=0}^n : 0 = t_0 < t_1 < \dots < t_n = 1\}$ , we characterize the property  $A_n(H) \leq \frac{c}{\sqrt{n}}$ , and give conditions for  $A_n(H) \leq \frac{c}{\sqrt{n}^\beta}$  and  $A_n(H) \geq \frac{1}{c\sqrt{n}^\beta}$  for  $\beta \in (0, 1)$ , both in terms of integrability properties of  $H$ . These results are applied to the approximation of stochastic integrals.

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## 1. Introduction

In this paper we estimate the size of the error which occurs when a stochastic integral is approximated discretely. To explain the problem in more detail, we assume a stochastic process  $X = (X_t)_{t \in [0, 1]}$  such that

$$dX_t = \sigma(X_t) dW_t \quad \text{with } X_0 \equiv x_0 > 0,$$

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where  $W = (W_t)_{t \in [0,1]}$  is the standard Brownian motion,  $\sigma$  satisfies certain regularity properties, and  $(\mathcal{F}_t)_{t \in [0,1]}$  is the augmentation of the filtration generated by  $W$ . It is of interest to approximate discretely a stochastic integral, which can be written as

$$f(X_1) = \mathbb{E}f(X_1) + \int_0^1 \lambda_u dX_u, \quad (1)$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomially bounded, Borel measurable function, and  $\lambda = (\lambda_t)_{t \in [0,1]}$  is a suitable adapted process. In order to compute the integrand  $\lambda_u$  we let  $F(t, x) := \mathbb{E}(f(X_1) | X_t = x)$  which satisfies  $F(1, x) = f(x)$  and (under the conditions of the paper)

$$\frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} = 0$$

for  $t \in [0, 1)$  and  $x \in \mathbb{R}$  or  $x \in (0, \infty)$ , depending on the setting. Applying Itô's formula yields that

$$f(X_1) = \mathbb{E}f(X_1) + \int_0^1 \frac{\partial F}{\partial x}(u, X_u) dX_u,$$

i.e. one has  $\lambda_u = \frac{\partial F}{\partial x}(u, X_u)$ .

We approximate  $f(X_1)$  by

$$\mathbb{E}f(X_1) + \sum_{i=1}^n \lambda_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}), \quad (2)$$

where  $\tau^n := (t_i)_{i=0}^n$  is a deterministic time net with  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ . Using this approximation instead of the original stochastic integral, we obtain an approximation error

$$f(X_1) - \mathbb{E}f(X_1) - \sum_{i=1}^n \lambda_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}). \quad (3)$$

We are interested in the minimal quadratic error under the constraint that the time net used in the approximation has  $n + 1$  time points. According to [3, Lemma 3.2 and its proof] (see also [6]), this error is equivalent to

$$a_n^X(Z) := \inf_{\tau^n} a_X(Z, \tau^n), \quad (4)$$

where  $a_X(Z, \tau^n) := \inf \left( \mathbb{E} |f(X_1) - \mathbb{E}f(X_1) - \sum_{i=1}^n v_{i-1}(X_{t_i} - X_{t_{i-1}})|^2 \right)^{\frac{1}{2}}$  and  $Z = f(X_1)$  with the infimum taken over all sequences  $v = (v_i)_{i=0}^{n-1}$  of  $\mathcal{F}_{t_i}$ -measurable step functions  $v_i : \Omega \rightarrow \mathbb{R}$ .

The approximation problem is of interest for at least two reasons.

- (a) In stochastic finance one would like to replace a continuously adjusted hedging portfolio in the Black–Scholes option pricing model by a discretely adjusted one, as portfolios can be adjusted in practice only finitely many times. If we consider the quadratic error which occurs in this replacement (and which we can interpret as risk in finance), then we end up with the approximation problem described above. In this case  $X = (X_t)_{t \in [0,1]}$  is an appropriate positive diffusion process,  $f : (0, \infty) \rightarrow [0, \infty)$  is a payoff function of a European type option, and  $\tau^n$  is the net of time points where the portfolio is rebalanced.

(b) The approximation introduced above yields an approximation of  $\int_0^t \lambda_u dX_u$  by  $\sum_{i=1}^n \lambda_{t_{i-1}}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t})$ . The point is that the approximation itself is a stochastic integral, but the integrand  $\lambda_u$  (which is usually hard to compute) is only computed  $n$  times, whereas the increments  $(X_{t_i \wedge t} - X_{t_{i-1} \wedge t})$  can be easily simulated (for example by using an Euler scheme).

There are several previous results concerning the error caused by the discrete approximation of stochastic integrals. Under certain conditions on  $Z$  and  $\sigma$ , C. and S. Geiss showed that if  $\tau^n = (\frac{i}{n})_{i=0}^n$  is the equidistant time net with cardinality  $n + 1$ , then one has that

$$a_X(Z, \tau^n) \leq \frac{c}{\sqrt{n}}$$

if and only if  $Z$  belongs to the Malliavin Sobolev space  $\mathbb{D}_{1,2}$  [3, Theorems 2.3, and 2.6]. Furthermore, they proved that there exists a constant  $c > 0$  such that  $a_n^X(Z) \geq \frac{1}{c\sqrt{n}}$  unless there are constants  $c_0$  and  $c_1$  such that  $Z = c_0 + c_1 X_1$  a.s. [3, Theorem 2.5] (if such constants do exist, then  $a_n^X(Z) = 0$ ). It is also known by [3, Theorem 2.9] and [7, Theorem 3.2] that there exists a constant  $c > 0$  such that  $a_n^X(Z) \leq \frac{c}{\sqrt{n}}$ , if  $Z$  has a certain polynomial smoothness measured by Besov spaces generated by real interpolation. In this case the rate  $\frac{1}{\sqrt{n}}$  is obtained by using suitable non-equidistant time nets.

M. Hujo showed in [9, Theorem 3] that for  $X$  being the Brownian motion or the geometric Brownian motion, there exist random variables  $Z = f(X_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  such that

$$\sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) = \infty,$$

which means that the approximation rate is not always  $\frac{1}{\sqrt{n}}$  even if the underlying process is the standard Brownian motion. However, there are no explicit known examples of such functions.

These results lead us to the question of how to characterize those  $Z = f(X_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  with

$$a_n^X(Z) \leq \frac{c}{\sqrt{n}} \quad \text{for some } c = c(Z) > 0.$$

According to the results from [3,7], rephrased as Theorem 4.7 below, there exists a constant  $c = c(\sigma) > 0$  such that

$$\frac{1}{c} a_X(Z, \tau) \leq \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H_X Z(t)^2 dt \right)^{\frac{1}{2}} \leq c a_X(Z, \tau),$$

where  $H_X Z(t) := \left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$ ,  $F : [0, 1) \times I \rightarrow \mathbb{R}$  is given by  $F(t, x) = \mathbb{E}(Z | X_t = x)$ ,  $f$  and  $X$  satisfy certain conditions and  $I \subset \mathbb{R}$  depends on  $X$ . Moreover, Lemma 4.6 implies that  $H_X Z$  is increasing, so we concentrate our investigations for some time on the quantity

$$A_n(H) := \inf_{\tau \in \mathcal{T}_n} \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}},$$

where the function  $H : [0, 1) \rightarrow [0, \infty)$  is increasing and

$$\mathcal{T}_n := \{\tau = (t_i)_{i=0}^n : 0 = t_0 < t_1 < \dots < t_n = 1\}.$$

Our first main result, [Theorem 2.4](#), says that

$$\inf_{\tau \in \mathcal{T}_n} \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{n}},$$

if and only if the function  $H$  is integrable. Moreover, in [Theorem 2.6](#) we give sufficient conditions for

$$\inf_{\tau \in \mathcal{T}_n} \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{n^\beta}}$$

and

$$\inf_{\tau \in \mathcal{T}_n} \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}} \geq \frac{1}{c\sqrt{n^\beta}},$$

where  $\beta \in (0, 1)$ , in terms of the growth rate of  $H : [0, 1] \rightarrow [0, \infty)$ .

These results can be applied to the setting introduced above and also to other situations, for example to the quadratic approximation of multi-dimensional stochastic integrals (see [\[10,15,17\]](#)).

**Remark 1.1.** Wasilkowski and Woźniakowski [\[16\]](#), Hertling [\[8\]](#) and Przybyłowicz [\[12\]](#) investigated the approximation error in a slightly different setting. Their results concern the approximation of stochastic integrals of the form

$$\mathcal{I}(g, B_t) = \int_0^1 g(t, B_t) dB_t, \quad g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$$

with the algorithm

$$\text{app}(g, B) = \rho(g^{i_1, j_1}(a_1, b_1), \dots, g^{i_k, j_k}(a_k, b_k), B_{t_1}, \dots, B_{t_\ell}),$$

where  $\rho$  is some function and  $g^{i,j} = \frac{\partial^{i+j}}{\partial s^i \partial x^j} g$  and  $t_i, a_i$ , and  $b_i$  can depend on the computed values of  $B$  and  $f$ . For  $n = k + \ell$ , the class of all algorithms of this form is denoted by  $\mathcal{A}_n$ . The approximation error is measured by

$$e(n, \mathcal{G}) = \inf_{\text{app} \in \mathcal{A}_n} \sup_{g \in \mathcal{G}} \sqrt{\mathbb{E}((\mathcal{I}(g, B) - \text{app}(g, B))^2)},$$

where  $\mathcal{G}$  is some class of functions. Wasilkowski and Woźniakowski showed in [\[16\]](#) that if  $\mathcal{G}$  is the class of smooth enough Lipschitz functions (w.r.t. to the first variable) such that for all  $g \in \mathcal{G}$ ,

$$\sup_{t \in [0,1], y \in \mathbb{R}} \left| \frac{\partial^2}{\partial y^2} g(t, y) \right| \leq K \quad \text{for some } K > 0,$$

then  $\frac{1}{cn} \leq e(n, \mathcal{G}) \leq \frac{c}{n}$  for some  $c > 0$ . It was also shown in [\[16\]](#) that for the class  $\mathcal{G}_{\text{Lip}}$  of Lipschitz functions (w.r.t. both variables) one has that  $e(n, \mathcal{G}_{\text{Lip}}) \leq \frac{c}{n^{1/2}}$ . Hertling [\[8\]](#) gave a corresponding lower bound.

The Lipschitz assumption is relatively strong and, for example, [\[3, Theorems 2.3 and 2.6\]](#) contains the same upper bound for a class of integrands which is not uniformly Lipschitz and for more general diffusions than the Brownian motion.

Przybyłowicz [12] showed that under strong smoothness assumptions one can obtain the error rate  $\frac{1}{n^{3/2}}$  or even  $\frac{1}{n^2}$ . In [12], Przybyłowicz also presented lower bounds for the approximation error for certain classes of functions.

**Remark 1.2.** Every continuous local martingale  $M$  such that its bracket process converges almost surely to infinity can be written as a time-changed Brownian motion [11, Theorem 41]. Having this in mind, the martingale  $\left(\int_0^t \lambda_u dX_u\right)_{t \in [0,1]}$  could be extended to a martingale  $(M_t)_{t \geq 0}$  with the above property and the approximation problem could be re-formulated as an approximation problem for a time-changed Brownian motion. The author did not follow this approach; however this should be of interest of further research.

The paper is organized as follows: In Section 2 we introduce the main results of the paper; their proofs can be found in Section 3. In Section 4 we apply the results of Section 2 to the one-dimensional stochastic setting. In particular, we give an example of random variables for which the approximation rate is  $\frac{1}{c\sqrt{n^\beta}} \leq a_n^X(Z) \leq \frac{c}{\sqrt{n^\beta}}$ , for  $\beta \in (0, 1)$  in the case where  $X$  is the standard Brownian motion or the geometric Brownian motion. In Section 5 the results of Section 2 are applied to the approximation of  $d$ -dimensional stochastic integrals where the underlying diffusion might have a drift.

## 2. Results

In this section we formulate the results. We start with the definition.

**Definition 2.1.** Let  $H : [0, 1) \rightarrow \mathbb{R}$  be a non-negative measurable function. If  $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$ , then we define

$$\begin{cases} A(H, \tau) := \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}}, \\ A_n(H) := \inf_{\tau \in \mathcal{T}_n} A(H, \tau). \end{cases}$$

**Remark 2.2.** For example, in the case where  $H$  is continuous and non-decreasing it follows from the definition of  $A_n(H)$  that  $A_n(H) < \infty$  if and only if the function  $H$  belongs to the Lorentz space  $L_{1,2}((0, 1))$ , that is

$$\left( \int_0^1 (s H^*(s))^2 \frac{ds}{s} \right)^{\frac{1}{2}} < \infty,$$

where  $H^*$  is a non-increasing rearrangement of  $H$  (in our case  $H^*(s) := H(1 - s)$ ).

**Definition 2.3.** We say that an increasing function  $H : [0, 1) \rightarrow [0, \infty)$  belongs to the set  $\mathcal{A}$  if and only if

$$\|H\|_{\mathcal{A}} := \sup_{n \in \mathbb{N}} \sqrt{n} A_n(H) < \infty,$$

and to the set  $\mathcal{H}$  if and only if

$$\|H\|_{\mathcal{H}} := \int_0^1 H(t) dt < \infty.$$

**Theorem 2.4.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be an increasing function. Then

$$\|H\|_{\mathcal{A}} \leq \|H\|_{\mathcal{H}} \leq \sqrt{2} \|H\|_{\mathcal{A}}.$$

**Remark 2.5.** The proof of Theorem 2.4 implies that  $I := \int_0^1 H(t) dt < \infty$  gives

$$A_n(H) \leq \frac{I}{\sqrt{n}} \quad \text{for all } n \in \mathbb{N}.$$

This rate can be obtained from regular sequences (see [13] and [14]) generated by  $H$ . Regular sequences generated by  $H$  are time nets  $\tau^n = (t_i^n)_{i=0}^n$  for which

$$\int_0^{t_i^n} H(t) dt = \frac{i}{n} \int_0^1 H(t) dt$$

for all  $i \in \{0, \dots, n\}$ .

Our second main result is:

**Theorem 2.6.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be an increasing function and suppose that  $\alpha \in (\frac{1}{2}, 1)$ . Then one has the following:

1. If there exists a constant  $c_1 \geq 1$  such that

$$H(t) \leq c_1 \frac{(1 - \log(1 - t))^{-\alpha}}{1 - t} \quad \text{for all } t \in [0, 1),$$

then

$$A_n(H) \leq \frac{c}{\sqrt{n^{2\alpha-1}}} \quad \text{for all } n \in \mathbb{N},$$

where  $c = c(\alpha) \geq 1$ .

2. If there exists  $s \in [0, 1)$  and a constant  $c_2 \geq 1$  such that

$$H(t) \geq \frac{1}{c_2} \frac{(1 - \log(1 - t))^{-\alpha}}{1 - t} \quad \text{for all } t \in [s, 1),$$

then

$$A_n(H) \geq \frac{1}{c} \frac{1}{\sqrt{n^{2\alpha-1}}} \quad \text{for all } n \in \mathbb{N},$$

where  $c = c(s, \alpha, c_2) \geq 1$ .

**Remark 2.7.** It follows from the arguments in [5, Lemma 4.14, Proposition 4.16] that if  $H$  is increasing and there are  $C \in (0, \infty)$ ,  $\alpha \in (1, \infty)$  with

$$H(t) \leq \frac{C}{[\alpha + \log(1 + \frac{1}{1-t})]^\alpha (1-t)}$$

for all  $t \in [0, 1)$ , then one has that

$$\|H\|_{\mathcal{A}} < \infty.$$

**Remark 2.8.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be a measurable function and  $\hat{H}(t) := \sup_{s \in [0, t]} H(s) < \infty$  for all  $t \in [0, 1)$ . Then the monotonicity properties of  $A_n(\cdot)$  imply the following:

(1)  $\|H\|_{\mathcal{A}} \leq \|\hat{H}\|_{\mathcal{H}}$  as a consequence of Lemma 3.1.

(2) If  $\hat{H}(t) \leq c_1 \frac{(1-\log(1-t))^{-\alpha}}{1-t}$  for all  $t \in [0, 1)$ , then

$$A_n(H) \leq \frac{c}{\sqrt{n^{2\alpha-1}}} \quad \text{for all } n \in \mathbb{N}.$$

### 3. Proof

In this section we prove [Theorems 2.4](#) and [2.6](#). To shorten the presentation, we use the notation  $A \sim_c B$  if there exists a constant  $c \geq 1$  such that  $\frac{1}{c}A \leq B \leq cA$ . Moreover, for the time net  $\tau \in \mathcal{T}_n$  we define

$$\|\tau\|_\infty := \max_{i \in \{1, \dots, n\}} \{t_i - t_{i-1}\}.$$

We prove [Theorem 2.4](#) using the following two lemmas concerning the connection between  $A_n(H)$  and  $\int_0^1 H(t)dt$ , where  $H$  is a non-negative and increasing function.

**Lemma 3.1.** *Let  $H : [0, T) \rightarrow [0, \infty)$ ,  $T > 0$ , be an increasing function such that*

$$I = \int_0^T H(t)dt < \infty.$$

*Then for all  $n \in \mathbb{N}$  there exists a sequence  $\tau^n = (t_i^n)_{i=0}^n$ ,  $0 = t_0^n < t_1^n < \dots < t_n^n = T$  such that*

$$\int_0^{t_i^n} H(t)dt = \frac{i}{n} I$$

*for all  $i \leq n$  and for this sequence it holds that*

$$\left( \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t) H(t)^2 dt \right)^{\frac{1}{2}} \leq \frac{I}{\sqrt{n}}.$$

**Proof.** The existence of the sequence  $(t_i^n)_{i=0}^n$  for which

$$\int_0^{t_i^n} H(t)dt = \frac{i}{n} I$$

follows from the continuity of the integral. Now we have

$$\begin{aligned} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t) H(t)^2 dt &= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} [(t_i^n - t) H(t)] H(t) dt \\ &\leq \frac{I}{n} \sum_{i=1}^n \sup_{t \in [t_{i-1}^n, t_i^n)} (t_i^n - t) H(t). \end{aligned}$$

Since  $H$  is increasing, it is clear that

$$(t_i^n - t) H(t) \leq \int_t^{t_i^n} H(s) ds \leq \int_{t_{i-1}^n}^{t_i^n} H(s) ds$$

for all  $t \in [t_{i-1}^n, t_i^n)$ . Hence

$$\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t) H(t)^2 dt \leq \frac{I}{n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} H(t) dt = \frac{I^2}{n}. \quad \square$$

**Lemma 3.2.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be an increasing function. If for all  $n \in \mathbb{N}$  there exists a time net  $\tau^n = (t_i^n)_{i=0}^n \in \mathcal{T}_n$  such that

$$A(H, \tau^n) \leq \frac{c}{\sqrt{n}}$$

for some fixed  $c > 0$ , then  $H$  is integrable and

$$\int_0^1 H(t) dt \leq \sqrt{2}c.$$

**Proof.** If  $A(H, \tau^n) = 0$ , then  $H \equiv 0$  and the claim is trivial. Assume then that  $A(H, \tau^n) > 0$ , which implies that  $H(t) > 0$  for some  $t \in [0, 1)$ . Let  $a := \inf\{t \in [0, 1) : H(t) > 0\}$  and  $\tilde{\tau}^n = \{a\} \cup \{t_i^n \in \tau^n : t_i^n > a\}$ . Since  $H$  is positive on  $(a, 1)$ , our assumption implies that  $\|\tilde{\tau}^n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Using the Cauchy–Schwarz inequality and the assumption  $A(H, \tau^n)^2 \leq \frac{c^2}{n}$  we see that

$$\begin{aligned} \left[ \sum_{i=1}^{n-1} H(t_{i-1}^n)(t_i^n - t_{i-1}^n) \right]^2 &\leq n \sum_{i=1}^{n-1} H^2(t_{i-1}^n)(t_i^n - t_{i-1}^n)^2 \\ &\leq 2n \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t) H(t)^2 dt \leq 2c^2. \end{aligned} \quad (5)$$

Suppose  $b \in (a, 1)$  and  $0 < \epsilon < \sqrt{c}$ . Choose  $n$  such that  $b < t_{n-1}^n$  and

$$\int_0^b H(t) dt < \sum_{i=1}^{n-1} H(t_{i-1}^n)(t_i^n - t_{i-1}^n) + \epsilon.$$

(We can choose  $n$  satisfying this, since the positivity of the function  $H$  on the interval  $(a, 1)$  implies that  $t_{n-1}^n \rightarrow 1$  and  $\|\tilde{\tau}^n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .) Now (5) gives that

$$\int_0^b H(t) dt \leq \sqrt{2}c + \epsilon$$

and since this is true for any  $b \in (a, 1)$  and any  $\epsilon > 0$ , we finally have

$$\int_0^1 H(t) dt \leq \sqrt{2}c. \quad \square$$

**Proof of Theorem 2.4.** We use Lemmas 3.1 and 3.2.  $\square$

**Lemma 3.3.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be a non-decreasing function. Then

$$A_n(H) \leq \inf_{T \in (0,1)} \left[ \frac{\left( \int_0^T H(t) dt \right)^2}{n-1} + \int_T^1 (1-t) H(t)^2 dt \right]^{\frac{1}{2}}$$

for all  $n \geq 2$ .

**Proof.** This is an immediate consequence of Lemma 3.1.  $\square$



**Remark 3.4.** The best rate that Lemma 3.3 can give is obtained by choosing  $T$  such that

$$\int_0^T H(t)dt = \sqrt{n-1} \left( \int_T^1 (1-t)H(t)^2 dt \right)^{1/2}.$$

However, it is not known whether Lemma 3.3 gives the optimal rate, i.e. we do not know whether the inequality

$$A_n(H)^2 \geq \frac{1}{c} \inf_{T \in (0,1)} \left[ \frac{\left( \int_0^T H(t)dt \right)^2}{n-1} + \int_T^1 (1-t)H(t)^2 dt \right] \quad (6)$$

holds. What we have is

$$A_n(H)^2 = \inf_{T \in (0,1)} \left[ A_{n-1}(H|[0, T])^2 + \int_T^1 (1-t)H(t)^2 dt \right],$$

where

$$A_{n-1}(H|[0, T])^2 := \inf_{0=t_0 < \dots < t_{n-1}=T} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - t)H(t)^2 dt.$$

In order to obtain inequality (6) we would need to know that there exists a constant  $c > 0$  such that

$$A_{n-1}(H|[0, T])^2 \geq \frac{1}{c} \frac{\left( \int_0^T H(t)dt \right)^2}{n-1},$$

for all  $n \geq 2$ , but we do not know whether this is true for all the functions that satisfy  $A_n(H) < \infty$ .

For the proof of Theorem 2.6, we need the following lemmas.

**Lemma 3.5.** Suppose  $\beta \in (0, 1)$ . Then there exists a constant  $c > 0$  such that

$$\frac{(1 - \log(1-t))^{-(1+\beta)}}{(1-t)^2} \sim_c \int_1^\infty z^{-\beta-2} (1-t)^{\frac{1}{z}-2} dz \quad \text{for all } t \in [0, 1).$$

**Proof.** Let  $\psi_\beta(t) = \frac{(1-\log(1-t))^{-(1+\beta)}}{(1-t)^2}$  and  $\varphi_\beta(t) = \int_1^\infty z^{-\beta-2} (1-t)^{\frac{1}{z}-2} dz$ . Choosing  $x = -\frac{\log(1-t)}{z}$ , we obtain

$$\begin{aligned} \varphi_\beta(t) &= \int_{-\log(1-t)}^0 \left( \frac{-\log(1-t)}{x} \right)^{-\beta-2} (1-t)^{-\frac{x}{\log(1-t)}-2} \frac{\log(1-t)dx}{x^2} \\ &= \frac{(-\log(1-t))^{-\beta-1}}{(1-t)^2} \int_0^{-\log(1-t)} x^\beta e^{-x} dx, \end{aligned}$$

since  $(1-t)^{\frac{1}{\log(1-t)}} = [e^{\log(1-t)}]^{\frac{1}{\log(1-t)}} = e$ .

The statement follows from

$$\lim_{t \rightarrow 1} \frac{\varphi_\beta(t)}{\psi_\beta(t)} = \int_0^\infty x^\beta e^{-x} dx \in (0, \infty). \quad \square$$

**Lemma 3.6** ([9, Lemma 7]). Suppose  $\theta \in [1, 2)$  and let  $H_\theta : [0, 1) \rightarrow [0, \infty)$  be given by

$$H_\theta(t) = \sqrt{(2 - \theta)(1 - t)^{-\theta}} \quad \text{for } t \in [0, 1).$$

Then

$$A_n(H_\theta)^2 \geq (\theta - 1)^{n-1}$$

for all  $n \in \{1, 2, \dots\}$ .

**Lemma 3.7.** Let  $H : [0, 1) \rightarrow [0, \infty)$  be an increasing function and  $\beta \in (0, 1)$ . If

$$H(t)^2 \geq \int_1^\infty z^{-\beta-2}(1-t)^{\frac{1}{z}-2} dz \quad \text{for all } t \in [0, 1),$$

then

$$A_n(H) \geq \frac{1}{c_\beta \sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N}$$

where  $c_\beta = \sqrt{\beta(4^{\beta+2} + 2^{\beta+2} + 1)e}$ .

**Proof.** Let  $g : [1, \infty) \times [0, 1) \rightarrow (0, \infty)$  be given by

$$g(z, t) = z^{-\beta-2}(1-t)^{\frac{1}{z}-2}.$$

Then

$$\frac{g(k, t)}{g(k+1, t)} = \left(1 + \frac{1}{k}\right)^{\beta+2} (1-t)^{\frac{1}{k(k+1)}} \leq 2^{\beta+2}$$

for all  $k \geq 1$  and  $t \in [0, 1)$ . We have

$$\frac{d}{dz} g(z, t) = (-\log(1-t) - (2+\beta)z) \frac{(1-t)^{\frac{1}{z}-2}}{z^{\beta+4}}$$

and it is easy to see that for any fixed  $t \in [0, 1)$  there exists  $k_t \geq 2$  such that  $g(z, t)$  is increasing for all  $z \leq k_t - 1$  and decreasing for all  $z \geq k_t$ . Hence

$$\int_1^\infty g(z, t) dz \geq \sum_{k=1}^{k_t-2} g(k, t) + \sum_{k=k_t+1}^\infty g(k, t),$$

where we treat an empty sum as zero. Since  $g(k, t) \leq 2^{\beta+2} g(k+1, t)$  for all  $k \geq 1$ , we have

$$g(k_t - 1, t) + g(k_t, t) + g(k_t + 1, t) \leq c_\beta g(k_t + 1, t),$$

with  $c_\beta := (4^{\beta+2} + 2^{\beta+2} + 1)$ , and therefore

$$\begin{aligned} \sum_{k=(k_t-1)}^\infty g(k, t) &= g(k_t - 1, t) + g(k_t, t) + g(k_t + 1, t) + \sum_{k=k_t+1}^\infty g(k+1, t) \\ &\leq c_\beta \sum_{k=k_t}^\infty g(k+1, t). \end{aligned}$$

This implies

$$\begin{aligned} \int_1^\infty g(z, t) dz &\geq \sum_{k=1}^{k_t-2} g(k, t) + \sum_{k=k_t}^\infty g(k+1, t) \\ &\geq \sum_{k=1}^{k_t-2} g(k, t) + \frac{1}{c_\beta} \sum_{k=(k_t-1)}^\infty g(k, t) \\ &\geq \frac{1}{c_\beta} \sum_{k=1}^\infty g(k, t) \end{aligned}$$

for all  $t \in [0, 1)$ .

Let  $a_k = 2 - \frac{1}{k}$  and  $p_k = k^{-(1+\beta)}$ . By assumption,

$$\begin{aligned} H(t)^2 &\geq \int_1^\infty g(z, t) dz \\ &\geq \frac{1}{c_\beta} \sum_{k=1}^\infty g(k, t) \\ &= \frac{1}{c_\beta} \sum_{k=1}^\infty \frac{1}{k^{\beta+1}} \frac{1}{k} (1-t)^{\frac{1}{k}-2} \\ &= \frac{1}{c_\beta} \sum_{k=1}^\infty p_k (2 - a_k) (1-t)^{-a_k}. \end{aligned}$$

Now

$$\begin{aligned} A_n(H)^2 &= \inf_{\tau \in \mathcal{T}_n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \\ &\geq \inf_{\tau \in \mathcal{T}_n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) \frac{1}{c_\beta} \sum_{k=1}^\infty p_k (2 - a_k) (1-t)^{-a_k} dt \\ &= \frac{1}{c_\beta} \inf_{\tau \in \mathcal{T}_n} \sum_{k=1}^\infty p_k \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) (2 - a_k) (1-t)^{-a_k} dt \\ &\geq \frac{1}{c_\beta} \sum_{k=1}^\infty p_k \inf_{\tau \in \mathcal{T}_n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) (2 - a_k) (1-t)^{-a_k} dt. \end{aligned}$$

To prove our claim it is enough to consider  $n \geq 2$ . We set

$$H_{a_k}(t) = \sqrt{(2 - a_k)(1 - t)^{-a_k}},$$

and now Lemma 3.6 implies that

$$\begin{aligned} A_n(H)^2 &\geq \frac{1}{c_\beta} \sum_{k=1}^\infty p_k (a_k - 1)^{n-1} \\ &= \frac{1}{c_\beta} \sum_{k=1}^\infty k^{-(1+\beta)} \left(1 - \frac{1}{k}\right)^{n-1} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{c_\beta e} \sum_{k=n}^{\infty} k^{-(1+\beta)} \\
&\geq \frac{1}{c_\beta e \beta} n^{-\beta} \\
&= \frac{1}{\tilde{c}_\beta n^\beta},
\end{aligned}$$

where  $\tilde{c}_\beta = e\beta c_\beta$ .  $\square$

**Lemma 3.8.** Suppose  $\beta \in (0, 1)$  and let  $H : [0, 1) \rightarrow [0, \infty)$  be an increasing function such that there exists a constant  $c_1 \geq 1$  for which

$$A_n(H+1) \geq \frac{1}{c_1 \sqrt{n}^\beta} \quad \text{for all } n \in \mathbb{N}.$$

Then there exists a constant  $c_2 \geq 1$  such that

$$A_n(H) \geq \frac{1}{c_2 \sqrt{n}^\beta} \quad \text{for all } n \in \mathbb{N}.$$

**Proof.** Assume first that  $n \geq \tilde{n} := (2^{\beta+1} c_1^2)^{\frac{1}{1-\beta}}$ . Then we have  $\frac{1}{2c_1^2(2n)^\beta} \geq \frac{1}{n}$  and since

$$A_{2n-1}(H+1)^2 \leq 2[A_n(H)^2 + A_n(1)^2] \leq 2 \left[ A_n(H)^2 + \frac{1}{2n} \right] \quad \text{for all } n \in \mathbb{N},$$

we get

$$A_n(H)^2 \geq \frac{1}{2c_1^2(2n-1)^\beta} - \frac{1}{2n} \geq \frac{1}{4c_1^2(2n)^\beta} = \frac{1}{\tilde{c}_2^2 n^\beta}$$

for all  $n \geq \tilde{n}$ , where  $\tilde{c}_2 = 2^{\frac{1+\beta}{2}} \sqrt{2} c_1$ .

If  $n < \tilde{n}$ , the computations above imply

$$A_n(H)^2 \geq A_{\lceil \tilde{n} \rceil}(H)^2 \geq \frac{1}{\tilde{c}_2^2 \lceil \tilde{n} \rceil^\beta} \geq \frac{1}{c_2^2 n^\beta},$$

where  $c_2 = \tilde{c}_2 \lceil \tilde{n} \rceil^{\frac{\beta}{2}}$  and  $\lceil \tilde{n} \rceil := \inf\{k \in \mathbb{Z} : \tilde{n} \leq k\}$ .  $\square$

**Proof of Theorem 2.6.** (1) Let  $T = 1 - e^{c_\alpha(n)}$ , where  $c_\alpha(n) = 1 - ((1-\alpha)n^{1-\alpha} + 1)^{\frac{1}{1-\alpha}}$ . Then

$$\begin{aligned}
\int_0^T H(t) dt &\leq c_1 \int_0^T \frac{(1 - \log(1-t))^{-\alpha}}{1-t} dt \\
&= \frac{c_1}{1-\alpha} [(1 - \log(1-T))^{1-\alpha} - 1] \\
&= c_1 n^{1-\alpha}
\end{aligned}$$

and

$$\int_T^1 (1-t)H(t)^2 dt \leq c_1^2 \int_T^1 \frac{(1 - \log(1-t))^{-2\alpha}}{1-t} dt$$

$$\begin{aligned}
&= \frac{c_1^2}{2\alpha - 1} (1 - \log(1 - T))^{1-2\alpha} \\
&= \frac{c_1^2}{2\alpha - 1} ((1 - \alpha)n^{1-\alpha} + 1)^{\frac{1-2\alpha}{1-\alpha}} \\
&\leq \frac{c_1^2(1 - \alpha)^{\frac{1-2\alpha}{1-\alpha}}}{2\alpha - 1} n^{1-2\alpha}
\end{aligned}$$

and hence Lemma 3.3 says that, for  $n \geq 2$ ,

$$\begin{aligned}
A_n(H) &\leq \left[ \frac{1}{n-1} \left( \int_0^T H(t) dt \right)^2 + \int_T^1 (1-t) H(t)^2 dt \right]^{1/2} \\
&\leq \left[ \frac{c_1^2}{n-1} n^{2-2\alpha} + c_1^2 \tilde{c}_\alpha n^{1-2\alpha} \right]^{1/2} \\
&\leq c_1 \frac{(2 + \tilde{c}_\alpha)^{\frac{1}{2}}}{\sqrt{n^{2\alpha-1}}},
\end{aligned}$$

where  $\tilde{c}_\alpha = \frac{(1-\alpha)^{\frac{1-2\alpha}{1-\alpha}}}{2\alpha-1}$ .

(2) Assume there exists a constant  $c_2 \geq 1$  such that

$$H(t) \geq \frac{(1 - \log(1 - t))^{-\alpha}}{c_2(1 - t)} \quad \text{for all } t \in [s, 1).$$

Then there exists a constant  $c_3 \geq 1$  such that

$$H(t) + 1 \geq \frac{(1 - \log(1 - t))^{-\alpha}}{c_3(1 - t)} \quad \text{for all } t \in [0, 1).$$

If we write  $\beta = 2\alpha - 1 \in (0, 1)$ , Lemma 3.5 implies that there exists a constant  $c_4 \geq 1$  such that

$$(H(t) + 1)^2 \geq \frac{1}{c_4} \int_1^\infty z^{-\beta-2} (1-t)^{\frac{1}{2}-2} dz \quad \text{for all } t \in [0, 1),$$

and Lemma 3.7 implies that there exists  $c_5 \geq 1$  such that

$$A_n(H + 1) \geq \frac{1}{c_5 \sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N}.$$

Finally, Lemma 3.8 implies the existence of a constant  $c \geq 1$  such that

$$A_n(H) \geq \frac{1}{c \sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N}. \quad \square$$

#### 4. Application: The optimal approximation rate of certain stochastic integrals

We recall the notation  $A \sim_c B$  if  $\frac{1}{c}A \leq B \leq cA$ , where  $A, B > 0$  and  $c \geq 1$ . Throughout the section, we assume a standard Brownian motion  $W = (W_t)_{t \in [0,1]}$  on a stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})$ , where  $(\mathcal{F}_t)_{t \in [0,1]}$  is the augmentation of the natural filtration of  $W$  and  $\mathcal{F} = \mathcal{F}_1$ . We let the process  $S = (S_t)_{t \in [0,1]}$  be the geometric Brownian motion, i.e.  $S_t = e^{W_t - \frac{t}{2}}$  for all  $t \in [0, 1]$ . The space of continuous functions continuously differentiable infinitely many

times, with bounded derivatives is denoted by  $\mathcal{C}_b^\infty(\mathbb{R})$ . Moreover, we let  $X = (X_t)_{t \in [0,1]}$  be a diffusion such that

$$dX_t = \sigma(X_t)dW_t \quad \text{with } X_0 \equiv x_0 \in \mathbb{R}, \quad (7)$$

where the process  $X$  is obtained through  $Y = (Y_t)_{t \in [0,1]}$  given as unique continuous solution of

$$dY_t = \hat{\sigma}(Y_t)dW_t + \hat{b}(Y_t)dt \quad \text{with } Y_0 \equiv y_0 \in \mathbb{R},$$

with  $0 < \epsilon_0 \leq \hat{\sigma} \in \mathcal{C}_b^\infty(\mathbb{R})$  and  $\hat{b} \in \mathcal{C}_b^\infty(\mathbb{R})$ , in the following two ways:

(a)  $y_0 = x_0 \in \mathbb{R}$ ,  $\hat{\sigma} := \sigma$ ,  $\hat{b} := 0$ ,  $X_t := Y_t$ ,

(b)  $y_0 = \log x_0$  with  $x_0 > 0$ ,

$$\hat{\sigma}(y) := \frac{\sigma(e^y)}{e^y}, \quad \hat{b}(y) := -\frac{1}{2}\hat{\sigma}(y)^2, \quad \text{and } X_t = e^{Y_t}.$$

We start with  $X$  by choosing  $\sigma$  and  $b$  such that  $\hat{\sigma}$  and  $\hat{b}$  satisfy the conditions above. Hence we can control the process  $X$  via the transition density of  $Y$ .

**Example 4.1.** If  $\sigma(x) = 1$ ,  $x \in \mathbb{R}$ , then the process  $X$  is the standard Brownian motion starting at  $x_0$  and we can use the case (a). If  $\sigma(x) = x$ ,  $x \in (0, \infty)$ , then  $X$  is the geometric Brownian motion and  $X$  can be recovered from  $Y$  using the case (b), where  $Y$  is a Brownian motion with drift.

**Definition 4.2.** Let  $\mathcal{C}_e$  be the linear space of Borel measurable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that there exists  $m > 0$  for which

$$\operatorname{ess\,sup}_{x \in \mathbb{R}} e^{-m|x|} |f(x)| < \infty.$$

Moreover, we define

$$\mathcal{C} := \{Z := f(Y_1) : \Omega \rightarrow \mathbb{R} \mid f \in \mathcal{C}_e \text{ and } Y \text{ as above}\}.$$

**Remark 4.3.** Using [Theorem 5.1](#) one can easily show that  $\mathcal{C} \subset L_2(\Omega, \mathcal{F}, \mathbb{P})$ .

**Definition 4.4.** Let  $X$  be a stochastic process as in (7) and assume that  $Z \in \mathcal{C}$  (or  $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  if  $X \in \{W, S\}$ ). Then

$$a_n^X(Z) := \inf_{\tau^n} a_X(Z, \tau^n), \quad (8)$$

where  $a_X(Z, \tau^n) := \inf \left( \mathbb{E} |Z - \mathbb{E} Z - \sum_{i=1}^n v_{i-1} (X_{t_i} - X_{t_{i-1}})|^2 \right)^{\frac{1}{2}}$  with the infimum taken over all sequences  $v = (v_i)_{i=0}^{n-1}$  of  $\mathcal{F}_{t_i}$ -measurable step functions  $v_i : \Omega \rightarrow \mathbb{R}$ .

The main tool for investigating the approximation problem in papers of C. Geiss, S. Geiss, and Hujo was the  $H$ -functional defined in the following way.

**Definition 4.5.** Let  $X$  be a stochastic process as in (7) and assume that  $Z \in \mathcal{C}$  (or  $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  if  $X \in \{W, S\}$ ). Then we set

$$H_X Z(t) := \left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} \quad \text{for all } t \in [0, 1), \quad (9)$$

where  $F : [0, 1) \times I \rightarrow \mathbb{R}$  is given by  $F(t, x) = \mathbb{E}(Z|X_t = x)$ , with  $I = \mathbb{R}$  in the case of (a) and  $I = (0, \infty)$  in the case of (b).

**Lemma 4.6** ([3, Lemma 5.3], [7, Lemma 3.9]). *The function  $H_X Z : [0, 1) \rightarrow [0, \infty)$  is continuous and increasing.*

In order to deduce from Theorem 2.4 a characterization of the approximation rate

$$a_n^X(Z) \leq \frac{c}{\sqrt{n}},$$

we need the following theorem.

**Theorem 4.7** ([3, Lemma 3.2] [7, Lemma 3.10]). *Let  $X$  be a stochastic process as in (7),  $Z \in \mathcal{C}$  (or  $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  if  $X \in \{W, S\}$ ) and  $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$ . Then*

$$a_X(Z, \tau) \sim_c \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H_X Z(t)^2 dt \right)^{\frac{1}{2}}$$

where  $c \geq 1$  is an absolute constant depending on  $\sigma$  only. Consequently,

$$a_n^X(Z) \sim_c A_n(H_X Z).$$

**Corollary 4.8.** *Let  $X$  be as in (7) and  $Z \in \mathcal{C}$  (or  $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  if  $X \in \{W, S\}$ ). Then*

$$\sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) \sim_c \int_0^1 \left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt,$$

where  $F : [0, 1) \times I \rightarrow \mathbb{R}$  is given by  $F(t, x) = \mathbb{E}(Z|X_t = x)$ , with  $I = \mathbb{R}$  in the case of (a) and  $I = (0, \infty)$  in the case of (b) and where the constant  $c \geq 1$  depends at most on  $\sigma$ .

**Proof.** Theorem 2.4 together with Lemma 4.6 and Theorem 4.7 gives the result immediately.  $\square$

**Remark 4.9.** Remark 2.5 implies that if  $\left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$  is integrable, then the regular sequences generated by  $\left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$  give the rate  $\frac{1}{\sqrt{n}}$ . Using these sequences, denoted by  $\tau_r^n$ , we have that if  $A := \int_0^1 \left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt < \infty$ , then

$$a_n^X(Z) \leq a_X(Z, \tau_r^n) \leq \frac{c_{4.7} A}{\sqrt{n}},$$

where  $c_{4.7} > 0$  is taken from Theorem 4.7 above.

One can also optimize over random time nets instead of deterministic ones considered here. The result [4, Theorem 1.1.] from C. and S. Geiss implies that  $\frac{1}{\sqrt{n}}$  is the best possible approximation rate also for the random time nets in the case where the underlying diffusion  $X$  is the Brownian motion  $W$  or the geometric Brownian motion  $S$ , and  $Z$  is not equal to  $c_0 + c_1 X_1$  a.s. for some  $c_0, c_1 \in \mathbb{R}$ . This means that if  $X \in \{W, S\}$ , the random time nets do not improve the approximation if the deterministic time nets already give the rate  $\frac{1}{\sqrt{n}}$ . According to this, Corollary 4.8 implies that if

$$\int_0^1 \left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt < \infty,$$

then the optimal approximation rate is  $\frac{1}{\sqrt{n}}$  also for the random time nets and this rate is obtained by using the regular sequences generated by  $\left\| \left( \sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$ .

Now we give for  $\beta \in (0, 1)$  an example such that

$$a_n^X(Z) \sim \frac{1}{\sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N},$$

in the case where  $X$  is a standard Brownian motion or the geometric Brownian motion. According to [Theorem 2.6](#), [Lemma 4.6](#) and [Theorem 4.7](#) it is sufficient to find a random variable  $Z = f_\alpha(W_1)$  such that

$$H_X Z(t) \sim \frac{(1 - \log(1 - t))^{-\alpha}}{1 - t},$$

where  $\alpha = \frac{\beta+1}{2}$ .

**Example 4.10.** Suppose  $\alpha \in (1/2, 1)$  and  $f_\alpha = \sum_{k=0}^{\infty} a_k \eta_k \in L_2(\gamma)$ , where  $\gamma$  is the Gaussian measure on  $\mathbb{R}$ , i.e.

$$d\gamma(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

and  $a = (a_k)_{k=0}^{\infty}$  is given by

$$a_k = \begin{cases} 0 & \text{if } k \in \{0, 1, 3\}, \\ \frac{1}{\sqrt{2}} & \text{if } k = 2, \\ \sqrt{\frac{k-2}{k(k-1)}} \log^{-\alpha}(k-2) & \text{if } k \geq 4, \end{cases}$$

and  $(\eta_k)_{k=0}^{\infty} \subset L_2(\gamma)$  is the complete orthonormal system of Hermite polynomials,

$$\eta_k(x) = \frac{(-1)^k}{\sqrt{k!}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}.$$

Then  $Z_\alpha := f_\alpha(W_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$  and it can be shown that

$$H_W Z_\alpha(t) = \left( 1 + \sum_{k=2}^{\infty} k \log^{-2\alpha}(k) t^k \right)^{1/2} \sim_{c_1} \frac{(1 - \log(1 - t))^{-\alpha}}{1 - t}$$

for all  $t \in [0, 1)$ , where  $c_1 \geq 1$  depends at most on  $\alpha$  (according to [Lemmas 4.11](#) and [4.12](#) below). Using [Lemma 4.12](#) it is easy to show that there exists a constant  $c_2 > 0$  such that

$$H_W Z_\alpha(t) \sim_{c_2} H_S Z_\alpha(t) \quad \text{for all } t \in (0, 1).$$

[Theorem 2.6](#) implies that there exists a constant  $c_3 \geq 1$  such that

$$\frac{1}{c_3 \sqrt{n^{2\alpha-1}}} \leq a_n^X(Z_\alpha) \leq \frac{c_3}{\sqrt{n^{2\alpha-1}}}$$



for all  $n \in \mathbb{N}$ , where  $X \in \{W, S\}$ . In other words, letting  $\beta \in (0, 1)$  and defining  $\alpha := \frac{\beta+1}{2}$  we have

$$a_n^X(Z_\alpha) \sim_{c_3} \frac{1}{\sqrt{n}^\beta} \quad \text{for all } n \in \mathbb{N}.$$

The following lemma should be known. For completeness and convenience of the reader we include a proof.

**Lemma 4.11.** *Suppose  $\beta > 1$ . Then for all  $t \in [0, 1)$ , one has that*

$$\frac{(1 - \log(1 - t))^{-\beta}}{(1 - t)^2} \sim_c 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k, \quad (10)$$

where the constant  $c \geq 1$  depends at most on  $\beta$ .

**Proof.** Let  $n \geq e^\beta$  be an integer,  $\epsilon \in [\frac{1}{n+1}, \frac{1}{n})$ , and  $t = e^{-\epsilon}$ . Since  $k \log^{-\beta}(k)$  is increasing if  $k \geq e^\beta$  and we assumed that  $n \geq e^\beta$ , we have

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k &\geq \sum_{k=n}^{2n} k \log^{-\beta}(k) (e^{-1/n})^k \\ &\geq \sum_{k=n}^{2n} n \log^{-\beta}(n) e^{-2} \\ &\geq e^{-2} n^2 \log^{-\beta}(n). \end{aligned}$$

Moreover,

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k &\leq 1 + \sum_{k=2}^n k \log^{-\beta}(k) + \sum_{m=1}^{\infty} \sum_{k=mn+1}^{(m+1)n} k \log^{-\beta}(k) e^{-\frac{mn}{n+1}} \\ &\leq c_\beta \sum_{k=2}^n n \log^{-\beta}(n) + \sum_{m=1}^{\infty} (m+1) n^2 \log^{-\beta}(n) e^{-\frac{mn}{n+1}} \\ &\leq c_\beta n^2 \log^{-\beta}(n) + n^2 \log^{-\beta}(n) \sum_{m=1}^{\infty} (m+1) e^{-m/2} \\ &\leq (c_\beta + c) n^2 \log^{-\beta}(n), \end{aligned}$$

where  $c_\beta$  depends at most on  $\beta$  and  $c = \sum_{m=1}^{\infty} (m+1) e^{-m/2}$ . This implies, for  $t = e^{-\epsilon}$  with  $\epsilon \in [\frac{1}{n+1}, \frac{1}{n})$ , that

$$1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k \sim_{c_1} n^2 \log^{-\beta}(n) \quad \text{for all } n \geq e^\beta,$$

where  $c_1 \geq 1$  is a constant depending at most on  $\beta$ . Adapting the constant  $c_1 > 0$ , we get this for  $n \geq 2$ .

Now we show that if  $n \geq 4$ , then

$$\frac{(1 - \log(1 - t))^{-\beta}}{(1 - t)^2} \sim_{c_2} n^2 \log^{-\beta}(n),$$

where  $c_2 \geq 2$  is a constant depending at most on  $\beta$ . Firstly, we have that  $\log(\frac{1}{t}) \sim_{c_3} \frac{1}{n}$ , where  $c_3 = \frac{5}{4}$ . Moreover

$$\log(u^{-1}) \sim_{c_4} 1 - u,$$

for all  $u \in [e^{-1/2}, 1]$ , where  $c_4 = [2(1 - e^{-\frac{1}{2}})]^{-1}$ . Hence

$$1 - t \sim_{c_5} \frac{1}{n},$$

where  $c_5 = \frac{5}{8}[1 - e^{-\frac{1}{2}}]^{-1}$ . Furthermore,

$$\frac{\log n}{2} \leq \log(n/c_5) \leq \log((1-t)^{-1}) \leq \log(c_5 n) \leq 2 \log n$$

since  $c_5 < 2$  and  $n \geq 4$ . Now

$$1 - \log(1-t) \sim_3 \log(n)$$

and hence

$$\frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \sim_{c_2} n^2 \log^{-\beta} n,$$

where  $c_2 = 3^\beta c_5^2$ .

If  $t > e^{-\frac{1}{4}}$ , the computations above imply that

$$\frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \sim_{c_2} n^2 \log^{-\beta} n \sim_{c_1} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k,$$

where  $n$  is such that  $e^{-\frac{1}{n}} < t \leq e^{-\frac{1}{n+1}}$ . If  $0 \leq t < e^{-\frac{1}{4}}$ , then one has that

$$1 \leq 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k \leq c_\beta,$$

where the constant  $c_\beta > 0$  depends only on  $\beta$ , and

$$\frac{1}{d_\beta} \leq \frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \leq d_\beta,$$

where the constant  $d_\beta > 0$  depends only on  $\beta$ . Hence

$$\frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \sim_c 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k$$

for all  $t \in [0, 1)$ , where the constant  $c \geq 1$  depends on  $\beta$ .  $\square$

**Lemma 4.12** ([7, Lemma 3.9]). For  $f = \sum_{k=0}^{\infty} a_k h_k \in L_2(\gamma)$ ,  $t \in [0, 1)$  and  $Z = f(W_1)$  one has that

$$H_W Z(t)^2 = \sum_{k=0}^{\infty} a_{k+2}^2 (k+2)(k+1) t^k,$$

$$H_S Z(t)^2 = \sum_{k=0}^{\infty} \left( a_{k+2} - \frac{a_{k+1}}{\sqrt{k+2}} \right)^2 (k+2)(k+1) t^k,$$

where  $W$  is a standard Brownian motion and  $S$  is the geometric Brownian motion. Moreover

$$\frac{1}{12} H_W Z(t)^2 - \frac{2}{3} (a_1^2 + a_2^2) \leq H_S Z(t)^2 \leq 4 H_W Z(t)^2 + 2 a_1^2.$$

## 5. Application: Approximation of certain $d$ -dimensional stochastic integrals with drift

We can apply [Theorems 2.4](#) and [2.6](#) also to the discrete time approximation of  $d$ -dimensional stochastic integrals considered by Zhang [[17](#)], Temam [[15](#)] and Hujo [[10](#)]. Our setting introduced below recalls for the convenience of the reader line by line the setting of [[10](#)], which generalizes the one-dimensional setting of [Section 4](#) to  $d$  dimensions.

We assume a stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})$ , where  $(\mathcal{F}_t)_{t \in [0,1]}$  is the augmentation of the natural filtration generated by the  $d$ -dimensional standard Brownian motion  $W = (W_t)_{t \in [0,1]}$  with  $\mathcal{F} = \mathcal{F}_1$ .

We consider a diffusion  $X = (X^1, \dots, X^d)$ , where

$$X_t^i = x_0^i + \int_0^t b_i(X_u) du + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_u) dW_u^j, \quad t \in [0, 1], \text{ a.s.} \quad (11)$$

for all  $i = 1, \dots, d$  and  $x_0 = (x_0^1, \dots, x_0^d)$ . We assume that  $X$  is obtained through  $Y$  given as the unique pathwise continuous solution of

$$Y_t^i = y_0^i + \int_0^t \hat{b}_i(Y_u) du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Y_u) dW_u^j, \quad t \in [0, 1], \text{ a.s.} \quad (12)$$

for all  $i = 1, \dots, d$ , where  $\hat{b}_i, \hat{\sigma}_{ij} \in C_b^\infty(\mathbb{R}^d)$  and  $(\hat{\sigma} \hat{\sigma}^T)_{ij}(x) = \sum_{k=1}^d \hat{\sigma}_{ik}(x) \hat{\sigma}_{jk}(x)$  is uniformly elliptic, i.e.

$$\sum_{i,j=1}^d (\hat{\sigma} \hat{\sigma}^T)_{ij}(x) \xi_i \xi_j \geq \lambda \|\xi\|^2, \quad \text{for all } x, \xi \in \mathbb{R}^d \text{ and some } \lambda > 0,$$

where  $\|\cdot\|$  is the Euclidean norm. Again, we assume that  $X$  is obtained through  $Y$  in one of the following two ways:

- (a)  $x_0 = y_0 \in \mathbb{R}^d$ ,  $\hat{b}_i(x) := b_i(x)$ ,  $\hat{\sigma}_{ij}(x) := \sigma_{ij}(x)$ , and  $X_t = Y_t$ ,
- (b)  $x_0 = e^{y_0} \in (0, \infty)^d$ ,  $\hat{b}_i(y) := \frac{b_i(e^y)}{e^{y_i}} - \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(y)$ ,  $\hat{\sigma}_{ij}(y) := \frac{\sigma_{ij}(e^y)}{e^{y_i}}$ , and  $X_t = e^{Y_t}$ .

Here and in the following  $e^y = (e^{y_1}, \dots, e^{y_d})$  for  $y = (y_1, \dots, y_d)$ . As in the one-dimensional case, we start with  $X$  and use  $Y$  just as a tool. Moreover, also in this setting (a) is related to the standard Brownian motion and (b) to the geometric Brownian motion.

Moreover, we assume that  $f : E \rightarrow \mathbb{R}$  is a Borel function such that for some  $q \in (0, \infty)$  and  $C > 0$  it holds that

$$|f(x)| \leq C(1 + \|x\|^q), \quad x \in E, \quad (13)$$

where the set  $E$  is defined by

$$E := \begin{cases} \mathbb{R}^d & \text{in case (a)} \\ (0, \infty)^d & \text{in case (b)}. \end{cases}$$

Finally, we define the function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$g(y) := \begin{cases} f(y) & \text{in case (a)} \\ f(e^y) & \text{in case (b).} \end{cases}$$

**Theorem 5.1** ([1, Theorem 8 on p. 263], [2, Theorem 5.4 on p. 149]). For  $\hat{b}$ ,  $\hat{\sigma}$  with  $\hat{\sigma}\hat{\sigma}^T$  uniformly elliptic, there exists a transition density  $\Gamma : (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty) \in \mathcal{C}^\infty$  such that  $\mathbb{P}(Y_t \in B) = \int_B \Gamma(t, y, \xi) d\xi$  for  $t \in (0, 1]$  and  $B \in \mathcal{B}(\mathbb{R}^d)$ , where  $Y = (Y_t)_{t \in [0, 1]}$  is the strong solution of stochastic differential equation (12) starting at  $y$ . Moreover, the following are satisfied:

(i) For  $(s, y, \xi) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$  we have that

$$\begin{aligned} \frac{\partial}{\partial s} \Gamma(s, y, \xi) &= \frac{1}{2} \sum_{k,l=1}^d \sum_{j=1}^d \hat{\sigma}_{kj}(y) \hat{\sigma}_{lj}(y) \frac{\partial^2}{\partial y_k \partial y_l} \Gamma(s, y, \xi) \\ &\quad + \sum_{i=1}^d \hat{b}_i(y) \frac{\partial}{\partial y_i} \Gamma(s, y, \xi). \end{aligned}$$

(ii) For  $a \in \{0, 1, 2, \dots\}$  and multi-indices  $b$  and  $c$  there exist positive constants  $C$  and  $D$ , depending only on  $a, b, c$  and  $d$  such that

$$\left| \frac{\partial^{a+|b|+|c|}}{\partial a \partial t \partial^b y \partial^c \xi} \Gamma(t, y, \xi) \right| \leq \frac{C}{t^{(d+2a+|b|+|c|)/2}} e^{-D \frac{\|y-\xi\|^2}{t}}.$$

If we apply Theorem 5.1 to the stochastic differential equation

$$\begin{cases} Z_t^i = Z_0^i + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u) dW_u^j & \text{in case (a),} \\ Z_t^i = Z_0^i - \int_0^t \left( \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(Z_u) \right) du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u) dW_u^j & \text{in case (b),} \end{cases}$$

we obtain a transition density  $\Gamma_0$  such that we can define the function  $G \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}^d)$  by

$$G(t, y) := \int_{\mathbb{R}^d} \Gamma_0(1-t, y, \xi) g(\xi) d\xi, \quad 0 \leq t < 1$$

and so

$$\begin{cases} \left( \frac{\partial}{\partial t} + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} \right) G(t, y) = 0 & \text{(a),} \\ \left( \frac{\partial}{\partial t} - \sum_{i=1}^d \left( \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(y) \right) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} \right) G(t, y) = 0 & \text{(b).} \end{cases}$$

We define the function  $F : E \rightarrow \mathbb{R}$  by setting

$$F(t, x) := \begin{cases} G(t, x), & \text{in case (a),} \\ G(t, \log(x)), & \text{in case (b),} \end{cases}$$

where  $\log x = (\log(x_1), \dots, \log(x_d))$ , and the operator  $\mathcal{L}$  by

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2} \sum_{k,l=1}^d L_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l},$$

where  $L_{kl}(x) = \sum_{j=1}^d \sigma_{kj}(x) \sigma_{lj}(x)$ . Now we have that

$$\mathcal{L}F(t, x) = 0 \text{ on } [0, 1) \times E,$$

and Itô's formula implies that

$$F(t, X_t) = F(0, X_0) + \sum_{k=1}^d \int_0^t \frac{\partial}{\partial x_k} F(u, X_u) dX_u^k, \text{ a.s. } t \in [0, 1).$$

From Theorem 5.1 we get that

$$F(t, X_t) \rightarrow f(X_1) \text{ in } L_2 \text{ as } t \nearrow 1$$

and

$$f(X_1) = F(0, X_0) + \sum_{k=1}^d \int_0^1 \frac{\partial}{\partial x_k} F(u, X_u) dX_u^k \text{ a.s.}$$

**Definition 5.2.** For  $f$ ,  $F$  and  $X$  as above we define

$$a_X^{\text{sim}}(f(X_1), \tau, s) := \left\| \sum_{i=1}^n \sum_{k=1}^d \int_{t_{i-1}^n \wedge s}^{t_i^n \wedge s} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right) dX_u^k \right\|_{L_2},$$

for all  $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$  and  $s \in [0, 1)$ .

**Definition 5.3.** We define  $H_X f$ ,  $H_X^* f : [0, 1) \rightarrow [0, \infty)$  by setting

$$H_X f(t) := \left( \sup_{k,l} \mathbb{E} \left[ L_{kk}(X_t) L_{ll}(X_t) \left| \frac{\partial^2}{\partial x_k \partial x_l} F(t, X_t) \right|^2 \right] \right)^{\frac{1}{2}} \text{ and}$$

$$H_X^* f(t) := \sup_{s \in [0, t]} H_X f(s).$$

Finally, we define functions  $Q_i : \mathbb{R}^d \rightarrow \mathbb{R}$  for  $i = 1, \dots, d$  by

$$Q_i(x) := \begin{cases} 1, & \text{in case (a)} \\ x_i & \text{in case (b).} \end{cases}$$

In this setting we have the following theorem, which refines [10, Theorem 1].

**Theorem 5.4.** Assume that for all  $x \in E$

$$\left| \frac{\partial^s}{\partial x_l^q \partial x_k^r} \sigma_{ij}(x) \right| \leq C_1 \frac{Q_i(x)}{Q_l^q(x) Q_k^r(x)}, \quad \text{where } q + r = s, \quad q, r, s \in \{0, 1, 2\},$$

$|b_i(x)| \leq C_1 Q_i(x)$  and  $L_{ii}(x) \geq \frac{1}{C_1} Q_i^2(x)$  for  $i \in \{1, \dots, d\}$  and some fixed  $C_1 > 0$ .

(1) If one has that

$$I_H := \int_0^1 H_X^* f(t) dt < \infty,$$

then

$$\inf_{\tau \in \mathcal{T}_n} \sup_{s \in [0,1]} a_X^{\text{sim}}(f(X_1), \tau, s) \leq \frac{D_1 I_H}{\sqrt{n}} \quad \text{for all } n \in \mathbb{N},$$

where  $D_1 = D_1(C_1, d) > 0$ .

(2) If there exists  $C_2 > 0$  and  $\alpha \in (\frac{1}{2}, 1)$  such that

$$H_X^* f(t) \leq C_2 \frac{(1 - \log(1 - t))^{-\alpha}}{1 - t} \quad \text{for all } t \in [0, 1),$$

then

$$\inf_{\tau \in \mathcal{T}_n} \sup_{s \in [0,1]} a_X^{\text{sim}}(f(X_1), \tau, s) \leq \frac{D_2}{\sqrt{n^{2\alpha-1}}} \quad \text{for all } n \in \mathbb{N},$$

where  $D_2 = D_2(C_1, C_2, \alpha, d) > 0$ .

**Proof of Theorem 5.4.** Hujo showed in the proof of [10, Theorem 1, p. 18] that under the assumptions of Theorem 5.4 we have that

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^n \sum_{k=1}^d \int_{t_{i-1}^n \wedge s}^{t_i^n \wedge s} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right) dX_u^k \right|^2 \\ & \leq c \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^t \sup_{k,l} \mathbb{E} \left[ L_{kk}(X_u) L_{ll}(X_u) \left| \frac{\partial^2}{\partial x_k \partial x_l} F(u, X_u) \right|^2 \right] du dt \\ & \leq c \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i - t) [H_X^* f(t)]^2 dt \end{aligned}$$

for any  $s \in [0, 1)$  and any time net  $\tau = (t_i^n)_{i=0}^n$ , where  $c = c(C_1, d)$ . Hence we can conclude by Theorems 2.4 and 2.6.  $\square$

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