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The Laguerre–Sobolev-type orthogonal polynomials

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Abstract

In this paper we study the asymptotic behaviour of polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Np^{(j)}(0)q^{(j)}(0),$$

where $N \in \mathbb{R}^+$ and $j \in \mathbb{N}$.

We will focus our attention on the outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler–Heine formula for the rescaled polynomials.

Keywords: Quasi-orthogonal polynomials; Laguerre polynomials; Relative asymptotics; Mehler–Heine formula; Zeros

1. Introduction

Let $(\mu_0, \mu_1, \ldots, \mu_j)$ be a vector of positive measures supported on the real line such that

$$\int_\Gamma |x|^n d\mu_k < \infty,$$

for $k = 0, 1, 2, \ldots, j$ and for every $n \in \mathbb{N}$.

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In the linear space $\mathbb{P}$ of polynomials with real coefficients we can define an inner product

$$\langle p, q\rangle = \sum_{k=0}^{j} \int_{\mathbb{R}} p^{(k)}(x) q^{(k)}(x) \mathrm{d}\mu_k, \tag{1}$$

$p, q \in \mathbb{P}$.

This inner product is known in the literature as a Sobolev inner product. The study of the sequences of monic polynomials orthogonal with respect to (1) has attracted the interest of many researchers during the last twenty years. One of the reasons is the fact that many properties of the standard polynomials ($j = 0$) are lost when (1) is considered. In particular, the existence of recurrence relations of low order only holds when the measures $\{\mu_k\}_{k=1}^{j}$ are discrete. Furthermore, the zeros can be complex or, if real, they can be located outside the convex hull of the union of the support of the measures $\{\mu_k\}_{k=0}^{j}$.

Most of the contributions deal with measures of bounded support (see [13] and [15] for a survey of the main results concerning the analytic properties of such polynomials).

If μ_0 has an unbounded support and $\{\mu_k\}_{k=1}^{j}$ are discrete measures very few results are known. From an algebraic point of view a first approach was made in [12] where, as an example, the case of $\mathrm{d}\mu_0 = \mathrm{e}^{-x^2}\mathrm{d}x$ is considered. When $j = 1$, for $\mathrm{d}\mu_0 = x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x + M_0\delta(x)$ and $\mathrm{d}\mu_1 = M_1\delta(x)$ in [8] a representation of the corresponding sequence of monic orthogonal polynomials in terms of Laguerre polynomials as well as its representation as hypergeometric series is found. Later on, in [2] the authors analyzed the outer relative asymptotics in terms of Laguerre polynomials $\{L_n^{\alpha}\}_{n\geq 0}$. Furthermore, a Mehler–Heine-type formula is obtained for such polynomials. As an application, some results concerning the behaviour of their zeros are obtained. For some more extra information see the survey [11].

For higher derivatives, i.e. $j > 1$, when $\mathrm{d}\mu_0 = x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x + M_0\delta(x), \mathrm{d}\mu_k = M_k\delta(x), k = 1, 2, \ldots, j$, in [7] an explicit expression for the Sobolev orthogonal polynomials in terms of classical Laguerre polynomials is given. Furthermore, their representation as a hypergeometric function and the holonomic second-order linear differential equation that such polynomials satisfy are obtained.

Unfortunately, no asymptotic results for such polynomials are known. More recently, in [1] the authors analyzed asymptotic properties when $\mathrm{d}\mu_0 = \mathrm{e}^{-x^2}\mathrm{d}x + M_0\delta(x), \mathrm{d}\mu_k = M_k\delta(x), k = 1, 2, 3$.

The structure of the paper is as follows. In Section 2 we present the basic background concerning classical Laguerre orthogonal polynomials. Section 3 deals with the connection formula between the monic polynomials $\{\mathcal{L}_n^{\alpha}\}_{n\geq 0}$ orthogonal with respect to the Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^{\infty} pq x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x + Np^{(j)}(0)q^{(j)}(0),$$

where $N \in \mathbb{R}^+$ and $j \in \mathbb{N}$, and the Laguerre polynomials $\left\{L_n^{\alpha+j+1}\right\}_{n\geq 0}$. In Section 4 we obtain the outer relative asymptotic of the polynomials $\{\mathcal{L}_n^{\alpha}\}_{n\geq 0}$ in terms of $\{L_n^{\alpha}\}_{n\geq 0}$ as well as a Mehler–Heine-type formula, i.e. the behaviour of $\frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}}$ on compact subsets of the complex plane, where $\widehat{\mathcal{L}}_n^{\alpha}(x) = \frac{(-1)^n}{n!}\mathcal{L}_n^{\alpha}(x)$.

This last result constitutes a partial answer to a question posed in [2] about the outer relative asymptotics and the existence of a Mehler–Heine-type formula for the corresponding sequence of Sobolev orthogonal polynomials.

2. Preliminaries

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be a linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\left\langle \mu, x^n \right\rangle = \mu_n, \quad n = 0, 1, 2 \ldots.$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is the n-th *moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if:

(i) The degree of P_n is n.
(ii) $\langle \mu, P_n(x)P_m(x) \rangle = 0, m \neq n$.
(iii) $\left\langle \mu, P_n^2(x) \right\rangle \neq 0, n = 0, 1, 2, \ldots.$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.* It is clear that for every sequence of orthogonal polynomials there exists a corresponding family of monic orthogonal polynomials. In the sequel we will work with monic polynomials.

The next theorem, whose proof appears in [5], gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 (*[5]*). *Let μ be a moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are non-singular.*

A moment functional such that there exists a corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* [5]. If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ, and $\mathrm{D}\mu$, the usual distributional derivative of μ, as follows:

$$\langle \phi\mu, p(x) \rangle = \langle \mu, \phi\,(x)\, p(x) \rangle\,, \qquad \langle D\mu, p(x) \rangle = -\left\langle \mu, p'(x) \right\rangle.$$

A sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is said to be *classical* if there exist polynomials ϕ and ψ, with $\deg \phi \leq 2$ and $\deg \psi = 1$, such that μ satisfies the Pearson differential equation

$$\mathrm{D}\,(\phi\mu) = \psi\mu.$$

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are extensively used in the literature taking into account their applications in mathematical physics. Indeed, one of their most popular applications is in the study of problems involving hypergeometric differential equations (see [3,6,9,16,18]).

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p, q \rangle_\alpha = \int_0^\infty pqx^\alpha \mathrm{e}^{-x}\mathrm{d}x, \quad \alpha > -1, p, q \in \mathbb{P}. \tag{2}$$

We need to summarize some properties of the monic Laguerre orthogonal polynomials that we will use in the sequel. The details of the proof can be found in [3,5,6,10,18].

Proposition 1. *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

(1) *For every $n \in \mathbb{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\,L_n^\alpha(x) + n\,(n+\alpha)\,L_{n-1}^\alpha(x) \tag{3}$$

with $L_0^\alpha(x) = 1$, $L_1^\alpha(x) = x - (\alpha+1)$.

(2) *For every $n \in \mathbb{N}$,*

$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n\,(n+\alpha)\,L_{n-1}^\alpha(x). \tag{4}$$

(3) *For every $n \in \mathbb{N}$, $L_n^\alpha(x)$ satisfies the differential equation*

$$xy'' + (\alpha+1-x)\,y' = -ny. \tag{5}$$

(4) *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}, \tag{6}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$, and $(a)_0 = 1$, is the Pochhammer symbol.

(5) *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{7}$$

(6) *For every $n \in \mathbb{N}$,*

$$\left\|L_n^\alpha\right\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{8}$$

(7) *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{9}$$

(8) *(Christoffel–Darboux formula) If*

$$K_n(x,y) = \sum_{j=0}^{n} \frac{L_j^\alpha(y)L_j^\alpha(x)}{\left\|L_j^\alpha\right\|_\alpha^2}$$

denotes the n-th kernel polynomial then, for every $n \in \mathbb{N}$,

$$K_n(x,y) = \frac{L_{n+1}^\alpha(x)L_n^\alpha(y) - L_{n+1}^\alpha(y)L_n^\alpha(x)}{x-y}\frac{1}{\left\|L_n^\alpha\right\|_\alpha^2}. \tag{10}$$

As a consequence, notice that

$$K_n(x,0) = \frac{L_n^\alpha(0)}{n!\Gamma(n+\alpha+1)}L_n^{\alpha+1}(x). \tag{11}$$

(9) *(The Mehler–Heine-type formula) Let J_α be the Bessel function of the first kind defined by*

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j(x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)}.$$

Then

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(x/(n+k))}{n^{\alpha}} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) \tag{12}$$

uniformly on compact subsets $\mathbb{C}$ *and, uniformly in* $k \in \mathbb{N} \cup \{0\}$. *Here* $\widehat{L}_n^{\alpha}(x) = \frac{(-1)^n}{n!} L_n^{\alpha}(x)$.

(10) *(Outer ratio asymptotics)*

$$\lim_{n\to\infty} \frac{\widehat{L}_{n+k}^{\alpha+t}(x)}{n^{t/2}\widehat{L}_{n+h}^{\alpha}(x)} = (-x)^{t/2}, \tag{13}$$

$k, h \in \mathbb{N}$, $t \in \mathbb{Z}$, *uniformly on compact subsets of* $\mathbb{C} \setminus [0, \infty)$ *(see* [2] *and* [18]*).*

We will use the following notation for the partial derivatives of $K_n(x, y)$:

$$\frac{\partial^{j+k}\left(K_n(x, y)\right)}{\partial^j x \partial^k y} = K_n^{(j,k)}(x, y).$$

If $p(x)$ is a polynomial with $\deg p \le n$, then we can write it as a linear combination of the Laguerre polynomials as follows:

$$p(x) = \sum_{k=0}^{n} \frac{\left\langle L_k^{\alpha}(x), p(x)\right\rangle_{\alpha}}{\left\| L_k^{\alpha}\right\|_{\alpha}^2} L_k^{\alpha}(x).$$

As a consequence,

$$p^{(j)}(y) = \sum_{k=0}^{n} \frac{\left\langle L_k^{\alpha}(x), p(x)\right\rangle_{\alpha}}{\left\| L_k^{\alpha}\right\|_{\alpha}^2} \left(L_k^{\alpha}\right)^{(j)}(y),$$

and, taking into account that

$$\begin{aligned}\left\langle K_n^{(0,j)}(x, y), p(x)\right\rangle_{\alpha} &= \left\langle \sum_{k=0}^{n} \frac{L_k^{\alpha}(x)\left(L_k^{\alpha}\right)^{(j)}(y)}{\left\| L_k^{\alpha}\right\|_{\alpha}^2}, p(x)\right\rangle_{\alpha} \\ &= \sum_{k=0}^{n} \frac{\left\langle L_k^{\alpha}(x), p(x)\right\rangle_{\alpha}}{\left\| L_k^{\alpha}\right\|_{\alpha}^2} \left(L_k^{\alpha}\right)^{(j)}(y),\end{aligned}$$

we get

$$\left\langle K_n^{(0,j)}(x, y), p(x)\right\rangle_{\alpha} = p^{(j)}(y). \tag{14}$$

Let $\{P_n\}_{n\ge 0}$ be a sequence of monic orthogonal polynomials. From the Christoffel Darboux formula (see [3,5,6,18]), we have

$$K_{n-1}(x, y) = \frac{1}{\|P_{n-1}\|^2} \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x - y}.$$

Calculating the j-th partial derivative with respect to y, we get

$$K_{n-1}^{(0,j)}(x, y) = \frac{1}{\|P_{n-1}\|^2} \left(P_n(x) \frac{\partial^j}{\partial y^j}\left(\frac{P_{n-1}(y)}{x - y}\right) - P_{n-1}(x) \frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x - y}\right)\right). \tag{15}$$

Using the Leibnitz rule

$$\frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x-y}\right)=\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}},$$

and replacing the last expression in (15), we obtain

$$\begin{aligned}K_{n-1}^{(0,j)}(x,y) &= \frac{1}{\|P_{n-1}\|^2}\left(P_n(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_{n-1}^{(k)}(y)}{(x-y)^{j-k+1}}-P_{n-1}(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}}\right)\\ &= \frac{j!}{\|P_{n-1}\|^2\,(x-y)^{j+1}}\\ &\quad\times\left(P_n(x)\sum_{k=0}^{j}\frac{1}{k!}P_{n-1}^{(k)}(y)(x-y)^k-P_{n-1}(x)\sum_{k=0}^{j}\frac{1}{k!}P_n^{(k)}(y)(x-y)^k\right).\end{aligned}$$

As a consequence,

$$K_{n-1}^{(0,j)}(x,0)=\frac{j!}{\|P_{n-1}\|^2\,x^{j+1}}\left(P_n(x)Q_j(x,0,P_{n-1})-P_{n-1}(x)Q_j(x,0,P_n)\right) \tag{16}$$

where $Q_j(x,0,P_{n-1})$ and $Q_j(x,0,P_n)$ denote the Taylor polynomials of degree j of the polynomials P_{n-1} and P_n around $x=0$, respectively.

Next, we will compute $K_{n-1}^{(0,j)}(x,0)$. Indeed

$$\begin{aligned}K_{n-1}^{(0,j)}(x,0) &= \frac{j!}{\|P_{n-1}\|^2\,x^{j+1}}\left[\left(P_n(0)+P_n'(0)x+\frac{P_n''(0)}{2!}x^2+\cdots+\frac{P_n^{(n)}(0)}{n!}x^n\right)\right.\\ &\quad\times\left(P_{n-1}(0)+P_{n-1}'(0)x+\frac{P_{n-1}''(0)}{2!}x^2+\cdots+\frac{P_{n-1}^{(j)}(0)}{j!}x^j\right)\\ &\quad-\left(P_{n-1}(0)+P_{n-1}'(0)x+\frac{P_{n-1}''(0)}{2!}x^2+\cdots+\frac{P_{n-1}^{(n-1)}(0)}{(n-1)!}x^{n-1}\right)\\ &\quad\left.\times\left(P_n(0)+P_n'(0)x+\frac{P_n''(0)}{2!}x^2+\cdots+\frac{P_n^{(j)}(0)}{j!}x^j\right)\right].\end{aligned} \tag{17}$$

If we make some computations in the last expression, the coefficients of the monomials of degree less than or equal to j inside the bracket are cancelled. Thus, when $x=0$, we have

$$K_{n-1}^{(0,j)}(0,0)=\frac{j!}{\|P_{n-1}\|^2}\left(P_{n-1}(0)\frac{P_n^{(j+1)}(0)}{(j+1)!}-P_n(0)\frac{P_{n-1}^{(j+1)}(0)}{(j+1)!}\right),$$

and

$$K_{n-1}^{(0,j)}(0,0)=\frac{1}{\|P_{n-1}\|^2\,(j+1)}\left(P_{n-1}(0)P_n^{(j+1)}(0)-P_n(0)P_{n-1}^{(j+1)}(0)\right). \tag{18}$$

In order to find $K_{n-1}^{(j,j)}(0,0)$, we just need to deduce in (17) the coefficient of x^{2j+1} inside the bracket which is

$$\begin{aligned}
&\left[\frac{P_{n-1}(0)}{0!}\frac{P_n^{(2j+1)}(0)}{(2j+1)!}+\frac{P'_{n-1}(0)}{1!}\frac{P_n^{(2j)}(0)}{(2j)!}+\cdots+\frac{P_{n-1}^{(j)}(0)}{j!}\frac{P_n^{(j+1)}(0)}{(j+1)!}\right]\\
&\quad-\left[\frac{P_n(0)}{0!}\frac{P_{n-1}^{(2j+1)}(0)}{(2j+1)!}+\frac{P'_n(0)}{1!}\frac{P_{n-1}^{(2j)}(0)}{(2j)!}+\cdots+\frac{P_n^{(j)}(0)}{j!}\frac{P_{n-1}^{(j+1)}(0)}{(j+1)!}\right]\\
&\quad=\frac{1}{(2j+1)!}\left[\left(P_{n-1}(0)P_n^{(2j+1)}(0)+P'_{n-1}(0)P_n^{(2j)}(0)\binom{2j+1}{1}+\cdots\right.\right.\\
&\quad\left.+\binom{2j+1}{j}P_{n-1}^{(j)}(0)P_n^{(j+1)}(0)\right)\\
&\quad-\left(P_n(0)P_{n-1}^{(2j+1)}(0)+P'_n(0)P_{n-1}^{(2j)}(0)\binom{2j+1}{1}+\cdots\right.\\
&\quad\left.\left.+\binom{2j+1}{j}P_n^{(j)}(0)P_{n-1}^{(j+1)}(0)\right)\right].
\end{aligned}$$

Furthermore, in the case of the Laguerre monic polynomials, we get

$$\begin{aligned}
K_{n-1}^{(j,j)}(0,0)&=\frac{(j!)^2}{(2j+1)!\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2}}\sum_{k=0}^{j}\binom{2j+1}{k}\\
&\quad\times\left[\left(L_{n-1}^{\alpha}\right)^{(k)}(0)(L_n^{\alpha})^{(2j+1-k)}(0)-\left(L_n^{\alpha}\right)^{(k)}(0)(L_{n-1}^{\alpha})^{(2j+1-k)}(0)\right]\\
&=\frac{(j!)^2}{(2j+1)!(n-1)!\Gamma(n+\alpha)}\sum_{k=0}^{j}\binom{2j+1}{k}\\
&\quad\times\left[(n-1)\cdots(n-k)L_{n-k-1}^{\alpha+k}(0)n(n-1)\cdots(n-2j+k)L_{n-2j-1+k}^{\alpha+2j+1-k}(0)\right.\\
&\quad\left.-\ n\cdots(n-k+1)L_{n-k}^{\alpha+k}(0)(n-1)\cdots(n-2j-1+k)L_{n-2j-2+k}^{\alpha+2j-k+1}(0)\right]\\
&=\sum_{k=0}^{j}\binom{2j+1}{k}\\
&\quad\times\frac{(j!)^2(2j-2k+1)n(n-1)\cdots(n-k+1)(n-1)\cdots(n-2j+k)\Gamma(n+\alpha+1)}{(2j+1)!(n-1)!\Gamma(\alpha+k+1)\Gamma(\alpha+2j-k+2)},
\end{aligned}\tag{19}$$

and as a consequence,

$$K_{n-1}^{(j,j)}(0,0)\sim C_{\alpha,j}\frac{\Gamma(n+\alpha+1)n^{2j+1}}{n!}\tag{20}$$

where $C_{\alpha,j}$ is a real constant number that depends of α and j. More precisely,

$$C_{\alpha,j}=\frac{(j!)^2}{(2j+1)!}\sum_{k=0}^{j}\binom{2j+1}{k}\frac{(2j-2k+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+2j+2-k)}.\tag{21}$$

In particular if $j = 1$

$$C_{\alpha,1} = \frac{1}{(\alpha+3)\,[\Gamma(\alpha+2)]^2}. \tag{22}$$

3. Connection formula

Let introduce the following Sobolev-type inner product:

$$\langle p, q\rangle_S = \int_0^\infty pqx^\alpha \mathrm{e}^{-x}\mathrm{d}x + Np^{(j)}(0)q^{(j)}(0), \tag{23}$$

where $N \in \mathbb{R}^+$ and $j \in \mathbb{N}$. Let $\{\mathcal{L}_n^\alpha\}_{n\geq 0}$ be the monic Laguerre–Sobolev-type orthogonal polynomials with respect to the above inner product.

Our aim is to obtain an explicit expression for these polynomials in terms of classical Laguerre polynomials. In order to do this, we will consider the Fourier expansion of $\mathcal{L}_n^\alpha$ in terms of $\{L_n^\alpha\}_{n\geq 0}$. Indeed

$$\mathcal{L}_n^\alpha(x) = L_n^\alpha(x) + \sum_{k=0}^{n-1} a_k^{(n)} L_k^\alpha(x),$$

where

$$a_k^{(n)} = \frac{\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_\alpha}{\left\| L_k^\alpha(x)\right\|_\alpha^2}.$$

But, from (23)

$$a_k^{(n)} = \frac{\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_S - N(\mathcal{L}_n^\alpha)^{(j)}(0)\left(L_k^\alpha\right)^{(j)}(0)}{\left\| L_k^\alpha(x)\right\|_\alpha^2},$$

and taking into account that $\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_S = 0$ for $k = 0, \ldots, n-1$, we get

$$a_k^{(n)} = -\frac{N(\mathcal{L}_n^\alpha)^{(j)}(0)\left(L_k^\alpha\right)^{(j)}(0)}{\left\| L_k^\alpha(x)\right\|_\alpha^2}.$$

As a consequence,

$$\begin{aligned}\mathcal{L}_n^\alpha(x) &= L_n^\alpha(x) - N(\mathcal{L}_n^\alpha)^{(j)}(0)\sum_{k=0}^{n-1}\frac{\left(L_k^\alpha\right)^{(j)}(0)L_k^\alpha(x)}{\left\| L_k^\alpha(x)\right\|_\alpha^2}\\ &= L_n^\alpha(x) - N(\mathcal{L}_n^\alpha)^{(j)}(0)K_{n-1}^{(0,j)}(x,0).\end{aligned} \tag{24}$$

Notice that for $n < j$, $\mathcal{L}_n^\alpha(x) = L_n^\alpha(x)$.

Next, we will express $K_{n-1}^{(0,j)}(x,0)$ as a linear combination of some Laguerre polynomials. Using the orthogonality of the Laguerre polynomials, we have

$$\frac{K_{n-1}^{(0,j)}(x,0)\left\| L_{n-1}^\alpha(x)\right\|_\alpha^2}{\left(L_{n-1}^\alpha\right)^{(j)}(0)} = L_{n-1}^{\alpha+j+1}(x) + \sum_{k=0}^{n-2} b_k^{(n)} L_k^{\alpha+j+1}(x),$$

where

$$b_k^{(n)} = \frac{\left\| L_{n-1}^{\alpha}(x) \right\|_{\alpha}^2 \left\langle K_{n-1}^{(0,j)}(x,0), L_k^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}}{\left(L_{n-1}^{\alpha}\right)^{(j)}(0) \left\| L_k^{\alpha+j+1}(x) \right\|_{\alpha+j+1}^2}.$$

Using (14) we get

$$\begin{aligned} K_{n-1}^{(0,j)}(x,0) &= \frac{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)}{\left\| L_{n-1}^{\alpha}(x) \right\|_{\alpha}^2} L_{n-1}^{\alpha+j+1}(x) + \frac{\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-2}^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}}{\left\| L_{n-2}^{\alpha+j+1}(x) \right\|_{\alpha+j+1}^2} L_{n-2}^{\alpha+j+1}(x) \\ &\quad + \cdots + \frac{\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-j-1}^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}}{\left\| L_{n-j-1}^{\alpha+j+1}(x) \right\|_{\alpha+j+1}^2} L_{n-j-1}^{\alpha+j+1}(x). \end{aligned} \tag{25}$$

Now we compute each one of the coefficients of the previous expression. From (4),

$$x L_{n-1}^{\alpha+1}(x) = L_n^{\alpha}(x) + (n+\alpha) L_{n-1}^{\alpha}(x),$$

and thus

$$\begin{aligned} x^j x L_{n-k-1}^{\alpha+j+1}(x) &= x^j \left(L_{n-k}^{\alpha+j}(x) + (n-k+\alpha+j) L_{n-k-1}^{\alpha+j}(x) \right) \\ &= x^{j-1} \Big[L_{n-k+1}^{\alpha+j-1}(x) + (n-k+j+\alpha) L_{n-k}^{\alpha+j-1}(x) \\ &\quad + (n-k+\alpha+j) \left(L_{n-k}^{\alpha+j-1}(x) + (n-k+\alpha+j-1) L_{n-k-1}^{\alpha+j-1}(x) \right) \Big] \\ &= x^{j-1} \Big[L_{n-k+1}^{\alpha+j-1}(x) + 2(n-k+\alpha+j) L_{n-k}^{\alpha+j-1}(x) \\ &\quad + (n-k+\alpha+j)(n-k+\alpha+j-1) L_{n-k-1}^{\alpha+j-1}(x) \Big]. \end{aligned}$$

Iterating the procedure, we will prove by induction that

$$x^{j+1} L_{n-k-1}^{\alpha+j+1}(x) = \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \, L_{n-k+j-r}^{\alpha}(x). \tag{26}$$

Indeed, assuming that the above expression holds for $j-1$, i.e.

$$x^j L_{n-k-1}^{\alpha+j}(x) = \sum_{r=0}^{j} \binom{j}{r} (n-k+\alpha+j-r)_r \, L_{n-k+j-r-1}^{\alpha}(x),$$

then

$$\begin{aligned} x^{j+1} L_{n-k-1}^{\alpha+j+1}(x) &= x \left(x^j L_{n-k-1}^{(\alpha+1)+j}(x) \right) \\ &= x \sum_{r=0}^{j} \binom{j}{r} (n-k+\alpha+j-r+1)_r \, L_{n-k+j-r-1}^{\alpha+1}(x) \\ &= \sum_{r=0}^{j} \binom{j}{r} (n-k+\alpha+j-r+1)_r \end{aligned}$$

$$
\begin{aligned}
&\times \left(L^{\alpha}_{n-k+j-r}(x) + (n-k+j-r+\alpha)L^{\alpha}_{n-k+j-r-1}(x)\right)\\
&= \sum_{r=0}^{j} \binom{j}{r} (n-k+\alpha+j-r+1)_r \, L^{\alpha}_{n-k+j-r}(x)\\
&\quad + \sum_{r=1}^{j+1} \binom{j}{r-1} (n-k+\alpha+j-r+1)_r \, L^{\alpha}_{n-k+j-r}(x)\\
&= L^{\alpha}_{n-k+j}(x) + \sum_{r=1}^{j} \left(\binom{j}{r} + \binom{j}{r-1}\right) (n-k+\alpha+j-r+1)_r \, L^{\alpha}_{n-k+j-r}(x)\\
&\quad + (n-k+\alpha)_{j+1} \, L^{\alpha}_{n-k-1}(x)\\
&= \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \, L^{\alpha}_{n-k+j-r}(x),
\end{aligned}
$$

and our results follows.

Now, we use this to compute the coefficients in (25). Using (26), for $1 \le k \le j$,

$$
\begin{aligned}
&\left\langle K^{(0,j)}_{n-1}(x,0), x^{j+1} L^{\alpha+j+1}_{n-k-1}(x)\right\rangle_{\alpha}\\
&= \left\langle K^{(0,j)}_{n-1}(x,0), \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \, L^{\alpha}_{n-k+j-r}(x)\right\rangle_{\alpha}\\
&= \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \left\langle K^{(0,j)}_{n-1}(x,0), L^{\alpha}_{n-k+j-r}(x)\right\rangle_{\alpha}.
\end{aligned}
$$

Therefore, everything comes down to calculating $\left\langle K^{(0,j)}_{n-1}(x,0), L^{\alpha}_{n-k+j-r}(x)\right\rangle_{\alpha}$ for $0 \le r \le j+1$. Using (2) and (23), we get

$$
\left\langle K^{(0,j)}_{n-1}(x,0), L^{\alpha}_{n-k+j-r}(x)\right\rangle_{\alpha} = 0,
$$

for $0 \le r \le j-k$, and

$$
\left\langle K^{(0,j)}_{n-1}(x,0), L^{\alpha}_{n-k+j-r}(x)\right\rangle_{\alpha} = \left(L^{\alpha}_{n-k+j-r}\right)^{(j)}(0) \quad j-k+1 \le r \le j+1.
$$

As a conclusion, for $1 \le k \le j$,

$$
\begin{aligned}
&\left\langle K^{(0,j)}_{n-1}(x,0), x^{j+1} L^{\alpha+j+1}_{n-k-1}(x)\right\rangle_{\alpha}\\
&= \sum_{r=j-k+1}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \left(L^{\alpha}_{n-k+j-r}\right)^{(j)}(0).
\end{aligned}
$$

Therefore

$$
K^{(0,j)}_{n-1}(x,0) = A^{(j)}_{n,1} L^{\alpha+j+1}_{n-1}(x) + A^{(j)}_{n,2} L^{\alpha+j+1}_{n-2}(x) + \cdots + A^{(j)}_{n,j+1} L^{\alpha+j+1}_{n-j-1}(x) \tag{27}
$$

where

$$A_{n,1}^{(j)} = \frac{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)}{\left\|L_{n-1}^{\alpha}(x)\right\|_{\alpha}^{2}}$$

and

$$A_{n,r}^{(j)} = \frac{1}{\left\|L_{n-r}^{\alpha+j+1}(x)\right\|_{\alpha+j+1}^{2}} \times \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} (n-r+\alpha+j-s+2)_s \left(L_{n-r+j-s+1}^{\alpha}\right)^{(j)}(0),$$

for $2 \le r \le j+1$.

In order to compute $A_{n,r}^{(j)}$, we use the next expression which is a consequence of (7):

$$\left(L_n^{\alpha}\right)^{(j)}(x) = \frac{n!}{(n-j)!} L_{n-j}^{\alpha+j}(x).$$

Indeed,

$$A_{n,1}^{(j)} = \frac{\frac{(n-1)!}{(n-j-1)!} L_{n-j-1}^{\alpha+j}(0)}{(n-1)!\Gamma(n+\alpha)} = \frac{(-1)^{n-j-1} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+j+1)}}{(n-j-1)!\Gamma(n+\alpha)} = \frac{(-1)^{n-j-1}}{(n-j-1)!\Gamma(\alpha+j+1)},$$

and

$$\begin{aligned} A_{n,r}^{(j)} &= \frac{1}{(n-r)!\Gamma(n+\alpha+j-r+2)} \\ &\quad \times \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} (n-r+\alpha+j-s+2)_s (n-r-s+2)_j \left(L_{n-r-s+1}^{\alpha+j}\right)(0) \\ &= \frac{1}{(n-r)!} \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} (n-r-s+2)_j \frac{(-1)^{n-r-s+1}}{\Gamma(\alpha+j+1)}, \end{aligned}$$

for $2 \le r \le j+1$.

Thus (24) becomes

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0) \sum_{r=1}^{j+1} A_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x). \tag{28}$$

Using this equality, we can compute $(\mathcal{L}_n^{\alpha})^{(j)}(0)$. Taking the j-th derivative on both sides,

$$\begin{aligned} \left(\mathcal{L}_n^{\alpha}\right)^{(j)}(x) &= (n-j+1)_j L_{n-j}^{\alpha+j}(x) \\ &\quad - N(\mathcal{L}_n^{\alpha})^{(j)}(0) \sum_{r=1}^{j+1} A_{n,r}^{(j)} (n-r-j+1)_j L_{n-r-j}^{\alpha+2j+1}(x), \end{aligned}$$

and thus

$$\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) = \frac{(n-j+1)_j L_{n-j}^{\alpha+j}(0)}{1 + N\sum_{r=1}^{j+1} A_{n,r}^{(j)}(n-r-j+1)_j D(n,\alpha,j,r)} = \frac{(n-j+1)_j(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1 + N\sum_{r=1}^{j+1} A_{n,r}^{(j)}(n-r-j+1)_j D(n,\alpha,j,r)},$$

with

$$D(n,\alpha,j,r) = \begin{cases} 0 & \text{if } r > n-j \\ (-1)^{n-r-j}\dfrac{\Gamma(n+\alpha+j-r+2)}{\Gamma(\alpha+2j+2)} & \text{if } r \le n-j \end{cases}.$$

As a consequence, we obtain:

Theorem 2. *Let $\{L_n^{\alpha}\}_{n\geq 0}$ be the monic Laguerre orthogonal polynomials and $\{\mathcal{L}_n^{\alpha}(x)\}_{n\geq 0}$ be the monic Laguerre–Sobolev-type orthogonal polynomials corresponding to the inner product defined in (23). Then, for every $n > j$,*

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0)\sum_{r=1}^{j+1} A_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x), \tag{29}$$

where

$$\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) = \frac{(n-j+1)_j(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1 + N\sum_{r=1}^{j+1} A_{n,r}^{(j)}(n-r-j+1)_j D(n,\alpha,j,r)},$$

$$D(n,\alpha,j,r) = \begin{cases} 0 & \text{if } n-r-j < 0 \\ (-1)^{n-r-j}\dfrac{\Gamma(n+\alpha+j-r+1)}{\Gamma(\alpha+2j+2)} & \text{if } n-r-j \ge 0 \end{cases},$$

$$A_{n,1}^{(j)} = \frac{(-1)^{n-j-1}}{(n-j-1)!\,\Gamma(\alpha+j+1)}$$

and, for $2 \le r \le j+1$,

$$A_{n,r}^{(j)} = \frac{1}{(n-r)!}\sum_{s=j-r+2}^{j+1}\binom{j+1}{s}\frac{(-1)^{n-r-s+1}(n-r-s+2)_j}{\Gamma(\alpha+j+1)}.$$

From (7), we get

$$L_n^{\alpha}(x) = \sum_{r=0}^{j+1}(n-r+1)_r\binom{j+1}{r}L_{n-r}^{\alpha+j+1}(x),$$

and using the notation of the above theorem, we obtain another equivalent expression:

Corollary 1. *For every $n \in \mathbb{N}$,*

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha+j+1}(x) + \sum_{r=1}^{j+1}\left[(n-r+1)_r \binom{j+1}{r} - N(\mathcal{L}_n^{\alpha})^{(j)}(0)A_{n,r}^{(j)}\right] L_{n-r}^{\alpha+j+1}(x). \quad (30)$$

If we define

$$C_{n,r}^{(j)} = (n-r+1)_r \binom{j+1}{r} - N(\mathcal{L}_n^{\alpha})^{(j)}(0)A_{n,r}^{(j)} \quad (31)$$

for $r = 1, \ldots, j+1$, we can write the Laguerre–Sobolev-type orthogonal polynomials as follows:

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha+j+1}(x) + \sum_{r=1}^{j+1} C_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x). \quad (32)$$

This means that $\left\{\mathcal{L}_n^{\alpha}(x)\right\}_{n\geq 0}$ is a quasi-orthogonal sequence of order $j+1$ with respect to the linear functional associated with the weight function $w_{\alpha+j+1}(x) = x^{\alpha+j+1}\mathrm{e}^{-x}$ (see [4]).

4. The zeros

In this section, we are going to prove that the zeros of the monic Laguerre–Sobolev-type orthogonal polynomials, are real, simple and interlace with the zeros of monic Laguerre orthogonal polynomials for $n \geq j$. The ideas of the proofs are the same as those used by Meijer in [14].

Theorem 3. *The monic Laguerre–Sobolev-type orthogonal polynomial $\mathcal{L}_n^{\alpha}(x)$ has n real simple zeros and at most one of them is outside of $(0, \infty)$.*

Proof. Let $\xi_1 < \xi_2 < \cdots < \xi_k$ be the positive zeros of $\mathcal{L}_n^{\alpha}(x)$ of odd multiplicity. Let

$$\varphi(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_k).$$

Thus $\varphi(x)\mathcal{L}_n^{\alpha}(x)$ does not change sign on $(0, \infty)$. Suppose that $\deg \varphi \leq n-2$; then, using the fact that $(x\varphi(x))^{(j)}(0) = j\varphi^{(j-1)}(0)$ we get

$$\left\langle \varphi(x), \mathcal{L}_n^{\alpha}(x)\right\rangle_S = \int_0^{\infty} \varphi(x)\mathcal{L}_n^{\alpha}(x)x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x + N\varphi^{(j)}(0)\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) = 0,$$

$$\left\langle x\varphi(x), \mathcal{L}_n^{\alpha}(x)\right\rangle_S = \int_0^{\infty} x\varphi(x)\mathcal{L}_n^{\alpha}(x)x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x + jN\varphi^{(j-1)}(0)\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) = 0.$$

Taking into account that the integrals in the last expressions are positive, then $\varphi^{(j)}(0)\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) < 0$ and $\varphi^{(j-1)}(0)\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) < 0$. This means that $\varphi^{(j)}(0)$ and $\varphi^{(j-1)}(0)$ have the same sign. But this is a contradiction with the well known fact that if $p(x)$ is a polynomial with simple zeros in $(0, \infty)$, then $p'(0)$ and $p(0)$ have different signs.

In other words, $\deg \varphi = n-1$ or $\deg \varphi = n$, which proves our statement. ∎

Theorem 4. *Let $\xi_1 < \xi_2 < \cdots < \xi_n$ be the zeros of the monic Laguerre–Sobolev-type orthogonal polynomials $\mathcal{L}_n^{\alpha}(x)$ and let $x_1 < x_2 < \cdots < x_n$ be the zeros of the monic Laguerre orthogonal polynomials $L_n^{\alpha}(x)$. Then $\xi_1 < x_1$ and $x_i < \xi_{i+1} < x_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

Proof. From the Gauss quadrature formula there exist positive constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \mathcal{L}_n^{\alpha}(x_i) x_i^r = \int_0^{\infty} x^r \mathcal{L}_n^{\alpha}(x) x^{\alpha} \mathrm{e}^{-x} \mathrm{d}x, \quad 0 \le r \le n-1,$$

i.e.,

$$\sum_{i=1}^{n} \lambda_i \mathcal{L}_n^{\alpha}(x_i) x_i^r = -j!N \left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) \delta_{j,r}, \quad 0 \le r \le n-1.$$

We consider the system of n linear equations in the n unknowns $\mathcal{L}_n^{\alpha}(x_1), \mathcal{L}_n^{\alpha}(x_2), \ldots, \mathcal{L}_n^{\alpha}(x_n)$. The determinant of the matrix of the coefficients in this linear system is

$$D = \lambda_1 \lambda_2 \cdots \lambda_n V(x_1, x_2, \ldots, x_n)$$

where $V(x_1, x_2, \ldots, x_n)$ denotes the Vandermonde determinant of $x_1, x_2, \ldots, x_n$. As a consequence, D is a positive real number. On the other hand, if for $1 \le i \le n$ we define

$$D_i = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_{i-1}^2 & x_{i+1}^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{j-1} & x_2^{j-1} & \cdots & x_{i-1}^{j-1} & x_{i+1}^{j-1} & \cdots & x_n^{j-1} \\ x_1^{j+1} & x_2^{j+1} & \cdots & x_{i-1}^{j+1} & x_{i+1}^{j+1} & \cdots & x_n^{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{i-1}^{n-1} & x_{i+1}^{n-1} & \cdots & x_n^{n-1} \end{vmatrix},$$

then

$$\mathcal{L}_n^{\alpha}(x_i) = (-1)^{i+j} j!N \left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) \frac{D_i}{\lambda_i V(x_1, x_2, \ldots, x_n)}.$$

But D_i is a positive real number according to problem 48 in page 43 in [17]. This means that $sign(\mathcal{L}_n^{\alpha}(x_i)) = (-1)^{i+j} sign((\mathcal{L}_n^{\alpha})^{(j)}(0)) = (-1)^{i+j} sign((L_n^{\alpha})^{(j)}(0)) = (-1)^{n+i}$. In other words, $\mathcal{L}_n^{\alpha}(x)$ has at least one zero in every interval (x_i, x_{i+1}) for $i = 1, 2, \ldots, n-1$. Finally, taking into account that the sign of $\mathcal{L}_n^{\alpha}(x_1)$ is $(-1)^{n+1}$, we conclude that $\xi_1 < x_1$. ■

5. Asymptotic behaviour

We will analyze the outer relative asymptotic behaviour of $\mathcal{L}_n^{\alpha}(x)$. From (24) we have

$$\frac{\mathcal{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 - \frac{N(\mathcal{L}_n^{\alpha})^{(j)}(0) K_{n-1}^{(0,j)}(x,0)}{L_n^{\alpha}(x)}.$$

But from (16), we get

$$\frac{\mathcal{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 - \frac{j!N(\mathcal{L}_n^{\alpha})^{(j)}(0) \left(L_n^{\alpha}(x) Q_j(x,0,L_{n-1}^{\alpha}) - L_{n-1}^{\alpha}(x) Q_j(x,0,L_n^{\alpha})\right)}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^2 x^{j+1} L_n^{\alpha}(x)},$$

where $Q_j(x,0,L_{n-1}^{\alpha})$ and $Q_j(x,0,L_n^{\alpha})$ are the Taylor polynomials of degree j of $L_{n-1}^{\alpha}(x)$ and $L_n^{\alpha}(x)$ around $x=0$, respectively. Therefore

$$\frac{\mathcal{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 - \frac{j!N(\mathcal{L}_n^{\alpha})^{(j)}(0)Q_j(x,0,L_{n-1}^{\alpha})}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2} x^{j+1}}\left(1-\frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})}\frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}\right). \tag{33}$$

First, we will study the behaviour of the following three expressions:

(i) $\dfrac{j!Q_j(x,0,L_{n-1}^{\alpha})}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2}}$,

(ii) $1-\dfrac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})}\dfrac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}$,

(iii) $(\mathcal{L}_n^{\alpha})^{(j)}(0)$.

For the first one,

$$\begin{aligned}\frac{j!Q_j(x,0,L_{n-1}^{\alpha})}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2}} &\sim \frac{j!\left(L_{n-1}^{\alpha}\right)^{(j)}(0)x^j}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2} j!}\\ &= \frac{(n-1)(n-2)\cdots(n-j)}{(n-1)!\Gamma(n+\alpha)}L_{n-1-j}^{\alpha+j}(0)x^j\\ &= \frac{(n-1)(n-2)\cdots(n-j)}{(n-1)!\Gamma(n+\alpha)}(-1)^{n-1-j}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+j+1)}x^j.\end{aligned}$$

As a consequence,

$$\frac{j!Q_j(x,0,L_{n-1}^{\alpha})}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2}x^{j+1}} \sim \frac{(-1)^{n-1-j}n^j}{x(n-1)!\Gamma(\alpha+j+1)}. \tag{34}$$

On the other hand,

$$\begin{aligned}1-\frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})}\frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)} &\sim 1+\frac{n(n-1)\cdots(n-j+1)\Gamma(n+\alpha+1)}{(n-1)(n-2)\cdots(n-j)\Gamma(n+\alpha)}\frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}\\ &= 1-\frac{n+\alpha}{n-j}\frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)},\end{aligned} \tag{35}$$

where $\widehat{L}_n^{\alpha}(x)$ denotes the n-th Laguerre polynomial with leading coefficient $\frac{(-1)^n}{n!}$.

Finally, from (9), (20), and (24) we get

$$\begin{aligned}(\mathcal{L}_n^{\alpha})^{(j)}(0) &= \frac{\left(L_n^{\alpha}\right)^{(j)}(0)}{1+NK_{n-1}^{(j,j)}(0,0)}\\ &\sim \frac{n(n-1)\cdots(n-j+1)(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1+NC_{\alpha,j}\frac{\Gamma(n+\alpha+1)n^{2j+1}}{n!}}.\end{aligned}$$

Thus

$$(\mathcal{L}_n^{\alpha})^{(j)}(0) \sim \frac{1}{\Gamma(\alpha+j+1)}\frac{(-1)^{n-j}n^j\Gamma(n+\alpha+1)(n-1)!}{(n-1)!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}}. \tag{36}$$

Using (13) in (35) we get

$$\lim_{n\to\infty}\left(1-\frac{Q_j(x,0,L_n^\alpha)}{Q_j(x,0,L_{n-1}^\alpha)}\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right)=0 \tag{37}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Then, from (33) and (34), we obtain

$$\begin{aligned}\frac{\mathcal{L}_n^\alpha(x)}{L_n^\alpha(x)}-1 &= -\frac{Nj!(\mathcal{L}_n^\alpha)^{(j)}(0)Q_j(x,0,L_{n-1}^\alpha)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2 x^{j+1}}\left(1-\frac{Q_j(x,0,L_n^\alpha)}{Q_j(x,0,L_{n-1}^\alpha)}\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right)\\ &\sim -\frac{N(-1)^{n-j}n^j\Gamma(n+\alpha+1)}{(n-1)!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}}\frac{(-1)^{n-1-j}n^j}{x\left(\Gamma(\alpha+j+1)\right)^2}\\ &\quad\times\left(1-\frac{Q_j(x,0,L_n^\alpha)}{Q_j(x,0,L_{n-1}^\alpha)}\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right)\\ &= \frac{N}{x\left(\Gamma(\alpha+j+1)\right)^2}\frac{n^{2j}\Gamma(n+\alpha+1)}{(n-1)!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}}\\ &\quad\times\left(1-\frac{Q_j(x,0,L_n^\alpha)}{Q_j(x,0,L_{n-1}^\alpha)}\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right)\end{aligned}$$

and therefore, using (37) we get:

Proposition 2.

$$\lim_{n\to\infty}\frac{\mathcal{L}_n^\alpha(x)}{L_n^\alpha(x)}=1 \tag{38}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Taking into account the Mehler–Heine formula (12), we will deduce an analogous Mehler–Heine formula for the Laguerre–Sobolev-type orthogonal polynomials.

First, we will find an expression for the j-th Taylor polynomial of $L_n^\alpha(x)$ replacing the variable x by x/n:

$$\begin{aligned}Q_j(x/n,0,L_n^\alpha) &= \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}+\frac{(-1)^{n-1}n\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)}\frac{x}{n}+\cdots\\ &\quad+\frac{(-1)^{n-j}n(n-1)\cdots(n-j+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)j!}\frac{x^j}{n^j}\\ &= \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\left[S_j^\alpha(x)-\frac{1}{n}R_j^\alpha(x)+\mathcal{O}(n^{-2})\right],\end{aligned}$$

where

$$\begin{aligned}S_j^\alpha(x) &= 1-\frac{x}{1!(\alpha+1)_1}+\cdots+\frac{(-1)^jx^j}{j!(\alpha+1)_j},\\ R_j^\alpha(x) &= \frac{x^2}{2!(\alpha+1)_2}-\frac{3x^3}{3!(\alpha+1)_3}+\cdots+\frac{(-1)^jx^j(j-1)j}{2j!(\alpha+1)_j}.\end{aligned}$$

Therefore

$$Q_j(x/n, 0, L_n^\alpha) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \left[S_j^\alpha(x) - \frac{1}{n} R_j^\alpha(x) + \mathcal{O}(n^{-2}) \right]. \tag{39}$$

In an analogous way, we conclude that

$$Q_j(x/n, 0, L_{n-1}^\alpha) = \frac{(-1)^{n-1} \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \left[S_j^\alpha(x) + \frac{1}{n} T_j^\alpha(x) + \mathcal{O}(n^{-2}) \right], \tag{40}$$

where

$$T_j^\alpha(x) = \frac{x}{1!(\alpha+1)} - \frac{3x^2}{2!(\alpha+1)_2} + \cdots + \frac{(-1)^{j-1} x^j j(j+1)}{2(\alpha+1)_j j!}.$$

From (16) and (24), we get

$$\mathcal{L}_n^\alpha(x) = L_n^\alpha(x) - \frac{Nj!(\mathcal{L}_n^\alpha)^{(j)}(0)}{\left\| L_{n-1}^\alpha \right\|_\alpha^2 x^{j+1}} \left[L_n^\alpha(x) Q_j(x, 0, L_{n-1}^\alpha) - L_{n-1}^\alpha(x) Q_j(x, 0, L_n^\alpha) \right],$$

and using (7) we have

$$\begin{aligned}
\mathcal{L}_n^\alpha(x) &= L_n^\alpha(x) - \frac{Nj!(\mathcal{L}_n^\alpha)^{(j)}(0)}{\left\| L_{n-1}^\alpha \right\|_\alpha^2 x^{j+1}} \Bigg[L_n^\alpha(x) Q_j(x, 0, L_{n-1}^\alpha) \\
&\quad - \frac{L_n^{\alpha-1}(x) - L_n^\alpha(x)}{n} Q_j(x, 0, L_n^\alpha) \Bigg] \\
&= L_n^\alpha(x) - \frac{Nj!(\mathcal{L}_n^\alpha)^{(j)}(0)}{\left\| L_{n-1}^\alpha \right\|_\alpha^2 x^{j+1}} \Bigg[\left(Q_j(x, 0, L_{n-1}^\alpha) + \frac{Q_j(x, 0, L_n^\alpha)}{n} \right) L_n^\alpha(x) \\
&\quad - \frac{L_n^{\alpha-1}(x)}{n} Q_j(x, 0, L_n^\alpha) \Bigg].
\end{aligned}$$

Multiplying both sides of the above expression by $\frac{(-1)^n}{n! n^\alpha}$ and using the change of variable $x \to x/n$, we get

$$\begin{aligned}
\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha} &= \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} - \frac{Nj!(\mathcal{L}_n^\alpha)^{(j)}(0) n^{j+1}}{\left\| L_{n-1}^\alpha \right\|_\alpha^2 x^{j+1}} \\
&\quad \times \Bigg[\left(Q_j(x/n, 0, L_{n-1}^\alpha) + \frac{Q_j(x/n, 0, L_n^\alpha)}{n} \right) \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} \\
&\quad - \frac{1}{n^2} \frac{\widehat{L}_n^{\alpha-1}(x/n)}{n^{\alpha-1}} Q_j(x/n, 0, L_n^\alpha) \Bigg],
\end{aligned}$$

and, from (8), (36), (39) and (40),

$$\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha} \sim \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} - \frac{\frac{Nj!}{\Gamma(\alpha+j+1)}\frac{(-1)^{n-j}n^j\Gamma(n+\alpha+1)(n-1)!}{(n-1)!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}}n^{j+1}}{(n-1)!\Gamma(n+\alpha)x^{j+1}}\frac{(-1)^n\Gamma(n+\alpha)}{\Gamma(\alpha+1)}$$
$$\times\left[\left(\frac{\alpha}{n}S_j^\alpha(x)-\frac{1}{n}T_j^\alpha(x)-\frac{1}{n}R_j^\alpha(x)+\mathcal{O}(n^{-2})\right)\frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha}\right.$$
$$\left.-\frac{\widehat{L}_n^{\alpha-1}(x/n)(n+\alpha)}{n^{\alpha+1}}\left(S_j^\alpha(x)-\frac{1}{n}R_j^\alpha(x)+\mathcal{O}(n^{-2})\right)\right].$$

As a consequence,

$$\lim_{n\to\infty}\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}) + \frac{(-1)^{j+1}j!}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+1)x^{j+1}}$$
$$\times\left[x^{-\alpha/2}J_\alpha(2\sqrt{x})\left(\alpha S_j^\alpha(x)-T_j^\alpha(x)-R_j^\alpha(x)\right)-x^{-(\alpha-1)/2}J_{\alpha-1}(2\sqrt{x})S_j^\alpha(x)\right],$$

uniformly on compact subsets of $\mathbb{C}$. Using the fact that

$$J_{\alpha-1}(2\sqrt{x})+J_{\alpha+1}(2\sqrt{x})=\frac{\alpha}{\sqrt{x}}J_\alpha(2\sqrt{x}), \tag{41}$$
$$T_j^\alpha(x)+R_j^\alpha(x)=\frac{x}{\alpha+1}S_{j-1}^{\alpha+1}(x),$$

and doing some computations we get:

Proposition 3.

$$\lim_{n\to\infty}\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x})\left[1+\frac{(-1)^j j!S_{j-1}^{\alpha+1}(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}\right]$$
$$-x^{-\alpha/2}J_{\alpha+1}(2\sqrt{x})\frac{(-1)^j j!S_j^\alpha(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+1)x^{j+1/2}} \tag{42}$$

uniformly on compact subsets of $\mathbb{C}$.

From (41), (42) becomes

$$\lim_{n\to\infty}\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x})\left[1+\frac{(-1)^j j!\left(S_{j-1}^{\alpha+1}(x)-S_j^\alpha(x)\right)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}\right]$$
$$-x^{-\alpha/2}J_{\alpha+2}(2\sqrt{x})\frac{(-1)^j j!S_j^\alpha(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}.$$

If $j=1$, then we deduce after some straightforward computations a result obtained in [2].

Now, we are going to find another important asymptotic behaviour of $\mathcal{L}_n^\alpha(x)$. Taking into account

$$\begin{aligned}\|\mathcal{L}_n^\alpha\|_S^2 &= \langle\mathcal{L}_n^\alpha, L_n^\alpha\rangle_S\\ &= \langle\mathcal{L}_n^\alpha, L_n^\alpha\rangle_\alpha + N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0)\\ &= \|L_n^\alpha\|_\alpha^2 + N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0),\end{aligned}$$

and, using the asymptotic behaviour of $\left(L_n^\alpha\right)^{(j)}(0)$ and (36) which show us the behaviour of $\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)$, we then get

$$\begin{aligned}
\frac{\left\|\mathcal{L}_n^\alpha\right\|_S^2}{\left\|L_n^\alpha\right\|_\alpha^2} &= 1+\frac{N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0)}{\left\|L_n^\alpha\right\|_\alpha^2} \\
&\sim 1+N\frac{(-1)^{n-j}n^j\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1+NC_{\alpha,j}\frac{\Gamma(n+\alpha+1)}{n!}n^{2j+1}}\frac{n^j(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{n!\Gamma(n+\alpha+1)} \\
&= 1+\frac{n^{2j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)^2}}{n!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j+1}} \\
&= 1+\mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Thus, we have proved:

Proposition 4.

$$\lim_{n\to\infty}\frac{\left\|\mathcal{L}_n^\alpha\right\|_S}{\left\|L_n^\alpha\right\|_\alpha}=1.$$

Finally, we present a result about the behaviour of the norm of the Laguerre–Sobolev-type orthogonal polynomials that is a consequence of the above proposition and the fact that $n^{-1}\left\|L_n^\alpha\right\|_\alpha^{1/n}\to \mathrm{e}^{-1}$ when $n\to\infty$.

Corollary 2.

$$\lim_{n\to\infty}n^{-1}\left\|\mathcal{L}_n^\alpha\right\|_S^{1/n}=\mathrm{e}^{-1}. \tag{43}$$

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