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On the log-concavity of the fractional integral of the sine function

Stamatis Koumandos

Department of Mathematics and Statistics, The University of Cyprus, P. O. Box 20537, 1678 Nicosia, Cyprus

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Abstract

We prove that the function

$$F_\lambda(x) := \int_0^x (x-t)^\lambda \sin t\, dt$$

is logarithmically concave on $(0, \infty)$ if and only if $\lambda \geq 2$. As a consequence, a Turán type inequality for certain Lommel functions of the first kind is obtained. Furthermore, some monotonicity properties of functions involving the remainders of the Taylor series expansion of the functions $\sin x$ and $\cos x$ are given.

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1. Introduction and results

A function $f : I \to (0, \infty)$ is called logarithmically concave (or log-concave, for short) on the interval I if $\log f$ is a concave function on I. If f is twice differentiable, the log-concavity

E-mail address: skoumand@ucy.ac.cy.

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of f on I is equivalent to $[f'(x)/f(x)]' \leq 0$ and, in turn, $f''(x)\,f(x) - [f'(x)]^2 \leq 0$ for all $x \in I$. Clearly, every positive and concave function is log-concave. The product of log-concave functions is log-concave, too. However, the sum of log-concave functions is not, in general, log-concave.

Log-concave functions appear frequently in many problems of classical analysis, probability theory and convex optimization. As it happens, many common probability distributions are log-concave [6]. The log-concavity of probability densities and of integrals involving probability densities has interesting qualitative implications in many areas of economics, in political science, in biology and in industrial engineering [4]. For further background information and applications of log-concave functions in both discrete and continuous setting, we refer to the recent survey paper [13].

Let $f : [0, \infty) \to \mathbb{R}$ be a locally integrable function. The fractional integral I_α, $\alpha > 0$, of f is defined by the formula

$$(I_\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, f(t)\, dt,$$

where $\Gamma(\alpha)$ is Euler's Gamma function. We refer to [3, p. 111] and [11, p. 98] for the definition, properties and applications of fractional integrals in the theory of special functions.

For $\lambda > 0$ we consider the fractional integral

$$F_\lambda(x) := \int_0^x (x-t)^\lambda\, \sin t\, dt, \qquad x > 0.$$

It should be mentioned that this is a positive function for all $x > 0$ and that $F_\lambda(x)$ can also be defined for $-1 < \lambda \leq 0$, but it is not strictly positive on $(0, \infty)$ for this range of λ, see Section 2.

The main result of this paper is the following.

Theorem 1.1. *The function $F_\lambda(x)$ is logarithmically concave on $(0, \infty)$, that is,*

$$F''_\lambda(x)\, F_\lambda(x) - [F'_\lambda(x)]^2 \leq 0, \quad \textit{for all } x > 0, \tag{1.1}$$

precisely when $\lambda \geq 2$. For $\lambda \geq 2$, equality occurs in (1.1) only when $\lambda = 2$ and $\tan \frac{x}{2} = \frac{x}{2}$.

We observe that

$$F_\lambda(x) = x^{\lambda+1} \int_0^1 (1-t)^\lambda\, \sin xt\, dt, \tag{1.2}$$

from which it follows that $F_\lambda(x)$ is infinitely often differentiable on $(0, \infty)$ for $\lambda > -1$.

We also have

$$F_\lambda(x) = \int_0^x t^\lambda\, \sin(x-t)\, dt = \sqrt{x}\, s_{\lambda+\frac{1}{2},\, \frac{1}{2}}(x), \tag{1.3}$$

where $s_{\mu,\nu}(z)$ is the Lommel function of the first kind. We recall that $s_{\mu,\nu}(z)$ is a particular solution of the inhomogeneous Bessel differential equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}.$$

It can be expressed in terms of a hypergeometric series

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\Big(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\Big).$$

For $\mu, \nu \in \mathbb{C}$ with $\Re(\mu \pm \nu + 1) > 0$ and $z \in \mathbb{A} := \mathbb{C} \setminus (-\infty, 0]$ we have the following integral representation

$$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z t^\mu J_\nu(t)\,dt - J_\nu(z)\int_0^z t^\mu Y_\nu(t)\,dt\right], \tag{1.4}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the usual Bessel functions (cf. [17, Sec. 10.7]).

Theorem 1.1 enables us to give a proof of the following Turán type inequality for some Lommel functions of the first kind.

Theorem 1.2. *The inequality*

$$\left(s_{a,\frac{1}{2}}(x)\right)^2 - s_{a-1,\frac{1}{2}}(x)s_{a+1,\frac{1}{2}}(x) \geq \frac{1}{\frac{1}{2}-a}\left(s_{a,\frac{1}{2}}(x)\right)^2 \tag{1.5}$$

holds true for all $x > 0$ *when* $a \geq \frac{3}{2}$. *This inequality fails to hold for appropriate* $x > 0$ *when* $-\frac{1}{2} < a < \frac{3}{2}$, $a \neq \frac{1}{2}$.

This quite settles a Conjecture posed and discussed in [5]. We note that inequality (1.5) is valid for all $x > 0$ when $a \in (-\frac{5}{2}, -\frac{1}{2})$, $a \neq -\frac{3}{2}$ according to the main result of [5]. The proof in this case is based on several properties of entire functions in the Laguerre–Pólya class. Results on Turán inequalities go back to the seminal work of Turán [16] and Szegő [15] for the case of Legendre polynomials. Since then various inequalities of this type for orthogonal polynomials and special functions have been obtained by many researchers. We refer the reader to [5,10] and the references given therein for background information and several results on Turán inequalities.

In the case where λ is a positive integer the function $F_\lambda(x)$ is closely related to the remainders of the Maclaurin series expansion of the functions $\sin x$ and $\cos x$. More specifically, we have

$$T_n(x) := \frac{1}{(2n)!}\int_0^x (x-t)^{2n}\,\sin t\,dt = (-1)^{n-1}\left(\cos x - \sum_{k=0}^{n}(-1)^k\,\frac{x^{2k}}{(2k)!}\right),$$

$$S_n(x) := \frac{1}{(2n+1)!}\int_0^x (x-t)^{2n+1}\,\sin t\,dt = (-1)^{n-1}\left(\sin x - \sum_{k=0}^{n}(-1)^k\,\frac{x^{2k+1}}{(2k+1)!}\right).$$

As an application of Theorem 1.1 we have the following.

Corollary 1.3. *For all* $n \geq 1$ *the functions* $T_n(x)$ *and* $S_n(x)$ *are positive, strictly increasing, convex and log-concave on* $(0, \infty)$.

Proof. It follows from Theorem 1.1 and the discussion in the beginning of Section 2. □

Corollary 1.4. *For all* $n \geq 1$ *the functions* $T_n(x)/S_n(x)$, $T_{n-1}(x)/T_n(x)$ *and* $S_{n-1}(x)/S_n(x)$ *are strictly decreasing on* $(0, \infty)$.

Proof. It follows from Corollary 1.3 that for $n \geq 1$ the function $S_n'(x)/S_n(x)$ is positive and strictly decreasing. Since $S_n'(x) = T_n(x)$ we infer that $T_n(x)/S_n(x)$ is strictly decreasing. Notice also that $T_n'(x) = S_{n-1}(x)$. Hence $T_{n-1}(x)/T_n(x) = \left(S_{n-1}'(x)/S_{n-1}(x)\right)\left(T_n'(x)/T_n(x)\right)$ is strictly decreasing by Corollary 1.3. We similarly have $S_{n-1}(x)/S_n(x) = \left(T_n'(x)/T_n(x)\right)\left(S_n'(x)/S_n(x)\right)$ is a strictly decreasing function. □

Since $F_\lambda'(x) = \lambda F_{\lambda-1}(x)$, for all $\lambda > 0$, Theorem 1.1 yields the following.

Corollary 1.5. *For all $\mu \geq 1$ we have that*

$$\left(1+\frac{1}{\mu}\right) F_\mu^2(x) - F_{\mu-1}(x)\, F_{\mu+1}(x) \geq 0, \quad \textit{for all } x > 0. \tag{1.6}$$

Equality occurs in the above only when $\mu = 1$ and $\tan \frac{x}{2} = \frac{x}{2}$.

In the case where μ is a positive integer the Turán type inequality (1.6) involves the remainders in the Maclaurin series expansions of the functions $\sin x$ and $\cos x$. It should be noted that inequality (1.6) fails to hold if the factor $1 + \frac{1}{\mu}$ is replaced by 1.

This paper is organized as follows. In the next section we give some preliminary reductions and proofs of some special cases of (1.1). In Section 3, we present a proof of Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Preliminary reductions

It is shown in [14,7] that inequality $s_{\mu,\frac{1}{2}}(x) > 0$ holds for all $x > 0$ precisely when $\mu > 1/2$. Using (1.3) we deduce that $F_\lambda(x) > 0$ for all $x > 0$ when $\lambda > 0$. Now by the elementary relation $F'_\lambda(x) = \lambda\, F_{\lambda-1}(x)$, it can be seen that for $\lambda \geq 2$ the function $F_\lambda(x)$ is positive, strictly increasing and convex on $(0, \infty)$.

The conclusion concerning the positivity of $F_\lambda(x)$, that is, the inequality

$$\int_0^x t^\lambda \sin(x-t)\, dt > 0, \quad x > 0,\ \lambda > 0, \tag{2.1}$$

has also been proved in [9, Prop. 3.1] in a simple and direct way.

It is relatively easy to see that the function $F_\lambda(x)$ is log-concave on $(0, \pi)$. Indeed, it follows readily that

$$g(x) := \int_0^1 (1-t)^\lambda \sin xt\, dt > 0, \qquad g''(x) = -\int_0^1 (1-t)^\lambda\, t^2 \sin xt\, dt < 0,$$
$$0 < x < \pi.$$

Therefore the function $g(x)$ is log-concave on $(0, \pi)$. It is straightforward that the function $h(x) := x^{\lambda+1}$ is log-concave. By (1.2) we conclude that $F_\lambda(x)$ is log-concave on $(0, \pi)$ as a product of log-concave functions.

In the case where $\lambda = 2$, inequality (1.1) reduces to an elementary one, *viz.*,

$$4 - 4x \sin x + x^2 \cos x - 4 \cos x + x^2 \geq 0, \quad \text{for } x > 0,$$

which, in turn, is equivalent to

$$\left(2 \sin\frac{x}{2} - x \cos\frac{x}{2}\right)^2 \geq 0.$$

We shall see in the next section that this is the only case where equality occurs in (1.1).

It is easy to verify the following recurrence relations

$$\frac{F'_\lambda(x)}{F_\lambda(x)} = \lambda\, \frac{F_{\lambda-1}(x)}{F_\lambda(x)}, \qquad \frac{F'_{\lambda-1}(x)}{F'_\lambda(x)} = \frac{\lambda-1}{\lambda}\, \frac{F_{\lambda-2}(x)}{F_{\lambda-1}(x)}. \tag{2.2}$$

The monotone form of l'Hospital rule, see [2,1,12] enables us to make one more reduction to the proof of Theorem 1.1 and it is given in Lemma 2.1.

Lemma 2.1. *Let $-\infty \le a < b \le \infty$ and let f and g be differentiable functions on (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Furthermore, suppose that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$ and f'/g' is (strictly) increasing (decreasing) on (a, b). Then the ratio f/g is (strictly) increasing (decreasing) too on (a, b).*

In view of this lemma and relations (2.2) we see that it is sufficient to prove Theorem 1.1 *only for* $2 < \lambda < 3$. The same reduction can be deduced in an equivalent way by applying Lemma 3 of [4, p. 466] quoted below.

Lemma 2.2. *Let $f : (a, b) \to (0, \infty)$ be a continuously differentiable function, $f(a) = \lim_{x\to a} f(x)$ and $F(x) = \int_a^x f(t)\,dt$. If f is log-concave on (a, b), then F is also log-concave on (a, b).*

On account of this, our reduction simply follows from the observation that

$$F_{\lambda+1}(x) = (\lambda + 1) \int_0^x F_\lambda(t)\,dt.$$

3. Proof of Theorem 1.1

We first prove that $F_\lambda(x)$ is log-concave on $[\pi, \infty)$, when $2 < \lambda < 3$. To this end, we use the following formula. For $0 < a < 1$ and $x > 0$ we have

$$\int_0^x t^a \sin(x - t)\,dt = x^a - \Gamma(a + 1) \cos\left(x - \frac{a\pi}{2}\right) + \frac{a}{\Gamma(1 - a)} \int_0^\infty e^{-xs} \frac{s^{1-a}}{s^2 + 1}\,ds. \tag{3.1}$$

This is obtained by combining formulas [8, p. 352, 3.389.6] and [8, p. 440, 3.768.5]. A short and direct proof of (3.1) is given in [9, Lemma 3.4].

Setting $\lambda = a + 2$, $0 < a < 1$, integrating twice by parts and using (3.1), we obtain

$$\begin{aligned} F_\lambda(x) &= x^{a+2} - (a + 1)(a + 2) \int_0^x t^a \sin(x - t)\,dt \\ &= (a + 1)(a + 2) \Big[\frac{x^{a+2}}{(a + 1)(a + 2)} - x^a + \Gamma(a + 1) \cos\left(x - \frac{a\pi}{2}\right) \\ &\quad - \frac{a}{\Gamma(1 - a)} \int_0^\infty e^{-xs} \frac{s^{1-a}}{s^2 + 1}\,ds \Big]. \end{aligned} \tag{3.2}$$

It is therefore sufficient to prove the log-concavity of the function in the square brackets in (3.2). For this purpose we shall apply the following elementary lemma.

Lemma 3.1. *Suppose that for all x in the interval I, we have that*

$$f(x) > 0, \qquad f'(x) > 0, \qquad f''(x) > 0$$

and that the function f is log-concave on I. If for all x in I the function φ satisfies

$$\varphi(x) > 0, \qquad \varphi'(x) < 0, \qquad \varphi''(x) > 0$$

and $f(x) - \varphi(x) > 0$, then $f - \varphi$ is log-concave on I.

Proof. We have

$$\begin{aligned}&\big(f''(x)-\varphi''(x)\big)\big(f(x)-\varphi(x)\big)-\big(f'(x)-\varphi'(x)\big)^2\\&=\big[f''(x)f(x)-(f'(x))^2\big]-f''(x)\varphi(x)-(f(x)-\varphi(x))\varphi''(x)-(\varphi'(x))^2\\&\quad+2f'(x)\varphi'(x)<0,\quad \forall x\in I,\end{aligned}$$

because every term in the second member of the equality above is negative in the interval in question. The proof is complete. □

Taking into account (3.2) we see that we have only to verify that the functions

$$f(x):=\frac{x^{a+2}}{(a+1)(a+2)}-x^a+\Gamma(a+1)\cos\Big(x-\frac{a\pi}{2}\Big) \tag{3.3}$$

and

$$\varphi(x):=\frac{a}{\Gamma(1-a)}\int_0^\infty e^{-xs}\,\frac{s^{1-a}}{s^2+1}\,ds, \tag{3.4}$$

satisfy the properties of Lemma 3.1 for $x\in[\pi,\infty)$.

We recall that a function $\varphi: I\to\mathbb{R}$ is called completely monotonic on I, if it has derivatives of all orders on I and satisfies $(-1)^n\varphi^{(n)}(x)>0$ for all $x\in I$ and $n\geq 0$. In the case where $I=(0,\infty)$, S. N. Bernstein [18, p. 160–161], gave the following characterization: φ is completely monotonic on $(0,\infty)$ if and only if there exists a nonnegative Borel measure m on $[0,\infty)$ such that $t\mapsto e^{-xt}$ is integrable with respect to m for all $x>0$ and $\varphi(x)=\int_0^\infty e^{-xt}\,dm(t)$.

It is readily seen that the function $\varphi(x)$ defined in (3.4) is completely monotonic and therefore strictly positive, decreasing and convex on $(0,\infty)$. The positivity of $F_\lambda(x)$ for $x>0$ implies that $f(x)-\varphi(x)>0$. This also gives that $f(x)>0$ for all $x>0$. On the other hand,

$$f''(x)=x^a+a(1-a)x^{a-2}-\Gamma(a+1)\cos\Big(x-\frac{a\pi}{2}\Big)>x^a-1>0,\quad \text{for all } x\geq\pi.$$

We have used the fact that $\Gamma(a+1)<1$, for $0<a<1$, which is well known.

Hence $f'(x)$ is strictly increasing on $[\pi,\infty)$, so that

$$f'(x)=\frac{x^{a+1}}{a+1}-ax^{a-1}-\Gamma(a+1)\sin\Big(x-\frac{a\pi}{2}\Big)\geq f'(\pi)>\frac{\pi^{a+1}}{a+1}-a\pi^{a-1}-1>0,$$

for $0<a<1$.

Next, we shall prove that the function $f(x)$ defined in (3.3) is log-concave on $[\pi,\infty)$.

Let

$$p(x):=\frac{x^{a+2}}{(a+1)(a+2)}-x^a.$$

It is easy to see that $p(x)$ is positive for $x\geq\pi$ and that

$$p''(x)p(x)-(p'(x))^2=-\frac{x^{2a+2}}{(a+1)^2(a+2)}-\frac{(2-2a)x^{2a}}{(a+1)(a+2)}-ax^{2a-2}<0, \tag{3.5}$$

for all $x>0$ and $0<a<1$. However, a stronger result holds true for $x\geq\pi$ and it is given in the lemma below.

Lemma 3.2. *For all $x \geq \pi$ we have*

$$\left[p'(x)\right]^2 - p''(x)p(x) - 1 > p(x) + p''(x) > 0. \tag{3.6}$$

Proof. Plainly, we have

$$p(x) + p''(x) = \frac{x^{a+2}}{(a+1)(a+2)} + a(1-a)x^{a-2} > 0,$$

for $x > 0$ and $0 < a < 1$. Let

$$\begin{aligned} K(x) := {} & \frac{x^{2a+2}}{(a+1)^2(a+2)} + \frac{(2-2a)x^{2a}}{(a+1)(a+2)} + ax^{2a-2} - 1 \\ & - \frac{x^{a+2}}{(a+1)(a+2)} - a(1-a)x^{a-2}. \end{aligned} \tag{3.7}$$

The first inequality in (3.6) amounts to showing that $K(x) > 0$ for $x \geq \pi$ and $0 < a < 1$. We have, in fact, that

$$\begin{aligned} K(x) > {} & \frac{x^{2a+2}}{(a+1)^2(a+2)} + \frac{(2-2a)x^{2a}}{(a+1)(a+2)} - 1 \\ & - \frac{x^{a+2}}{(a+1)(a+2)} - a(1-a)x^{a-2} := L(x). \end{aligned} \tag{3.8}$$

It is clear that $L(x)$ is a strictly increasing function of x in $[\pi, \infty)$. It is therefore sufficient to show that $L(\pi) > 0$ for $0 < a < 1$.

We have

$$\begin{aligned} \Phi(a) : = \pi^{2-a}\,(a+1)^2\,(a+2)\,L(\pi) = {} & \pi^{a+4} + 2(1-a^2)\pi^{a+2} - \pi^{2-a}\,(a+1)^2(a+2) \\ & - \pi^4(a+1) - a(1-a)(a+1)^2(a+2). \end{aligned}$$

In order to prove that $\Phi(a) > 0$ for $0 < a < 1$, we distinguish two cases: (I) $0 < a \leq 0.8$ and (II) $0.8 < a < 1$.

In the first case we write

$$\Phi(a) = \pi^a \rho(a) - \pi^{-a}\sigma(a) - \tau(a),$$

where

$$\begin{aligned} \rho(a) &:= \pi^4 + 2\pi^2(1-a^2), \\ \sigma(a) &:= \pi^2(a+1)^2\,(a+2), \\ \tau(a) &:= \pi^4(a+1) + a(1-a^2)(a+1)(a+2). \end{aligned}$$

A simple computation gives

$$\rho(a) - \sigma(a) = -\pi^2 a^3 - 6\pi^2 a^2 - 5\pi^2 a + \pi^4.$$

This is clearly strictly decreasing for $a \in (0,\ 1)$ and $\rho(0.8) - \sigma(0.8) = 14.97\ldots$. Consequently, we have that

$$\rho(a) - \sigma(a) > 0, \quad 0 < a \leq 0.8. \tag{3.9}$$

Using this, it is not hard to see that

$$\begin{aligned}\Phi(a) &= \big(\rho(a) - \sigma(a)\big)\,\cosh(a\,\log\pi) + \big(\rho(a) + \sigma(a)\big)\,\sinh(a\,\log\pi) - \tau(a)\\ &\ge \rho(a) - \sigma(a) + \big(\rho(a) + \sigma(a)\big)\,(a\,\log\pi) - \tau(a)\\ &= a\,\big[a^4 + (3 + \pi^2\,\log\pi)\,a^3 + (-\pi^2 + 1 + 2\pi^2\,\log\pi)\,a^2\\ &\quad + (-3 - 6\pi^2 + 5\pi^2 \log\pi)\,a - 2 - 5\pi^2 - \pi^4 + (4\pi^2 + \pi^4)\log\pi\big] > 0.\end{aligned}$$

In the case where $0.8 < a < 1$, we have

$$\Phi(a) \ge \pi^{a+4} - \pi^{2-a}\,(a+1)^2(a+2) - \pi^4(a+1) - (a+1)^2(a+2) =: \omega(a).$$

We observe that

$$\begin{aligned}\pi^a\,\omega(a) &\ge \pi^{a+4}(\pi^a - a - 1) - (a+1)^2(a+2)(\pi^2 + \pi)\\ &\ge \pi^{a+4}(\pi^a - a - 1) - 12\,(\pi^2 + \pi) =: \xi(a).\end{aligned}$$

It is easily seen that the function $\xi(a)$ is strictly increasing on $(0,\ 1)$ with $\xi(0.8) = 13.936\ldots$.

Combining the above we conclude that $\Phi(a) > 0$ for all $0 < a < 1$ and complete the proof of the lemma. $\square$

We now turn to establish the log-concavity of the function $f(x)$ of (3.3) on the interval $[\pi,\ \infty)$. For simplicity we set $\kappa := \Gamma(a+1)$ and $X := x - \frac{a\pi}{2}$. Recall that for $0 < a < 1$ we have $0 < \kappa < 1$.

An elementary computation yields

$$\begin{aligned}f''(x)f(x) - (f'(x))^2 &= p''(x)p(x) - (p'(x))^2 - \kappa^2\\ &\quad + \kappa\,\Big[\big(p''(x) - p(x)\big)\,\cos X + 2\,p'(x)\,\sin X\Big]\\ &\le p''(x)p(x) - (p'(x))^2 - \kappa^2\\ &\quad + \kappa\,\sqrt{\big(p''(x) - p(x)\big)^2 + 4(p'(x))^2}.\end{aligned} \tag{3.10}$$

Taking into account Eq. (3.5) we see that, in order to prove that the last expression in (3.10) is negative, it suffices to prove that

$$\big[(p'(x))^2 - p''(x)p(x) + \kappa^2\big]^2 > \kappa^2\,\big[\big(p''(x) - p(x)\big)^2 + 4(p'(x))^2\big]. \tag{3.11}$$

Some elementary manipulations show that (3.11) is equivalent to

$$\big[(p'(x))^2 - p''(x)p(x) - \kappa^2\big]^2 > \kappa^2\,\big[p(x) + p''(x)\big]^2. \tag{3.12}$$

According to Lemma 3.2 we have that

$$\big[p'(x)\big]^2 - p''(x)p(x) - \kappa^2 > \big[p'(x)\big]^2 - p''(x)p(x) - 1 > 0, \quad \text{for } x \ge \pi.$$

Therefore, the desired inequality (3.12) is equivalent to

$$\big[p'(x)\big]^2 - p''(x)p(x) - \kappa^2 > \kappa\,\big[p(x) + p''(x)\big]. \tag{3.13}$$

Using again Lemma 3.2 we obtain

$$\kappa^2 + \kappa\,\big[p(x) + p''(x)\big] < 1 + p(x) + p''(x) < \big[p'(x)\big]^2 - p''(x)p(x),$$

which establishes (3.13) and completes the proof of (1.1). It follows from the above that (1.1) holds as strict inequality when $\lambda > 2$.

Finally, we need to verify that (1.1) fails to hold for $1 < \lambda < 2$ and appropriate $x > 0$. Indeed, from the first equality in (3.2) we have

$$F_\lambda(x) = x^\lambda - \lambda\,(\lambda - 1)\,F_{\lambda-2}(x).$$

For $0 < \mu < 1$ we have a formula similar to (3.1), *viz.*,

$$\int_0^x \frac{\sin(x-t)}{t^\mu}\,dt = -\Gamma(1-\mu)\,\cos\Big(x + \frac{\pi}{2}\,\mu\Big) + \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-xs}\,\frac{s^{\mu-1}}{s^2+1}\,ds,$$

(cf. [9, Lemma 3.3]). Setting $\mu = 2 - \lambda$ we find that

$$x^{-\lambda}\left[F_\lambda''(x)\,F_\lambda(x) - [F_\lambda'(x)]^2\right] \sim -\lambda\,x^{\lambda-2} + \Gamma(\lambda+1)\,\cos\Big(x - \frac{\lambda\pi}{2}\Big), \quad \text{as } x \to \infty.$$

Therefore (1.1) cannot hold for all $x > 0$ in the case where $1 < \lambda < 2$.

The proof of Theorem 1.1 is complete.

Remark 3.3. In the case where $0 < \lambda < 1$, the function $F_{\lambda-1}(x)$ is well defined and by (1.3) we have $F_{\lambda-1}(x) = \sqrt{x}\,s_{\lambda-\frac{1}{2},\frac{1}{2}}(x)$, but, this function has infinitely many changes of sign in $(0, \infty)$ according to a Theorem of J. Steinig in [14]. The latter can also be seen by (3.1), which for $0 < \lambda < 1$ yields

$$F_{\lambda-1}(x) \sim \Gamma(\lambda)\,\sin\Big(x - \frac{\lambda\pi}{2}\Big), \quad \text{as } x \to \infty.$$

On the other hand, (2.1) implies that $F_\lambda(x) > 0$ for all $x > 0$. Therefore, the ratio

$$\frac{F_{\lambda-1}(x)}{F_\lambda(x)},$$

cannot be monotonic on $(0, \infty)$ for this range of λ. For $\lambda = 1$, this ratio is equal to the elementary function $\frac{1-\cos x}{x-\sin x}$, which is clearly nonnegative and it is easy to see that it is not monotonic on $(0, \infty)$.

4. Proof of Theorem 1.2

Using the recurrence relation

$$s_{\mu,\nu}'(z) + \frac{\nu}{z}s_{\mu,\nu}(z) = (\mu + \nu - 1)s_{\mu-1,\nu-1}(z),$$

and the symmetry property

$$s_{\mu,-\nu}(z) = s_{\mu,\nu}(z),$$

[17, p. 348], we obtain for $a > -\frac{1}{2},\ a \neq \frac{1}{2}$

$$\begin{aligned} s_{a,\frac{1}{2}}'(x)\,s_{a+1,\frac{1}{2}}(x) - s_{a,\frac{1}{2}}(x)\,s_{a+1,\frac{1}{2}}'(x) &= \Big(a - \frac{1}{2}\Big)s_{a-1,\frac{1}{2}}(x)\,s_{a+1,\frac{1}{2}}(x) \\ &\quad - \Big(a + \frac{1}{2}\Big)\Big(s_{a,\frac{1}{2}}(x)\Big)^2 \\ &= \Big(\frac{1}{2} - a\Big)\left[\Delta_a(x) - \frac{1}{\frac{1}{2} - a}\Big(s_{a,\frac{1}{2}}(x)\Big)^2\right], \end{aligned}$$

where

$$\Delta_a(x) := \Big(s_{a,\frac{1}{2}}(x)\Big)^2 - s_{a-1,\frac{1}{2}}(x)s_{a+1,\frac{1}{2}}(x).$$

Hence,

$$\left(\frac{s_{a,\frac{1}{2}}(x)}{s_{a+1,\frac{1}{2}}(x)}\right)' = \frac{\frac{1}{2}-a}{\Big(s_{a+1,\frac{1}{2}}(x)\Big)^2}\left[\Delta_a(x) - \frac{1}{\frac{1}{2}-a}\Big(s_{a,\frac{1}{2}}(x)\Big)^2\right].$$

Accordingly, for $a > \frac{1}{2}$ inequality (1.5) is equivalent to

$$\left(\frac{s_{a,\frac{1}{2}}(x)}{s_{a+1,\frac{1}{2}}(x)}\right)' \leq 0, \quad \text{for all } x > 0. \tag{4.1}$$

In view of (1.3) and (2.2) we have

$$\frac{\sqrt{x}\, s_{a,\frac{1}{2}}(x)}{\sqrt{x}\, s_{a+1,\frac{1}{2}}(x)} = \frac{F_{a-\frac{1}{2}}(x)}{F_{a+\frac{1}{2}}(x)} = \frac{1}{a+\frac{1}{2}}\,\frac{F'_{a+\frac{1}{2}}(x)}{F_{a+\frac{1}{2}}(x)}. \tag{4.2}$$

This shows that the desired inequality (4.1) follows from (1.1) for all $a \geq \frac{3}{2}$. Theorem 1.1 also implies that inequality (1.5) fails to hold for appropriate positive values of x when $a \in (\frac{1}{2}, \frac{3}{2})$. Finally, for $-\frac{1}{2} < a < \frac{1}{2}$ inequality (1.5) is equivalent to the reverse inequality (4.1). But, the ratio $\frac{s_{a,\frac{1}{2}}(x)}{s_{a+1,\frac{1}{2}}(x)}$ cannot be monotonic on $(0, \infty)$ because $s_{a+1,\frac{1}{2}}(x) > 0$, for all $x > 0$, while $s_{a,\frac{1}{2}}(x)$ has infinitely many changes of sign in $(0, \infty)$ according to the aforementioned Theorem of Steinig.

The proof of Theorem 1.2 is complete.

Remark 4.1. The functional bound for the determinant $\Delta_a(x)$ provided by (1.5) is clearly negative for $a \geq \frac{3}{2}$. However, the function $\Delta_a(x)$ itself, changes sign infinitely often in $(0, \infty)$ for all $a > -\frac{1}{2}, a \neq \frac{1}{2}$.

Indeed, setting $\lambda = a + \frac{1}{2}$ and using the asymptotic formulas, as $x \to \infty$, for the Lommel functions of the first kind, see [14, p. 126], together with the recurrence formula

$$s_{a+2,\frac{1}{2}}(x) = x^{a+1} - \left[(a+1)^2 - \frac{1}{4}\right] s_{a,\frac{1}{2}}(x),$$

see [17, p. 348], we find that for all $\lambda > 0$, $\lambda \neq 1$ we have

$$x^{1-\lambda}\,\Delta_{\lambda-\frac{1}{2}}(x) \sim \frac{\Gamma(\lambda)}{1-\lambda}\cos\Big(x - \frac{\lambda\pi}{2}\Big), \quad \text{as } x \to \infty.$$

Set $\mu = \lambda - 1$ and notice that for all $\mu > 0$ we have

$$x\,\Delta_{\lambda-\frac{1}{2}}(x) = F_\mu^2(x) - F_{\mu-1}(x)\,F_{\mu+1}(x).$$

Consequently, inequality (1.6) fails to hold for appropriate $x > 0$ and $\mu \geq 1$, when the factor $1 + \frac{1}{\mu}$ is replaced by 1.

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References

[1] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Inequalities for quasiconformal mappings in space, Pacific J. Math. 160 (1) (1993) 1–18.
[2] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.
[3] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[4] M. Bagnoli, T. Bergstrom, Log-concave probability and its applications, Econom. Theory 26 (2) (2005) 445–469.
[5] Á. Baricz, S. Koumandos, Turán type inequalities for some Lommel functions of the first kind, in: Proc. Edinburgh Math. Soc., available on CJO2015. http://dx.doi.org/10.1017/S0013091515000413.
[6] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[7] G. Gasper, Positive integrals of Bessel functions, SIAM J. Math. Anal. 6 (5) (1975) 868–881.
[8] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Seventh ed., Elsevier Academic Press, New York, 2007.
[9] S. Koumandos, Positive trigonometric integrals associated with some Lommel functions of the first kind (submitted for publication).
[10] S. Koumandos, H.L. Pedersen, Turán type inequalities for the partial sums of the generating functions of Bernoulli and Euler numbers, Math. Nachr. 285 (2012) 2129–2156.
[11] A.M. Mathai, H.J. Haubold, Special Functions for Applied Scientists, Springer, 2008.
[12] I. Pinelis, L'Hospital's rules for monotonicity, with applications, J. Inequal. Pure Appl. Math. 3 (1) (2002) 5. Art. 5.
[13] A. Saumard, J.A. Wellner, Log-concavity and Strong Log-concavity: a review, 2014. arXiv: 1404.5886v1.
[14] J. Steinig, The sign of Lommel's function, Trans. Amer. Math. Soc. 163 (1972) 123–129.
[15] G. Szegő, On an inequality of P. Turán concerning Legendre polynomials, Bull. Amer. Math. Soc. 54 (1948) 401–405. Also in: Collected papers of G. Szegő, Ed. by Richard Askey; 3 (1982) 69–73, Birkhäuser-Boston, Basel, Stuttgart.
[16] P. Turán, On the zeros of the polynomials of Legendre, Čas. Pest. Mat. Fys. 75 (1950) 113–122.
[17] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944.
[18] D.V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1946.