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Pólya-type criteria for conditional strict positive definiteness of functions on spheres

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Abstract

Identifying (conditionally) strictly positive definite functions is of great importance as they allow the unique solution of certain interpolation problems. We introduce new sufficient (and some necessary) conditions for functions to be conditionally strictly positive definite on all spheres \mathbb{S}^{d-1} , $d > 2$, only employing monotonicity properties. For strictly positive definite and conditionally negative definite functions on all spheres we give a characterisation in terms of monotonicity properties. Further, we prove that multiply monotone functions are positive definite on spheres up to a certain dimension.
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1. Introduction

Given a continuous function $F : A \rightarrow \mathbb{R}$ on a domain A , a frequent goal is to approximate it by interpolation at prescribed points or centres $\xi \in \Xi \subset A$ or otherwise from finite-dimensional linear spaces of simpler functions, $s : A \rightarrow \mathbb{R}$, say. Other approximations such as the very useful quasi-interpolation are possible, see e.g. the paper by Dai and the first author, [12], but we address classical interpolation here.

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In this paper, we will consider the case $A = \mathbb{S}^{d-1}$ with \mathbb{S}^{d-1} being the $(d-1)$ -dimensional sphere embedded in \mathbb{R}^d defined by

$$\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\},$$

but we will also refer to the case $A := \mathbb{R}^d$ in which the definitions of conditionally positive definite functions were originally introduced. A useful and by now quite popular approach is to interpolate using linear combinations of shifts of radially symmetric functions whose radial part, call it $\phi : [0, \infty) \rightarrow \mathbb{R}$, is such that the aforementioned interpolation problem

$$s(\zeta) := \sum_{\xi \in \Xi} \lambda_\xi \phi(\text{dist}(\zeta, \xi)) = F(\zeta), \quad \forall \zeta \in \Xi,$$

is solved, $\text{dist}(\cdot, \cdot)$ between ξ and ζ being the Euclidean norm $\|\xi - \zeta\|_2$ for $A = \mathbb{R}^d$ and the geodesic distance $d(\xi, \zeta) = \arccos(\xi^T \zeta)$, with T being the transpose operator, for $A = \mathbb{S}^{d-1}$.

As an immediate consequence, we are faced with the question whether this interpolation problem is – uniquely – solvable. Micchelli [28], Powell [33] and the first of this paper's author together with Micchelli [13], see also the book [11], among others, considered this question for Euclidean spaces.

A generalisation, which is well established in the Euclidean setting, is achieved for $A = \mathbb{S}^{d-1}$ by adding low order spherical harmonics (this should be compared with the polynomials in the Euclidean setting).

A class of functions for which the interpolation problem is uniquely solvable for any distinct point set $\Xi \subset A$ is the class of strictly positive definite functions, and conditionally strictly positive definite functions if low order spherical harmonics are added to the interpolant.

Definition 1. A function $\phi : [0, \pi] \rightarrow \mathbb{R}$ is conditionally strictly positive definite of order m on the sphere \mathbb{S}^{d-1} (in notation $CSPD_m(\mathbb{S}^{d-1})$) if the matrix $\Phi_\Xi = \{\phi(d(\xi, \zeta))\}_{\xi, \zeta \in \Xi}$ is positive definite on the subspace

$$W_{m-1} = \left\{ \lambda \in \mathbb{R}^{|\Xi|} : \sum_{\xi \in \Xi} \lambda_\xi Y(\xi) = 0 \text{ for all } Y \in \mathcal{H}_{m-1}(\mathbb{S}^{d-1}) \right\},$$

where $\mathcal{H}_{m-1}(\mathbb{S}^{d-1}) := \cup_{j=0}^{m-1} H_j(\mathbb{S}^{d-1})$ and $H_j(\mathbb{S}^{d-1})$ is the class of spherical harmonics of order j . For $m = 0$ the function class is just called the set of strictly positive definite functions ($SPD(\mathbb{S}^{d-1})$).

A spherical harmonic of order j on \mathbb{S}^{d-1} is the restriction of a j -homogeneous and harmonic polynomial on \mathbb{R}^d of degree j to \mathbb{S}^{d-1} . For a full definition and further details on spherical harmonics we refer the reader to Müller's fundamental book [29].

It can easily be verified that such functions allow the interpolation of any function on the sphere on any set of data sites Ξ , containing a unisolvent subset with respect to $\mathcal{H}_{m-1}(\mathbb{S}^{d-1})$, using an interpolant of the form

$$s(\zeta) = \sum_{\xi \in \Xi} \lambda_\xi \phi(d(\zeta, \xi)) + Y(\zeta), \quad Y \in \mathcal{H}_{m-1}(\mathbb{S}^{d-1}).$$

The coefficients have to satisfy the additional condition that

$$\sum_{\xi \in \Xi} \lambda_\xi Y(\xi) = 0 \text{ for all } Y \in \mathcal{H}_{m-1}(\mathbb{S}^{d-1}).$$

Besides enlarging the class of applicable basis functions, this definition allows the reproduction of low order spherical harmonics by interpolants. It is obvious that these function classes satisfy the nestedness conditions

$$SPD(\mathbb{S}^{d-1}) \supset SPD(\mathbb{S}^{(d+1)-1}) \supset \dots \supset SPD(\mathbb{S}^\infty)$$

and

$$SPD(\mathbb{S}^{d-1}) = CSPD_0(\mathbb{S}^{d-1}) \subset \dots \subset CSPD_m(\mathbb{S}^{d-1}).$$

The described interpolation problem on the sphere has received more and more attention during the last years, even though the theory was in fact started by Schoenberg in 1942. Then the investigation of interpolation with Euclidean “radial” basis functions which were restricted to the sphere was discussed, for example in [15] for general embedded manifolds or in [21,30] for the sphere. This work also resulted in error estimates for the interpolation with this class of basis functions.

More recently, functions which are not restrictions of functions defined originally on Euclidean space were described and error estimates were developed, e.g. [2–4,6,7,9,18,22,24,31,36,40]. The increased number of publications on the topic is a result of its importance in different fields of applied mathematics, for example in geostatistics, where the functions are referred to as isotropic covariance functions on spheres, and approximation theory where the functions are called positive definite spherical or zonal functions.

The characterisation and description of those positive definite spherical basis functions is nevertheless a topic of ongoing research. Further, only a few of the named contributions study conditionally positive definite functions, e.g. [4,21,27]. Also, recently the concept of intrinsic random functions on \mathbb{S}^2 was introduced in [20] and is the geostatistical counterpart of conditionally positive definite functions on \mathbb{S}^{d-1} , which is likely to increase interest in the topic.

This contribution establishes new results on conditionally positive definite functions on spheres, which corresponds to the results we have in Euclidean spaces. Parts of these results were contained in the second author’s PhD thesis, [23].

Especially, the following fundamental result is known. It was developed in three steps, first by Schoenberg in [34] for $k = 0$ and as a sufficient condition, then by Micchelli in [28] for $k = 1$, and then the necessity was proven by Guo et al. in [19].

Theorem 1.1. *A continuous function ϕ on $[0, \infty)$ is conditionally strictly positive definite of order m on \mathbb{R}^d , for any integer $d \in \mathbb{N}$, if and only if g with $\phi(r) = g(r^2)$ is completely monotone of order m on $(0, \infty)$ and*

$$g^{(m)}(t) \not\equiv \text{const.}$$

In Section 2 we study (conditional) strict positive definiteness on the sphere and derive a similar sufficient condition for conditionally positive definite functions on all spheres \mathbb{S}^{d-1} with $d > 2$. The results generalise a characterisation of Ma described in [24] for positive definite functions. To be precise, we prove in Theorem 2.8 that a function $\phi = \varphi(\cos(\cdot))$ for which

$$\varphi(-x) - \varphi(x) \text{ and } \varphi(x) + \varphi(-x), \quad x \in [0, 1),$$

are absolutely monotone of order k and no polynomials is strictly conditionally positive definite of this order. Further we prove that this condition is necessary and sufficient for strictly positive definite functions (Theorem 2.5) and conditionally negative definite functions (Theorem 2.6).

Special cases of the above provide that absolute monotonicity of the function $\varphi \in C([-1, 1])$ on $(-1, 1)$ implies conditional strict positive definiteness of $\varphi(\cos \cdot)$ if φ is no polynomial.

In Section 3 we prove that multiple monotonicity of ϕ on $(0, \infty)$ implies strict positive definiteness up to a certain dimension. In addition, we give sufficient conditions employing the monotonicity of the function $\phi : [0, \pi) \rightarrow \mathbb{R}$. These are presented in Theorem 3.3–Lemma 6. For example they allow to establish the use of the function

$$\phi(d(\xi, \zeta)) = \sqrt{c^2 + d(\xi, \zeta)}, \quad c > 0,$$

the multiquadric as a spherical basis function.

A result of Gneiting is generalised showing that all functions $\phi \in C([0, \pi])$ which are completely monotone on $(0, \pi)$ and no linear polynomials are strictly positive definite on all spheres. Similar results are presented for conditionally positive definite functions of order 1.

As one example we study the function classes

$$SPD(\mathbb{S}^\infty) \cap \mathbb{P}_1^2 \text{ and } (CSPD_1(\mathbb{S}^\infty) \setminus SPD(\mathbb{S}^\infty)) \cap \mathbb{P}_1^2.$$

1.1. Definitions of the monotonicity properties

We will denote in what follows the class of n -times continuously differentiable functions on A with $C^n(A)$ and the class of smooth functions on A with $C^\infty(A)$.

Definition 2. A function $f \in C^{n-2}(I)$ defined on an interval I of reals, also including the infinite set $I = \mathbb{R}_{\geq 0}$, is called n -times monotone (or multiply monotone) on I if

$$(-1)^j f^{(j)}(t) \geq 0, \quad \forall t \in I,$$

and $(-1)^j f^{(j)}$ is non-increasing and convex for $j = 0, 1, \dots, n-2$. Here, $n > 1$ is an integer. For $n = 1$ we require $f \in C(I)$ to be non-negative and non-increasing; then it is called (once) monotone.

In the fundamental article [38], Williamson proved the existence of a representation for these functions if $I = \mathbb{R}_+$. It is a simple integral like in the Bernstein–Widder representation theorem of completely monotone functions.

Theorem 1.2. Every function which is n -times monotone on $I = \mathbb{R}_+$ has a representation of the form

$$f(t) = \int_0^\infty (1 - t\beta)_+^{n-1} d\gamma(\beta), \quad t > 0, \quad (1)$$

where γ is a non-decreasing measure which is bounded from below.

Multiply monotone functions were first described in the context of radial basis functions by Micchelli and the first author. We cite the main result from [10].

Theorem 1.3. A function $\phi \in C([0, \infty))$, which is n -times monotone on $(0, \infty)$ but not constant, is strictly positive definite on \mathbb{R}^d for all dimensions d with $n \geq \lfloor d/2 \rfloor + 2$.

The multiply monotone functions contain a subset of functions which are n -times monotone for any n , the completely monotone functions. We set from now on $\mathbb{N}_m := \{m, m+1, m+2, \dots\}$, for an integer m .

Definition 3. A function $g : I \rightarrow \mathbb{R}$ is said to be completely monotone of order m on an interval I of reals, if and only if it is in $C^\infty(I)$ and

$$(-1)^j g^{(j)}(t) \geq 0, \quad \forall t \in I, \quad \forall j \in \mathbb{N}_m,$$

holds. For $m = 0$ these functions are called completely monotone.

We also need to define the property of absolute monotonicity which is of special interest for spherical interpolation.

Definition 4. A function f is called absolutely monotone of order m on an interval I of reals if $f \in C^\infty(I)$ and

$$f^{(j)}(t) \geq 0, \quad \text{for all } j \in \mathbb{N}_m, \quad t \in I.$$

For $m = 0$ the function is just called absolutely monotone.

1.2. Conditionally strictly positive definite functions on spheres

We will investigate the monotonicity of two kinds of spherical basis functions; namely, the functions

$$\phi : [0, \pi] \rightarrow \mathbb{R},$$

which are radial basis functions on the sphere when used as $\phi(d(\xi, \zeta))$ (see Section 3), and the functions $\varphi : [-1, 1] \rightarrow \mathbb{R}$ satisfying $\varphi(\xi^T \zeta) = \phi(d(\xi, \zeta))$ (see Section 2). Functions of this form are sometimes referred to as dot product kernels.

In the second case we will always have to say that $\varphi(\cos \cdot)$ is (conditionally) strictly positive definite, but we point out that this somewhat unwieldy description could be circumvented by an alternative definition of (conditional) positive definiteness which uses the inner product instead of the geodesic distance. However, we decided on the first described variant to avoid competing definitions of positive definiteness.

All of the following results on positive definiteness are based on the famous representation theorem by Schoenberg [35] for positive definite functions on spheres, which was then extended to a characterisation of strictly positive definite functions on spheres by Chen, Menegatto and Sun [14].

Theorem 1.4 (Schoenberg/Chen et al.). A function $\phi : [0, \pi] \rightarrow \mathbb{R}$ is strictly positive definite on \mathbb{S}^{d-1} ($SPD(\mathbb{S}^{d-1})$), $d > 2$, if and only if it can be represented as

$$\phi(r) = \sum_{k=0}^{\infty} a_{k,d} C_k^\lambda(\cos(r)), \quad r \in [0, \pi], \quad (2)$$

where $a_{k,d} \geq 0$, for all k , $\sum_{k=0}^{\infty} a_{k,d} < \infty$, $\lambda := (d-2)/2$, and infinitely many of the coefficients $a_{k,d}$ with odd indices and infinitely many of the coefficients with even indices are positive, and finally the C_k^λ are the Gegenbauer polynomials (as in [1, 22.9.2]).

The case $d = 2$ of the circle allows for the same series representation but the strict positive definiteness is only given if the indices k , for which $a_{k,2} > 0$ holds, intersect with every arithmetic progression.

The summability of the coefficients is given for all functions $\phi \in C([0, \pi])$ with non-negative coefficients. The series then converges absolutely and uniformly on $[0, \pi]$. For

conditionally positive definite functions a similar condition was described for example by [27, Theorem 4.6].

Theorem 1.5. *Every continuous $\phi : [0, \pi] \rightarrow \mathbb{R}$ with an infinite expansion or representation*

$$\phi(r) = \sum_{k=0}^{\infty} a_{k,d} C_k^{\lambda}(\cos(r)), \quad r \in [0, \pi],$$

where $a_{k,d} \geq 0$ for all $k \geq m$, $\lambda = \frac{d-2}{2} > 0$ and $\sum_{k=m}^{\infty} a_{k,d} < \infty$ is strictly conditionally positive definite of order m ($CSPD_m(\mathbb{S}^{d-1})$) if infinitely many coefficients $a_{k,d}$ with odd k and infinitely many coefficients with even k are positive.

The Gegenbauer coefficients of a function $\phi : [0, \pi] \rightarrow \mathbb{R}$ satisfying $\phi(\xi^T \zeta) = \phi(d(\xi, \zeta))$ for $\lambda > 0$, are given by

$$\begin{aligned} a_{k,d} &= \frac{1}{h_k^{\lambda}} \int_{-1}^1 \phi(x) C_k^{\lambda}(x) (1-x^2)^{\lambda-\frac{1}{2}} dx \\ &= \frac{1}{h_k^{\lambda}} \int_0^{\pi} \phi(\theta) C_k^{\lambda}(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta, \end{aligned}$$

where

$$h_k^{\lambda} = \frac{1}{(k+\lambda)} \frac{2^{1-2\lambda} \pi \Gamma(k+2\lambda)}{k! (\Gamma(\lambda))^2}. \quad (3)$$

Schoenberg proved in the same paper [35] that functions which are positive definite on all spheres can be represented as described in the following theorem. He also showed that these functions are exactly the functions which are positive definite on the real Hilbert sphere \mathbb{S}^{∞} . The full characterisation of strictly positive definite functions on \mathbb{S}^{∞} was later proven by Menegatto in [26]; he also gave a characterisation of conditionally negative definite functions (Theorem 2.7 [26]), which we adapt to fit into our notation of conditionally positive definite functions.

Theorem 1.6. *A function ϕ is positive definite on \mathbb{S}^{d-1} for all $d > 1$ if and only if it has the form*

$$\phi(r) = \sum_{k=0}^{\infty} a_k (\cos(r))^k, \quad r \in [0, \pi], \quad (4)$$

where $a_k \geq 0$ for all k , $0 \neq \sum_{k=0}^{\infty} a_k < \infty$. It is strictly positive definite on \mathbb{S}^{∞} ($SPD(\mathbb{S}^{\infty})$) if and only if the above holds and additionally $a_k > 0$ for infinitely many even and infinitely many odd values of k .

Theorem 1.7. *A function ϕ is conditionally strictly positive definite of order one on \mathbb{S}^{∞} ($CSPD(\mathbb{S}^{\infty})$) if and only if it has the form*

$$\phi(r) = \sum_{k=0}^{\infty} a_k (\cos(r))^k, \quad r \in [0, \pi], \quad (5)$$

where $a_k \geq 0$ for all $k \geq 1$, $\sum_{k=0}^{\infty} a_k < \infty$ and $a_k > 0$ for infinitely many even and infinitely many odd values of k .

2. Monotonicity properties of functions depending on the dot product

From this representation we can immediately deduce that for an analytic function $f : \mathbb{R} \rightarrow \mathbb{R}$, which satisfies our conditions in [Theorem 1.6](#), we have

$$f(x) = \phi(\arccos(x)) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k.$$

Therefore f will be absolutely monotone on $[0, \infty)$ if the series converges on $[0, \infty)$.

Furthermore, if a function $\tilde{\phi}$ is conditionally positive definite of order m on \mathbb{R}^d for all dimensions d , then we have that

$$g(\cdot) = \tilde{\phi}(\sqrt{\cdot})$$

is completely monotone of order m (see [Theorem 1.1](#)). Using the connection of the Euclidean distance and the geodesic distance, that is $\|\xi - \zeta\|_2 = \sqrt{2 - 2\xi^T \zeta}$, we get that the restriction of this basis function $\tilde{\phi}$ to the sphere, dependent on the inner product is

$$\varphi(\cdot) = \tilde{\phi}(\sqrt{2 - 2\cdot}) = g(2 - 2\cdot).$$

The function φ is obviously conditionally positive definite of order m on \mathbb{S}^{d-1} under the assumptions above. It follows that if g is completely monotone of order m on $[0, \infty)$, then φ will be absolutely monotone of order m on $(-\infty, 1]$. This gives us a reason to investigate the monotonicity of the function φ . The latter argument can be reversed and gives a criterion for positive definiteness on all spheres. The theorem was also proven by Beatson and zu Castell (as is known from private communication, 2017).

Theorem 2.1. *Let $\varphi \in C((-\infty, 1])$ be absolutely monotone of order m on $(-\infty, 1)$ and no polynomial. Then $\varphi(\cos(\cdot))$ is strictly conditionally positive definite of order m on \mathbb{S}^{d-1} for all d .*

For $m = 0$, the function is even strictly positive definite and the functions with $m = 1$ will result in non-singular interpolation matrices if $\varphi(1) < 0$.

In the next section we will prove a less restricted version of this result. The analogue of this theorem for n -times absolutely monotone functions is not true, as is to be seen in the next (counter-)example.

Example 1. The function $\varphi(x) = x_+^{\mu-1}$ is n -times absolute monotone on $(-\infty, 1)$ for μ larger than n , but $\varphi(\cos(\cdot))$ is not positive definite. The $(\mu + 2)$ -nd coefficients in the Gegenbauer expansion can be computed as

$$\begin{aligned} a_{\mu+2,d} &= \frac{1}{h_{\mu+2}^\lambda} \int_{-1}^1 \varphi(x) C_{\mu+2}^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx \\ &= \frac{(\mu+2+\lambda)(\mu+2)! \Gamma(\lambda) 2^{-\mu} \Gamma(\mu)}{\Gamma(\frac{3}{2} + \mu + \lambda) \Gamma(-\frac{1}{2})} \\ &= - \frac{(\mu+2+\lambda)(\mu+2)! \Gamma(\lambda) \Gamma(\mu)}{\Gamma(\frac{3}{2} + \mu + \lambda) 2^{\mu+1} \sqrt{\pi}} < 0, \end{aligned}$$

according to (18.17.37) in [\[32\]](#) together with [\(3\)](#). The negativity mentioned in the last display is due to the negative factor of the Gamma-function $\Gamma(-1/2) = -2\sqrt{\pi}$ in the denominator.

Now, however, we shall see that it is also possible to derive a monotonicity result for *finite* (multiple) monotonicity.

Lemma 1. *If*

$$\tilde{\phi}(x) = \varphi\left(1 - \frac{x^2}{2}\right), \quad x \geq 0,$$

is continuous on $[0, \infty)$ and n -times monotone on $(0, \infty)$, then $\varphi(\cos(\cdot))$ is positive definite on \mathbb{S}^{d-1} so long as $n \geq \lfloor d/2 \rfloor + 2$.

Proof. According to Theorem 1.3, $\tilde{\phi}$ is positive definite on \mathbb{R}^d for $n \geq \lfloor d/2 \rfloor + 2$ and so is its restriction to the sphere which can also be represented as

$$\tilde{\phi}(\sqrt{2 - 2\cos(r)}) = \varphi(\cos(r)).$$

This is positive definite on \mathbb{S}^{d-1} for $d \leq 2n - 3$. Thus, according to the definition of the lower brackets, the integral part of $d/2$ can be at most $n - 2$. \square

Using the computation of the coefficients of the expansion we can prove another sufficient condition for positive definiteness on all spheres other than the circle.

Theorem 2.2. *Let $\varphi \in C^\infty([-1, 1])$ be absolutely monotone of order m on $[-1, 1]$ and let it be no polynomial. Then the function $\varphi(\cos \cdot)$ is conditionally strictly positive definite of order m on \mathbb{S}^{d-1} for all $d \geq 3$.*

Proof. We compute the coefficients $a_{k,d}$ of the function $\varphi(\cos(\cdot))$ by repeatedly (by k -times) applying integration by parts, namely, using the Pochhammer symbol $(\cdot)_k$,

$$\begin{aligned} a_{k,d} &= \frac{1}{h_k^\lambda} \int_{-1}^1 \varphi(x) C_k^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx \\ &= \frac{1}{h_k^\lambda} \int_{-1}^1 \varphi(x) \frac{(2\lambda)_k}{(-2)^k (\lambda + \frac{1}{2})_k k!} \frac{\partial^k}{\partial x^k} (1-x^2)^{k+\lambda-\frac{1}{2}} dx \\ &= \frac{1}{h_k^\lambda} \int_{-1}^1 \frac{(2\lambda)_k}{2^k (\lambda + \frac{1}{2})_k k!} \varphi^{(k)}(x) (1-x^2)^{k+\lambda-\frac{1}{2}} dx > 0, \quad \text{for all } k \geq m. \end{aligned}$$

This establishes the assertion according to Schoenberg's famous results (Theorems 1.4 and 1.5). \square

This theorem gives a sufficient condition for (conditionally) positive definiteness on all spheres which is easy to verify. Indeed, the theorem is allowing us to give a number of new conditionally positive definite basis functions, for example the function class

$$\varphi(x) = (-1)^m (2-x)^{m-\varepsilon}, \quad x \in [-1, 1],$$

where $m \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, which is absolutely monotone of order m and therefore conditionally positive definite of order m on \mathbb{S}^{d-1} for any $d \geq 3$.

The connection between the monotonicity of a function and this function being analytic, which we can deduce from the above theorem, was first described by Bernstein and later proven in a more general setting by McHugh. We cite his result from [25] in Theorem 2.3, without proof. First, we need a definition:

Definition 5. We say that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is regularly monotonic if $f \in C^\infty((a, b))$ and each derivative is of a fixed sign on the whole interval.

This definition includes complete monotonicity as well as absolute monotonicity.

Theorem 2.3. *If the function f is regularly monotonic on (a, b) , then f is analytic on (a, b) .*

From [Theorem 1.6](#) it follows that all functions which are strictly positive definite on \mathbb{S}^∞ are absolutely monotone on the interval $[0, 1)$ and are no polynomials, the monotonicity is in this case necessary but not sufficient. One counter-example is the function

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^{2k},$$

which is absolutely monotone but does not satisfy the condition of Menegatto [\[26\]](#) for strict positive definite functions on the Hilbert sphere.

The function above is positive definite but not strictly positive definite, yet it can easily be transformed into a strictly positive definite function if an analytic extension is known on a slightly larger interval.

Lemma 2. *For every function $\varphi(\cos(\cdot))$ which is positive definite on \mathbb{S}^∞ and for which φ is no polynomial and possesses an analytic extension to $(-1, 1 + \varepsilon)$, $\varepsilon > 0$, we have that*

$$\varphi(\cos(\cdot) + \varepsilon) \in SPD(\mathbb{S}^\infty).$$

Proof. The lemma follows immediately from the characterisation of Menegatto cited as [Theorem 1.6](#). \square

The lemma allows to deduce strict positive definiteness for a list of shifts of trigonometric functions (as dot product kernels):

$$\begin{aligned} \sec(\cdot + \varepsilon), \tan(\cdot + \varepsilon), & \quad \varepsilon \in \left(0, \frac{\pi}{2} - 1\right), \\ \cosh(\cdot + \varepsilon), \sinh(\cdot + \varepsilon), & \quad \varepsilon > 0. \end{aligned}$$

A necessary and sufficient condition for strict positive definiteness and strict conditional positive definiteness of order one can be derived from the results of Ma in [\[24\]](#), who proved a similar characterisation for isotropic covariance functions, which are, as a function class, identical to our (non-strictly) positive definite functions.

Theorem 2.4 ([\[24, Theorem 1\]](#)). *For a continuous function $\varphi : [-1, 1] \rightarrow \mathbb{R}$, $\varphi \in C^\infty((-1, 1))$, the composite function $\varphi(\cos \cdot)$ is positive definite on all spheres if and only if $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are absolutely monotone on $[0, 1)$.*

The generalisation for strictly positive definite functions reads

Theorem 2.5. *For a continuous function $\varphi : [-1, 1] \rightarrow \mathbb{R}$, $\varphi \in C^\infty((-1, 1))$, the composite function $\varphi(\cos \cdot)$ is strictly positive definite on all spheres if and only if $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are absolutely monotone on $[0, 1)$ and are no polynomials.*

Proof. Let $\varphi(\cos \cdot)$ be strictly positive definite. Then it is of the form (4) with $a_k > 0$ for infinitely many even and infinitely many odd values of k . Therefore both

$$\varphi(x) + \varphi(-x) = \sum_{k=0}^{\infty} 2a_{2k}x^{2k} \quad (6)$$

and

$$\varphi(x) - \varphi(-x) = \sum_{k=0}^{\infty} 2a_{2k+1}x^{2k+1} \quad (7)$$

are absolutely monotone and are no polynomials. Conversely, if $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are absolutely monotone on $[0, 1]$, then the result of Ma (Theorem 2.4) proves them to have a representation as in (4) with positive coefficients. Further infinitely many coefficients with even and odd indices are positive, because neither of the expressions (6) or (7) are polynomials. \square

Using the characterisation of conditionally positive definite functions of order one by Menegatto we can further establish

Theorem 2.6. *For a continuous function $\varphi : [-1, 1] \rightarrow \mathbb{R}$, $\varphi \in C^\infty((-1, 1))$, $\varphi(\cos \cdot)$ is conditionally strictly positive definite of order 1 on all spheres if and only if $\varphi(x) + \varphi(-x)$ is absolutely monotone of order 1 on $[0, 1]$ and $\varphi(x) - \varphi(-x)$ is absolutely monotone on $[0, 1]$ and neither of them are polynomials.*

Proof. The proof follows using the insight that for every strictly positive definite function of order one given in the form (5) the function

$$\tilde{\varphi}(\cos(r)) = \varphi(\cos(r)) - \varphi(0) = \sum_{k=1}^{\infty} a_k \cos(r)^k$$

is strictly positive definite. In combination with the last theorem the result is proved. \square

We observe that again interpolation using the above function is always uniquely possible without adding a constant if $\varphi(1) \leq 0$.

A sufficient condition for conditionally positive definiteness of higher order on all spheres using Theorem 1.6 is as follows.

Theorem 2.7. *For any function $\varphi : [-1, 1] \rightarrow \mathbb{R}$ which has a representation for all arguments of the form*

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where $a_k \geq 0$ for $k \geq m$, $\sum_{k=0}^{\infty} a_k < \infty$ and $a_k > 0$ for infinitely many even and infinitely many odd values of k , $\varphi(\cos \cdot)$ is conditionally strictly positive definite of order m on \mathbb{S}^{d-1} for all $d > 2$.

Proof. The result follows immediately from Lemma 1 from [8], where the relation

$$x^n = \frac{n! \Gamma(\lambda)}{2^n \Gamma(2\lambda)} \sum_{0 \leq 2k \leq n} \frac{(n-2k+\lambda) \Gamma(n-2k+2\lambda)}{k!(n-2k)! \Gamma(n-k+\lambda+1)} \frac{C_{n-2k}^\lambda(x)}{C_{n-2k}^\lambda(1)}$$

is established for all positive λ , the factors of the Gegenbauer coefficients being all positive for $\lambda \geq \frac{1}{2}$. \square

Example 2. An interesting consequence of the above is the conditionally strict positive definiteness of certain multiples of hypergeometric functions; these functions were recently discussed in the context of strict positive definiteness of multivariate functions on spheres in [17]. Let

$$\varphi(x) = (-1)^m {}_1F_0(a; ; cx) = (-1)^m \sum_{k=0}^{\infty} \frac{(a)_k c^k}{k!} x^k = (-1)^m (1 - cx)^{-a}, \quad c > 0, \quad m \in \mathbb{Z}_+,$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ which has for a fixed constant a the convergence radius 1. Then $\varphi(\cos \cdot)$ is an element of $CSPD_m(\mathbb{S}^{d-1})$ for all $d > 2$ if $a \in (-m, -m+1)$. Further for the confluent hypergeometric limit function

$$\varphi(x) = (-1)^m {}_0F_1(; b; cx) = (-1)^m \sum_{k=0}^{\infty} \frac{c^k}{(b)_k k!} x^k, \quad m \in \mathbb{Z}_+, \quad c > 0,$$

the function $\varphi(\cos(\cdot))$ is in $SPD(\mathbb{S}^\infty)$ for $b > 0$, $m = 0$, and in $CSPD_m(\mathbb{S}^{d-1})$, for $d > 2$, if $b \in (-m, -m+1)$. The results can be generalised to other classes of hypergeometric functions like Kummer's confluent functions ${}_1F_1$.

To provide criteria for these types of functions which only demand certain monotonicity properties we need some preliminary results.

Lemma 3. Let $g \in C^\infty([0, a])$ be absolutely monotone of order m on $[0, a]$, then

$$\tilde{g}(x) = \sum_{j=0}^{m-1} \frac{|g^{(j)}(0)|}{j!} x^j + g(x)$$

is absolutely monotone on $[0, a]$.

Proof. We prove the result via induction over the order of absolute monotonicity. For absolutely monotone functions the statement is trivial, the sum above being empty. Let us assume the statement holds for absolutely monotone functions of order m and let g be absolute monotone of order $m+1$. Then $f = g'$ is absolutely monotone of order m and

$$\tilde{f}(x) = \sum_{j=0}^{m-1} \frac{|g^{(j+1)}(0)|}{j!} x^j + g'(x)$$

is absolutely monotone by induction hypothesis. We can now define

$$g_1(x) = \sum_{j=0}^{m-1} \frac{|g^{(j+1)}(0)|}{(j+1)!} x^{j+1} + g(x) = \sum_{j=1}^m \frac{|g^{(j)}(0)|}{j!} x^j + g(x),$$

which satisfies $g'_1 = \tilde{f}$ and therefore $g_1 + |g_1(0)| = \tilde{g}$ is absolutely monotone. \square

Lemma 4. A continuous function g on $[-a, a]$, $g \in C^\infty((-a, a))$, for $a > 0$, has a representation of the form

$$g(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{8}$$

with $a_k \geq 0$, for all $k \geq m$, $\sum_{k=0}^{\infty} a_k a^k < \infty$, if and only if $g(x) - g(-x)$ and $g(x) + g(-x)$ are absolutely monotone of order m on $[0, a)$.

Proof. The first direction follows immediately since, if g has the form (8), then

$$g(x) + g(-x) = \sum_{k=0}^{\infty} 2a_{2k}x^{2k} \quad (9)$$

and

$$g(x) - g(-x) = \sum_{k=0}^{\infty} 2a_{2k+1}x^{2k+1} \quad (10)$$

are absolutely monotone of order m on $[0, a)$. For establishing the second direction we define the functions $f_1(x) := g(x) + g(-x)$ and $f_2(x) := g(x) - g(-x)$. We can easily compute

$$f_1^{(n)}(0) = \begin{cases} 2g^{(n)}(0), & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd,} \end{cases}$$

and

$$f_2^{(n)}(0) = \begin{cases} 0, & \text{for } n \text{ even,} \\ 2g^{(n)}(0), & \text{for } n \text{ odd,} \end{cases}$$

and set the values $c_k := 2|g^{(k)}(0)|$. Applying the above lemma, we find that

$$\begin{aligned} \tilde{f}_1(x) &= \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{c_{2j}}{(2j)!} x^{2j} + g(x) + g(-x) \\ &= \underbrace{\left(\frac{1}{2} \sum_{j=0}^m \frac{c_j}{j!} x^j + g(x) \right)}_{:= \tilde{g}} + \left(\frac{1}{2} \sum_{j=0}^m \frac{c_j}{j!} (-x)^j + g(-x) \right) \\ &= \tilde{g}(x) + \tilde{g}(-x) \end{aligned}$$

is absolutely monotone. The same arguments yield that $\tilde{f}_2(x) = \tilde{g}(x) - \tilde{g}(-x)$ is absolutely monotone. We can represent \tilde{f}_1, \tilde{f}_2 as power series about zero employing Theorem 3a in [37] on p. 146 finding:

$$\tilde{f}_1(x) = \sum_{k=0}^{\infty} a_k^1 x^k$$

and

$$\tilde{f}_2(x) = \sum_{k=0}^{\infty} a_k^2 x^k,$$

with $a_k^1, a_k^2 \geq 0$ and $a_k^1 = f_1^{(k)}(0)$, $a_k^2 = f_2^{(k)}(0)$ for $k > m$ and

$$a_{2k}^1 = c_{2k} + 2g^{(2k)}(0),$$

$$a_{2k+1}^2 = c_{2k+1} + 2g^{(2k+1)}(0)$$

and $a_{2k+1}^1 = 0 = a_{2k}^2$ for the indices smaller than m . Then

$$\tilde{g}(x) = \begin{cases} \frac{1}{2} \left(\tilde{f}_1(x) + \tilde{f}_2(x) \right) = \frac{1}{2} \sum_{k=0}^m \frac{c_k}{k!} x^k + \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k, & x \in [0, a], \\ \frac{1}{2} \left(\tilde{f}_1(-x) - \tilde{f}_2(-x) \right) = \frac{1}{2} \sum_{k=0}^m \frac{c_k}{k!} x^k + \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k, & x \in [-a, 0], \end{cases}$$

has a power series representation with non-negative coefficients, and the result follows by setting $g(x) = \tilde{g}(x) - \frac{1}{2} \sum_{j=0}^m \frac{c_j}{j!} x^j$. \square

Theorem 2.8. *Let $\varphi : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function in $C^\infty((-1, 1))$ and suppose $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are absolutely monotone of order m and no polynomials on $[0, 1)$, then $\varphi(\cos \cdot)$ is conditionally strictly positive definite of order m on \mathbb{S}^{d-1} for all $d > 2$.*

Proof. Using the results of Lemma 4 we find a representation of the form (8) for φ and since $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are no polynomials each of the series (9) and (10) will possess infinitely many non-negative coefficients. \square

If the representation in Theorem 2.7 would be proven to be necessary, then our results would immediately allow a characterisation of the conditionally strictly positive definite function of order m on the sphere \mathbb{S}^{d-1} in terms of monotonicity. We state this as

Conjecture 2.9. *Let $\varphi : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function. For this $\varphi(\cos(\cdot))$ is conditionally strictly positive definite of order m on \mathbb{S}^{d-1} for all $d > 2$ if and only if $\varphi \in C^\infty((-1, 1))$ and $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ are absolutely monotone of order m on $[0, 1)$ and they are no polynomials.*

3. Monotonicity properties of functions depending on the geodesic distance

We now turn to the investigation of the monotonicity properties of the functions $\phi : [0, \pi] \rightarrow \mathbb{R}$ used in connection to the geodesic distance.

Most recently Gneiting, Beatson et al. stated Pólya criteria for positive definiteness of functions on the sphere. Here is a short list of available results.

- Beatson et al. in [5] stated in a conjecture a sufficient condition for all d for the positive definiteness of compactly supported basis functions. They proved this result for $d \leq 8$.
- Gneiting in [16] generalised the conjecture for functions that are not compactly supported, and he proved it furthermore for $d \leq 8$.
- Both conjectures can now be proven using the results of Xu [39]. The article includes the proof of Beatson et al.'s conjecture.
- The result of Xu was generalised by Nie and Ma [31] proving that all functions $\phi : [0, \infty) \rightarrow \mathbb{R}$ which are compactly supported in $[0, \pi]$ and positive definite in \mathbb{R}^d are also positive definite on \mathbb{S}^d , for d odd.

We state the result for multiply monotone functions which is a slight change to the conjecture of Gneiting (Theorem 6 in [16] for $d \leq 3$), and we shall give the proof using the result of Xu.

Theorem 3.1. *Suppose that $\phi \in C([0, \infty))$ is n -times monotone on $(0, \infty)$, $n \geq 3$ and not constant. Then its restriction $\phi|_{[0, \pi]}$ is strictly positive definite on \mathbb{S}^{d-1} , for d at most $2n - 3$.*

Proof. Using the Williamson representation (cited as [Theorem 1.2](#)) in the computation of the Gegenbauer coefficients we get

$$\begin{aligned}
 a_{k,d} &= \frac{1}{h_k^\lambda} \int_0^\pi \phi(\theta) C_k^\lambda(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta \\
 &= \frac{1}{h_k^\lambda} \int_0^\pi \int_0^{\frac{1}{\theta}} (1 - \theta\beta)_+^{n-1} d\gamma(\beta) C_k^\lambda(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta \\
 &= \frac{1}{h_k^\lambda} \int_0^\infty \int_0^\pi (1 - \theta\beta)_+^{n-1} C_k^\lambda(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta d\gamma(\beta) \\
 &= \frac{1}{h_k^\lambda} \underbrace{\int_{\frac{1}{\pi}}^\infty \int_0^{\frac{1}{\beta}} (1 - \theta\beta)_+^{n-1} C_k^\lambda(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta d\gamma(\beta)}_{(a)} \\
 &\quad + \frac{1}{h_k^\lambda} \underbrace{\int_0^{\frac{1}{\pi}} \int_0^\pi (1 - \theta\beta)_+^{n-1} C_k^\lambda(\cos(\theta)) (\sin(\theta))^{2\lambda} d\theta d\gamma(\beta)}_{(b)},
 \end{aligned}$$

where we have also used the definition of the truncated power functions to redefine the limits of integration.

Expression (a) was shown to be strictly positive for $n - 1 \geq \lceil \frac{d}{2} \rceil$ in [[39](#), Theorem 2]. Moreover, the part (b) inside the above integral is non-negative for all k and positive for infinitely many even and infinitely many odd values of k because the functions $\phi(r) = (1 - \theta r)_+^{n-1}$ are strictly positive definite on the sphere \mathbb{S}^{d-1} for all d if $0 < \theta < \frac{1}{\pi}$ and $n \geq 3$ [Lemma 4, [16](#)], since ϕ is not constant $\text{supp}(\gamma) \neq \{0\}$ and $\gamma(A) > 0$ for some $A \subset (0, \infty)$. \square

We also cite the result of Gneiting concerning completely monotone functions [[16](#), Theorem 7].

Theorem 3.2. Suppose that the function $\phi : [0, \infty) \rightarrow \mathbb{R}$ is completely monotone on $(0, \infty)$ with $\phi(0) = 1$ and not constant. Then the restriction $\tilde{\phi} = \phi|_{[0, \pi]}$ is strictly positive definite on the sphere \mathbb{S}^{d-1} for any $d \geq 2$.

We introduce an alternative condition which requires only the complete monotonicity on the interval $(0, \frac{\pi}{2}]$ while imposing slightly stronger conditions on the smoothness of the function.

Theorem 3.3. Let $\phi : [0, \pi] \rightarrow \mathbb{R}$ be a continuous function which can be represented as a convergent power series with centre at $\frac{\pi}{2}$; if ϕ is completely monotone on $(0, \pi/2]$ and $\phi^{(j)}(\frac{\pi}{2}) \neq 0$ for at least one $j > 1$ even and at least one odd j , then ϕ is strictly positive definite on \mathbb{S}^∞ .

Proof. We can represent the function as

$$\phi(r) = \sum_{j=0}^{\infty} \frac{1}{j!} \phi^{(j)}\left(\frac{\pi}{2}\right) \left(r - \frac{\pi}{2}\right)^j, \quad \text{with } (-1)^j \phi^{(j)}\left(\frac{\pi}{2}\right) \geq 0,$$

because of the required monotonicity. Then we compute the series representation of $\varphi(x) = \phi(\arccos(x))$, $x \in [-1, 1]$, using the power series of the

$$\arccos(x) = \frac{\pi}{2} - \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k+1}}{4^k(2k+1)}.$$

It follows that

$$\begin{aligned} \varphi(x) &= \phi(\arccos(x)) = \sum_{j=0}^{\infty} \frac{1}{j!} \phi^{(j)}\left(\frac{\pi}{2}\right) \left(-\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k+1}}{4^k(2k+1)}\right)^j \\ &= \sum_{\ell=0}^{\infty} a_{\ell} x^{\ell}. \end{aligned}$$

And now it follows that $a_{\ell} > 0$ for infinitely many even and infinitely many odd values of ℓ , namely from

$$(-1)^j \phi^{(j)}\left(\frac{\pi}{2}\right) > 0$$

for at least one even and one odd value of $j \geq 1$. The summability of the coefficients follows from the required convergence radii of the expansions of the functions. Now applying [Theorem 1.6](#) yields the strict positive definiteness. \square

Example 3. We can now characterise the class of polynomials $p \in \mathbb{P}^2(t)|_{[0,\pi]}$ which satisfy $p \in SPD(\mathbb{S}^{\infty})$. Namely,

$$p(r) = ar^2 + br + c \in SPD(\mathbb{S}^{\infty})$$

if and only if

$$a > 0, \quad b < -\pi a, \quad c \geq -\left(\frac{\pi^2}{4}a + \frac{\pi}{2}b\right).$$

The next result is again a generalisation of a result for positive definite functions derived by Ma in [\[24\]](#) which allows us to express the above smoothness condition using absolute monotonicity.

Lemma 5. A continuous function $\phi : [0, \pi] \rightarrow \mathbb{R}$ can be represented as a convergent power series with centre at $\frac{\pi}{2}$ and ϕ is completely monotone of order m on $(0, \frac{\pi}{2}]$ if and only if $\phi(\frac{\pi}{2} - r) + \phi(\frac{\pi}{2} + r)$ and $\phi(\frac{\pi}{2} - r) - \phi(\frac{\pi}{2} + r)$ are absolutely monotone of order m on $(0, \frac{\pi}{2}]$ and $\phi \in C^{\infty}((0, \pi))$.

Proof. The result can easily be verified by applying [Lemma 4](#) to $\phi(\frac{\pi}{2} - r)$. \square

We can now rewrite the last theorem using the new condition.

Theorem 3.4. A continuous function ϕ on $[0, \pi]$, $\phi \in C^{\infty}((0, \pi))$, is strictly positive definite on \mathbb{S}^{∞} if both $\phi(\frac{\pi}{2} - r) + \phi(\frac{\pi}{2} + r)$ and $\phi(\frac{\pi}{2} - r) - \phi(\frac{\pi}{2} + r)$ are absolutely monotone on $(0, \frac{\pi}{2}]$ and not in $\mathbb{P}_1^1|_{[0,\pi/2]}$.

To provide a stricter but easier verifiable criterion we state

Theorem 3.5. If $\phi : [0, \pi] \rightarrow \mathbb{R}$ is continuous on $[0, \pi]$ and completely monotone on $(0, \pi)$, then it is sufficient for the strict positive definiteness on \mathbb{S}^{∞} that $\phi \notin \mathbb{P}_1^1|_{[0,\pi]}$.

Proof. Since ϕ is completely monotone on $[0, \pi]$, [Theorem 2.3](#) implies that it possesses the representation necessary to apply [Theorem 3.3](#). \square

Example 4. This theorem allows to verify easily the positive definiteness of functions such as:

1. $\phi(r) = e^{-\alpha r}$, $\alpha > 0$,
2. $\phi(r) = \frac{1}{c^2 + \alpha r}$, for $c, \alpha > 0$.

We also state a result for completely monotone functions of order one.

Theorem 3.6. *A continuous function ϕ on $[0, \pi]$, $\phi \in C^\infty((0, \pi))$, is conditionally strictly positive definite of order 1 on \mathbb{S}^∞ if both $\phi\left(\frac{\pi}{2} - r\right) + \phi\left(\frac{\pi}{2} + r\right)$ and $\phi\left(\frac{\pi}{2} - r\right) - \phi\left(\frac{\pi}{2} + r\right)$ are absolutely monotone of order 1 on $(0, \frac{\pi}{2}]$ and not in $\mathbb{P}_1|_{[0, \pi/2]}$. If, on top of our other conditions, the function is non-positive at the origin, the associated interpolation matrix will be non-singular even without any constants added to the interpolant and without side-conditions.*

Proof. For a function ϕ as described, [Lemma 5](#) yields a series representation that we shall now use. We define the function $\tilde{\phi}(r) = \phi(r) - \phi\left(\frac{\pi}{2}\right)$, then [Theorem 3.3](#) applies to $\tilde{\phi}$ which therefore is strictly positive definite for spheres of arbitrary dimensions d . Also

$$\tilde{\phi}(x) = \tilde{\phi}(\arccos(x)) = \sum_{\ell=1}^{\infty} a_\ell x^\ell = \varphi(x) - \phi\left(\frac{\pi}{2}\right),$$

and thereby $\varphi(x) = \sum_{\ell=1}^{\infty} a_\ell x^\ell + \phi\left(\frac{\pi}{2}\right)$. Therefore ϕ is conditionally positive definite of order one according to [Theorem 2.7](#).

For the statement about $\phi(0) \leq 0$, namely that the according interpolation matrix will be non-singular even without any constants added to the interpolant and without side-conditions, see the classical argument that in this case the trace of the interpolation matrix is non-positive, therefore so is the sum of its eigenvalues, and thus – all but one eigenvalue being positive – the missing one must be negative. Thus the interpolation matrix is regular. \square

Of course in this theorem the condition can be replaced by the representation condition according to [Lemma 5](#).

Example 5. We can now characterise the class of conditionally positive definite polynomials $p \in \mathbb{P}_2^2(t)|_{[0, \pi]}$ which satisfy $p \in \text{CSPD}_1(\mathbb{S}^\infty) \setminus \text{SPD}(\mathbb{S}^\infty)$. Namely,

$$p(r) = ar^2 + br + c \in \text{CSPD}_1(\mathbb{S}^\infty) \setminus \text{SPD}(\mathbb{S}^\infty)$$

if and only if

$$a > 0, \quad b < -\pi a, \quad c < -\left(\frac{\pi^2}{4}a + \frac{\pi}{2}b\right).$$

If in addition $c < 0$ then interpolation using the above function is possible without adding a constant to the interpolant.

We can again avoid the two conditions in [Theorem 3.6](#) which are harder to test by making use of [Theorem 2.3](#) and giving up the necessity property.

Theorem 3.7. Let $\phi : [0, \pi] \rightarrow \mathbb{R}$ be a continuous function which is completely monotone of order 1 on $(0, \pi)$, $\phi''(\frac{\pi}{2}) \neq 0$ and $\phi(t) \neq 0$, $\forall t \in [0, \pi]$, then ϕ is conditionally strictly positive definite of order one on \mathbb{S}^∞ .

Proof. From Theorem 2.3 we can deduce that the function has a series expansion of the form required in Theorem 3.6. The positive definiteness follows from that theorem. \square

Example 6. Examples of the functions for which non-singularity of the interpolation matrix can be proven using the above results are

$$\phi(r) = -(c + r\alpha)^\beta, \quad c \geq 0, \quad \alpha > 0, \quad 0 < \beta < 1,$$

and

$$\phi(r) = -\log(1 + r),$$

which both are completely monotone of order 1 with $\phi''(\frac{\pi}{2}) \neq 0$ and $\phi(r) \neq 0$.

A further generalisation of Theorem 3.6 to higher orders of conditional positive definiteness cannot be expected. An example is the function

$$\phi(r) = a \left(r - \frac{\pi}{2}\right)^2 + b \left(r - \frac{\pi}{2}\right) + c,$$

with $a > 0$ and $b < 0$, for which both $\phi(\frac{\pi}{2} - r) + \phi(\frac{\pi}{2} + r)$ and $\phi(\frac{\pi}{2} - r) - \phi(\frac{\pi}{2} + r)$ are absolutely monotone of order 2 on $(0, \frac{\pi}{2}]$ and not in $\mathbb{P}_1^I|_{[0, \pi/2]}$. Simple computation confirms that the third coefficient of the series expansion of $\varphi(x) = \phi(\arccos(x))$ at zero is negative.

Finally we collect necessary conditions for ϕ to be positive definite for arbitrary d .

Lemma 6. For $\phi : [0, \pi] \rightarrow \mathbb{R}$ to be strictly positive definite on all spheres it is necessary that $\phi \in C^\infty((0, \pi))$, continuous on $[0, \pi]$ and once monotone on $(0, \pi/2]$.

Proof. Since for every positive definite ϕ , the function $\varphi(x) = \phi(\arccos(x))$ has a representation of the form (4) we know $\varphi \in C^\infty((-1, 1))$. From $\varphi(x) \in C^\infty((-1, 1))$, together with the series representation of the cos, we deduce that $\phi = \varphi(\cos(\cdot)) \in C^\infty((0, \pi))$. Also we saw in Theorem 1.6 that it is necessary that φ be absolutely monotone on $[0, 1]$. Since

$$\varphi(x) = \phi(\arccos(x)) \geq 0$$

for $x \in [0, 1]$ we need that $\phi(r) \geq 0$ for $r \in (0, \pi/2]$. This is because the image of $[0, 1]$ under the arccos is $(0, \pi/2]$. Also

$$\varphi'(x) = \phi'(\arccos(x)) \cdot (\arccos)'(x) \geq 0, \quad \forall x \in [0, 1],$$

can only be satisfied if $\phi'(r) \leq 0$ for all $r \in (0, \pi/2]$ since $(\arccos)'(x) \leq 0$ for all $x \in [0, 1]$. \square

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