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A best constant for bivariate Bernstein and Szász-Mirakyan operators[☆]

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Abstract

For classical Bernstein operators over the unit square, we obtain the best uniform constant in preservation of the usual I_∞ -modulus of continuity, at the same time we show that it coincides with the corresponding best uniform constant for bivariate Szász operators. The result validates a conjecture stated in a previous paper. The proof involves both probabilistic and analytic arguments, as well as numerical computation of some specific values.

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1. Introduction and main result

For $n, k = 1, 2, \dots$ let $B_n^{\langle k \rangle} := B_n \otimes \dots \otimes B_n$ be the tensor product of k copies of the classical Bernstein operator over the interval $[0, 1]$ given by

$$B_n f(x) := \sum_{k=0}^n f(k/n) \binom{n}{k} x^k (1-x)^{n-k},$$

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and denote by $C_n^{\langle k \rangle}(\delta)$ the best constant in preservation of the usual modulus of continuity for the l_∞ -norm in \mathbb{R}^k , that is

$$C_n^{\langle k \rangle}(\delta) := \sup_{f \in \mathcal{F}_k} \frac{\omega(B_n^{\langle k \rangle} f; \delta)}{\omega(f; \delta)}, \quad 0 < \delta \leq 1,$$

where \mathcal{F}_k is the set of all real non-constant bounded functions on $[0, 1]^k$, and $\omega(f; \cdot)$ stands for the above-mentioned modulus of continuity, i.e.,

$$\omega(f; \delta) := \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in [0, 1]^k, \|\mathbf{x} - \mathbf{y}\|_\infty \leq \delta\}, \quad \delta \geq 0.$$

On the other hand, for $t > 0$ and $k = 1, 2, \dots$, let $S_t^{\langle k \rangle} := S_t \otimes \dots \otimes S_t$ be the tensor product of k copies of the Szász–Mirakyan operator S_t over the interval $[0, \infty)$ given by

$$S_t g(x) := \sum_{k=0}^{\infty} g(k/t) e^{-tx} \frac{(tx)^k}{k!},$$

and denote by $D_t^{\langle k \rangle}(\delta)$ the corresponding best constant for such an operator, i.e.,

$$D_t^{\langle k \rangle}(\delta) := \sup_{g \in \mathcal{G}_k} \frac{\omega(S_t^{\langle k \rangle} g; \delta)}{\omega(g; \delta)}, \quad \delta > 0,$$

where \mathcal{G}_k is the set of all real non-constant functions g on $[0, \infty)^k$ such that $\omega(g; 1) < \infty$ (or, equivalently, $\omega(g; \delta) < \infty$ for all $\delta > 0$).

Many facts about $C_n^{\langle k \rangle}(\delta)$ and $D_t^{\langle k \rangle}(\delta)$ are known in the literature. In [4], the reader can find explicit probabilistic formulae for such constants (to be used in the next section) which generalize the one-dimensional formulae given in [1]. From the formula for $D_t^{\langle k \rangle}(\delta)$, it follows that

$$D_t^{\langle k \rangle}(\delta) = D_1^{\langle k \rangle}(t\delta), \quad t, \delta > 0,$$

implying that the best uniform constant $\sup_{\delta > 0} D_t^{\langle k \rangle}(\delta)$ only depends upon the dimension k . The facts

$$\sup_{0 < \delta \leq 1} C_n^{\langle 1 \rangle}(\delta) = 2 \quad (n \geq 1) \quad \text{and} \quad \sup_{\delta > 0} D_t^{\langle 1 \rangle}(\delta) = 2 - e^{-1} \quad (t > 0)$$

were respectively established in [2,6]. It was shown in [5] that, for $k \geq 3$,

$$\sup_{0 < \delta \leq 1} C_n^{\langle k \rangle}(\delta) = k = \sup_{\delta > 0} D_1^{\langle k \rangle}(\delta), \quad n \geq 1,$$

that is, the best uniform constants coincide with the dimension, and (as in the one-dimensional case) the one for $B_n^{\langle k \rangle}$ does not depend upon the parameter n , while, in the case $k = 2$, the value of $\sup_{0 < \delta \leq 1} C_n^{\langle 2 \rangle}(\delta)$ depends upon n , and both the values of $\sup_{n \geq 1} \sup_{0 < \delta \leq 1} C_n^{\langle 2 \rangle}(\delta)$ and $\sup_{\delta > 0} D_1^{\langle 2 \rangle}(\delta)$ lie in the interval $[2, 5/2]$. As for the exact value of these quantities, on the basis of certain computational evidence, it was conjectured the following.

Theorem. *We have*

$$\begin{aligned} \sup_{\delta > 0} D_1^{\langle 2 \rangle}(\delta) &= \sup_{n \geq 1} \sup_{0 < \delta \leq 1} C_n^{\langle 2 \rangle}(\delta) \\ &= 1 - e^{-2} + \sum_{j=0}^{\infty} \left[1 - e^{-2} \left(\sum_{i=0}^j \frac{1}{i!} \right)^2 \right] = 2.3884423 \dots \end{aligned} \quad (1)$$

In the present paper, we give a theoretical proof of this result.

2. Auxiliary results

In this section, we introduce some notations, restate the preceding theorem in a more convenient form for our purposes, and collect some necessary auxiliary results.

We set, for $n \geq 1$ and $0 < x \leq n$,

$$C_n(x) := C_n^{\langle 2 \rangle}(x/n).$$

We recall that, according to the formulae in [3], we have

$$C_n(x) = E \left\lceil \frac{\eta_n(x)}{x} \right\rceil,$$

where E denotes mathematical expectation, $\lceil \cdot \rceil$ is the ceiling function, and $\eta_n(x) := \eta_n'(x) \vee \eta_n''(x)$ is the maximum of two independent integer-valued random variables $\eta_n'(x)$ and $\eta_n''(x)$ having the same binomial distribution given by

$$P(\eta_n'(x) = k) = p_{n,k}(x) := \begin{cases} \binom{n}{k} \left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n-k} & k = 0, 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

We also denote by

$$C_n^*(x) := P(\eta_n(x) > 0) + \frac{E\eta_n(x)}{x} = 1 - p_{n,0}^2(x) + \frac{E\eta_n(x)}{x},$$

and recall that

$$\begin{aligned} E\eta_n(x) &= \sum_{k=1}^{\infty} k P(\eta_n(x) = k) = \sum_{k=1}^{\infty} P(\eta_n(x) \geq k) \\ &= \sum_{k=0}^{n-1} \left[1 - \left(\sum_{j=0}^k p_{n,j}(x) \right)^2 \right]. \end{aligned}$$

Similarly, we set, for $x > 0$,

$$D(x) := D_1^{\langle 2 \rangle}(x) = E \left\lceil \frac{\xi(x)}{x} \right\rceil,$$

and

$$\begin{aligned} D^*(x) &:= P(\zeta(x) > 0) + \frac{E\zeta(x)}{x} \\ &= 1 - \pi_0^2(x) + \frac{1}{x} \sum_{k=0}^{\infty} \left[1 - \left(\sum_{j=0}^k \pi_j(x) \right)^2 \right], \end{aligned}$$

where $\zeta(x) := \zeta'(x) \vee \zeta''(x)$ is the maximum of two independent random variables $\zeta'(x)$ and $\zeta''(x)$ having the same Poisson distribution of parameter x , i.e.,

$$P(\zeta'(x) = k) = \pi_k(x) := e^{-x} \frac{x^k}{k!}, \quad k = 0, 1, 2, \dots$$

With the preceding notations, it is clear that (1) can be rewritten as follows:

$$\sup_{n \geq 1} \sup_{0 < x \leq n} C_n(x) = \sup_{x > 0} D(x) = D^*(1). \quad (2)$$

The point is that the functions $C_n(\cdot)$ and $D(\cdot)$ are quite irregular and hardly tractable, but $C_n^*(\cdot)$ and $D^*(\cdot)$ are fairly smooth. The following lemmas collect the necessary facts for the proof of (2) given in the next section.

Lemma 1. *We have*

(a)

$$C_n(x) \leq C_n^*(x), \quad n \geq 1, \quad 0 < x \leq n.$$

(b)

$$D(x) \leq D^*(x), \quad x > 0.$$

Proof. Both inequalities are nothing but particular cases of the inequality

$$E[U] \leq P(U > 0) + EU,$$

which holds true for every nonnegative random variable U . \square

Lemma 2. *We have*

$$C_n^*(x) \leq D^*(x), \quad n \geq 1, \quad 0 < x \leq n.$$

Proof. Fix $n \geq 1$ and $x \in (0, n]$, and denote by

$$a_{n,k} := P(\eta_n'(x) \leq k) = \sum_{j=0}^k p_{n,j}(x),$$

$$b_k := P(\zeta'(x) \leq k) = \sum_{j=0}^k \pi_j(x),$$

for $k = 0, 1, 2, \dots$. From some results of Anderson and Samuels [3, Corollary 2.1], there is an integer $r \geq 1$ such that

$$\begin{aligned} a_{n,k} &\leq b_k, & 0 \leq k \leq r-1, \\ a_{n,k} &\geq b_k, & k \geq r. \end{aligned} \quad (3)$$

We have

$$\begin{aligned} E\xi(x) - E\eta_n(x) &= \sum_{k=0}^{\infty} (1 - b_k^2) - \sum_{k=0}^{\infty} (1 - a_{n,k}^2) \\ &= \sum_{k=0}^{\infty} (a_{n,k}^2 - b_k^2) \\ &= a_{n,0}^2 - b_0^2 + \sum_{k=1}^{\infty} (a_{n,k} + b_k)(a_{n,k} - b_k) \\ &\geq a_{n,0}^2 - b_0^2 + (a_{n,r} + b_r) \sum_{k=1}^{\infty} (a_{n,k} - b_k), \end{aligned}$$

the inequality by (3) and the fact that the sequence $\{a_{n,k} + b_k : k \geq 0\}$ is nondecreasing. Since

$$\begin{aligned} \sum_{k=1}^{\infty} (a_{n,k} - b_k) &= \left[\sum_{k=0}^{\infty} (1 - b_k) - \sum_{k=0}^{\infty} (1 - a_{n,k}) \right] - (a_{n,0} - b_0) \\ &= [E\xi'(x) - E\eta_n'(x)] - (a_{n,0} - b_0) \\ &= -(a_{n,0} - b_0) \end{aligned}$$

(the last equality because $E\xi'(x) = x = E\eta_n'(x)$), and

$$a_{n,r} + b_r \geq a_{n,1} + b_1 \geq (1+x)(a_{n,0} + b_0),$$

we finally obtain that

$$E\xi(x) - E\eta_n(x) \geq x(b_0^2 - a_{n,0}^2),$$

which is another way to express the conclusion. \square

Lemma 3. *We have:*

(a)

$$x \frac{d}{dx} E\eta_n(x) = E\eta_n(x) - \sum_{k=1}^n k p_{n,k}^2(x), \quad n \geq 1, \quad x \in (0, n].$$

(b)

$$x \frac{d}{dx} E\xi(x) = E\xi(x) - \sum_{k=1}^{\infty} k \pi_k^2(x), \quad x > 0.$$

Proof. Let $n \geq 1$ be fixed. It is immediate that we have, for $x \in (0, n]$ and $k \geq 0$,

$$x \frac{d}{dx} p_{n,k}(x) = k p_{n,k}(x) - (k+1) p_{n,k+1}(x),$$

implying that

$$x \frac{d}{dx} \sum_{j=0}^k p_{n,j}(x) = -(k+1)p_{n,k+1}(x),$$

and, therefore,

$$\begin{aligned} x \frac{d}{dx} \left(\sum_{j=0}^k p_{n,j}(x) \right)^2 &= -(k+1)2P(\eta'_n(x) \leq k)P(\eta''_n(x) = k+1) \\ &= -(k+1)[P(\eta'_n(x) \leq k)P(\eta''_n(x) = k+1) \\ &\quad + P(\eta''_n(x) \leq k)P(\eta'_n(x) = k+1)] \\ &= -(k+1)[P(\eta_n(x) = k+1) - p_{n,k+1}^2(x)]. \end{aligned}$$

We conclude that

$$\begin{aligned} x \frac{d}{dx} E\eta_n(x) &= x \frac{d}{dx} \sum_{k=0}^{n-1} \left[1 - \left(\sum_{j=0}^k p_{n,j}(x) \right)^2 \right] \\ &= \sum_{k=1}^{\infty} kP(\eta_n(x) = k) - \sum_{k=1}^n kp_{n,k}^2(x) \\ &= E\eta_n(x) - \sum_{k=1}^n kp_{n,k}^2(x), \end{aligned}$$

showing part (a). The proof of (b) is achieved in the same way, by starting from the fact that we have

$$x \frac{d}{dx} \pi_k(x) = k\pi_k(x) - (k+1)\pi_{k+1}(x),$$

for all $x > 0$ and $k \geq 0$. \square

Lemma 4. (a) For each $n \geq 1$, the function $x^{-1}E\eta_n(x)$ is decreasing in $(0, n]$.

(b) The function $x^{-1}E\xi(x)$ is decreasing in $(0, \infty)$.

(c) The function $D^*(\cdot)$ is increasing in $(0, 1]$ and decreasing in $[3/2, \infty)$.

Proof. From the preceding lemma, we have

$$\frac{d}{dx} \frac{E\eta_n(x)}{x} = -\frac{1}{x^2} \sum_{k=1}^n kp_{n,k}^2(x) < 0, \quad n \geq 1, \quad x \in (0, n],$$

and

$$\frac{d}{dx} \frac{E\xi(x)}{x} = -\frac{1}{x^2} \sum_{k=1}^{\infty} k\pi_k^2(x) = -\sum_{k=0}^{\infty} \frac{1}{k+1} \pi_k^2(x) < 0, \quad x > 0, \quad (4)$$

showing parts (a) and (b). From (4), we also have, for $x > 0$,

$$\begin{aligned}\frac{d}{dx}D^*(x) &= \frac{d}{dx}\left[1 - \pi_0^2(x) + \frac{E\xi(x)}{x}\right] \\ &= 2\pi_0^2(x) - \sum_{k=0}^{\infty} \frac{1}{k+1} \pi_k^2(x) \\ &= \pi_0^2(x) - \sum_{k=1}^{\infty} \frac{1}{k+1} \pi_k^2(x) \\ &= e^{-2x} \left[1 - \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{x^k}{k!}\right)^2\right].\end{aligned}$$

If $x \in (0, 1]$, we have

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{x^k}{k!}\right)^2 \leq \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{1}{k!}\right)^2 < \frac{1}{2}(e-1) < 1,$$

implying that $\frac{d}{dx}D^*(x) > 0$, while, for $x \geq 3/2$,

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{x^k}{k!}\right)^2 \geq \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{(3/2)^k}{k!}\right)^2 > \frac{1}{2} \left(\frac{3}{2}\right)^2 > 1,$$

which implies that $\frac{d}{dx}D^*(x) < 0$. This shows part (c), and completes the proof of the lemma. \square

Lemma 5. *We have:*

- (a) $\lim_{x \uparrow 1} C_n(x) = C_n^*(1)$, $n \geq 1$.
- (b) $\lim_{n \rightarrow \infty} C_n^*(1) = D^*(1)$.
- (c) $\lim_{x \uparrow 1} D(x) = D^*(1)$.
- (d) $E\xi(1) = D^*(1) - 1 + e^{-2} = 1.52377761 \dots$.
- (e) $D^*(1.55) = 2.38835554 \dots$.

Proof. We have, for $n \geq 1$,

$$\begin{aligned}\lim_{x \uparrow 1} C_n(x) &= \lim_{x \uparrow 1} \sum_{k=1}^n \left[\frac{k}{x}\right] P(\eta_n(x) = k) = \sum_{k=1}^n (k+1) P(\eta_n(1) = k) \\ &= P(\eta_n(1) > 0) + E\eta_n(1) = C_n^*(1),\end{aligned}$$

showing part (a). Part (c) is shown in the same way, and we omit the details. Part (b) readily follows from the fact that

$$\lim_{n \rightarrow \infty} p_{n,k}(1) = \pi_k(1), \quad k = 0, 1, 2, \dots$$

(i.e., the Poisson approximation to the binomial distribution). Finally, parts (d) and (e) merely are numerical computations. \square

3. Proof of the theorem

Recall the numerical value of $D^*(1)$ appearing in (1). We have, by Lemmas 1, 2, 4(c) and 5(e),

$$D(x) \leq D^*(x) \leq D^*(1), \quad 0 < x \leq 1,$$

$$C_n(x) \leq C_n^*(x) \leq D^*(x) \leq D^*(1), \quad 0 < x \leq 1, \quad n \geq 1,$$

$$D(x) \leq D^*(x) \leq D^*(1.55) < D^*(1), \quad x \geq 1.55,$$

$$C_n(x) \leq C_n^*(x) \leq D^*(x) \leq D^*(1.55) < D^*(1), \quad n \geq 2, \quad 1.55 \leq x \leq n.$$

Let $1 < x < 1.55$. Using the fact that the random variable $\xi(x)$ is integer-valued, and Lemmas 4(b) and 5(d), we obtain

$$\begin{aligned} D(x) &\leq E[\xi(x)] = E\xi(x) \leq xE\xi(1) \\ &\leq 1.55E\xi(1) = 2.361855\dots < D^*(1), \end{aligned}$$

and, analogously, by Lemmas 2 and 4(a),

$$C_n(x) \leq E[\eta_n(x)] = E\eta_n(x) \leq xE\eta_n(1) \leq xE\xi(1) < D^*(1), \quad n \geq 2.$$

From all the above, we conclude that

$$\sup_{n \geq 1} \sup_{0 < x \leq n} C_n(x) \leq D^*(1) \quad \text{and} \quad \sup_{x > 0} D(x) \leq D^*(1). \quad (5)$$

Finally, we have, by Lemma 5(c)

$$\sup_{x > 0} D(x) \geq \lim_{x \uparrow 1} D(x) = D^*(1),$$

and, by Lemma 5(a,b),

$$\sup_{n \geq 1} \sup_{0 < x \leq n} C_n(x) \geq \lim_{n \rightarrow \infty} \lim_{x \uparrow 1} C_n(x) = D^*(1),$$

showing that the inequalities in (5) actually are equalities, and finishing the proof of (2).

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