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Minimal shape-preserving projections in tensor product spaces

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Abstract

Let X denote a (real) Banach space. If $P : X \to X$ is a linear operator and $S \subset X$ such that $PS \subset S$ then we say that S is invariant under P. In the case that P is a projection and S is a cone we say that P is a *shape-preserving projection* (relative to S) whenever P leaves S invariant. If we assume that cone S has a particular structure then, given a finite-dimensional subspace $V \subset X$, we can describe, in geometric terms, the set of all shape-preserving projections (relative to S) from X onto V. From here (assuming that such projections exist), we can then look for those shape-preserving projections $P : X \to V$ of the minimal operator norm; that is, we look for *minimal shape-preserving projections*.

If $P_i : X_i \to V_i$ is a minimal shape-preserving projection (relative to S_i) defined on Banach space X_i for $i = 1, 2$ then it is obvious that $P_1 \otimes P_2$ is a shape-preserving projection (relative to $S_1 \otimes S_2$) on $X_1 \otimes X_2$. But is it true that $P_1 \otimes P_2$ must have minimal norm? In this paper we show that in general this need not be the case (note that this is somewhat unexpected since, in the standard minimal projection setting, the tensor of two minimal projections is always minimal). We also identify a collection of operators in which $P_1 \otimes P_2$ is always a minimal shape-preserving projection (within that collection). This result is then applied to a (well-known) special case to reveal a (non-trivial) situation in which $P_1 \otimes P_2$ is indeed a minimal shape-preserving projection (among all possible shape-preserving projections).

Keywords: Tensor product; Minimal projection; Shape-preserving projection

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1. Introduction

We say that set S *is invariant* under mapping P if $Ps \in S$ for every $s \in S$ (or more briefly as $PS \subset S$). The pairing of mappings with their invariant sets is a basic problem, with variations appearing throughout the mathematical literature. As an initial refinement of this problem, one may take P as a linear operator (defined on Banach space X) and S as a *cone* – a convex set, closed under nonnegative scalar multiplication; in this setting, P is said to be a cone -preserving map if $PS \subset S$ (see [12] for an overview) (of course if S induces a proper lattice structure on X then P can be regarded as a *positive operator*, as in [11]).

To further refine, suppose the elements of (cone) $S \subset X$ possess a common characteristic, or *shape*, and let $\mathcal{P}$ denote a collection of operators on X of (fixed) finite rank. This setting gives rise to two fundamental questions (and provides a starting point for this paper): does there exist $P \in \mathcal{P}$ leaving S invariant (i.e., what can be said about the existence of a *shape-preserving operator* in $\mathcal{P}$) and, if such an operator exists, determine the smallest possible norm of an element in $\mathcal{P}$ which preserves S (i.e. find a *minimal shape-preserving operator* in $\mathcal{P}$).

In the paper [5], these two questions were addressed in the case when P is a projection operator (P^2 is the identity map) mapping $X = C^L[0, 1]$ (Lth continuously differentiable functions) onto finite-dimensional polynomial spaces. For a large class of cones (or shapes) S, the existence of shape-preserving projections was established and formulas for minimal shape-preserving projections (and their norms) were given. The goal of this paper is to extend the results of [5] to tensor product spaces (tensor product spaces are a natural setting in which to construct minimal shape-preserving projections of multi-variate functions; general results in this direction are contained in the (upcoming) paper [6]). One would expect that such a generalization would utilize the main result from [4], which says (roughly speaking) that tensor product of two minimal projections is again a minimal projection (on a tensor product space). Unfortunately this is not true (in general) in the setting of minimal shape-preserving projections. Indeed, we demonstrate, in Example 4.1, a case in which the tensor product of two minimal shape-preserving projections does not have minimal norm. However, in the specific setting of [5], we can prove the main result from [4] and, consequently, construct minimal shape-preserving projections for tensor product spaces involving $C^L[0, 1]$.

In the Sections 2 and 3 we give basic definitions and results from, respectively, tensor product theory and shape-preserving projection theory. Section 4 contains the main results, with Example 4.1 demonstrating that the tensor product of two minimal shape-preserving projections need not have minimal norm and Theorem 4.6 accomplishing the goal of generalizing (to tensor product spaces) the results from [5].

2. Preliminaries from tensor product theory

Definition 2.1. Let X, Y be two real Banach spaces. Fix $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in Y$. Then $\sum_{i=1}^{k} x_i \otimes y_i$ denotes an operator from X^* into Y defined by

$$\left(\sum_{i=1}^{k} x_i \otimes y_i\right) f = \sum_{i=1}^{k} f(x_i) y_i.$$

In particular, $(x \otimes y) f = f(x) y$ for any $x \in X$, $y \in Y$ and $f \in X^*$. Then $X \otimes Y$ denote the space of all finite-dimensional operators from X^* into Y (we put into one equivalence class all different representations of any fixed operator).

Definition 2.2. Let α denote any norm on $X \otimes Y$. Then $X \otimes_\alpha Y$ means the completion of $X \otimes Y$ (in the sense of Banach spaces).

Definition 2.3. Let α be any norm on $X \otimes Y.\alpha$ is called a *cross-norm* if

$$\alpha(x \otimes y) = \|x\| \cdot \|y\|$$

for any $x \in X$, $y \in Y$.

Definition 2.4. Let α be any norm on $X \otimes Y$. For $f \in X^*$, $g \in Y^*$ define

$$(f \otimes g)\left(\sum_{i=1}^{k} x_i \otimes y_i\right) = \sum_{i=1}^{k} f(x_i)g(y_i).$$

α is called *reasonable* if

$$\alpha^*(f \otimes g) = \|f\| \cdot \|g\|$$

for any $f \in X^*$, $g \in Y^*$, where α^* denote the norm in $(X \otimes_\alpha Y)^*$.

Definition 2.5. Let X and Y be Banach spaces. Assume $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Then we can define a linear operator $A \otimes B$ by

$$(A \otimes B)(x \otimes y) = A(x) \otimes B(y).$$

A norm α on $X \otimes Y$ is called *uniform* if

$$\|A \otimes B\|_\alpha \leq \|A\| \cdot \|B\|$$

for any $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$.

To the end of the paper, unless otherwise stated, we assume that any norm α on $X \otimes Y$ is a reasonable cross-norm.
We use $X \otimes_\lambda Y$ and $X \otimes_\gamma Y$ to denote, respectively, the injective tensor product and the projective tensor product of X and Y. Both λ and γ are uniform, reasonable cross-norms (see e.g. [2] Lemma 1.6, 1.8 and 1.12]).

Definition 2.6 (*see e.g. [[2], Def.1.45, p.27*).] Let X, Y be Banach spaces. For $1 \leq p \leq \infty$ the p-nuclear norm of $z \in X \otimes Y$ is defined by:

$$\alpha_p(z) = \inf\left\{\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} a_q(y_1, \ldots, y_n) : z = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

Here q is so chosen that $1/p + 1/q = 1$ and

$$a_q(y_1, \ldots, y_n) = \sup\left\{\left(\sum_{i=1}^{n} |f(y_i)|^q\right)^{1/q} : f \in S_{X^*}\right\}.$$

If $q = \infty$, then

$$a_q(y_1, \ldots, y_n) = \sup\{max_{1 \leq i \leq n}|f(y_i)| : f \in S_{X^*}\}.$$

By [[2], Lemma 1.46, p.27] the p-nuclear norm is a reasonable cross-norm. Observe that by [[2], Lemma 1.44, p. 27] for any $B \in \mathcal{L}(Y)$

$$a_q(By_1, \ldots, By_n) \le \|B\| a_q(y_1, \ldots, y_n).$$

Hence α_p is a uniform, reasonable cross-norm. By a result of [9] we have:

$$L_p(S) \otimes_{\alpha_p} L_p(T) = L_p(S \times T)$$

where S and T are finite measure spaces.

3. Preliminaries from shape-preserving projection theory

For a given real Banach space X and subspace $V \subset X$, let $\mathcal{P}(X, V)$($\mathcal{P}$ for brevity) denote the set of all continuous linear projections from X onto V.

Definition 3.1. Let X be a (fixed) Banach space and $V \subset X$ a (fixed) n-dimensional subspace. Let $S \subset X$ denote a closed cone. We say that $x \in X$ *has shape* (*in the sense of* S) whenever $x \in S$. If $P \in \mathcal{P}$ and $PS \subset S$ then we say P is a *shape-preserving projection*; we denote the set of all such projections by $\mathcal{P}_S(X, V)$ ($\mathcal{P}_S$ for brevity). For a given cone S, define $S^* = \{\phi \in X^* \mid \phi(x) \ge 0 \; \forall x \in S\}$. We will refer to S^* as the *dual cone* of S.

In a (real) topological vector space, a *cone* K is a convex set, closed under nonnegative scalar multiplication. K is *pointed* if it contains no lines. For $\phi \in K$, let $[\phi]^+ := \{\alpha\phi \mid \alpha \ge 0\}$. We say $[\phi]^+$ is an *extreme ray* of K if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of K. When K is a closed, pointed cone of finite dimension we always have $K = \mathrm{co}(E(K))$ (this need not be the case when K is infinite-dimensional; indeed, we note in [7] that it is possible that $E(K) = \emptyset$ despite K being closed and pointed).

Definition 3.2 (*[8]*). Let X be a Hausdorff topological vector space over $\mathbb{R}$ and let X^* be the topological dual of X. We say that a pointed closed cone $K \subset X^*$ is *simplicial* if K can be recovered from its extreme rays, (i.e., $K = \overline{co}(E(K))$) and the set of extreme rays of K form an independent set (independent in the sense that any generalized representing measure for $x \in K$ supported on $E(K)$ must be a representing measure.)

Proposition 3.1 (*[8]*). *A pointed closed cone $K \subset X^*$ of finite dimension n is simplicial iff K has exactly n extreme rays.*

The following result gives necessary and sufficient conditions for the existence of a shape-preserving projection.

Theorem 3.1 (*[8]*). *Suppose S^* simplicial. Then $\mathcal{P}_S \ne \emptyset$ if and only if $S^*_{|V}$ is simplicial.*

Without the assumption that S^* simplicial we still have one direction of above characterization; and it is this result that will be of most use to us.

Corollary 3.1 (*[8]*). *If $S^*_{|V}$ is simplicial then $\mathcal{P}_S \ne \emptyset$.*

4. Main results

Let X be a (real) Banach space, $V \subset X$ an m-dimensional subspace and $S \subset X$ a closed cone, $S \neq \{0\}$. As in Definition 3.1, set

$$S^* = \{\phi \in X^* : \phi(x) \geq 0 \text{ for any } x \in S\}.$$

With Corollary 3.1 in mind, we assume throughout this paper that $S^*|_V$ is simplicial, i.e.

$$S^*|_V = conv([g_1], \ldots, [g_l]),$$

where $l = dim(S^*|_V)$ and for $i = 1, \ldots, l[g_i]$ are extreme rays determined by linearly independent $g_1, \ldots, g_l \in V^*$, each with unit norm.

Lemma 4.1. *Let $V_o = V \cap (S^*)^{\perp}$. Choose a subspace $V_1 \subset V$ such that*

$$V = V_o \oplus V_1.$$

Then $\dim(V_1) = l$. *Moreover, there exist $\{v_1, \ldots v_l\} \subset S$ a basis of V_1 such that $g_i(v_j) = \delta_{ij}$.*

Proof. By definition of S^*, for any fixed $v \in V_1$ if $g_j(v) = 0$ then $v = 0$. Since $g_1, \ldots, g_l$ are linearly independent, there exist $v_1, \ldots, v_l \in V_1$ such that $g_j(v_i) = \delta_{ij}$ for $i, j = 1, \ldots, l$. Now fix $i \in \{1, \ldots, l\}$. Since $g_j(v_i) \geq 0$ for $j = 1, \ldots, l$ and $S^*|_V$ is simplicial, $f(v_i) \geq 0$ for any $f \in S^*$. Hence $v_i \in S$. ∎

Lemma 4.2. *Let $V_o = V \cap (S^*)^{\perp}$. Let $w_{l+1}, \ldots, w_m$ be a fixed basis of V_o. Choose a subspace $V_1 \subset V$ such that $V = V_0 \oplus V_1$. Let $v_1, \ldots, v_l \in S$ be a basis of V_1. Define*

$$\mathcal{P}_{S,V_1} = \{P \in \mathcal{P}(X, V_1) : P(S) \subset S, P|_{V_o} = 0\},$$
$$\mathcal{P}_{V_1}(X, V_o) = \{P \in \mathcal{P}(X, V_o) : P|_{V_1} = 0\}$$

and

$$\mathcal{L}_V(X, V_o) = \{L \in \mathcal{L}(X, V_o) : L|_V = 0\}.$$

Then

$$\mathcal{P}_S = \mathcal{P}_{V_1}(X, V_o) + \mathcal{P}_{S,V_1} = P_o + \mathcal{L}_V(X, V_o) + \mathcal{P}_{S,V_1},$$

where P_o is a fixed element from $\mathcal{P}_{V_1}(X, V_o)$.

Proof. Take any $P \in \mathcal{P}_S$. Since P is a projection onto V and $w_{l+1}, \ldots w_m, v_1, \ldots, v_l$ is a basis of V,

$$P = \sum_{j=l+1}^{m} \phi_j(\cdot) w_j + \sum_{j=1}^{l} \psi_j(\cdot) v_j.$$

Here for $j = l+1, \ldots, m, \phi_j \in X^*, \phi_i(w_j) = \delta_{ij}$ for $i, j = l+1, \ldots, m$ and $\phi_i|_{V_1} = 0$ for $i = l+1, \ldots, m$. Analogously, for $j = 1, \ldots, l, \psi_j \in X^*, \psi_i(v_j) = \delta_{ij}$ for $i, j = 1, \ldots, l$ and $\psi_i|_{V_o} = 0$ for $i = 1, \ldots, l$. Moreover, since $P \in \mathcal{P}_S$, $\psi_j \in S^*$ for $j = 1, \ldots, l$. Indeed, if $\psi_j \notin S^*$ for some $j = 1, \ldots, l$, then $\psi_j(x) < 0$ for some $x \in S$. Fix $f \in S^*$ with $f|_{V_1} = g_j$. Since $g_i(v_j) = \delta_{ij}$ and $f|_{V_o} = 0$,

$$f(Px) = \psi_j(x) f(v_j) = \psi_j(x) g_j(v_j) = \psi_j(x) < 0.$$

Hence $Px \notin S$; a contradiction. Set

$$P_1 = \sum_{j=l+1}^{m} \phi_j(\cdot)w_j$$

and

$$P_2 = \sum_{j=1}^{l} \psi_j(\cdot)v_j.$$

It is clear that $P_1 \in \mathcal{P}_{V_1}(X, V_o)$ and $P_2 \in \mathcal{P}_{S,V_1}$. Hence $P = P_1 + P_2 \in \mathcal{P}_{V_1}(X, V_o) + \mathcal{P}_{S,V_1}$.
Now assume $P = P_1+P_2$, where $P_1 \in \mathcal{P}_{V_1}(X, V_o)$ and $P_2 \in \mathcal{P}_{S,V_1}$. Hence $P_2 = \sum_{j=1}^{l} f_j(x)v_j$, where $f_j \in S^*$ for $j = 1, \dots, l$. It is clear that P is a projection onto V. Fix $x \in S$ and $f \in S^*$. Note that

$$f(Px) = f(P_1x) + f(P_2x) = f\left(\sum_{j=1}^{l} f_j(x)v_j\right) \geq 0$$

since $v_j \in S$ and $f_j \in S^*$ for $j = 1, \dots, l$. Hence $P \in \mathcal{P}_S$, as required.
Notice that for any fixed $P_o \in \mathcal{P}_{V_1}(X, V_o)$

$$\mathcal{P}_{V_1}(X, V_o) = P_o + \mathcal{L}_V(X, V_o),$$

which completes the proof. ■

Now we fix some notation concerning tensor product case. Let X_1, X_2 be two Banach spaces. Let for $i = 1, 2 W_i \subset X_i$ be and m_i-dimensional subspace and let $S_i \subset X_i$ be a k_i-dimensional cone. Assume that $S_i^*|_{W_i}$ is simplicial and $dim S_i^*|_{W_i} = l_i$ for $i = 1, 2$. Let g_{ij} for $j = 1, \dots, l_i$ be the elements from W_i^* of norm one which determine the extreme rays of $S_i^*|_{W_i}$. Denote for $i = 1, 2$

$$W_{io} = W_i \cap (S_i^*)^{\perp}$$

and fix w_{ij}, $j = 1, \dots, m_i - l_i$ a basis of W_{io}. Fix for $i = 1, 2$ an l_i-dimensional subspace of W_i, W_{i1} given by (Lemma 4.1) such that

$$W_i = W_{io} \oplus W_{i1}.$$

Assume that v_{ij} is a basis of W_{i1} satisfying $g_{iu}(v_{ij}) = \delta_{uj}$ for $j, u = 1, \dots, l_i$, $i = 1, 2$. Now we are ready to define the corresponding notions concerning tensor product. Let α be a fixed reasonable cross-norm on $X_1 \otimes X_2$. Define:

$$X = X_1 \otimes_\alpha X_2, \tag{1}$$

and

$$V = W_1 \otimes_\alpha W_2. \tag{2}$$

We define

$$S_1^* \otimes S_2^* = \{s_1 \otimes s_2 \mid s_1 \in S_1^*,\ s_2 \in S_2^*\}.$$

From this definition, note that $\mathrm{co}\,(S_1^* \otimes S_2^*)$ (the convex hull) is a cone in $X_1^* \otimes X_2^*$. This allows us to define the following cone in X:

$$S = (\mathrm{co}\,(S_1^* \otimes S_2^*))^* \cap (X_1 \otimes_\alpha X_2). \tag{3}$$

Note that

$$V \cap (S^*)^{\perp} = V_o = W_{1o} \otimes W_{2o} + W_{1o} \otimes W_{21} + W_{11} \otimes W_{2o} \tag{4}$$

and

$$V_1 = W_{11} \otimes W_{21}, \tag{5}$$

where V_1 is a subspace of V determined by Lemma 4.1. Note that $dim(V) = m_1 \cdot m_2$ and $dim(S^*|_V) = l_1 \cdot l_2$.

Lemma 4.3. *Assume that X_1, X_2 are finite-dimensional. Then $S_1^* \otimes S_2^*$ is a closed set.*

Proof. Let $f_k = g_k \otimes h_k \in S_1^* \otimes S_2^*$ converges to $f \in X_1^* \otimes X_2^*$. Since $\|f_k\| = \|g_k\|\|h_k\|$, without loss of generality, we can assume that $\|h_k\| = \|g_k\| = \sqrt{\|f_k\|}$. Since $X_1 \otimes X_2$ is finite-dimensional, and $\|f_k - f\| \to 0$, we can assume that $g_k \to g \in S_1^*$ and $h_k \to h \in S_2^*$. Note that,

$$\|g \otimes h - g_k \otimes h_k\| \le \|g \otimes (h - h_k)\| + \|(g - g_k) \otimes h_k\| \le \|g\|\|h - h_k\| + M\|g - g_k\|.$$

Hence $\|f_k - g \otimes h\| \to 0$ which gives that $f = g \otimes h \in S_1^* \otimes S_2^*$, as required. ■

Note that Lemma 4.3 remains valid when X_1, X_2 are reflexive separable spaces.

Lemma 4.4. *Let X_1, X_2 be finite-dimensional. Assume that for $i = 1, 2$ and $f \in S_i^* \setminus \{0\}$ there exists $x_i \in X_i$ with $f(x_i) > 0$. Then*

$$S^* = \mathrm{co}\,(S_1^* \otimes S_2^*).$$

Proof. Since $S = (S_1^* \otimes S_2^*)^*$, $S^* = cl(conv(S_1^* \otimes S_2^*))$. Hence we only need to show that $conv(S_1^* \otimes S_2^*)$ is a closed set. Let $f \in cl(conv(S_1^* \otimes S_2^*))$. Choose a sequence $f_k \in conv(S_1^* \otimes S_2^*)$ such that $\|f_k - f\| \to 0$. Since $dim(S_i) = k_i$, for $i = 1, 2$, $dim(S_1^* \otimes S_2^*) = k_1 k_2$. By the Carathéodory theorem, for any $k \in \mathbb{N}$,

$$f_k = \sum_{j=1}^{k_1 k_2 + 1} a_{jk} f_{jk},$$

where $a_{jk} \ge 0$ and $f_{jk} \in S_1^* \otimes S_2^*$. Now we show that there exists $M > 0$ such that $\|f_{jk} a_{jk}\| < M$ for any $j = 1, \ldots, k_1 k_2 + 1$ and $k \in \mathbb{N}$. If not, passing to a subsequence if necessary, we can assume that

$$\|a_{1k} f_{1k}\| \to \infty$$

and for any $j = 1, \ldots, k_1 k_2 + 1$,

$$\limsup \frac{\|a_{jk} f_{jk}\|}{\|a_{1k} f_{1k}\|} < \infty.$$

Consequently, passing to a subsequence, if necessary, we can assume that for any $j = 1, \ldots, k_1 k_2 + 1$,

$$\lim_k \frac{a_{jk} f_{jk}}{\|a_{1k} f_{1k}\|} = g_j.$$

By Lemma 4.3, $g_j \in S_1^* \otimes S_2^*$. Notice that

$$0 = \lim_k \frac{f_k}{\|a_{1k} f_{1k}\|} = \sum_{j=1}^{k_1 k_2+1} g_j.$$

Since $\|g_1\| = 1$, and $g_1 = h_1 \otimes h_2$ there exists $x_1 \in X_1$ and $x_2 \in X_2$ with $h_i(x_i) > 0, i = 1, 2$. Hence

$$0 = \sum_{j=1}^{k_1 k_2+1} g_j(x_1 \otimes x_2) \geq h_1(x_1) h_2(x_2) > 0;$$

a contradiction. Passing to a subsequence, if necessary we get by Lemma 4.3

$$a_{jk} f_{jk} \to m_j \in S_1^* \otimes S_2^*.$$

Consequently,

$$f = 1/(k_1 k_2 + 1) \sum_{j=1}^{k_1 k_2+1} (k_1 k_2 + 1) m_j \in \mathrm{co}\,(S_1^* \otimes S_2^*),$$

as required. ■

Lemma 4.5. *Let X be a normed space and let $v_1, \ldots, v_k \in X$. Let $P : X \to X$ be a continuous, linear operator given by:*

$$Px = \sum_{i=1}^{k} \left(\sum_{j=1}^{k_i} a_{ij} f_{ij}(x) \right) v_i,$$

where for any $i \in \{1, \ldots, k\}$ $\sum_{j=1}^{k_i} a_{ij} = 1$ and $f_{ij} \in X^*$ for any i, j. Set for $i \in \{1, \ldots, k\}$ $D_i = \{a_{i1}, \ldots, a_{ik_i}\}$ and

$$D = \prod_{i=1}^{k} D_i.$$

Then

$$P = \sum_{(j_1, \ldots j_k) \in D} (a_{1j_1} \cdot \ldots \cdot a_{kj_k}) \left(\sum_{i=1}^{k} f_{ij_i}(\cdot) v_i \right). \tag{6}$$

Proof. Note that for any $i \in \{1, \ldots, k\}$, and $j \in \{1, \ldots, k_i\}$,

$$a_{ij} = a_{ij} \prod_{l \neq i} \left(\sum_{j=1}^{k_l} a_{lj} \right).$$

Applying the above equality to each a_{ij}, we get that P can also be represented by the right-side of (6), which completes the proof. ■

Theorem 4.1. *Assume that X_1, X_2 are finite-dimensional Banach spaces. Let $X = X_1 \otimes_\alpha X_2$, where α is a reasonable cross-norm on X. Assume that V, V_1, V_o, and S are given by (2)–(5).*

Additionally, assume that for $i = 1, 2$ *and* $f \in S_i^* \setminus \{0\}$ *there exists* $x_i \in X_i$ *with* $f(x_i) > 0$. *Then*

$$\mathcal{P}_S = \mathcal{L}_V(X, V_o) + \mathrm{co}\,(\mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}).$$

Proof. Fix $Q_o \in \mathcal{P}_S$. By Lemma 4.2

$$Q_o = P_o + P_1,$$

where $P_o \in \mathcal{P}_{V_1}(X, V_o)$ and $P_1 \in \mathcal{P}_{S,V_1}$. Let for $i = 1, 2, C_i = \{1, \ldots, l_i\}$, where $l_i = dim(S_i^*|_{W_i})$. Notice that by definition of $\mathcal{P}_{S,V_1}$,

$$P_1 = \sum_{(i,j)\in C_1\times C_2} \phi_{ij}(\cdot) v_{1i} \otimes v_{2j},$$

where $\phi_{ij} \in S^*$,

$$\phi_{ij}(w_{1l} \otimes w_{2k}) = \delta_{il}\delta_{jk}, \tag{7}$$

where $i = 1, \ldots, l_1 l = 1, \ldots, m_1 j = 1, \ldots, l_2$ and $k = 1, \ldots, m_2$. Here for $i = 1, 2, \{w_{il}, l = 1, \ldots, m_i\}$ is a fixed basis of W_{i1} such that $w_{il} = v_{il}$ for $l = 1, \ldots, l_i$ and $\{w_{il} : l = l_i + 1, \ldots, m_i\}$ is a fixed basis of W_{io}. By Lemma 4.4 for any $(i, j) \in C_1 \times C_2$

$$\phi_{ij} = \sum_{(l,k)\in D_{ij}} a_{lk}^{ij} f_{li} \otimes h_{kj}.$$

Here $D_{ij} \subset \mathbb{N}^2$ is a finite set $a_{lk}^{ij} > 0$,

$$\sum_{(l,k)\in D_{ij}} a_{lk}^{ij} = 1 \tag{8}$$

and $f_{li} \in S_1^* \setminus \{0\}$, $g_{kj} \in S_2^* \setminus \{0\}$. By (7)

$$\left(\sum_{(l,k)\in D_{ij}} a_{lk}^{ij} f_{li} \otimes h_{kj}\right)(v_{1i} \otimes v_{2j}) = 1 \tag{9}$$

and

$$\left(\sum_{(l,k)\in D_{ij}} a_{lk}^{ij} f_{li} \otimes h_{kj}\right)(v_{1p} \otimes v_{2q}) = 0 \tag{10}$$

if $(p, q) \neq (i, j)$. Since $f_{li} \in S_1^* \setminus \{0\}$, $g_{kj} \in S_2^* \setminus \{0\}$, by (10) for any $(l, k) \in D_{ij}$

$$f_{li} \otimes h_{kj}(v_{1p} \otimes v_{2q}) = 0 \tag{11}$$

if $(p, q) \neq (i, j)$. By (9) and (11), without loss of generality, we can assume that $f_{li}(v_{1i}) > 0$ and $h_{kj}(v_{2j}) > 0$ for any $(l, k) \in D_{ij}$. Consequently, by (8), (10) and (11) and definition of V_o, we can assume that for any $(l, k) \in D_{ij}$,

$$f_{li}(v_{1i}) = 1, \qquad h_{kj}(v_{2j}) = 1 \tag{12}$$

and

$$f_{li}(v_{1m}) = 0, \qquad h_{kj}(v_{2u}) = 0 \tag{13}$$

for $m \in \{1, \ldots, l_1\}, m \neq i$ and $u \in \{1, \ldots, l_2\}, u \neq j$. By Lemma 4.5 applied to P_1 and the above considerations

$$P_1 \in conv(\mathcal{P}_{S_1, W_{11}} \otimes \mathcal{P}_{S_2, W_{21}}). \tag{14}$$

Hence

$$P_1 = \sum_{j=1}^{k} b_j (P_{1j} \otimes P_{2j}), \tag{15}$$

where for $i = 1, 2$, and $j = 1, \ldots, k P_{ij} \in \mathcal{P}_{S_i, W_{i1}}, b_j \geq 0, \sum_{j=1}^{k} b_j = 1$.
Now for $i = 1, 2$ fix $Q_i \in \mathcal{P}_{W_{i1}}(X_i, W_{io})$. Let for $i = 1, 2$ and $j = 1, \ldots, k Q_{ij} = Q_i + P_{ij}$. Note that by definition of W_{io},

$$Q_{ij} \in \mathcal{P}_{S_i}(X_i, W_i) \tag{16}$$

for $i = 1, 2$. By (15),

$$Q_o = P_o - \sum_{j=1}^{k} b_j (Q_1 \otimes P_{2j} + P_{1j} \otimes Q_2 + Q_1 \otimes Q_2) + \sum_{j=1}^{k} b_j (Q_{1j} \otimes Q_{2j}).$$

Now we show that for any $j = 1, \ldots, k$

$$L_j = Q_1 \otimes P_{2j} + P_{1j} \otimes Q_2 + Q_1 \otimes Q_2 \in \mathcal{P}_{V_1}(X, V_o).$$

Let

$$U_1 = \{1, \ldots, l_1\} \times \{l_2 + 1, \ldots, m_2\} \cup \{l_1 + 1, \ldots, m_1\} \times \{1, , \ldots, l_2\}$$

$$\cup \{l_1 + 1, \ldots, m_1\} \times \{l_2 + 1, \ldots, m_2\} \tag{17}$$

and

$$U_2 = \{1, \ldots, l_1\} \times \{1, \ldots, l_2\}. \tag{18}$$

Fix $(u, l) \in U_1$. Note that for any $j = 1, \ldots, k$, by definition of Q_1, Q_2 and P_{ij},

$$L_j (w_{1u} \otimes w_{2l}) = w_{1u} \otimes w_{2l}.$$

Analogously for any $(u, l) \in U_2$,

$$L_j (w_{1u} \otimes w_{2l}) = 0.$$

This shows that for any $j = 1, \ldots k L_j \in \mathcal{P}_{V_1}(X, V)$. Hence

$$F_1 = P_o - \sum_{j=1}^{k} b_j L_j \in \mathcal{L}_V(X, V_o).$$

Consequently, by (16),

$$Q_o = F_1 + \sum_{j=1}^{k} b_j (Q_{1j} \otimes Q_{2j}) \in \mathcal{L}_V(X, V_o) + \mathrm{co}\,(\mathcal{P}_{S_1} \otimes P_{S_2}),$$

as required.

Now assume that $Q_o \in \mathcal{L}_V(X, V_o) + conv(\mathcal{P}_{S_1} \otimes P_{S_2})$. Hence

$$Q_o = F_1 + \sum_{j=1}^{k} b_j (Q_{1j} \otimes Q_{2j}),$$

where $F_1 \in \mathcal{L}_V(X, V_o)$ and $Q_{ij} \in \mathcal{P}_{S_i}$. By definition, $Q_o \in \mathcal{P}(X, V)$. We show that $Q_o \in \mathcal{P}_S$. By Lemma 4.2 applied to X_i, S_i and Q_{ij} $i = 1, 2$, $j = 1, \ldots, k$

$$Q_{ij} = L_{ij} + P_{ij},$$

where $L_{ij} \in \mathcal{P}_{W_{i1}}(X_i, W_{io})$ and $P_{ij} \in \mathcal{P}_{S_i, W_{i1}}$. Hence

$$Q_o = F_1 + \sum_{j=1}^{k} b_j (L_{1j} \otimes L_{2j} + L_{1j} \otimes P_{2j} + P_{1j} \otimes L_{2j}) + \sum_{j=1}^{k} b_j (P_{1j} \otimes P_{2j}),$$

where for any $j = 1, \ldots, k b_j \geq 0$ and $\sum_{j=1}^{k} b_j = 1$. Note that by definition and Lemma 4.4

$$F_1 + \sum_{j=1}^{k} b_j (L_{1j} \otimes L_{2j} + L_{1j} \otimes P_{2j} + P_{1j} \otimes L_{2j}) \in \mathcal{P}_{V_1}(X, V_o)$$

and

$$\sum_{j=1}^{k} b_j (P_{1j} \otimes P_{2j}) \in \mathcal{P}_{S, V_1}.$$

By Lemma 4.2 applied to X, S, V_1, V_o, $Q_o \in \mathcal{P}_S$, which completes the proof. ■

Now for a Banach space X and its closed subspace V let

$$\lambda(V, X) = \inf\{\|P\| : P \in \mathcal{P}(X, V)\}.$$

Analogously, if $S \subset X$ is a cone, we define

$$\lambda_S(V, X) = \inf\{\|P\| : P \in \mathcal{P}_S(X, V)\}.$$

In both cases we assume that infimum is taken over nonempty sets.

Theorem 4.2. *Let X_1, X_2 be finite-dimensional Banach spaces. Assume that for $i = 1, 2$ and $f \in S_i^* \setminus \{0\}$ there exists $x_i \in X_i$ with $f(x_i) > 0$. Let for $i = 1, 2$,*

$$\mathcal{P}_{S_i} = \mathcal{P}_{W_{i1}}(X_i, W_{io}) + \mathcal{P}_{S_i, W_{i1}}$$

where

$$\mathcal{P}_{W_i}(X_i, W_{io}) = \{P \in \mathcal{P}(X_i, W_{io}) : P|_{W_i} = 0\}$$

and

$$\mathcal{P}_{S_i, W_{i1}} = \{P \in \mathcal{P}(X_i, W_{i1}) : P(S_i) \subset S_i, P|_{W_{io}} = 0\}.$$

Assume that P_1 is a minimal projection in $\mathcal{P}_{S_1}$ and P_2 is a minimal projection in $\mathcal{P}_{S_2}$. Let for $i = 1, 2$, $P_i = Q_i + R_i$, where $Q_i \in \mathcal{P}_{W_{i1}}(X_i, W_{io})$ and $R_i \in \mathcal{P}_{S_i, W_{i1}}$. Also define for $i = 1, 2$

$$\mathcal{W}_i = \mathcal{L}(X_i, W_{io}) - \mathcal{P}_{S_i, W_{i1}}.$$

Then

$$\lambda_S(X, V) \geq dist(Q_1, \mathcal{W}_1) dist(Q_2, \mathcal{W}_2).$$

Proof. Since for $i = 1, 2$, $\mathcal{W}_i$ is finite-dimensional, there exists $Z_i \in \mathcal{W}_i$ such that

$$\|Q_i - Z_i\| = dist(Q_i, \mathcal{W}_i).$$

Set for $i = 1, 2$

$$C_i = \{(x^*, x) \in S_{X_i^*} \times S_{X_i} : x^*((Q_i - Z_i)x) = \|Q_i - Z_i\|\}.$$

Note that for $i = 1, 2$, $\mathcal{W}_i$ is a closed, convex subset of $\mathcal{L}(X_i)$. By the separation theorem applied to the open ball in $\mathcal{L}(X_1)$ of radius $dist(Q_1, \mathcal{W}_1)$ with a center at Q_1 and $\mathcal{W}_1$ there exists $F_1 \in (\mathcal{L}(X_1))^*$, $\|F_1\| = 1$, such that

$$F_1(Q_1 - Q) \geq F_1(Q_1 - Z_1) = \|Q_1 - Z_1\| \tag{19}$$

for any $Q \in \mathcal{W}_i$. Analogously, there exists $F_2 \in (\mathcal{L}(X_2))^*$, $\|F_2\| = 1$, such that

$$F_2(Q_2 - Q) \geq F_2(Q_2 - Z_2) = \|Q_2 - Z_2\| \tag{20}$$

for any $Q \in \mathcal{W}_2$. By the Choquet Theorem (see e.g. [10,3]) for $i = 1, 2$, there exists a probabilistic Borel measure μ_i supported on C_i such that

$$F_i(L) = \int_{C_i} x^*(Lx) \mathrm{d}\mu_i(x^*, x)$$

for any $L \in \mathcal{L}(X_i)$. Let us define a functional T on $\mathcal{L}(X)$ by

$$T(L) = \int_{C_1 \times C_2} (x_1^* \otimes x_2^*) L(x_1 \otimes x_2) \mathrm{d}(\mu_1 \otimes \mu_2)(x_1^*, x_1, x_2^*, x_2).$$

Since α is a reasonable, cross-norm, and μ_i is a probabilistic measure for $i = 1, 2$, $\|T\| \leq 1$. Note that by the Fubini theorem and definition of F_1 and F_2,

$$T((Q_1 - Z_1) \otimes (Q_2 - Z_2)) = F_1(Q_1 - Z_1) F_2(Q_2 - Z_2) = \|Q_1 - Z_1\| \|Q_2 - Z_2\|.$$

Now we show that

$$T(R) \geq \|Q_1 - Z_1\| \|Q_2 - Z_2\|$$

for any $R \in \mathcal{P}_S$. First assume that $R = R_1 \otimes R_2$, where $R_1 \in \mathcal{P}_{S_1}$ and $R_2 \in \mathcal{P}_{S_2}$. Note that by the Fubini Theorem,

$$\begin{aligned} T(R) &= T(R_1 \otimes R_2) \\ &= \int_{C_1 \times C_2} x_1^*(R_1 x_1) x_2^*(R_2 x_2) \mathrm{d}(\mu_1 \otimes \mu_2)(x_1^*, x_1, x_2^*, x_2) \\ &= F_1(R_1) F_2(R_2). \end{aligned}$$

Since for $i = 1, 2$, $Q_i - R_i \in \mathcal{W}_i$, by (19) and (20),

$$\begin{aligned} F_1(R_1) F_2(R_2) &= F_1(Q_1 - (Q_1 - R_1)) F_2(Q_2 - (Q_2 - R_2)) \\ &\geq F_1(Q_1 - Z_1) F_2(Q_2 - Z_2) = \|Q_1 - Z_1\| \cdot \|Q_2 - Z_2\|. \end{aligned}$$

Now take any $R \in \mathcal{P}_S$. By Theorem 4.1,

$$R = L + C,$$

where $L \in \mathcal{L}_V(X, V_o)$ and $C \in conv(\mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2})$. Consequently, by the previous reasoning,

$$T(R) = T(L) + T(C) \geq T(L) + \|Q_1 - Z_1\| \cdot \|Q_2 - Z_2\|.$$

To end the proof, it is necessary to show that for any $L \in \mathcal{L}_V(X, V)$

$$T(L) = 0. \tag{21}$$

Notice that for $i = 1, 2$, $L - Z_i \in \mathcal{W}_i$ and $-(L + Z_i) \in \mathcal{W}_i$. Hence the functionals F_i defined by (19) and (20) satisfy

$$F_i(L_i) = 0 \tag{22}$$

for any $L_i \in \mathcal{L}(X_i, W_{io})$. Note that $\mathcal{L}_V(X, V_o)$ is spanned by mappings of type

$$Lx = f(x)w, \tag{23}$$

where $x \in X$, $f \in X^*$, $f|_V = 0$ and $w \in V_o$. By definition of V_o, we can assume that

$$w = w_{1l} \otimes w_{2k}, \tag{24}$$

where $(l, k) \in U_1$ (see (17)). Hence $l \geq l_1 + 1$ or $k \geq l_2 + 1$. Since X_i $i = 1, 2$ are finite-dimensional, we only can consider $f = h_1 \otimes h_2$, where $h_i \in X_i^*$ and $h_1|_{W_1} = 0$ or $h_2|_{W_2} = 0$. Notice that if $k \geq l_2 + 1$, the mapping $L_2 = h_2(\cdot)w_{2k} \in \mathcal{L}(X_2, W_{2o})$. By (22)

$$F_2(L_2) = 0. \tag{25}$$

If $l \geq l_1 + 1$, the mapping $L_1 = h_1(\cdot)w_{1l} \in \mathcal{L}(X_1, W_{1o})$. Hence again by (22),

$$F_1(L_1) = 0. \tag{26}$$

Since $L = L_1 \otimes L_2$, by (25), (26) and the Fubini theorem,

$$T(L) = T(L_1 \otimes L_2) = (F_1 \otimes F_2)(L) = F_1(L_1)F_2(L_2) = 0,$$

which proves (21). The proof of our theorem is fully complete. ■

Now we proof a version of Theorem 4.2 in the case of arbitrary Banach spaces X_1, X_2. First we need the following two lemmas which permit us to approximate infinite-dimensional case by, considered in Theorem 4.2, finite-dimensional situation.

Lemma 4.6. *Let X be an infinite-dimensional Banach space, $V \subset X$ be its finite-dimensional subspace and $S \subset X$ a closed convex cone. Assume that $\mathcal{P}_S(X, V) \neq \emptyset$. Let $\{X_b\}_{b \in B}$ be a directed (by inclusion) family of finite-dimensional subspaces of X such that $V \subset X_b$ for any $b \in B$,*

$$cl\left(\sum_{b \in B} X_b\right) = X$$

and

$$cl\left(\sum_{b \in B} (X_b \cap S)\right) = S.$$

Assume that $P_o \in \mathcal{P}_S(X, V)$, $P_o = Q_o + R_o$, where $Q_o \in \mathcal{P}_{V_1}(X, V_o)$ and $R_o \in \mathcal{P}_{S,V_1}$, (see Lemma 4.2). Let $\mathcal{W} = \mathcal{L}(X, V_o) - \mathcal{P}_{S,V_1}$ and for $b \in B$, $\mathcal{W}_b = \mathcal{L}(X_b, V_o) - (\mathcal{P}_{S,V_1})|_{X_b}$. Then

$$dist(Q_o, \mathcal{W}) = \sup_{b \in B} dist(Q_o|_{X_b}, \mathcal{W}_b).$$

Proof. Note that for any $b \in B$ and $L \in \mathcal{W}$,

$$\|Q_o - L\| \geq \|(Q_o - L)|_{X_b}\| \geq dist(Q_o|_{X_b}, \mathcal{W}_b)$$

which gives immediately that

$$dist(Q_o, \mathcal{W}) \geq \sup_{b \in B} dist(Q_o|_{X_b}, \mathcal{W}_b).$$

To prove the converse, assume on the contrary that

$$dist(Q_o, \mathcal{W}) > \sup_{b \in B} dist(Q_o|_{X_b}, \mathcal{W}_b) + d$$

for some $d > 0$. Let $L_b \in \mathcal{W}_b$ be so chosen that $\|Q_o|_{X_b} - L_b\| \leq dist(Q_o, \mathcal{W}) - d/2$. Define

$$\mathcal{U} = \prod_{x \in \bigcup_{b \in B} X_b} B_V(0, \|x\|(dist(Q_o, \mathcal{W}) - d/2)),$$

where $B_V(0, r)$ denotes the closed ball in V with radius r and center at 0. Let us equip $\mathcal{U}$ with the Tychonoff topology, where in each $B_V(0, \|x\|(dist(Q_o, \mathcal{W}) - d/2))$ we consider the topology determined by the norm. Since V is finite-dimensional, by the Tychonoff theorem, $\mathcal{U}$ is a compact set. Define for any $b \in B$, $Z_b : \bigcup_{c \in B} X_c \to V$ by

$$Z_b x = Q_o|_{X_b} x - L_b x$$

if $x \in X_b$ and $Z_b x = 0$ in the opposite case. Note that $Z_b \in \mathcal{U}$ for any $b \in B$. Define for $b \in B$

$$\mathcal{D}_b = cl(\{Z_c : c \in B, c \geq b\}),$$

where $c \geq b$ means that $X_b \subset X_c$. Since $\{X_b\}_{b \in B}$ is directed by inclusion,

$$\bigcap_{n=1}^{k} \mathcal{D}_{b_k} \neq \emptyset$$

for any $k \in \mathbb{N}$ and $b_1, \ldots, b_k \in B$. Since $\mathcal{U}$ is a compact set,

$$\bigcap_{b \in B} \mathcal{D}_b \neq \emptyset.$$

Take any $Z \in \bigcap_{b \in B} \mathcal{D}_b$. First we show that Z is a linear mapping. To do this, fix $x, y \in \bigcup_{b \in B} X_b$. Then there exists $b \in B$, such that $x, y \in X_c$ for any $c \geq b$. Fix $\epsilon > 0$. Since $Z \in \mathcal{D}_b$, there exists $d \geq b$ such that $\|Zx - Z_d x\| \leq \epsilon/3$, $\|Zy - Z_d y\| \leq \epsilon/3$ and $\|Z(x+y) - Z_d(x+y)\| \leq \epsilon/3$. Since

$$Z_d(x + y) = Z_d x + Z_d y,$$

$\|Z(x + y) - Zx - Zy\| \leq \epsilon$. This shows that $Z(x + y) = Zx + Zy$. Analogously we can demonstrate that $Z(\alpha x) = \alpha P x$ for any $x \in \bigcup_{b \in B} X_b$ and $\alpha \in \mathbb{R}$. Hence Z is linear. Moreover, by definition of mappings Z_b,

$$\sup \left\{ \|Zx\| : x \in \bigcup_{b \in B} X_b, \|x\| = 1 \right\} \leq dist(Q_o, \mathcal{W}) - d/2.$$

Hence Z is a continuous, linear mapping. Since $cl(\sum_{b \in B} X_b) = X$ and V is finite-dimensional, we can extend definition of Z onto the whole X. It is clear that

$$\|Z\| \leq dist(Q_o, \mathcal{W}) - d/2.$$

To get a contradiction we show that $Q_o - Z \in \mathcal{W}$. To do this, fix $x \in X_b \cap S$. Note that we can find a sequence $\{b_k\} \subset \mathcal{D}_b$, such that

$$L_{b_k} x \to (Q_o - Z)x.$$

Since S is closed and $L_{b_k} x \in S$, $(Q_o - Z)x \in S$ too. Since $cl(\sum_{b \in B}(X_b \cap S)) = S$, $(Q_o - Z)x \in S$ for any $x \in S$. This shows that $Q_o - Z \in \mathcal{W}$. The proof is complete. ■

Reasoning in an similar way, we can prove the following

Lemma 4.7. *Let X be an infinite-dimensional Banach space, $V \subset X$ be its finite-dimensional subspace and $S \subset X$ a closed convex cone. Assume that $\mathcal{P}_S(X, V) \neq \emptyset$. Let $\{X_b\}_{b \in B}$ be a directed (by inclusion) family of finite-dimensional subspaces of X such that $V \subset X_b$ for any $b \in B$,*

$$cl\left(\sum_{b \in B} X_b\right) = X$$

and

$$cl\left(\sum_{b \in B} (X_b \cap S)\right) = S.$$

Then

$$\lambda_S(V, X) = \sup_{b \in B} \lambda_S(V, X_b).$$

Theorem 4.3. *For $i = 1, 2$ assume X_i, W_i and S_i satisfy the assumptions of Lemma 4.7. Assume further that for $i = 1, 2$ there exists a directed (by inclusion) family $\{X_b^i\}_{b \in B_i}$ of finite-dimensional subspaces of X_i such that $W_i \subset X_b^i$ for any $b \in B_i$ which satisfies the assumptions of Lemma 4.7. Assume for $i = 1, 2$ the cones S_i are such that for $f \in S_i^* \setminus \{0\}$ there exists $x_i \in X_i$ with $f(x_i) > 0$. For $i = 1, 2$, let*

$$\mathcal{P}_{S_i} = \mathcal{P}_{W_{i1}}(X_i, W_{io}) + \mathcal{P}_{S_i, W_{i1}}$$

where

$$\mathcal{P}_{W_i}(X_i, W_{io}) = \{P \in \mathcal{P}(X_i, W_{io}) : P|_{W_i} = 0\}$$

and

$$\mathcal{P}_{S_i, W_{i1}} = \{P \in \mathcal{P}(X_i, W_{i1}) : P(S_i) \subset S_i, P|_{W_{io}} = 0\}.$$

Assume that P_1 is a minimal projection in $\mathcal{P}_{S_1}$ and P_2 is a minimal projection in $\mathcal{P}_{S_2}$. Let for $i = 1, 2$, $P_i = Q_i + R_i$, where $Q_i \in \mathcal{P}_{W_{i1}}(X_i, W_{io})$ and $R_i \in \mathcal{P}_{S_i, W_{i1}}$. Also define for $i = 1, 2$

$$\mathcal{W}_i = \mathcal{L}(X_i, W_{io}) - \mathcal{P}_{S_i, W_{i1}}.$$

Then

$$\lambda_S(X, V) \geq dist(Q_1, \mathcal{W}_1) dist(Q_2, \mathcal{W}_2).$$

Proof. Let for any $i = 1, 2$ and $b \in B_i$, $P^i_b \in \mathcal{P}_{S_i}(X^i_b, W_i)$ be so chosen that

$$\|P^i_b\| = \lambda_{S_i}(W_i, X^i_b).$$

This is possible, since $dim(X^i_b) < \infty$. Let $P^i_b = Q^i_b + R^i_b$, where $Q^i_b \in \mathcal{P}_{S_i}(X^i_b, W_i)$ and $R^i_b \in \mathcal{P}_{S_i,W_{i1}}(X^i_b, W_{i1})$. By Theorem 4.2, for any $(b_1, b_2) \in B_1 \times B_2$,

$$\lambda_S(X^1_{b_1} \otimes_\alpha X^2_{b_2}, V) \geq dist(Q^1_{b_1}, \mathcal{W}_{1b})dist(Q^2_{b_2}, \mathcal{W}_{2b})$$

(compare with Lemma 4.7). By 4.7 applied to X and $\{X^1_{b_1} \otimes_\alpha X^2_{b_2} : (b_1, b_2) \in B_1 \times B_2, \}$

$$\lambda_S(V, X) = \sup\{\lambda_S(X^1_{b_1} \otimes_\alpha X^2_{b_2}, V) : (b_1, b_2) \in B_1 \times B_2\}$$

Notice that for $i = 1, 2$ and $b_i \in B_i$,

$$dist(Q^i_{b_i}, \mathcal{W}_i) = dist(Q^i|_{X_{b_i}}, \mathcal{W}_{ib}).$$

Hence by Lemma 4.6,

$$\begin{aligned}\lambda_S(V, X) &\geq \sup_{b_1\in B_1} dist(Q^1|_{X_{b_1}}, \mathcal{W}_{1b}) \sup_{b_2\in B_2} dist(Q^2|_{X_{b_2}}, \mathcal{W}_{2b})\\ &= dist(Q^1, \mathcal{W}_1)dist(Q_2, \mathcal{W}_2).\end{aligned}$$

The proof is complete. ■

From Theorem 4.3 one can get immediately.

Theorem 4.4. *Assume that W_1, S_1, X_1, P_1 and W_2, S_2, X_2, P_2 satisfy the assumptions of Theorem 4.3. Assume furthermore that α is a reasonable, uniform cross-norm and for $i = 1, 2$,*

$$dist(Q_i, \mathcal{W}_i) = dist(Q_i, \mathcal{L}_{W_i}(X_i, W_{io}) - \mathcal{P}_{S_i,W_{1i}}).$$

Then

$$\lambda_S(V, X) = \lambda_{S_1}(W_1, X_1)\lambda_{S_2}(W_2, X_2).$$

Proof. By Theorem 4.3,

$$\begin{aligned}\lambda_S(V, X) &\geq dist(Q_1, \mathcal{W}_1)dist(Q_2, \mathcal{W}_2)\\ &= dist(Q_1, \mathcal{L}_{W_1}(X_1, W_{1o}) - \mathcal{P}_{S_1,W_{11}})dist(Q_2, \mathcal{L}_{W_2}(X_2, W_{2o}) - \mathcal{P}_{S_2,W_{21}}).\end{aligned}$$

Since $\|P_1\| = \lambda_{S_1}(W_1, X_1)$ and $\|P_2\| = \lambda_{S_2}(W_1, X_1)$,

$$dist(Q_i, \mathcal{L}_{W_i}(X_i, W_{oi}) - \mathcal{P}_{S_i,W_{1i}}) = \|P_i\|$$

for $i = 1, 2$. Consequently,

$$\lambda_S(V, X) \geq \|P_1\|\|P_2\|.$$

Since α is uniform,

$$\|P_1 \otimes P_2\| = \|P_1\|\|P_2\|.$$

Since $P_1 \otimes P_2 \in \mathcal{P}_S$,

$$\lambda_S(V, X) \leq \|P_1\|\|P_2\| = \lambda_{S_1}(W_1, X_1)\lambda_{S_2}(W_2, X_2).$$

The proof is complete. ■

Reasoning as in the proof of Theorems 4.2 and 4.4 we can show that, in general, $P_1 \otimes P_2$ is a projection of minimal norm in a smaller set than $\mathcal{P}_S$.

Theorem 4.5. *Assume that W_1, S_1, X_1 and W_2, S_2, X_2 satisfy the assumptions of Theorem 4.3. Assume that P_1 is a minimal projection in $\mathcal{P}_{S_1}$ and P_2 is a minimal projection in $\mathcal{P}_{S_2}$. Then $P_1 \otimes P_2$ is a minimal projection in*

$$\mathcal{W} = \mathcal{L}(X_1, W_1) \otimes \mathcal{L}_{W_2}(X_2, W_{2o}) + \mathcal{L}_{W_1}(X_1, W_{1o}) \otimes \mathcal{L}(X_2, W_2) + conv(\mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}).$$

Proof. Applying the reasoning from Theorems 4.2 and 4.3, we only need to show that $T(L) = 0$ for any $L \in \mathcal{L}(X_1, W_1) \otimes \mathcal{L}_{W_2}(X_2, W_{2o}) + \mathcal{L}_{W_1}(X_1, W_{1o}) \otimes \mathcal{L}(X_2, W_2)$. But this is true since $F_i(K) = 0$ for any $K \in \mathcal{L}_{W_i}(X_i, W_{io})$, $i = 1, 2$. ■

In general $P_1 \otimes P_2$ is not a minimal projection in $\mathcal{P}_S$, as demonstrated in the next example.

Example 4.1. Let $X_1 = (\mathbb{R}^3, \|\cdot\|_e)$, where $\|\cdot\|_e$ denote the Euclidean norm. Let $W_1 = span[e_1, e_2]$, where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Let for $x \in \mathbb{R}^3$ and $i = 1, 2, 3$, $f_i(x) = x_i$ and $g_2(x) = x_2 + x_3$. Define

$$S_1 = \{x \in \mathbb{R}^3 : g_2(x) \geq 0\}.$$

Notice that in our case $W_{10} = span[e_1]$ and $W_{11} = span[e_2]$. We show that a projection

$$P_1 x = f_1(x)e_1 + g_2(x)e_2 = (x_1, x_2 + x_3, 0)$$

is a minimal projection in $\mathcal{P}_{S_1}$. It is clear that $P_1 \in \mathcal{P}_{S_1}$. Since the function $x \to x_1^2 + (x_2 + x_3)^2$ restricted to the unit Euclidean sphere in $\mathbb{R}^3$, achieves its maximum at $x = (0, 1/\sqrt{2}, 1/\sqrt{2})$, $\|P_1\| = \sqrt{2}$. Take any $R \in \mathcal{P}_{S_1}$. Note that

$$Rx = g_1(x)e_1 + g_2(x)e_2,$$

where $g_1(e_1) = 1$ and $g_1(e_2) = 0$. Hence

$$\|R\| \geq \|R(0, 1/\sqrt{2}, 1/\sqrt{2})\| \geq \sqrt{2} = \|P_1\|,$$

which shows our claim.

Now we define X_2, W_2 and S_2. Let for $m \in \mathbb{N}$, F_m denote the classical Fourier projection from $C_o(2\pi)$ onto π_m, where π_m denotes the space of all trigonometric polynomials of degree $\leq m$. Choose $m \in \mathbb{N}$ such that

$$\|F_m\|\|P_1\| > \|F_m\| + 2. \tag{27}$$

This is possible since $\|F_m\| \geq (4/\pi^2)\log(m)$ (see e.g. [1], p. 213) and $\|P_1\| = \sqrt{2}$. Let $X_2 = C_o(2\pi)$ and

$$S_2 = \{p \in X_2 : \int_0^{2\pi} \sin(mt)p(t)\mathrm{d}t \geq 0\}.$$

Let $P_2 = F_m$. Note that

$$P_2 = R_2 + h_m(\cdot)(\sin(m\cdot)),$$

where

$$h_m(p) = (1/\pi)\int_0^{2\pi} p(t)\sin(mt)\mathrm{d}t$$

and R_2 is portion of F_m without the $\sin(m\cdot)$ term. Hence $P_2 \in \mathcal{P}_{S_2}$. Moreover P_2 is a minimal projection in $\mathcal{P}_{S_2}$, since P_2 is a minimal projection in $\mathcal{P}(C_o(2\pi), \pi_m)$. Notice that in our case

$$W_{2o} = span[\cos(j\cdot), j = 0, \dots, m, \sin(j\cdot), j = 1, \dots, m-1]$$

and

$$W_{21} = span[\sin(m\cdot)].$$

Now we show that $P_1 \otimes P_2$ is not a minimal projection in $X_1 \otimes_\alpha X_2$, for any reasonable, uniform cross-norm α. Let us define

$$Z = (f_1(\cdot)e_1) \otimes R_2 + (f_2(\cdot)e_2) \otimes R_2 + (f_1(\cdot)e_1) \otimes (h_m(\cdot)\sin(m\cdot))$$

and

$$D = (g_2(\cdot)e_2) \otimes (h_m(\cdot)\sin(m\cdot)).$$

Set $Q = Z + D$. It is clear that $Q \in \mathcal{P}_S$. Let

$$D_1 = (f_2(\cdot)e_2) \otimes (h_m(\cdot)\sin(m\cdot))$$

and

$$P_o x = f_1(x)e_1 + f_2(x)e_2 = (x_1, x_2, 0)$$

Note that, by (27),

$$\begin{aligned} \|Q\| &= \|Z + D_1 + D - D_1\| \le \|Z + D_1\| + \|D - D_1\| \\ &= \|P_o \otimes P_2\| + \|D - D_1\| \le \|P_o\|\|P_2\| + 2 \\ &= \|P_2\| + 2 < \|P_1\|\|P_2\| = \|P_1 \otimes P_2\|, \end{aligned}$$

since $\|P_o\| = 1$. Hence $P_1 \otimes P_2$ is not a minimal projection in $\mathcal{P}_S$, as required.

Now we show an application of Theorem 4.4. Let for $i = 1, 2$ $X_i = (C^{L_i}[0,1], \|\cdot\|_{L_i})$, denote the set of all L_i-times continuously differentiable functions on $[0, 1]$ normed by

$$\|f\|_{L_i} = \max_{j=0,\dots,L_i} \|f^{(j)}\|_{\sup}.$$

Put $W_1 = \pi_{n_1}$ and $W_2 = \pi_{n_2}$, where π_l denote the space of all algebraic polynomials of degree $\le l$ restricted to $[0, 1]$. Assume that $2 \le n_i \le L_i$ for $i = 1, 2$. Let for $i = 1, 2$, $\sigma_i = (\sigma_{io}, \dots, \sigma_{in_i})$ be an $(n_i + 1)$-tuple with $\sigma_{ij} \in \{0, 1\}$, such that $\sigma_i \ne 0$. Define

$$S_i = \{f \in X_i : \sigma_{ij} f^{(j)} \ge 0, j = 0, \dots, L_i\}. \tag{28}$$

Also set

$$m_i = \min\{j : \sigma_{ij} = 1\} \quad \text{and} \quad M_i = \max\{j : \sigma_{ij} = 1\}. \tag{29}$$

Assume that

$$m_i \le M_i - 1 \tag{30}$$

for $i = 1, 2$. In ([5], Th. 2.1) it has been shown that $\mathcal{P}_{S_i} \ne \emptyset$ if and only if $M_i \ge n_i - 1$ and $\sigma_{ij} = 1$ for $j = m_i, \dots, M_i$. Moreover, minimal projections $P_i \in \mathcal{P}_{S_i}$ have been determined

([5], Th. 2.3, 2.4 and 2.5) and it has been shown that

$$\|P_i\| = \sum_{j=0}^{n_i-m_i-1} 1/j!. \tag{31}$$

Let

$$S = (\mathrm{co}\,((S_{\sigma_1})^* \otimes (S_{\sigma_2})^*))^* \cap (X_1 \otimes_\alpha X_2).$$

Now, we show applying Theorem 4.4, that $P_1 \otimes_\alpha P_2$ is a minimal projection in $X_1 \otimes_\alpha X_2$ for any uniform, reasonable cross-norm α.

Theorem 4.6. *Let $P_i \in \mathcal{P}_{S_i}$ be minimal projections for $i = 1, 2$. Then $P_1 \otimes_\alpha P_2$ is a minimal projection in $\mathcal{P}_S$ for any reasonable, uniform cross-norm α.*

Proof. The goal is to show that P_1 and P_2 satisfy the assumptions of Theorem 4.4. The majority of this effort is contained in demonstrating that

$$\|P_i\| = dist(Q_i, \mathcal{W}_i),$$

for $i = 1, 2$. To accomplish this, we will borrow extensively from [5]. Indeed, to conform to the notation of this paper, let $\hat{L}_i$ and $\hat{n}_i$ denote the positive integers such that

$$L_i = \hat{L}_i + m_i \quad \text{and} \quad n_i = \hat{n}_i + m_i.$$

Then (as in Theorems 2.3, 2.4 and 2.5 from [5]) $P_{m_i,\hat{n}_i+m_i}$ denotes the operator obtained by m_i applications of

$$(P_{j+1,\hat{n}_i+1} f)(x) = \frac{f(0) + f(1)}{2} + \int_0^x (P_{j,\hat{n}_i} f')(t)\,\mathrm{d}t - \frac{1}{2}\int_0^1 (P_{j,\hat{n}_i} f')(t)\,\mathrm{d}t$$

starting with $P_{0,\hat{n}_i}$ given by

$$P_{0,\hat{n}_i} = \delta_0 \otimes 1 + \delta_0^1 \otimes \frac{x}{1!} + \cdots + \delta_0^{\hat{n}_i-1} \otimes \left(\frac{x^{\hat{n}_i-1}}{(\hat{n}_i-1)!} - \frac{x^{\hat{n}_i}}{(\hat{n}_i)!}\right) + \delta_1^{\hat{n}_i-1} \otimes \frac{x^{\hat{n}_i}}{(\hat{n}_i)!}.$$

From Theorem 2.5 of [5] we know $P_{m_i,\hat{n}_i+m_i}$ is minimal in $\mathcal{P}_{S_i}$ and thus

$$\|P_i\| = \|P_{m_i,\hat{n}_i+m_i}\| \tag{32}$$

for $i = 1, 2$. Now let $P_i = Q_i + R_i$ (compare with Theorem 4.2). Notice that for any $L \in \mathcal{W}_i$ we have

$$\begin{aligned} \|Q_i - L\| &= \sup_{f\in B(X_i)} \max\{|(Q_i f - Lf)^{(j)}(x)| : j = 0, \ldots, n_i, x \in [0, 1]\} && (33)\\ &\geq \sup_{f\in B(X_i)} \max\{|(Q_i f - Lf)^{(m_i)}(x)| : x \in [0, 1]\} && (34)\\ &= \sup_{f\in B(X_i)} \max\{|(Lf)^{(m_i)}(x)| : x \in [0, 1]\}. && (35) \end{aligned}$$

Note that $L \in \mathcal{W}_i$ implies we can write $L = \Lambda - R$ where

$$\Lambda \in \mathcal{L}(X_i, W_{i0}) \quad \text{and} \quad R \in \mathcal{P}_{S_i, W_{i1}}.$$

Additionally, notice that for any $f \in X_i$ we have

$$\max\{|(Lf)^{(m_i)}(x)| : x \in [0,1]\} = \max\{|(Rf)^{(m_i)}(x)| : x \in [0,1]\}. \tag{36}$$

Finally, choose any $P \in \mathcal{P}_{W_{i1}}(X_i, W_{i0})$ and define $Q = P + R$. From the definition of Q we have for every $f \in B(X_i)$,

$$\max\{|(Rf)^{(m_i)}(x)| : x \in [0,1]\} = \max\{|(Qf)^{(m_i)}(x)| : x \in [0,1]\}. \tag{37}$$

Moreover, it is clear that $Q \in \mathcal{P}_{S_i}$ and thus (as a consequence of equations (34), (35) and (36) from [5]) we also find

$$\sup_{f \in B(X_i)} \max\{|(Qf)^{(m_i)}(x)| : x \in [0,1]\} = \|P_{m_i, \hat{n}_i + m_i}\|. \tag{38}$$

Combining (33)–(38) with (32) we are led to conclude that

$$\|Q_i - L\| \geq \|P_i\|$$

and therefore $\|P_i\| = dist(Q_i, \mathcal{W}_i)$. Now define for $b \in \mathbb{N}$ and $i = 1, 2 X_b^i = \pi_b$. Observe that for $i = 1, 2 \{X_b^i\}_{b > n_i}$ satisfies the assumptions of Lemmas 4.6, 4.7 and Theorem 4.2. Consequently all the assumptions of Theorem 4.4 are satisfied. The proof is complete. ■

By Theorem 4.6 and ([5], Th. 2.3 and 2.5) and (31) we immediately get

Theorem 4.7.

$$\lambda_S(\pi_{n_1} \otimes_\alpha \pi_{n_2}, C^{L_1}[0,1] \otimes_\alpha C^{L_2}[0,1]) = \left(\sum_{j=0}^{n_1 - m_1 - 1} 1/j!\right)\left(\sum_{j=0}^{n_2 - m_2 - 1} 1/j!\right).$$

Note 4.1. By ([5], Th. 2.3 and 2.4) Theorems 4.6 and 4.7 remain true for a large class of norms in $C^{L_i}[0,1]$ different from $\|\cdot\|_{L_i}$ for $i = 1, 2$. Indeed, if $\{t_i\}$ is a countable, dense subset of $[0,1]$ such that $t_o = 0$ and $t_1 = 1$, then for $i = 1, 2$ we can define $\|\cdot\|_{L_i,k}$ by:

$$\|f\|_{L_i,k} = \max_{j=0,\dots,L_i} A_{jk}(f),$$

where

$$A_{jk}(f) = \max_{u=0,\dots,k} \{|f^{(j)}(t_u)| : j = 0, \dots, L_i - 1\}$$

and

$$A_{L_i k} = \|f^{(L_i)}\|_{\sup}.$$

Note 4.2. Theorems 4.2–4.4, 4.6 and 4.7 can be easily generalized by induction to the case of tensor product of more than two Banach spaces.

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