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Mergelyan's Theorem for Zero Free Functions

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Abstract

The paper gives a partial answer to a question posed by Johan Andersson and Paul Gauthier. Namely we prove that every function f continuous on a locally connected compact set K with connected complement, analytic in its interior and not vanishing in the interior can be approximated by polynomials with zeros outside of K.

Key words: Mergelyan's Theorem, polynomial approximation, logarithmically continuous functions, locally connected compact sets, Peano spaces.
MSC2010: 30E10, 41A10, 46J10, 54D30, 54F15, 54F35, 42C05.

1 Introduction

Let K be a compact subset of $\mathbb{C}$ and K° be the set of all interior point of K. We denote by $A = A(K)$ the Banach algebra of all continuous functions on K which are analytic in K°. This algebra is equipped with the standard norm

$$\|f\| = \sup_{z\in K} |f(z)|. \tag{1}$$

A well-known theorem of Mergelyan claims that any function in $A(K)$ can be approximated in the norm (1) by polynomials in z if $G = \mathbb{C} \setminus K$ is connected. An interesting question was raised recently by J. Andersson [1] and proposed by P. Gauthier for a discussion at the workshop "Stability, hyperbolicity, and zero location of functions", December 5-9, 2011, organized by Petter Branden, George Csordas, Olga Holtz, and Mikhael Tyaglov in the American Institute of Mathematics.

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Question. Suppose that $f \in A(K)$ has no zeros in K° and $\epsilon > 0$. Is there a polynomial p with no zeros on K such that $\|f - p\| < \epsilon$?

For finite chains of Jordan domains a positive answer to this question was obtained by by Gauthier and Knese in [2]. The purpose of the present paper is to give a positive answer to this question for the more general class of locally connected compact sets with connected complement on the complex plain $\mathbb{C}$.

Let e^A denote the functions of the form e^h, where $h \in A$. For the sets being considered, the problem is equivalent to the possibility of approximation by functions in A^{-1}. It is shown in the paper that f can be approximated by functions in e^A.

Definition 1.1 *We say that $f \in A(K)$ is logarithmically continuous if there is a single valued branch of $\log f$ on K which is continuous at every point z_0 of K such that $f(z_0) \neq 0$ and*

$$\lim_{z \to z_0} \log f(z) = \infty \,, \tag{2}$$

if $f(z_0) = 0$.

For example, if K is the closed unit disc then

$$f(z) = (1 - z) \exp\left\{-\frac{1+z}{1-z}\right\}$$

is logarithmically continuous. Logarithmically continuous functions cannot vanish in K°.

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2 The Main Theorem

Theorem 2.1 *For every logarithmically continuous f in $A(K)$ and $\epsilon > 0$ there is a function g in $e^{A(K)}$ and hence invertible in $A(K)$ such that $\|f - g\| < \epsilon$.*

PROOF. In what follows we assume that $|f| < 0.5$ on K so that $\log f$ takes values in the half-plane $\{z : \mathfrak{Re}\, z < 0\}$. Let $0 < \varepsilon < 0.5$ and $K_\varepsilon = \{z \in K : |f(z)| \geq \varepsilon\}$. Then $\log f$ is a continuous function on K_ε and therefore its modulus is bounded by a positive constant $M_\varepsilon > 1$ on K_ε. There exists δ,

$0 < \delta < 0.1$ such that

$$\delta M_\varepsilon^2 < \varepsilon \quad \text{and} \quad \delta \log \frac{1}{\varepsilon} < 0.5 \tag{3}$$

Consider the function

$$f_\delta(z) = \begin{cases} \exp\left\{\dfrac{\log f}{1 - \delta \log f}\right\} & \text{if } f(z) \neq 0\,; \\ \exp\left\{-\dfrac{1}{\delta}\right\} & \text{if } f(z) = 0\,. \end{cases}$$

If $f(z_0) \neq 0$ then f_δ is continuous at z_0 by the logarithmic continuity of $\log f$. If $f(z_0) = 0$ then by (2)

$$\lim_{z \to z_0} f_\delta(z) = \exp\left\{-\frac{1}{\delta}\right\}, \tag{4}$$

implying that f_δ is analytic in K° and is continuous on K. Hence f_δ is invertible in $A(K)$. We have

$$f_\delta - f = e^{\frac{\log f}{1-\delta \log f}} - e^{\log f} = f\left\{e^{\frac{\log f}{1-\delta\log f} - \log f} - 1\right\} = f\left\{e^{\frac{\delta \log^2 f}{1-\delta \log f}} - 1\right\}. \tag{5}$$

For every $|w| \leq M_\varepsilon$ (see (3)):

$$\left|\frac{\delta w^2}{1 - \delta w}\right| < \frac{\varepsilon}{1 - \frac{\varepsilon}{M_\varepsilon}} < \frac{\varepsilon}{1 - \varepsilon} < 2\varepsilon < 1.$$

If $0 < |u| < 1$ then

$$|e^u - 1| = \left|\sum_{k=1}^{\infty} \frac{u^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|u|^k}{k!} = e^{|u|} - 1 < e|u|. \tag{6}$$

Hence

$$\sup_{z \in K_\varepsilon} |f_\delta(z) - f(z)| \leq 0.5 \cdot 2\varepsilon \cdot e \leq 3\varepsilon. \tag{7}$$

Next

$$\sup_{z \in K \setminus K_\varepsilon} |f_\delta(z) - f(z)| \leq \sup_{z \in K \setminus K_\varepsilon} |f(z)| + \sup_{z \in K \setminus K_\varepsilon} |f_\delta(z)| \leq \varepsilon + \sup_{z \in K \setminus K_\varepsilon} |f_\delta(z)|. \tag{8}$$

The linear fractional transform

$$w \longmapsto \frac{w}{1 - \delta w} \tag{9}$$

maps the real line onto itself and the left-half plane onto the disc with the diameter $[-1/\delta, 0]$. The function $\log f$ maps $K \setminus K_\varepsilon$ into the half-plane $\{w :$

$\mathfrak{Re}\, w < \log \varepsilon\}$. The linear fractional transform (9) maps this half-plane onto the disc with the diameter $[-1/\delta, (\log \varepsilon)/(1 - \delta \log \varepsilon)$. But by the choice of δ (see (3))

$$\frac{1}{1 - \delta \log \varepsilon} > \frac{1}{1 + 0.5} = \frac{2}{3}.$$

It follows that

$$\mathfrak{Re}\left(\frac{\log f}{1 - \delta \log f}\right) < \frac{2}{3} \log \varepsilon$$

on $K \setminus K_\varepsilon$. Hence $|f_\delta| < \varepsilon^{\frac{2}{3}}$.

By (7) and (8) we therefore obtain

$$||f - f_\delta|| < 4\varepsilon + \varepsilon^{\frac{2}{3}} \qquad \square.$$

Corollary 2.2 *Suppose that K is a compact subset of $\mathbb{C}$ with connected complement, $f \in A(K)$ is a logarithmically continuous function and $\epsilon > 0$. Then there a polynomial p with no zeros on K such that $\|f - p\| < \epsilon$.*

PROOF. By Theorem 2.1 for every $\varepsilon > 0$ there is an invertible element g in $A(K)$ such that $||f - g|| < \epsilon/2$. Since g is an invertible element of $A(K)$ there is $\epsilon_1 < \epsilon/2$ such that $\inf_{z \in K} |g(z)| > \epsilon_1$. Since $\mathbf{C} \setminus K$ is connected by Mergelyan's theorem there is a polynomial p such that $||g - p|| < \epsilon_1$ implying that p has no roots on K. It follows that $||f - p|| < \epsilon$.

3 Applications

Theorem 3.1 *Let K be a locally connected compact set with connected complement. Then every function $f \in A(K)$ not vanishing at K° is logarithmically continuous.*

PROOF. Every locally connected compact set is a union of a finite number of disjoint connected and locally connected compact sets. A compact, connected, locally connected metric space is called a Peano space ([3], Definition 31.1). Every Peano set is arcwise connected ([3], Theorem 31.2). Therefore it is sufficient to prove the theorem for the case when K is a Peano space.

Let $f \in A(K)$ be a function analytic in K° and not vanishing in K°. Let $Z(f) = \{z \in K : f(z) = 0\}$ be the zero set of f. By the Hahn-Mazurkiewich Theorem (see [3], Theorem 31.5) the Peano space K is a continuous image

$\gamma : \mathbf{I} \to K$ of the unit interval $\mathbf{I}$. Then $C = \gamma^{-1}(Z(f))$ is a closed subset of $\mathbf{I}$. Hence its complement

$$\mathbf{I} \setminus C = \bigcup_{j \geq 1} I_j \tag{10}$$

is an at most countable union of open intervals I_j. We assume that the set of indexes is linearly ordered in a such a way that $i < j$ implies that $|I_i| \geq |I_j|$. Here $|I|$ stands for the length of the interval I.

Every point $p \in K \setminus Z(f)$ is contained in a maximal path connected component V of the set $K \setminus Z(f)$. Given j any component V either does not intersect $\gamma(I_j)$ or contains it completely. By (10) any component V contains at least one path connected set $\gamma(I_j)$. It follows that the number of connected components V is at most countable. Therefore we can represent $K \setminus Z(f)$ as a disjoint union

$$K \setminus Z(f) = \bigcup_{s \geq 1} V_s \tag{11}$$

of connected components V_s. We assume that V_s contains a marked point p_s.

Lemma 3.2 *For every $s \geq 1$ and every choice of the branch $\log f(p_s)$ the function $\log f$ extends continuously to V_s.*

PROOF. Any point p in V_s is connected to p_s by an arc I. We define $\log f(p)$ as the result of continuation of a continuous function $\log f$ along I. To show that the result does not depend on the arc consider two arcs I and J in V_s connecting p_s with $p \in V_s$. The intersection $F = I \cap J$ is a closed subset of $V_s \subset K \setminus Z(f)$. Then $I \cup J$ can be represented as the union of F with at most a countable number of loops made by I and J outside of F. Each such loop is a closed Jordan curve which by Jordan's Theorem separates the plane into two open connected sets. Since the complement of K is connected the inside parts of loops are located in the open components of K. Open components of K do not intersect $Z(f)$. It follows that the inside parts of the above loops lie in V_s.

The arc I is a homeomorphic image of the unit interval $\mathbf{I}$. Therefore the points of F can be ordered by this parametrization. Let q be the greatest point of F at which both branches of $\log f$ coincide. Since different branches differ by $2\pi i n$, where n is a nonzero integer, and f is continuous we see that both continuations must coincide at q: $\log f(q)|_I = \log f(q)|_J$. If $q = p$ then there is nothing to prove. If it is not the case and q is on the boundary of some loop then $\log f$ is analytic inside the loop and is continuous on its boundary which results in equal values of the continuation of the $\log f$ at the other intersection of this very loop with F, which contradicts the maximal choice of q. If $q \neq p$ and is not on the boundary of any loop, then it is a cluster point of the loops corresponding to bigger values of the parameterization of I. But

f is continuous. Therefore it changes very little if we move a little bit to the right along J or I. Therefore $\log f$ also cannot jump by $2\pi in$, $n \neq 0$, implying that q is not the greatest point of F. This proves the lemma. $\square$

By Lemma 3.2 the function $\log f$ is defined on every V_s and therefore on $K \setminus Z(f)$. We extend $\log f$ to be equal ∞ on $Z(f)$.

Let us prove that f is logarithmically continuous on K. Let $\{q_j\}_{j\geq 1}$ be a sequence in K such that $\lim_j q_j = q$. If $f(q) = 0$ then $\lim_j f(q_j) = f(q) = 0$ and therefore $\lim_j \log f(q_j) = \infty$. Suppose that $f(q) \neq 0$. Then by (11) there is a unique component V_s containing this q. By (10) we can choose the maximal family of complementary intervals I_{k_j} such that

$$V_s = \bigcup_j \gamma(I_{k_j}). \tag{12}$$

The set $B_q = \{t \in \mathbf{I} : q = \gamma(t)\}$ is a compact subset of $\mathbf{I} \setminus C$ covered by intervals I_{k_j}. Therefore this compact set can be covered by a finite system of such intervals:

$$B_q = \{t \in \mathbf{I} : q = \gamma(t)\} \subset \bigcup_{r=1}^{d} I_{k_{j_r}}. \tag{13}$$

Since $f(q) \neq 0$ we conclude that $q_j \in K \setminus Z(f)$ starting from some place. Let t_j be any solution to the equation $q_j = q(t_j)$. Let a be any accumulation point of the sequence $\{t_j\}$ in $\mathbf{I}$. Then $q = \gamma(a)$ and a is in the compact set B_q covered by d open intervals $I_{k_{j_r}}$. It follows that the sequence $\{t_j\}$ is inside of this finite union of intervals starting from some place. Otherwise there would be a subsequence of $\{t_j\}$ outside of this finite union of open intervals but having an accumulation point in B_q which is impossible. Hence the corresponding $\{q_j\}$ are inside of V_s starting from the same place. But $\log f$ was defined as a continuous function in V_s. It follows that $\lim_j \log f(q_j) = \log f(q)$. $\square$

Corollary 3.3 *Let K be a locally connected compact set with connected complement. Then every function $f \in A(K)$ not vanishing at K° can be arbitrarily closely approximated by polynomials with roots outside K.*

PROOF. By Theorem 3.1 such an f is logarithmically continuous. The proof is completed by Corollary 2.2. $\square$

To conclude let us consider an example. Take a semi-disc in the lower complex half-plane constructed on $[0,1]$ as on a diameter. Then extend this domain with rectangular octagonal "tunnels" starting on closed intervals $[1/(2n+1), 1/2n]$ for $n = 1, 2, 3, ...$ and surrounding the vertical segment $0 \leq y \leq 1$ from both sides and approaching it as $n \to \infty$. Then clearly $f(z) = z - i/2$ is not logarithmically continuous for this compact set. Still it can be approximated with polynomials with roots outside K.

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