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Choquet order for spectra of higher Lamé operators and orthogonal polynomials

Julius Borcea

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

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Abstract

We establish a hierarchy of weighted majorization relations for the singularities of generalized Lamé equations and the zeros of their Van Vleck and Heine–Stieltjes polynomials as well as for multiparameter spectral polynomials of higher Lamé operators. These relations translate into natural dilation and subordination properties in the Choquet order for certain probability measures associated with the aforementioned polynomials. As a consequence we obtain new inequalities for the moments and logarithmic potentials of the corresponding root-counting measures and their weak-* limits in the semi-classical and various thermodynamic asymptotic regimes. We also prove analogous results for systems of orthogonal polynomials such as Jacobi polynomials.

MSC: primary 34L20; secondary 30C15; 33C45; 60E15

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1. Introduction

The *generalized Lamé equation* in algebraic form is the second order differential equation

$$Q_2(z)y''(z) + Q_1(z)y'(z) + Q_0(z)y(z) = 0, \tag{1.1}$$

where $Q_2, Q_1, Q_0 \in \mathbb{C}[z]$ with $\deg Q_2 = p$, $\deg Q_1 = p - 1$, $\deg Q_0 \leqslant p - 2$. Particularly important cases are $p = 2$ and 3, which correspond to the hypergeometric differential equation

E-mail address: julius@math.su.se.

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and Heun's equation, respectively (cf. [23]). The classical Heine–Stieltjes multiparameter spectral problem deals with the so-called *Lamé solutions of the first kind* (of given degree and type) to Eq. (1.1) and may be formulated as follows: given $Q_2(z)$, $Q_1(z)$ as above and $n \in \mathbb{N}$ find a polynomial $V(z)$ of degree at most $p-2$ and a polynomial $S(z)$ of degree n such that (1.1) holds for $Q_0(z) = V(z)$ and $y(z) = S(z)$. If such $V(z)$ and $S(z)$ exist we say that (1.1) is *n-solvable*. A generalized Lamé equation is *solvable* if it is n-solvable for all $n \in \mathbb{N}$. The coefficients $V(z)$ are called *Van Vleck polynomials* and the corresponding solutions $S(z)$ are known as *Heine–Stieltjes polynomials*. These two classes are also referred to as *Lamé polynomials* or *generalized spectral polynomials* for (1.1).

There are several known sufficient conditions for the solvability of Eq. (1.1). For instance, Heine [18] proved that for any $n \in \mathbb{N}$ there exist at most

$$\sigma(n) := \binom{n+p-2}{n}$$

different Van Vleck polynomials $V(z)$ for which (1.1) has a polynomial solution $y(z) = S(z)$ of degree n. Heine's text is written in a traditional XIXth century style German and the exact statements it contains seem to have created some confusion (cf. [22]). Szegö [26, Section 6.8] quotes this result and adds that "Heine asserts that, in general, there are exactly $\sigma(n)$ determinations of this kind". As explained in [8,10], Heine actually showed that if the coefficients of $Q_2(z)$ and $Q_1(z)$ are *algebraically independent*—that is, these coefficients satisfy no algebraic equation with integer coefficients—then (1.1) is solvable. Moreover, if this is the case then for any $n \in \mathbb{N}$ there exist exactly $\sigma(n)$ different Van Vleck polynomials $V(z)$ of degree $p-2$ and the same number of corresponding monic Heine–Stieltjes polynomials $S(z)$ of degree n. An explicit characterization of the exceptional cases when this number is strictly less than $\sigma(n)$ seems to be lacking for the moment [22]. In the general case, Heine's arguments imply that (1.1) is n-solvable for all sufficiently large n [8].

The solvability of (1.1) has been established under various other assumptions, most notably when $Q_2(z)$ and $Q_1(z)$ have strictly interlacing real zeros and the leading coefficient of $Q_1(z)$ is positive. This case is particularly interesting from a physical point of view and has attracted a lot of attention in recent years. Indeed, differential equations of the form (1.1) whose coefficients satisfy the above condition arise naturally when separating variables in the Laplace equation in spherical coordinates and yield important examples of quantum completely integrable systems such as generalized (real or complex) Gaudin spin chains [11–13]. A fundamental result of Stieltjes [25]—also known as the Heine–Stieltjes theorem [26]—asserts that if $Q_2(z)$ and $Q_1(z)$ have strictly interlacing real zeros and $Q_1(z)$ has positive leading coefficient then for each $n \in \mathbb{N}$ there are exactly $\sigma(n)$ different Van Vleck polynomials $V(z)$ of degree $p-2$ and the same number of corresponding monic Heine–Stieltjes polynomials $S(z)$ of degree n. The latter are given by all possible ways of distributing the zeros of $S(z)$ in the $p-1$ open intervals defined by the zeros of $Q_2(z)$. Stieltjes actually showed that the zeros of $S(z)$ are the coordinates of the equilibrium points of a certain electrostatic potential. Similar results have recently been obtained in cases when $Q_2(z)$ has all real zeros and the residues in the partial fractional decomposition of $Q_1(z)Q_2(z)^{-1}$ have mixed signs [15,17].

Let us assume that $Q_2(z)$ and $Q_1(z)$ are such that

$$Q_2(z) = \prod_{l=1}^{p}(z-\zeta_l) \quad \text{and} \quad \frac{Q_1(z)}{Q_2(z)} = \sum_{l=1}^{p} \frac{a_l}{z-\zeta_l},$$
$$\text{where } \zeta_l \in \mathbb{C} \text{ and } a_l > 0,\ 1 \leqslant l \leqslant p. \tag{1.2}$$

Note that if $Q_2(z)$ and $Q_1(z)$ are as above and Eq. (1.1) is solvable then any Van Vleck polynomial is of degree exactly $p-2$. Pólya [24]—and Klein and Bôcher before him (cf. [15])—showed that in this case the zeros of all Van Vleck and Heine–Stieltjes polynomials lie in the convex hull of $\zeta_1, \dots, \zeta_p$. Extensions of this Gauss–Lucas type theorem to cases when the residues a_i are not necessarily positive real numbers as well as various other results on the location of zeros of Lamé polynomials have since been obtained [1,20,27]. In this paper we show that much more is actually true. Namely, if (1.2) holds then the zeros of any Van Vleck polynomial together with those of a corresponding Heine–Stieltjes polynomial and the zeros of $Q_2(z)$ satisfy a *weighted majorization relation* in the sense of [4] (see Section 2). This amounts to a dilation property—equivalently, a subordination relation in the Choquet order—for certain probability measures associated with the generalized spectral polynomials and the singularities of Eq. (1.1). A precise statement of this result is given in Theorem 3. As a consequence we obtain new inequalities for the moments and logarithmic potentials associated with the root-counting measures of Lamé polynomials and we establish similar properties in the thermodynamic ($p \to \infty$) and semi-classical ($n \to \infty$) asymptotic regimes (Corollaries 1–3). These results hold in the greatest possible generality and require no additional assumptions besides (1.2). Therefore, Theorem 3 and Corollary 1 apply whenever Eq. (1.1) is n-solvable while Corollaries 2–3 make sense in all cases when (1.1) is solvable and the considered limits exist (see Section 3 for several concrete examples). In the special case when $\zeta_i \in \mathbb{R}$, $1 \leqslant i \leqslant p$, our results are a natural complement to those of [10–13,23] dealing with asymptotic distributions, limiting level-spacings and mean densities of zeros of Lamé polynomials.

Various extensions of the Heine–Stieltjes multiparameter spectral problem to higher order linear ordinary differential operators with polynomial coefficients have been studied in [8,10]. In particular, if $Q_2(z)$ and $Q_1(z)$ are as in (1.2) and $k \geqslant 2$ then one may consider an operator of the form

$$\mathfrak{d}(z) = Q_2(z)\frac{d^k}{dz^k} + Q_1(z)\frac{d^{k-1}}{dz^{k-1}}. \tag{1.3}$$

As in [8], we call $\mathfrak{d}(z)$ a *higher order generalized Lamé operator* or a *higher Lamé operator* for short, provided that its *Fuchs index* $r := p-k$ is non-negative. If $r = 0$ then $\mathfrak{d}(z)$ is a so-called *hypergeometric type operator*. Such operators and their polynomial eigenfunctions have important applications to the study of the Bochner–Krall problem and exactly solvable models (see, e.g., [2,8,10] and references therein). The multiparameter spectral problem for a higher Lamé operator $\mathfrak{d}(z)$ is as follows: given $n \in \mathbb{N}$ find a polynomial $V(z)$ of degree at most r such that the equation

$$\mathfrak{d}(z)y(z) + V(z)y(z) = 0 \tag{1.4}$$

has a polynomial solution $y(z) = S(z)$ of degree n. One can then define the notions of *n-solvability*, *solvability*, *higher Van Vleck* and *Heine–Stieltjes polynomials*—that is, *higher spectral polynomials* or *Lamé polynomials*—corresponding to (1.4) by analogy with the terminology used for (1.1). Several sufficient conditions for the solvability of (1.4) that extend those of Heine for (1.1) were recently obtained in [8] (see Section 3). We show that whenever Eq. (1.4) is solvable its singularities and the zeros of all corresponding higher Lamé polynomials satisfy weighted majorization relations (Theorem 4) and we establish natural analogs of Corollaries 1–3 for the higher order case (Corollaries 4–6).

Our methods also yield interesting applications of the Choquet order/weighted majorization to the theory of orthogonal polynomials. In particular, we prove appropriate versions of the aforementioned results for classical orthogonal polynomials such as (ultraspherical) Gegenbauer polynomials, (associated) Legendre polynomials, Chebyshev polynomials and indeed any family of Jacobi polynomials (see Section 3.3).

This paper is organized as follows. In Section 2 we recall the notion of weighted multivariate majorization from [4] as well as the definition and properties of the Choquet order for non-negative Radon measures. We state and prove our main results in Section 3. In Section 4 we give several generalizations and discuss some related problems.

2. Weighted majorization and the Choquet order

The majorization preorder on n-tuples of real numbers—also known as the strong spectral order, vector majorization or classical majorization—essentially quantifies the intuitive notion that the components of a real n-vector are less spread out than the components of another such vector. Several matrix versions of this notion have been proposed and studied in various contexts [21]. A weighted multivariate extension of both classical and matrix majorization was introduced in [4]. In the special case of complex n-vectors the definition of [4] is as follows. For $m \in \mathbb{N}$ set

$$\mathcal{A}_m = \left\{ \mathbf{a} = (a_1, \ldots, a_m) \,\middle|\, a_i \in [0,1], 1 \leqslant i \leqslant m, \sum_{i=1}^{m} a_i = 1 \right\},$$

$$\mathbb{X}_m = \mathbb{C}^m \times \mathcal{A}_m, \quad \mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_m. \tag{2.1}$$

Denote by $\operatorname{conv}(\Omega)$ the (closed) convex hull of a (bounded) set $\Omega \subset \mathbb{C}$ and by X^T the transpose of a (row) vector $X = (x_1, \ldots, x_m) \in \mathbb{C}^m$. We frequently write $\operatorname{conv}(X)$ for $\operatorname{conv}(\{x_1, \ldots, x_m\})$. Let $\mathbb{M}^{\mathrm{rs}}_{m,n}$ be the set of all row stochastic $m \times n$ matrices.

Defintion 1. The pair $(X, \mathbf{a}) \in \mathbb{X}_m$ is said to be *weightly majorized* by the pair $(Y, \mathbf{b}) \in \mathbb{X}_n$, denoted $(X, \mathbf{a}) \prec (Y, \mathbf{b})$, if there exists a matrix $R \in \mathbb{M}^{\mathrm{rs}}_{m,n}$ such that

$$\tilde{X}^T = R\tilde{Y}^T \quad \text{and} \quad \mathbf{b} = \mathbf{a}R,$$

where $\tilde{X}^T$ and $\tilde{Y}^T$ are obtained by some (and then any) ordering of the coordinates of X^T and Y^T, respectively.

Remark 1. Note that if $(X, \mathbf{a}) \prec (Y, \mathbf{b})$ then $X \in \operatorname{conv}(Y)^m$ and the $\mathbf{a}$-barycenter of X must coincide with the $\mathbf{b}$-barycenter of Y, that is, $\sum_{i=1}^m a_i x_i = \sum_{j=1}^n b_j y_j$. Moreover, it is clear that the weighted majorization relation is both reflexive and transitive, which makes it a preorder on $\mathbb{X}$. One can also show that for every $m \in \mathbb{N}$ this preorder induces a partial order on the orbit space $\mathbb{C}^m/\Sigma_m$, where Σ_m is the symmetric group on m elements.

The following characterization of the weighted majorization relation may be found in [4, Theorem 1].

Theorem 1. *Let* $(X, \mathbf{a}) \in \mathbb{X}_m$ *and* $(Y, \mathbf{b}) \in \mathbb{X}_n$, *where* $X = (x_1, \ldots, x_m) \in \mathbb{C}^m$, $\mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{A}_m$, $Y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ *and* $\mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{A}_n$. *The following conditions* *are equivalent*:

(i) *for any* (*continuous*) *convex function* $f : \mathbb{C} \to \mathbb{R}$ *one has*

$$\sum_{i=1}^{m} a_i f(x_i) \leqslant \sum_{j=1}^{n} b_j f(y_j);$$

(ii) *the relation* $(X, \mathbf{a}) \prec (Y, \mathbf{b})$ *holds.*

Remark 2. If $(X, \mathbf{a}) \prec (Y, \mathbf{b})$ then the inequality in Theorem 1(i) holds for every convex function f defined on $\mathrm{conv}(Y)$.

There is a natural connection between the weighted multivariate majorization relation and the Choquet order for non-negative Radon measures. The latter has been studied in the general context of locally convex separable topological vector spaces in e.g., [14] and subsequent papers. For measures defined on compact subsets of the complex plane the Choquet order and the main results of [14] may be described as follows. Let K be a convex compact subset of $\mathbb{C}$, denote by $\mathcal{C}(K)$ the space of real continuous functions on K and let $\mathcal{P}(K)$ the subset of $\mathcal{C}(K)$ consisting of convex functions. If μ is a non-negative Radon measure on K and f is a function on K one defines $\mu(f) = \int_K f(y) d\mu(y)$. The mass of μ is therefore $\mu(1) = \int_K d\mu(y)$ and if $\mu(1) > 0$ then the barycenter of μ is the point $r(\mu) := \mu(1)^{-1} \int_K y\, d\mu(y)$.

Defintion 2. Given two non-negative Radon measures μ and ν on K one says that ν *dominates* μ *in the Choquet order* or that ν is a *dilation* of μ, denoted $\mu \prec \nu$, if $\mu(f) \leqslant \nu(f)$ for any $f \in \mathcal{P}(K)$.

The use of the term "dilation" in Definition 2 is motivated by Definition 3 and Theorem 2(iii). To formulate this result we need a few more concepts and notations. Let $\mathcal{M}(K)$ be the set of all probability measures with $\mathrm{supp}\, \mu \subseteq K$. Note that if $\mu \in \mathcal{M}(K)$ then its barycenter $r(\mu)$ lies in K.

Remark 3. As is well known, the set $\mathcal{M}(K)$ equipped with the weak-* topology is a sequentially compact Hausdorff space. This will allow us to choose a convergent subsequence from any sequence of measures belonging to $\mathcal{M}(K)$.

Defintion 3. A *dilation* on K is a weakly Borel measurable $\mathcal{M}(K)$-valued function on K that inverts the barycenter mapping. In other words, a map $T : K \to \mathcal{M}(K)$, $x \mapsto T_x$, is a dilation on K if $r(T_x) = x$ for all $x \in K$ and the real-valued function on K given by $x \mapsto T_x(f)$ is borelian for any $f \in \mathcal{C}(K)$.

If T is a dilation on K then for any non-negative Radon measure μ on K one can define a new such measure $\nu := T(\mu)$ by setting

$$\nu(f) = \int_K T_x(f)\, d\mu(x), \qquad f \in \mathcal{C}(K). \tag{2.2}$$

It is not difficult to show that the real-valued function on K given by $x \mapsto T_x(f)$ is borelian and bounded whenever f is a bounded borelian real-valued function on K and that (2.2) actually holds for all such functions (cf. [14]). The main results of [14] provide various descriptions of the

Choquet order in a quite general setting. In the case discussed above these may be summarized as follows (see also [16]).

Theorem 2. *If K is a convex compact subset of* $\mathbb{C}$ *and* $\mu, \nu \in \mathcal{M}(K)$ *then the following conditions are equivalent*:

(i) $\mu \prec \nu$;
(ii) *for every convex combination* $\mu = \sum_{i=1}^{n} \lambda_i \mu_i$ *with* $\mu_i \in \mathcal{M}(K)$, $1 \leqslant i \leqslant n$, *there exists a corresponding convex combination* $\nu = \sum_{i=1}^{n} \lambda_i \nu_i$ *such that* $\nu_i \in \mathcal{M}(K)$ *and* $r(\nu_i) = r(\mu_i)$, $1 \leqslant i \leqslant n$;
(iii) $\nu = T(\mu)$, *where T is a dilation on K*.

Remark 4. If the conditions in Theorem 2 hold then $\operatorname{supp}(\mu) \subseteq \operatorname{conv}(\operatorname{supp}(\nu))$. This may be viewed as a "continuous" version of the corresponding result for weighted majorization (cf. Remark 1).

3. Main results and proofs

Given a complex polynomial P of degree $d \geqslant 1$ we let $\mathcal{Z}(P)$ be the d-tuple (or multiset) consisting of the zeros of P, where it is understood that each zero occurs as many times as its multiplicity. In particular, $|\mathcal{Z}(P)| = d$. To P we associate its root-counting measure, namely the (finite) real probability measure given by

$$\mu_P = |\mathcal{Z}(P)|^{-1} \sum_{\zeta \in \mathcal{Z}(P)} \delta_\zeta,$$

where δ_ζ is the Dirac measure supported at ζ. The symbol $\vee$ is used below for the concatenation operation, that is, if $(x_1, \dots, x_m) \in \mathbb{C}^m$ and $(y_1, \dots, y_n) \in \mathbb{C}^n$ then

$$(x_1, \dots, x_m) \vee (y_1, \dots, y_n) = (x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{C}^{m+n},$$

and the "all ones" vector is denoted by $\mathbf{1}_m = (1, \dots, 1) \in \mathbb{R}^m$.

3.1. Generalized Lamé operators

Suppose that $n \geqslant 2$ is an integer such that (1.1) is n-solvable and that $Q_2(z), Q_1(z)$ satisfy (1.2). Let $S(z)$ be a Heine–Stieltjes polynomial of degree n corresponding to a Van Vleck polynomial $V(z)$, so that $\deg V = p - 2$ (cf. Section 1). Let $\alpha = \alpha(n, p) := n - 1 + \sum_{l=1}^{p} a_l$ and define the following weight vectors:

$$\mathbf{a} = \frac{\alpha}{(p-1)\alpha + n - 1}\mathbf{1}_{p-2}, \quad \mathbf{b} = \frac{\alpha + n - 1}{n[(p-1)\alpha + n - 1]}\mathbf{1}_n,$$
$$\mathbf{c} = \left(\frac{\alpha - a_1}{(p-1)\alpha + n - 1}, \dots, \frac{\alpha - a_p}{(p-1)\alpha + n - 1}\right). \tag{3.1}$$

Recall (1.2) and note that $\alpha > 1$, $\mathbf{c} \in \mathcal{A}_p$ while $\mathbf{a} \vee \mathbf{b} \in \mathcal{A}_{n+p-2}$. Finally, set

$$\mathcal{Z}(V) = (v_1, \dots, v_{p-2}), \quad \mathcal{Z}(S) = (s_1, \dots, s_n). \tag{3.2}$$

We can now state our first main result.

Theorem 3. *With the above notations and assumptions the inequality*

$$\sum_{i=1}^{p-2} f(v_i) + \left[1 - \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{\alpha}\right)\right] \sum_{j=1}^{n} f(s_j) \leqslant \sum_{l=1}^{p} \left(1 - \frac{a_l}{\alpha}\right) f(\zeta_l) \tag{3.3}$$

holds for any convex function $f : \mathbb{C} \to \mathbb{R}$ *and if equality occurs in* (3.3) *for some strictly convex function f then the zeros of* Q_2 *must be collinear. Equivalently,*

$$\big(\mathcal{Z}(V) \vee \mathcal{Z}(S), \mathbf{a} \vee \mathbf{b}\big) \prec \big(\mathcal{Z}(Q_2), \mathbf{c}\big).$$

Thus there exists a matrix $R = R(n, p) \in \mathbb{M}^{\mathrm{rs}}_{n+p-2,p}$ *such that*

$$\big(\mathcal{Z}(V) \vee \mathcal{Z}(S)\big)^T = R\mathcal{Z}(Q_2)^T \quad \textit{and} \quad \mathbf{c} = (\mathbf{a} \vee \mathbf{b})R.$$

Remark 5. As pointed out in Section 1, the only requirements for Theorem 3 are that (1.2) holds and Eq. (1.1) is n-solvable. For instance, Stieltjes' theorem shows that (1.1) is always solvable if $\mathcal{Z}(Q_2) \subset \mathbb{R}$ while Heine's result [18] asserts that the same is true whenever $Q_2(z)$ and $Q_1(z)$ are algebraically independent.

Note that in particular Theorem 3 immediately implies the Pólya–Klein–Bôcher result mentioned in Section 1, namely $\mathcal{Z}(V) \cup \mathcal{Z}(S) \subseteq \mathrm{conv}(\mathcal{Z}(Q_2))$ (cf. Remark 1). Now given a compact set $K \subset \mathbb{C}$ and $\mu \in \mathcal{M}(K)$ let

$$\mu^{(m)} := \int |w|^m \, d\mu(w), \quad m \in \mathbb{Z}_+,$$

denote the moments of μ. As is well known, the logarithmic potential of μ

$$U^{\mu}(z) = \int \log|z - w| \, d\mu(w)$$

is subharmonic in $\mathbb{C}$ and $U^{\mu}(z) = -\infty$ for every atom z of μ.

Clearly, (3.3) may be reformulated in terms of the Choquet order for atomic probability measures with finite point spectrum:

Corollary 1. *In the situation of the preceding theorem one has*

$$\frac{(p-2)\alpha}{(p-1)\alpha + n - 1}\mu_V + \frac{\alpha + n - 1}{(p-1)\alpha + n - 1}\mu_S \prec \tilde{\mu}_{Q_2},$$

where μ_V *and* μ_S *are the root-counting measures of V and S, respectively, while* $\tilde{\mu}_{Q_2} \in \mathcal{M}(\mathrm{conv}$ $(\mathcal{Z}(Q_2)))$ *is defined by*

$$\mathrm{supp}\big(\tilde{\mu}_{Q_2}\big) = \mathcal{Z}(Q_2) = \{\zeta_l\}_{l=1}^{p} \quad \textit{and} \quad \tilde{\mu}_{Q_2}(\{\zeta_l\}) = \frac{\alpha - a_l}{(p-1)\alpha + n - 1}$$

for $1 \leqslant l \leqslant p$. *In particular,*

$$\frac{(p-2)\alpha}{(p-1)\alpha + n - 1}\mu_V^{(m)} + \frac{\alpha + n - 1}{(p-1)\alpha + n - 1}\mu_S^{(m)} \leqslant \tilde{\mu}_{Q_2}^{(m)}$$

for all $m \in \mathbb{Z}_+$ *and*

$$\frac{(p-2)\alpha}{(p-1)\alpha+n-1}U^{\mu_V}(z)+\frac{\alpha+n-1}{(p-1)\alpha+n-1}U^{\mu_S}(z)\geqslant U^{\tilde{\mu}_{Q_2}}(z)$$

whenever $z \in \mathbb{C}\backslash\mathrm{conv}(\mathcal{Z}(Q_2))$.

In the semi-classical asymptotic regime ($n \to \infty$) Theorem 3 yields:

Corollary 2. *Assume that* (1.2) *holds and that* (1.1) *is solvable. Let* $\{S_n(z)\}_{n\in\mathbb{N}}$ *be a sequence of monic Heine–Stieltjes polynomials such that* $\deg S_n = n$, $n \in \mathbb{N}$, *and* $\{V_n(z)\}_{n\in\mathbb{N}}$ *be a corresponding sequence of Van Vleck polynomials with* $\deg V_n = p-2$ *normalized so that each* V_n *is monic. Then*

$$\frac{p-2}{p}\mu_V^* + \frac{2}{p}\mu_S^* \prec \tilde{\mu}_{Q_2},$$

where $\mu_V^* = \lim^*_{\Lambda\ni n\to\infty}\mu_{V_n}$ *and* $\mu_S^* = \lim^*_{\Lambda\ni n\to\infty}\mu_{S_n}$ *for an appropriately chosen* $\Lambda \subset \mathbb{N}$. *Equivalently, there exists a dilation* T *on* $\mathrm{conv}(\mathcal{Z}(Q_2))$ *such that* $p\tilde{\mu}_{Q_2} = T\big((p-2)\mu_V^* + 2\mu_S^*\big)$. *In particular,*

$$\frac{p-2}{p}\mu_V^{*(m)} + \frac{2}{p}\mu_S^{*(m)} \leqslant \tilde{\mu}_{Q_2}^{(m)}, \quad m \in \mathbb{Z}_+,$$

and

$$\frac{p-2}{p}U^{\mu_V^*}(z) + \frac{2}{p}U^{\mu_S^*}(z) \geqslant U^{\tilde{\mu}_{Q_2}}(z)$$

for any $z \in \mathbb{C}\backslash\mathrm{conv}(\mathcal{Z}(Q_2))$.

Finally, we may also let $p \to \infty$ and consider various so-called thermodynamic asymptotic regimes (cf., e.g., [11–13]). In this case we get the following result.

Corollary 3. *Let* $\{Q_{2,p}(z)\}_{p=2}^{\infty}$ *and* $\{Q_{1,p}(z)\}_{p=2}^{\infty}$ *be two sequences of polynomials such that for all* $p \geqslant 2$ *the pair* $(Q_{2,p}(z), Q_{1,p}(z))$ *satisfies* (1.2) *and the corresponding Eq.* (1.1) *is solvable. Assume further that* $K \subset \mathbb{C}$ *is a compact set with* $\mathcal{Z}(Q_{2,p}) \subset K$, $p \geqslant 2$, *and that* $\{S_{p,n}(z)\}_{n\in\mathbb{N}}$, *respectively* $\{V_{p,n}(z)\}_{n\in\mathbb{N}}$, *is a sequence of monic Heine–Stieltjes polynomials, respectively normalized Van Vleck polynomials, associated with the resulting system of equations such that* $\deg S_{p,n} = n$, $\deg V_{p,n} = p-2$ *and* $V_{p,n}$ *is monic for all* $p \geqslant 2$ *and* $n \in \mathbb{N}$. *Then*

$${}_*\mu_V \prec {}_*\mu_{Q_2},$$

where ${}_*\mu_V = \lim^*_{\Gamma\ni p\to\infty}\mu_{V_{p,n(p)}}$ *and* ${}_*\mu_{Q_2} = \lim^*_{\Gamma\ni p\to\infty}\mu_{Q_{2,p}}$ *for an appropriately chosen* $\Gamma \subset \mathbb{N}$. *Hence there exists a dilation* T *on* K *such that* ${}_*\mu_{Q_2} = T\big({}_*\mu_V\big)$. *In particular,*

$${}_*\mu_V^{(m)} \leqslant {}_*\mu_{Q_2}^{(m)}, \quad m \in \mathbb{Z}_+,$$

while

$$U^{{}_*\mu_V}(z) \geqslant U^{{}_*\mu_{Q_2}}(z)$$

whenever $z \in \mathbb{C}\backslash K$.

3.2. Higher Lamé operators

Let now $k \geqslant 2$ be a fixed integer and consider an order k generalized Lamé operator $\mathfrak{d}(z)$ with Fuchs index $r := p - k$ as in (1.3) and the corresponding multiparameter spectral problem (1.4). Assume that the latter is n-solvable and that $(S(z), V(z))$ is a pair of (higher) spectral polynomials with $\deg S = n$ and $\deg V = r$ satisfying (1.4). Let $\alpha_k = \alpha(n, p, k) := n - k + 1 + \sum_{l=1}^{p} a_l$ and define the following weight vectors:

$$\mathbf{a}_k = \frac{\alpha_k}{(p-1)\alpha_k + n - k + 1}\mathbf{1}_r, \quad \mathbf{b}_k = \frac{(k-1)\alpha_k + n - k + 1}{n[(p-1)\alpha_k + n - k + 1]}\mathbf{1}_n,$$

$$\mathbf{c}_k = \left(\frac{\alpha_k - a_1}{(p-1)\alpha_k + n - k + 1}, \dots, \frac{\alpha_k - a_p}{(p-1)\alpha_k + n - k + 1}\right). \tag{3.4}$$

One clearly has $\alpha_k > 1$, $\mathbf{c}_k \in \mathcal{A}_p$, $\mathbf{a}_k \vee \mathbf{b}_k \in \mathcal{A}_{n+r}$, $\mathbf{a}_2 = \mathbf{a}$, $\mathbf{b}_2 = \mathbf{b}$ and $\mathbf{c}_2 = \mathbf{c}$. Since in this case $\deg V = r$ we adapt notation (3.2) to the current situation simply by setting $\mathcal{Z}(V) = (v_1, \dots, v_r)$.

The analog of Theorem 3 for higher Lamé operators reads as follows.

Theorem 4. *Under the above assumptions the inequality*

$$\sum_{i=1}^{r} f(v_i) + \left[1 - \left(1 - \frac{k-1}{n}\right)\left(1 - \frac{1}{\alpha_k}\right)\right]\sum_{j=1}^{n} f(s_j) \leqslant \sum_{l=1}^{p}\left(1 - \frac{a_l}{\alpha_k}\right) f(\zeta_l) \tag{3.5}$$

holds for any convex function $f : \mathbb{C} \to \mathbb{R}$ and if equality occurs in (3.5) *for some strictly convex function f then the ζ_l's must be collinear. Equivalently,*

$$\big(\mathcal{Z}(V) \vee \mathcal{Z}(S), \mathbf{a}_k \vee \mathbf{b}_k\big) \prec \big(\mathcal{Z}(Q_2), \mathbf{c}_k\big).$$

Thus there exists a matrix $R_k = R(n, p, k) \in \mathbb{M}^{\mathrm{rs}}_{n+r,p}$ such that

$$\big(\mathcal{Z}(V) \vee \mathcal{Z}(S)\big)^T = R_k \mathcal{Z}(Q_2)^T \quad \textit{and} \quad \mathbf{c}_k = (\mathbf{a}_k \vee \mathbf{b}_k) R_k.$$

Remark 6. The proof of Theorem 4 actually yields an inequality stronger than (3.5) involving all four zero sets $\mathcal{Z}(V)$, $\mathcal{Z}(S)$, $\mathcal{Z}(S^{(k-1)})$ and $\mathcal{Z}(Q_2)$ (see (3.10)).

Remark 7. Theorem 4 applies to all situations when $Q_2(z)$, $Q_1(z)$ satisfy (1.2) and Eq. (1.4) is n-solvable. By [8, Theorem 5] this is always true for all sufficiently large n. Moreover, it was shown in [8] that (1.4) is n-solvable for any $n \in \mathbb{N}$ in each of the following cases: (i) $Q_2(z)$ and $Q_1(z)$ are algebraically independent, (ii) $\mathcal{Z}(Q_2) \subset \mathbb{R}$, (iii) $\mathfrak{d}(z)$ is a hyperbolicity preserving operator (HPO for short), i.e., it maps polynomials with all real zeros to polynomials with all real zeros. A complete classification of all HPOs was recently obtained in [6] (see also [7]).

Natural extensions of Corollaries 1–3 to higher Lamé operators are as follows.

Corollary 4. *In the situation of Theorem* 4 *one has*

$$\frac{(p-k)\alpha_k}{(p-1)\alpha_k + n - 1}\mu_V + \frac{(k-1)\alpha_k + n - 1}{(p-1)\alpha_k + n - 1}\mu_S \prec \tilde{\mu}_{Q_2},$$

where μ_V and μ_S are the root-counting measures of V and S, respectively, while $\tilde{\mu}_{Q_2} \in \mathcal{M}(\mathrm{conv}(\mathcal{Z}(Q_2)))$ is defined by

$$\mathrm{supp}\big(\tilde{\mu}_{Q_2}\big) = \mathcal{Z}(Q_2) = \{\zeta_l\}_{l=1}^{p} \quad \textit{and} \quad \tilde{\mu}_{Q_2}(\{\zeta_l\}) = \frac{\alpha_k - a_l}{(p-1)\alpha_k + n - 1}$$

for $1 \leqslant l \leqslant p$. In particular,

$$\frac{(p-k)\alpha_k}{(p-1)\alpha_k + n - 1}\mu_V^{(m)} + \frac{(k-1)\alpha_k + n - 1}{(p-1)\alpha_k + n - 1}\mu_S^{(m)} \leqslant \tilde{\mu}_{Q_2}^{(m)}$$

for all $m \in \mathbb{Z}_+$ and

$$\frac{(p-k)\alpha_k}{(p-1)\alpha_k + n - 1}U^{\mu_V}(z) + \frac{(k-1)\alpha_k + n - 1}{(p-1)\alpha_k + n - 1}U^{\mu_S}(z) \geqslant U^{\tilde{\mu}_{Q_2}}(z)$$

whenever $z \in \mathbb{C}\setminus\mathrm{conv}(\mathcal{Z}(Q_2))$.

Corollary 5. *Assume that* (1.2) *holds and that* (1.4) *is solvable. Let $\{S_n(z)\}_{n\in\mathbb{N}}$ be a sequence of monic higher Heine–Stieltjes polynomials such that* $\deg S_n = n$, $n \in \mathbb{N}$, *and $\{V_n(z)\}_{n\in\mathbb{N}}$ be a corresponding sequence of higher Van Vleck polynomials with* $\deg V_n = p - 2$ *normalized so that each V_n is monic. Then*

$$\frac{p-k}{p}\mu_V^* + \frac{k}{p}\mu_S^* \prec \tilde{\mu}_{Q_2},$$

where $\mu_V^* = \lim^*_{\Lambda \ni n \to \infty} \mu_{V_n}$ and $\mu_S^* = \lim^*_{\Lambda \ni n \to \infty} \mu_{S_n}$ for an appropriately chosen $\Lambda \subset \mathbb{N}$. Equivalently, there exists a dilation T on $\mathrm{conv}(\mathcal{Z}(Q_2))$ *such that $p\tilde{\mu}_{Q_2} = T\big((p-k)\mu_V^* + k\mu_S^*\big)$. In particular,*

$$\frac{p-k}{p}\mu_V^{*(m)} + \frac{k}{p}\mu_S^{*(m)} \leqslant \tilde{\mu}_{Q_2}^{(m)}, \quad m \in \mathbb{Z}_+,$$

and

$$\frac{p-k}{p}U^{\mu_V^*}(z) + \frac{k}{p}U^{\mu_S^*}(z) \geqslant U^{\tilde{\mu}_{Q_2}}(z)$$

for any $z \in \mathbb{C}\setminus\mathrm{conv}(\mathcal{Z}(Q_2))$.

Corollary 6. *Let $\{Q_{2,p}(z)\}_{p=2}^{\infty}$ and $\{Q_{1,p}(z)\}_{p=2}^{\infty}$ be two sequences of polynomials such that for all $p \geqslant 2$ the pair $(Q_{2,p}(z), Q_{1,p}(z))$ satisfies* (1.2) *and the corresponding higher Lamé equation* (1.4) *is solvable. Assume further that $K \subset \mathbb{C}$ is a compact set with $\mathcal{Z}(Q_{2,p}) \subset K$, $p \geqslant 2$, and that $\{S_{p,n}(z)\}_{n\in\mathbb{N}}$, respectively $\{V_{p,n}(z)\}_{n\in\mathbb{N}}$, is a sequence of monic higher Heine–Stieltjes polynomials, respectively normalized higher Van Vleck polynomials, associated with the resulting system of higher Lamé equations such that* $\deg S_{p,n} = n$, $\deg V_{p,n} = p - 2$ *and $V_{p,n}$ is monic for all $p \geqslant 2$ and $n \in \mathbb{N}$. Then*

$${}_*\mu_V \prec {}_*\mu_{Q_2},$$

where ${}_*\mu_V = \lim^*_{\Gamma \ni p \to \infty} \mu_{V_{p,n(p)}}$ and ${}_*\mu_{Q_2} = \lim^*_{\Gamma \ni p \to \infty} \mu_{Q_{2,p}}$ for an appropriately chosen $\Gamma \subset \mathbb{N}$. Hence there exists a dilation T on K such that ${}_*\mu_{Q_2} = T\big({}_*\mu_V\big)$. In particular,

$${}_*\mu_V^{(m)} \leqslant {}_*\mu_{Q_2}^{(m)}, \quad m \in \mathbb{Z}_+,$$

while

$$U^{*\mu_V}(z) \geqslant U^{*\mu_{Q_2}}(z)$$

whenever $z \in \mathbb{C} \setminus K$.

3.3. Orthogonal polynomials

As we shall now explain, the above results have interesting yet apparently unknown analogs for important classes of orthogonal polynomials such as Jacobi polynomials. Recall that the latter are defined by

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\left[(1-z)^{n+\alpha}(1+z)^{n+\beta}\right],$$

where $\alpha > -1, \beta > -1$. For special values of the parameters α and β one gets (up to a normalizing factor) all the other classical Jacobi-like polynomials including (ultraspherical) Gegenbauer polynomials, (associated) Legendre polynomials and Chebyshev polynomials of the first or second kind. The relative location and asymptotic behaviour of the zeros of Jacobi polynomials have been of permanent interest in view of their important role as nodes of Gaussian quadrature formulae and their nice electrostatic interpretation (Bethe Ansatz) [26].

We prove the following result.

Theorem 5. *Let* $\mathcal{Z}\left(P_n^{(\alpha,\beta)}\right) = \{\zeta_{n,i}\}_{i=1}^n$ *be the zero set of* $P_n^{(\alpha,\beta)}$. *Then*

$$\frac{1}{n}\sum_{i=1}^n f(\zeta_{n,i}) \leqslant \frac{(n+\beta)f(1)+(n+\alpha)f(-1)}{2n+\alpha+\beta}$$

for any convex function $f : [-1, 1] \to \mathbb{R}$.

Remark 8. The proof of Theorem 5 yields in fact an even stronger relation that involves both zero sets $\mathcal{Z}\left(P_n^{(\alpha,\beta)}\right)$ and $\mathcal{Z}\left(P_n^{(\alpha,\beta)\prime}\right)$, see (3.11) in Section 3.4.

Remark 9. Note that if $\mu_{P_n^{(\alpha,\beta)}}$ denotes the root-counting measure of $P_n^{(\alpha,\beta)}$ then Theorem 5 may be restated in terms of the Choquet order simply as

$$\mu_{P_n^{(\alpha,\beta)}} \prec \frac{n+\beta}{2n+\alpha+\beta}\delta_1 + \frac{n+\alpha}{2n+\alpha+\beta}\delta_{-1}.$$

In particular, by letting $n \to \infty$ we get

$$\mu^*_{\alpha,\beta} \prec \tfrac{1}{2}(\delta_1 + \delta_{-1}), \tag{3.6}$$

where $\mu^*_{\alpha,\beta} := \lim^*_{n\to\infty} \mu_{P_n^{(\alpha,\beta)}}$ is the *-limiting distribution of the zeros of $P_n^{(\alpha,\beta)}$. As is well known (see, e.g., [26]) the latter is the (uniform) arcsine distribution and thus (3.6) may be rewritten as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(z)}{\sqrt{1-z^2}}\,dz \leqslant \frac{f(1)+f(-1)}{2}$$

for any convex function $f : [-1, 1] \to \mathbb{R}$. Although elementary, the above inequality is not completely obvious; arguably the most direct way of proving it is to note that it actually amounts to showing that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{|z-c|}{\sqrt{1-z^2}}\,dz \leqslant \max(1, |c|)$$

for any $c \in \mathbb{R}$, which is a trivial exercise.

3.4. Proofs

Fix an integer $m \geqslant 2$ and let z_i, $1 \leqslant i \leqslant m$, be (not necessarily distinct) points in the complex plane that do not coalesce into a single one. Given $\tau_i > 0$, $1 \leqslant i \leqslant m$, such that $\sum_{i=1}^{m}\tau_i = 1$ we define a meromorphic function

$$\Phi(z) = \sum_{i=1}^{m}\frac{\tau_i}{z - z_i}. \tag{3.7}$$

Functions of this type are sometimes called generalized derivatives in the sense of Sz.-Nagy (see [4]) and may be interpreted as the resulting electrostatic force of a planar charge configuration (cf. [4]). One of the key ingredients in our proofs is [4, Theorem 2], which we restate as follows:

Lemma 1. *Let Φ be as in* (3.7) *and denote its zeros by w_j, $1 \leqslant j \leqslant m-1$, where it is understood that z_i counts as a "zero" of Φ of multiplicity $m_i - 1$ if it occurs precisely m_i times in* (3.7). *Then*

$$\sum_{j=1}^{m-1} f(w_j) \leqslant \sum_{i=1}^{m}(1-\tau_i) f(z_i) \tag{3.8}$$

for any convex function $f : \mathbb{C} \to \mathbb{R}$.

Remark 10. The arguments in [4] further imply that if f is a strictly convex function such that equality is attained in (3.8) then the z_i's must be collinear.

We emphasize an important special case of Lemma 1:

Corollary 7. *If $P \in \mathbb{C}[z]$ is such that* $\deg P = d \geqslant 2$ *then*

$$(d-i+1)\sum_{w\in\mathcal{Z}(P^{(i)})} f(w) \leqslant (d-i)\sum_{z\in\mathcal{Z}(P^{(i-1)})} f(z)$$

for any convex function $f : \mathbb{C} \to \mathbb{R}$ and $1 \leqslant i \leqslant d-1$.

To prove Theorem 4 let $(S(z), V(z))$ be a pair of (higher) spectral polynomials of $\mathfrak{d}(z)$ as in Section 3.2, set

$$\mathcal{Z}\big(S^{(i)}\big) = \big(s_1^{(i)}, \ldots, s_{n-i}^{(i)}\big), \quad 1 \leqslant i \leqslant n-1,$$

and note that by (1.2) Eq. (1.4) may be rewritten as

$$-\frac{V(z)S(z)}{Q_2(z)S^{(k-1)}(z)} = \frac{S^{(k)}(z)}{S^{(k-1)}(z)} + \frac{Q_1(z)}{Q_2(z)} = \sum_{j=1}^{n-k+1} \frac{1}{z - s_j^{(k-1)}} + \sum_{l=1}^{p} \frac{a_l}{z - \zeta_l}. \tag{3.9}$$

Since $\mathcal{Z}(VS) = \mathcal{Z}(V) \vee \mathcal{Z}(S)$ (by the convention made at the beginning of Section 3) and $a_l > 0$, $1 \leqslant l \leqslant p$, we deduce from Lemma 1 that for any convex function $f : \mathbb{C} \to \mathbb{R}$ one has

$$\sum_{i=1}^{r} f(v_i) + \sum_{j=1}^{n} f(s_j) \leqslant \left(1 - \frac{1}{\alpha_k}\right) \sum_{j=1}^{n-k+1} f\left(s_j^{(k-1)}\right) + \sum_{l=1}^{p} \left(1 - \frac{a_l}{\alpha_k}\right) f(\zeta_l). \tag{3.10}$$

On the other hand, by Corollary 7 we know that

$$n \sum_{j=1}^{n-k+1} f\left(s_j^{(k-1)}\right) \leqslant (n-k+1) \sum_{j=1}^{n} f(s_j),$$

which combined with (3.10) yields (3.5) after some straightforward computations. Theorem 3 follows from the above simply by letting $k = 2$.

Since functions of the type $\mathbb{C} \ni w \mapsto |w|^m$, $m \in \mathbb{Z}_+$, and $\operatorname{conv}(\mathcal{Z}(Q_2)) \ni w \mapsto -\log|z - w|$, $z \notin \operatorname{conv}(\mathcal{Z}(Q_2))$, are convex, Corollaries 1 and 4 are immediate consequences of Theorems 3 and 4, respectively. Corollaries 2 and 5 follow from Corollaries 1 and 4, respectively, by using Remark 3 and the expression for α_k (i.e., $\alpha_k = \alpha(n, p, k) = n - k + 1 + \sum_{l=1}^{p} a_l$) and noticing that

$$\frac{(p-k)\alpha_k(n,p,k)}{(p-1)\alpha_k(n,p,k) + n - 1} \to \frac{p-k}{p} \quad \text{and} \quad \frac{(k-1)\alpha_k(n,p,k) + n - 1}{(p-1)\alpha_k(n,p,k) + n - 1} \to \frac{k}{p}$$

as $n \to \infty$, p being fixed. To prove Corollary 6 (hence also Corollary 3) we use again Corollary 4 and Remark 3 together with the fact that for any $\Gamma \subset \mathbb{N}$ such that each of the sequences $\left\{\mu_{V_{p,n(p)}}\right\}_{p \in \Gamma}$, $\left\{\mu_{S_{p,n(p)}}\right\}_{p \in \Gamma}$ and $\left\{\mu_{Q_{2,p}}\right\}_{p \in \Gamma}$ weak-* converges the following holds:

$$\lim_{\Gamma \ni p \to \infty} \frac{(p-k)\alpha_k(n,p,k)}{(p-1)\alpha_k(n,p,k) + n(p) - 1} = 1,$$

$$\frac{k-1}{p-1} \leqslant \frac{(p-k)\alpha_k(n,p,k) + n(p) - 1}{(p-1)\alpha_k(n,p,k) + n(p) - 1} \leqslant \frac{k}{p} \quad \text{if } p \geqslant k,$$

$$\begin{aligned}\left|\tilde{\mu}_{Q_{2,p}}(f) - \mu_{Q_{2,p}}(f)\right| &\leqslant \sum_{l=1}^{p} \frac{|\alpha_k(n,p,k) - pa_l - n(p) + 1|}{p[(p-1)\alpha_k(n,p,k) + n(p) - 1]} \max_{z \in K} |f(z)| \\ &\leqslant \frac{2}{p-1} \max_{z \in K} |f(z)|,\end{aligned}$$

where K is a compact subset of $\mathbb{C}$ as in Corollary 6, $f : K \to \mathbb{R}$ is any (continuous) convex function, $\mu_{Q_{2,p}}$ denotes the root-counting measure of $Q_{2,p}$ and $\tilde{\mu}_{Q_{2,p}} \in \mathcal{M}(K)$ is defined by

$$\operatorname{supp}\left(\tilde{\mu}_{Q_{2,p}}\right) = \mathcal{Z}(Q_{2,p}) = \{\zeta_{l,p}\}_{l=1}^{p},$$

$$\tilde{\mu}_{Q_{2,p}}(\{\zeta_{l,p}\}) = \frac{\alpha_k(n,p,k) - a_l}{(p-1)\alpha_k(n,p,k) + n(p) - 1}, \quad 1 \leqslant l \leqslant p.$$

Turning to the proof of Theorem 5 recall first (cf., e.g., [26]) that $P_n^{(\alpha,\beta)}$ satisfies Jacobi's equation, i.e., the homogeneous second order linear differential equation

$$(1-z^2)y''(z)+[\beta-\alpha-(\alpha+\beta+2)z]y'(z)+n(n+\alpha+\beta+1)y(z)=0.$$

Therefore, if $\mathcal{Z}\left(P_n^{(\alpha,\beta)'}\right)=\{\zeta'_{n,j}\}_{j=1}^{n-1}$ denotes the zero set of $P_n^{(\alpha,\beta)'}$ then

$$\begin{aligned}-\frac{n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(z)}{(1-z^2)P_n^{(\alpha,\beta)'}(z)}&=\frac{P_n^{(\alpha,\beta)''}(z)}{P_n^{(\alpha,\beta)'}(z)}+\frac{\beta-\alpha-(\alpha+\beta+2)z}{1-z^2}\\&=\sum_{j=1}^{n-1}\frac{1}{z-\zeta'_{n,j}}+\frac{\alpha+1}{z-1}+\frac{\beta+1}{z+1}.\end{aligned}$$

Since $\alpha>-1$ and $\beta>-1$ we may apply Lemma 1 to get

$$\begin{aligned}(n+\alpha+\beta+1)\sum_{i=1}^{n}f(\zeta_{n,i})\leqslant{}&(n+\alpha+\beta)\sum_{j=1}^{n-1}f\left(\zeta'_{n,j}\right)+(n+\beta)f(1)\\&+(n+\alpha)f(-1)\end{aligned}\tag{3.11}$$

for any (continuous) convex function $f:[-1,1]\to\mathbb{R}$ (cf. Remark 2 in Section 2). Now by Corollary 7 we know that

$$n\sum_{j=1}^{n-1}f\left(\zeta'_{n,j}\right)\leqslant(n-1)\sum_{i=1}^{n}f(\zeta_{n,i}),$$

which together with (3.11) immediately gives the desired conclusion.

4. Further results and related problems

1. One can actually obtain stronger albeit somewhat less transparent versions of Theorems 3–4 and establish convex domination relations for vectors whose coordinates are symmetric functions on (subsets of) $\mathcal{Z}(V)\vee\mathcal{Z}(S)$ and $\mathcal{Z}(S^{(k-1)})\vee\mathcal{Z}(Q_2)$, respectively. Indeed, given $d\in\mathbb{N}$ and $e\in\mathbb{Z}_+$ with $e\leqslant d$ let $\Pi_{d,e}$ denote the eth elementary symmetric function on d elements. Then [4, Corollary 3] shows that the zeros and poles of the function Φ defined in (3.7) satisfy the following inequalities.

Lemma 2. *Let Φ be as in* (3.7) *and denote its zeros by w_j, $1\leqslant j\leqslant m-1$, where as before it is understood that z_i counts as a "zero" of Φ of multiplicity m_i-1 if it occurs precisely m_i times in* (3.7). *Then*

$$\begin{aligned}&\sum_{1\leqslant j_1<\cdots<j_d\leqslant m-1}f\left(\Pi_{d,e}(w_{j_1},\ldots,w_{j_d})\right)\\&\quad\leqslant\sum_{1\leqslant i_1<\cdots<i_d\leqslant m}\left(1-\sum_{l=1}^{d}\tau_{i_l}\right)f\left(\Pi_{d,e}(z_{i_1},\ldots,z_{i_d})\right)\end{aligned}$$

for any convex function $f:\mathbb{C}\to\mathbb{R}$, $d\in\{1,\ldots,m-1\}$ and $e\in\mathbb{Z}_+$, $e\leqslant d$.

Note that Lemma 1 corresponds to the case when $e = d = 1$ in Lemma 2. Using the latter with $m = n - k + p + 1$ and

$$\tau_j = \begin{cases} \alpha(n, p, k)^{-1}, & 1 \leqslant j \leqslant n - k + 1, \\ \alpha(n, p, k)^{-1} a_{j-n+k-1}, & n - k + 2 \leqslant j \leqslant n - k + p + 1, \end{cases}$$

$$\{z_i\}_{i=1}^{n-k+p+1} = \mathcal{Z}(S^{(k-1)}) \vee \mathcal{Z}(Q_2), \quad \{w_j\}_{j=1}^{n-k+p} = \mathcal{Z}(V) \vee \mathcal{Z}(S)$$

together with (3.9) one can then deduce weighted majorization relations of the aforementioned type that strengthen (3.10) in various ways. For special choices of the function f one can slightly simplify these relations by first separating elements in $\mathcal{Z}(S^{(k-1)})$ from those in $\mathcal{Z}(Q_2)$ (once the terms occurring in the right-hand side of the above inequality are appropriately regrouped) and then using Corollary 7 in order to estimate from above all resulting expressions that contain elements in $\mathcal{Z}(S^{(k-1)})$ by means of similar expressions involving only elements in $\mathcal{Z}(S)$. Such simplifications can be made e.g., for multiplicative convex functions of the form $f(z) = |z|^q$, $q \in \mathbb{Z}_+$.

2. In view of the above results it would be interesting to know whether similar properties with respect to the Choquet order also hold for spectral polynomials of more general classes of (Lamé-like) operators. Let

$$\mathfrak{d}(z) = \sum_{i=m}^{k} Q_i(z) \frac{d^i}{dz^i} \tag{4.1}$$

be a linear ordinary differential operator of order k with polynomial coefficients. Following the terminology that we already employed for (1.3) (cf. [8]) we call $\mathfrak{d}(z)$ a *higher Lamé operator* if its *Fuchs index* $r := \max_{m \leqslant i \leqslant k}(\deg Q_i - i)$ is non-negative. If $r = 0$ then $\mathfrak{d}(z)$ is usually referred to as an *exactly solvable operator* in the physics literature. A higher Lamé operator $\mathfrak{d}(z)$ given by (4.1) is said to be *non-degenerate* if $\deg Q_k = k + r$, which is equivalent to the (quite natural) requirement that $\mathfrak{d}(z)$ has either a regular or a regular singular point at ∞. For such an operator one may then consider the multiparameter spectral problem stated in (1.4) and the corresponding notions of (n-)solvability and higher Lamé (i.e., Van Vleck and Heine–Stieltjes) polynomials. A systematic study of the latter was recently made in [8]. In particular, in [8] it was shown that a non-degenerate higher Lamé operator $\mathfrak{d}(z)$ is n-solvable for all sufficiently large n and it was further proved that if the coefficients of $\mathfrak{d}(z)$ are algebraically independent then for any $n \in \mathbb{N}$ there are exactly $\binom{n+r}{n}$ Van Vleck polynomials and as many degree n Heine–Stieltjes polynomials, thus generalizing Heine's result (cf. Remarks 5 and 7).

Problem 1. Extend Theorems 3 and 4 to non-degenerate higher Lamé operators (subject to appropriate conditions).

An important class of linear operators which seems particularly well suited for Problem 1 consists of non-degenerate higher Lamé operators that also preserve hyperbolicity (HPOs). Indeed, as we already mentioned in Remark 7 a complete classification of all HPOs—i.e., linear operators T on $\mathbb{R}[z]$ such that $T(P(z))$ has all real zeros whenever $P \in \mathbb{R}[z]$ has all real zeros—was recently obtained in [6]. Moreover, various properties and characterizations of HPOs that belong to the Weyl algebra $\mathcal{A}_1$ (that is, operators of the form (4.1)) were established in [7]. For instance, in [7] it was shown that the coefficients $Q_i(z)$ of such an operator have all real zeros and satisfy interlacing properties like those in (1.2). (Note, e.g., that if $Q_2(z)$, $Q_1(z)$ are as in (1.2) and $\mathcal{Z}(Q_2) \subset \mathbb{R}$

then the corresponding operator $\mathfrak{d}(z)$ given by (1.3) is an HPO.) Furthermore, in [8] it was proved that if a non-degenerate higher Lamé operator $\mathfrak{d}(z)$ is also an HPO then $\mathfrak{d}(z)$ is solvable and all its Van Vleck and Heine–Stieltjes polynomials have simple real zeros. Finally, [3, Conjecture 1] claims that HPOs either preserve or reverse the Choquet order on real-zero polynomials and that in particular, if $\mathfrak{d}(z)$ is an HPO of the form (4.1) then $\mathcal{Z}\big(\mathfrak{d}(z)P(z)\big) \prec \mathcal{Z}\big(\mathfrak{d}(z)Q(z)\big)$ whenever $P, Q \in \mathbb{R}[z]$, $\deg P = \deg Q$, $\mathcal{Z}(P), \mathcal{Z}(Q) \subset \mathbb{R}$, $\mathcal{Z}(P) \prec \mathcal{Z}(Q)$. For results supporting this conjecture, see [3,5,9]. Problem 1 should therefore be particularly interesting for non-degenerate higher Lamé operators of HPO type.

3. Let $P_n(z)$, $n \in \mathbb{Z}_+$, be polynomials orthogonal with respect to a weight function ω supported on a (finite or infinite) interval $[a, b]$ with $\omega(z) > 0$, $z \in (a, b)$. It is well known that such polynomial families satisfy a 3-term recurrence relation

$$zP_n(z) = a_{n+1}P_{n+1}(z) + b_nP_n(z) + a_nP_{n-1}(z)$$

and that under some mild assumptions on the weight function ω (see, e.g., [19]) they also satisfy a differential recurrence relation of the type

$$P_n'(z) = A_n(z)P_{n-1}(z) - B_n(z)P_n(z)$$

and a second order differential equation of the form

$$P_n''(z) + R_n(z)P_n'(z) + S_n(z)P_n(z) = 0,$$

where $\{A_n(z)\}_{n\in\mathbb{Z}_+}$, $\{B_n(z)\}_{n\in\mathbb{Z}_+}$, $\{R_n(z)\}_{n\in\mathbb{Z}_+}$, $\{S_n(z)\}_{n\in\mathbb{Z}_+}$ are certain function sequences. It is therefore natural to ask the following questions.

Problem 2. Investigate whether there are weighted majorization relations similar to those in Theorem 5 and/or weighted majorization relations involving the zeros of any two (or three) consecutive terms for

(i) Laguerre-like and Hermite-like polynomials;
(ii) (appropriate classes of) general orthogonal polynomials.

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