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Analytic approximation in L^p and coinvariant subspaces of the Hardy space

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Abstract

We generalize a classical result by A. Macintyre and W. Rogosinski on best H^p-approximation in L^p of rational functions. For each inner function θ we give a description of H^p-badly approximable functions in $\bar{\theta}H^p$.

Keywords: Analytic approximation; Coinvariant subspaces

1. Introduction

The classical problem of best analytic approximation in L^p on the unit circle $\mathbb{T}$ reads as follows: given a function $g \in L^p$, find a function p_g in the Hardy space H^p such that

$$\|g - p_g\|_{L^p} = \operatorname{dist}_{L^p}(g, H^p).$$

In 1920, F. Riesz proved [10] that best H^1-approximation in L^1 of a trigonometric polynomial of degree n is an analytic polynomial of degree at most n. His result was generalized in 1950 by A. Macintyre and W. Rogosinski [6], who treat the problem of best analytic approximation in L^p for rational functions with finite number of poles in the open unit disk.

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Theorem 1 (*A. Macintyre, W. Rogosinski*)**.** *Let $1 \leqslant p \leqslant \infty$, and let g be a rational function with n poles β_i in $|z| < 1$, each counted according to multiplicity. Then best H^p-approximation p_g of the function g exists uniquely. Moreover, there exist $n - 1$ numbers α_i with $|\alpha_i| \leqslant 1$ such that*

$$g - p_g = \text{const} \cdot {\prod}' \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \prod_1^{n-1} (1 - \bar{\alpha}_i z)^{2/p} \prod_1^{n} \frac{1 - \bar{\beta}_i z}{z - \beta_i} (1 - \bar{\beta}_i z)^{-2/p}, \tag{1}$$

where $\prod'$ is extended over all, some, or none of the α_i with $|\alpha_i| < 1$.

Among other things, this result shows that best H^1-approximation of a rational function is a rational function as well. The same holds for best H^∞-approximation.

In 1953, W. Rogosinski and H. Shapiro [11] presented a uniform approach to the problem of best analytic approximation in L^p based on duality for classes H^p. Their paper contains a refined (but still rather complicated) proof of Theorem 1.

The matrix-valued case of the problem of best analytic approximation has been studied extensively in the past years. In particular, V. Peller and V. Vasyunin [9] consider this problem for rational matrix-valued functions motivated by applications in H^∞—Control Theory. A survey of results related to best analytic approximation in L^p of matrix-valued functions can be found in L. Baratchart, F. Nazarov, V. Peller [1].

Our aim in this note is to give a short proof of Theorem 1 and present its analogue in more general situation. We will consider the problem of best analytic approximation for functions of the form h/θ, where $h \in H^p$ and θ is an inner function. If θ is a finite Blaschke product we are in the setting of Theorem 1. In general, functions of the form h/θ may have much more complex behavior near the unit circle than rational functions. To be more specific, we need some definitions.

A bounded analytic function θ in the open unit disk is called inner if $|\theta| = 1$ almost everywhere on the unit circle $\mathbb{T}$ in the sense of angular boundary values. Given an inner function θ, define the coinvariant subspace K_θ^p of the Hardy space H^p by the formula $K_\theta^p = H^p \cap \bar{z}\theta\overline{H^p}$. Here and in what follows we identify the Hardy space H^p in the open unit disk $\mathbb{D}$ with the corresponding subspace of the space L^p on the unit circle $\mathbb{T}$ via angular boundary values. All the information we need about Hardy spaces is available in Sections II and IV of [3]. Basic theory of coinvariant subspaces K_θ^p can be found in [7,12].

Our main result is the following.

Theorem 2. *Let θ be an inner function and let $1 \leqslant p < \infty$. Take a function $g \in \bar{\theta}H^p$ and denote by p_g its best H^p-approximation. The function $g - p_g$ can be uniquely represented in the form $g - p_g = c \cdot \bar{\theta} I F^{2/p}$, where $c = \operatorname{dist}_{L^p}(g, H^p)$, F is an outer function in K_θ^2 of unit norm, $F(0) > 0$, and I is an inner function such that $IF \in K_\theta^2$.*

Taking $\theta = z^n$ and $p = 1$ in Theorem 2, we get the mentioned result by F. Riesz: trigonometric polynomials are preserved under best analytic approximation in L^1. Our paper contains the fourth proof of this fact (see Section 3); previous proofs can be found in [10,6,5]. Similarly, Theorem 1 follows from Theorem 2 by taking the function θ to be a finite Blaschke product. The choice $\theta = e^{iaz}$ leads to the following fact.

Theorem 3. *Let g be a function in $L^1(\mathbb{R})$ with compact support of Fourier transform:* $\operatorname{supp} \hat{g} \subset [-a, a]$. *Then we have* $\operatorname{supp} \hat{p}_g \subset [0, a]$ *for best H^1-approximation p_g of the function g.*

Proofs of Theorems 1, 2, 3 are given in Sections 3, 2, 4, correspondingly. In Section 5 we discuss how the problem of best analytic approximation in L^p for functions from $\bar{\theta}H^p$ can be reduced to a special problem of interpolation.

2. Proof of Theorem 2

We need the following known result from [2].

Lemma 2.1 (*K. Dyakonov*). *A nonnegative function φ can be represented in the form $\varphi = |F|^2$ for some outer function $F \in K^2_\theta$ if and only if $\varphi \in z\bar{\theta}H^1$.*

The proof is included for completeness.

Proof. Let φ be a function of the form $\varphi = |F|^2$, where $F \in H^2 \cap \bar{z}\theta\overline{H^2}$. Take a function $G \in H^2$ such that $F = \bar{z}\theta\bar{G}$. We have $\varphi = z\bar{\theta}GF \in z\bar{\theta}H^1$, as required.

Conversely, consider a nonnegative function $\varphi \in z\bar{\theta}H^1$. Since θ is unimodular on the unit circle $\mathbb{T}$, we have $\log\varphi \in L^1$. Let F be the outer function in H^2 with modulus $\sqrt{\varphi}$ on $\mathbb{T}$. We have $\bar{z}\theta|F|^2 \in H^1$. Hence, $\bar{z}\theta|F|^2 = IF^2$ for an inner function I. Thus, the function $F = \bar{z}\theta\overline{IF}$ belongs to the subspace $\bar{z}\theta\overline{H^2}$. It follows that $F \in K^2_\theta$, which completes the proof. □

Proof of Theorem 2. Let g be a function in the subspace $\bar{\theta}H^p$, where $1 \leqslant p < \infty$. Denote by p' the conjugate exponent to p. There exist functions $p_g \in H^p$, $h_g \in zH^{p'}$ satisfying

$$\|g - p_g\|_{L^p} = \mathrm{dist}_{L^p}(g, H^p) = \int_{\mathbb{T}} (g - p_g)h_g\, dm, \quad \|h_g\|_{L^{p'}} = 1, \tag{2}$$

where m denotes the normalized Lebesgue measure on $\mathbb{T}$. This well-known fact was first established in [11]; its modern proof can be found, e.g., in Section IV of [3]. Denote by f the function $g - p_g \in \bar{\theta}H^p$ and set $c = \|f\|_{L^p} = \mathrm{dist}_{L^p}(g, H^p)$. It follows from (2) that we have equality in the Hölder inequality $\|fh_g\|_{L^1} \leqslant \|f\|_{L^p}\|h_g\|_{L^{p'}}$. Therefore, $fh_g = c^{1-p}\cdot|f|^p$.

The function fh_g belongs to the subspace $z\bar{\theta}H^1$. Hence, the function $|f|^p$ belongs to $z\bar{\theta}H^1$ as well, and we see from Lemma 2.1 that $|f|^p = c^p|F|^2$ for an outer function $F \in K^2_\theta$ of unit norm. We may assume that $F(0) > 0$. The function θf lies in H^p and has modulus $c|F|^{2/p}$. It follows that $\theta f = cIF^{2/p}$ for an inner function I. Let us prove that $IF \in K^2_\theta$. By the construction, we have

$$c^p|F|^2 = |f|^p = c^{p-1}fh_g = c^p\cdot\bar{\theta}IF^{2/p}h_g. \tag{3}$$

Hence, the function $h_g \in zH^{p'}$ has the form $h_g = zJF^{2/p'}$ where J is an inner function. From (3) we get the formula $z\bar{\theta}IJF = \bar{F}$. This yields the fact that $IF \in \bar{z}\theta\overline{H^2}$. Thus, the inclusion $IF \in K^2_\theta$ is proved. By the construction, $f = c\cdot\bar{\theta}IF^{2/p}$.

Now prove that functions I, F in the statement of the theorem are determined uniquely. For $1 \leqslant p < \infty$, best H^p-approximation p_g of the function g is unique; see [11] or Section IV in [3]. Hence, the function $c\cdot IF^{2/p} = \theta(g - p_g)$ is determined uniquely. It remains to use uniqueness in the inner–outer factorization for functions in H^p. □

Remark 2.1. In the case $p = \infty$, Theorem 2 holds provided the dual extremal function $h_g \in zH^1$ in formula (2) exists. Indeed, under this assumption best H^∞-approximation p_g is unique and we get from (2) that $fh_g = c|h_g|$, where $f = g - p_g$ and $c = \|f\|_\infty = \mathrm{dist}_{L^\infty}(f, H^\infty)$. As above, there exists an outer function $F \in K^2_\theta$ such that $|h_g| = |F|^2$. Hence, $fh_g = c|F|^2$ and we have $z\bar{\theta} I F^2 = c|F|^2$ for some inner function I. It follows that $IF \in K^2_\theta$ and $f = c\bar{\theta} I$, as required.

It can be shown that the dual extremal function h_g exists for every continuous function g on the unit circle $\mathbb{T}$; see [4] or Section IV in [3]. In particular, it exists for every rational function with poles in the open unit disk. This will allow us to prove Theorem 1 in the case $p = \infty$; see details in the next section.

Remark 2.2. As we have seen in the proof of Theorem 2, the dual extremal function h_g to the function g is given by the formula $h_g = zJF^{2/p'}$, where J is the inner function such that $IJF = \bar{z}\theta\bar{F}$. It can be shown that every inner function U for which $UF \in K^2_\theta$ is a divisor of the function IJ; see Theorem 2 in [2].

3. Proof of Theorem 1

Let us first prove the classical result by F. Riesz on best analytic approximation in L^1 of trigonometric polynomials. By a trigonometric (correspondingly, analytic) polynomial of degree n we mean a linear combination of harmonics z^k, $|k| \leqslant n$ (correspondingly, $0 \leqslant k \leqslant n$). Every trigonometric polynomial can be regarded as a rational function with multiple pole at the origin. Hence, the result below can be readily obtained from Theorem 1. However, we would like to give a separate proof as an example of using Theorem 2.

Proposition 3.1. *Let g be a trigonometric polynomial of degree $n \geqslant 1$ and let p_g be its best H^1-approximation. Then p_g is an analytic polynomial of degree at most n. Moreover, the function $g - p_g$ has the form*

$$g - p_g = \text{const} \cdot \bar{z}^n \prod_1^K (1 - \bar{\lambda}_k z)(z - \lambda_k) \prod_1^M (1 - \bar{\mu}_m z)^2, \tag{4}$$

where $|\lambda_k| < 1$, $|\mu_m| \leqslant 1$, and $K + M \leqslant n - 1$.

Proof. Consider the inner function $\theta_n = z^n$. By the assumption, $g \in \bar{\theta}_n H^1 \cap \theta_n \overline{H^1}$. The coinvariant subspace $K^2_{\theta_n}$ consists of analytic polynomials of degree at most $n - 1$. It follows from Theorem 2 that $g - p_g = \bar{z}^n I F^2$, where F is an analytic polynomial of degree at most $n - 1$ and without zeros in the open unit disk; I is a finite Blaschke product such that IF is an analytic polynomial of degree at most $n - 1$. Denote by λ_k the zeros of I and by $1/\bar{\mu}_m$ those zeros of F that are not poles of I, taking into account multiplicities. It is now evident that the function $g - p_g$ is of form (4). Since g and the right side in (4) are trigonometric polynomials of degree at most n, the function p_g is an analytic polynomial of degree at most n. □

Proof of Theorem 1. Let $1 \leqslant p \leqslant \infty$, and let g be a rational function with n poles β_i in the open unit disk, each counted according to multiplicity. Then $g = h/B$, where $h \in H^p$ and B is the Blaschke product with zeros β_i,

$$B = \prod_{i=1}^n \frac{z - \beta_i}{1 - \bar{\beta}_i z}.$$

On the unit circle $\mathbb{T}$ we have $g = \bar{B}h$. Let p_g denote best H^p-approximation of g. By Theorem 2 (see also Remark 2.1 for the case $p = \infty$), the function $g - p_g$ can be uniquely represented in the form $g - p_g = c\bar{B} I F^{2/p}$, where F is an outer function in K_B^2 and I is an inner function such that $IF \in K_B^2$.

It follows from the definition of K_B^2 that every function $f \in K_B^2$ has the form P_f / Q_B, where $Q_B = \prod_{i=1}^n (1 - \bar{\beta}_i z)$ and P_f is an analytic polynomial of degree at most $n - 1$. Since the function F is outer, the polynomial P_F has no zeros in the open unit disk. Let us write it in the form $P_F = c_1 \cdot \prod_1^{n-1} (1 - \bar{\alpha}_i z)$, where c_1 is a constant and $|\alpha_i| \leqslant 1$ (if $\deg P_F < n - 1$, we let some of α_i's equal to zero). By the construction, $IF \in K_B^2$. Hence, we have $I = \prod' \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}$, where the product $\prod'$ is extended over all, some, or none of the α_i with $|\alpha_i| < 1$. This yields formula (1). The theorem is proved. □

Remark 3.1. The dual extremal function h_g to the function g has the form

$$h_g = c_2 \cdot z \prod\nolimits'' \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \prod_1^{n-1} (1 - \bar{\alpha}_i z)^{2/p'} \prod_1^{n} (1 - \bar{\beta}_i z)^{-2/p'},$$

where $\prod''$ is complementary to $\prod'$ with respect to the α_i with $|\alpha_i| < 1$ and c_2 is a constant. Indeed, this follows from Remark 2.2.

4. Proof of Theorem 3

A bounded analytic function θ in the upper half-plane $\mathbb{C}_+$ of the complex plane $\mathbb{C}$ is called inner if $|\theta| = 1$ almost everywhere on the real line $\mathbb{R}$ in the sense of angular boundary values. Coinvariant subspaces of the Hardy space $\mathcal{H}^p$ in $\mathbb{C}_+$ have the form $\mathcal{K}_\theta^p = \mathcal{H}^p \cap \theta \overline{\mathcal{H}^p}$. Theorem 2 holds for functions g in $\bar{\theta}\mathcal{H}^p$, as can be easily seen from its proof. We will deduce Theorem 3 from the following more general result.

Proposition 4.1. *Let θ be an inner function in $\mathbb{C}_+$ and let $g \in \bar{\theta}\mathcal{H}^1 \cap \theta\overline{\mathcal{H}^1}$. Then we have $p_g \in \mathcal{K}_\theta^1$ for best $\mathcal{H}^1$-approximation p_g of g.*

Proof. By Theorem 2, we have $g - p_g = \bar{\theta} I F^2$, where F, IF are functions in $\mathcal{K}_\theta^2$. Hence, the function $g - p_g$ belongs to the subspace

$$\bar{\theta} \cdot (\mathcal{H}^2 \cap \theta\overline{\mathcal{H}^2}) \cdot (\mathcal{H}^2 \cap \theta\overline{\mathcal{H}^2}) \subset \bar{\theta} \cdot (\mathcal{H}^1 \cap \theta^2\overline{\mathcal{H}^1}) \subset \bar{\theta}\mathcal{H}^1 \cap \theta\overline{\mathcal{H}^1}.$$

It follows that the function p_g lies in the subspace $\mathcal{K}_\theta^1 = \mathcal{H}^1 \cap \theta\overline{\mathcal{H}^1}$. □

Proof of Theorem 3. Consider the inner function $S^a : z \mapsto e^{iaz}$ in the upper half-plane $\mathbb{C}_+$. A function f in $L^1(\mathbb{R})$ belongs to the Hardy space $\mathcal{H}^1$ if and only if $\operatorname{supp} \hat{f} \subset [0, +\infty)$. It follows that every function $g \in L^1(\mathbb{R})$ with $\operatorname{supp} \hat{g} \subset [-a, a]$ belongs to the subspace $\overline{S^a}\mathcal{H}^1 \cap S^a\overline{\mathcal{H}^1}$. By Proposition 4.1, we have $p_g \in \mathcal{H}^1 \cap S^a\overline{\mathcal{H}^1}$. Hence, $\operatorname{supp} \hat{g} \subset [0, a]$ and the result follows. □

5. Interpolation problems related to best analytic approximation

The problem of best H^p-approximation for functions in $\bar{\theta}H^p$ can be rewritten in the following form: given a function $g \in H^p$, find a function $h \in H^p$ such that the norm $\|g - \theta h\|_{L^p}$ is minimal. This is the problem of *constrained interpolation in H^p* with respect to the inner function θ. An account of results related to constrained interpolation in H^∞ is available in Chapter 3 of [8]. We will consider the same problem in H^p, $1 \leqslant p \leqslant \infty$. Our observations are in line with [6], where the problem of best analytic approximation in L^p for rational functions is reduced to a problem of interpolation. For θ is an inner function, define the class $E_{\theta,p}$ by the formula

$$E_{\theta,p} = \{cIF^{2/p} : \ c \in \mathbb{C}, \ I \text{ is inner}, \ F \text{ is outer}, \ IF \in K^2_\theta\}. \tag{5}$$

If p is finite, there is no need in the constant c in (5). We say that a function $f_2 \in H^p$ interpolates a function $f_1 \in H^p$ with respect to the inner function θ if $f_1 - f_2 \in \theta H^p$. For example, f_1 interpolates f_2 with respect to z^n if and only if $f_1^{(k)}(0) = f_2^{(k)}(0)$ for all integers $0 \leqslant k \leqslant n-1$. Another example: for θ is a Blaschke product with simple zeros Λ, f_1 interpolates f_2 with respect to θ if and only if $f_1(\lambda) = f_2(\lambda)$ for all $\lambda \in \Lambda$.

The main result of this section is the following.

Proposition 5.1. *Let $1 \leqslant p < +\infty$ and let θ be an inner function. Each function $f_1 \in H^p$ can be interpolated by a unique function $f_2 \in E_{\theta,p}$ with respect to θ. Moreover, we have $\|f_2\|_{L^p} = \mathrm{dist}_{H^p}(f_1, \theta H^p)$.*

Proof. Take a function $f_1 \in H^p$ and set $g = \bar{\theta} f_1$. Let p_g denote best H^p-approximation of g. By Theorem 2, we have $g - p_g = c \cdot \bar{\theta} I F^{2/p}$, where the function $f_2 = c \cdot IF^{2/p}$ belongs to the class $E_{\theta,p}$. Note that $f_1 - f_2 \in \theta H^p$. Hence, the function f_2 interpolates f_1 with respect to the inner function θ. By the construction, we have $\|f_2\|_{L^p} = \mathrm{dist}_{H^p}(f_1, \theta H^p)$.

Let us now prove that the interpolating function f_2 is unique. Suppose that there is an another function $f_2^* = c^* I^* F^{*2/p}$ in $E_{\theta,p}$ that interpolates f_1 with respect to θ. We may assume that functions F, F^* are of unit norm in K^2_θ and have positive values at the origin. Let also $c > 0$, $c^* > 0$. Consider the inner function J such that $IJF = \bar{z}\theta\bar{F}$. Since $cIF^{2/p} - c^* I^* F^{*2/p}$ lies in θH^p, we have

$$c = \int_{\mathbb{T}} c\bar{\theta} I F^{2/p} \cdot zJF^{2/p'}\, dm = \int_{\mathbb{T}} c^*\bar{\theta} I^* F^{*2/p} \cdot zJF^{2/p'}\, dm \leqslant c^*. \tag{6}$$

Symmetric argument tells us that $c_* \leqslant c$. Hence, we have equality in (6). It follows that outer functions F, F^* have the same modulus on $\mathbb{T}$. Since $F(0) > 0$ and $F^*(0) > 0$, we have $F = F^*$. Again by equality in (6), inner functions I and I^* have the same argument on $\mathbb{T}$. Hence, $I = I^*$ and $f_2 = f_2^*$. $\square$

A function $g \in L^p$ is called H^p-badly approximable if the zero function is the best analytic approximation of g in L^p. Theorem 2 and Proposition 5.1 allow us to describe all H^p-badly approximable functions in $\bar{\theta} H^p$, where θ is an inner function and $1 \leqslant p < \infty$.

Proposition 5.2. *Let $1 \leqslant p < \infty$. A function $g \in \bar{\theta} H^p$ is H^p-badly approximable if and only if $\theta g \in E_{\theta,p}$.*

Proof. By Theorem 2, we have $\theta g \in E_{\theta,p}$ for every H^p-badly approximable function $g \in \bar{\theta} H^p$. Conversely, take a function $g \in \bar{\theta} E_{\theta,p}$ and consider its best H^p-approximation p_g. Set $f_1 = \theta g$. The function $f_2 = f_1 - \theta p_g$ interpolates f_1 with respect to the inner function θ. By Theorem 2, we have $f_2 \in E_{\theta,p}$. Hence, two functions $f_1, f_2 \in E_{\theta,p}$ interpolate the function f_1 with respect to θ. It follows from Proposition 5.1 that $f_1 = f_2$ and so $p_g = 0$. $\square$

We conclude this section with some examples.

Example 1. The class $E_{z^n,1}$ consists of polynomials of the form

$$\text{const} \cdot \prod_1^K (1 - \bar{\lambda}_k z)(z - \lambda_k) \prod_1^M (1 - \bar{\mu}_m z)^2, \tag{7}$$

where $|\lambda_k| < 1$, $|\mu_m| \leqslant 1$, and $K + M \leqslant n - 1$. As to the author knowledge, the problem of *constructive* interpolation by polynomials of form (7) with respect to z^n is open until now. A detailed discussion of this problem can be found in Section 5 of [6].

Example 2. Let us compute the upper bound of quantities $|f(0) + f'(0)|$ over all $f \in H^1$ of unit norm. By duality and Proposition 5.1, this problem reduces to interpolation of $1 + z$ with respect to z^2 by a polynomial in $E_{z^2,1}$. It is easy to see that the polynomial $\frac{1}{4}(2 + z)^2$ solves this problem. Hence, we have

$$\sup \left\{ |f(0) + f'(0)| : \ f \in H^1, \|f\|_{H^1} = 1 \right\} = \frac{1}{4} \int_{\mathbb{T}} |z + 2|^2 \, dm = \frac{5}{4}.$$

The general problem of calculation of $\sup \left\{ |a_0 f(0) + a_1 f'(0)| : \ f \in H^1, \|f\|_{H^1} = 1 \right\}$ is solved in Section 5 of [6].

Example 3. The problem of constrained interpolation in H^∞ with respect to the inner function z^n is the classical Schur problem. It can be stated as follows: given n numbers $a_0, \dots, a_{n-1}$, find a function $f \in H^\infty$ of minimal norm such that $f^{(k)}(0)/k! = a_k$ for all k. This problem can be solved constructive; see [13] or Paragraph 3.4.2.(ii).(b) in [8]. The solution has the form

$$\text{const} \cdot \prod_{k=1}^K \frac{z - \lambda_k}{1 - \bar{\lambda}_k z}, \quad |\lambda_k| < 1, \ K \leqslant n - 1. \tag{8}$$

Clearly, class $E_{z^n,\infty}$ consists of functions of form (8). This agrees well with Proposition 5.1 (its proof works in the case $p = \infty$ if there exists the dual extremal function to the function $\bar{\theta} f_1$). Similarly, the classical Nevanlinna–Pick problem reduces to interpolation by functions of form (8) with respect to a finite Blaschke product. For more information, see Chapter 3 in [8].

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