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Biorthogonal Rational Functions and the Generalized Eigenvalue Problem

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We present some general results concerning so-called biorthogonal polynomials of R_{II} type introduced by M. Ismail and D. Masson. These polynomials give rise to a pair of rational functions which are biorthogonal with respect to a linear functional. It is shown that these rational functions naturally appear as eigenvectors of the generalized eigenvalue problem for two arbitrary tri-diagonal matrices. We study spectral transformations of these functions leading to a rational modification of the linear functional. An analogue of the Christoffel–Darboux formula is obtained. © 1999 Academic Press

Key Words: biorthogonal rational functions; generalized eigenvalue problem; Christoffel–Darboux formula; spectral transformations.

1. INTRODUCTION

The ordinary orthogonal polynomials (OOP) $P_n(x)$ are defined through the three-term recurrence relation (TRR)

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \tag{1.1}$$

and initial conditions

$$P_0(x) = 1, \qquad P_1(x) = x - b_0. \tag{1.2}$$

The polynomials $P_n(x)$ are monic, i.e., $P_n(x) = x^n + O(x^{n-1})$. It is well known (see, e.g., [5]) that (under some restrictions upon the recurrence coefficients) there exists a linear functional $\mathscr{L}$ defined on the space of all polynomials via moments $c_n = \mathscr{L}\{x^n\}$ such that the orthogonality condition

$$\mathscr{L}\{x^j P_n(x)\} = 0, \qquad 0 \leqslant j < n, \tag{1.3}$$

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takes place. This condition can be rewritten in terms of orthogonality of polynomials $P_n(x)$ themselves

$$\mathscr{L}\{P_m(x)P_n(x)\} = h_n \delta_{nm}, \tag{1.4}$$

where $h_n = u_1 u_2 \cdots u_n$ is a normalization constant.

Biorthogonal polynomials (BOP) are defined through three-term recurrence relations but with a more complicated dependence on the spectral parameter x.

For example, so-called Laurent BOP are defined through the recurrence relation [3, 10]

$$P_{n+1}(z) + d_n P_n(z) = z(P_n(z) + b_n P_{n-1}(z)), \tag{1.5}$$

with initial conditions

$$P_0(z) = 1, \qquad P_1(z) = z - d_0. \tag{1.6}$$

It is obvious from (1.5) and (1.6) that $P_n(z)$ are again monic polynomials: $P_n(z) = z^n + O(z^{n-1})$. It can be shown [10] that there exists a *linear Laurent functional* $\mathscr{L}$ defined on all Laurent series through the moments $c_n = \mathscr{L}\{z^n\}$, $n = 0, \pm 1, \pm 2, \ldots$ such that the orthogonality condition

$$\mathscr{L}\{z^{-j} P_n(z)\} = 0, \qquad 0 \leqslant j < n, \tag{1.7}$$

holds. This orthogonality condition can be rewritten in terms of the *biorthogonality* condition for the polynomials themselves

$$\mathscr{L}\{P_n(z)\, Q_m(1/z)\} = 0, \qquad n \neq m, \tag{1.8}$$

where $Q_m(z)$ are *adjacent* LBP defined by the relation $Q_n(z) = z^n(P_{n+1}(1/z) - z^{-1} P_n(1/z))$, $Q_0 = 1$.

We see that in contrast to the case of OOP the orthogonality condition (1.8) involves *two different* systems of the polynomials $P_n(z)$ and $Q_n(z)$ and hence we deal rather with biorthogonality instead of orthogonality.

J. Wilson [19] constructed remarkable explicit families of biorthogonal rational functions expressed via generalized hypergeometric functions ${}_9F_8$. This result lead Ismail and Masson to considering general families of polynomials satisfying the following three-term recurrence relation [12]

$$P_{n+1}(z) + \rho_n(v_n - z)\, P_n(z) + u_n(z - a_n)(z - b_n)\, P_{n-1}(z) = 0, \tag{1.9}$$

with initial conditions

$$P_0(z) = 1, \qquad P_1(z) = \rho_0(z - v_0). \tag{1.10}$$

For convenience we introduce the notations

$$A_0 = B_0 = 1, \qquad A_n(z) = \prod_{k=1}^{n} (z - a_k), \qquad B_n(z) = \prod_{k=1}^{n} (z - b_k). \tag{1.11}$$

Assume that the following restrictions are fulfilled:

$$P_n(a_k) \neq 0, \qquad P_n(b_k) \neq 0, \qquad u_n \neq 0, \qquad \text{for all} \quad n, k. \tag{1.12}$$

Then it can be shown [12] that a linear functional $\mathscr{L}$ exists defined on the space of rational functions of the kind $F(z) = z^j/(A_m(z)\, B_n(z))$ such that the orthogonality relation

$$\mathscr{L}\left\{\frac{z^j P_n(z)}{A_n(z)\, B_n(z)}\right\} = 0, \qquad 0 \leqslant j < n \tag{1.13}$$

holds. Following the definition of [12] we will call (1.9) the recurrence relation of R_{II} type. Corresponding polynomials $P_n(z)$ will be also called polynomials of type R_{II}.

The purpose of this paper is twofold: first, we would like to elucidate the origin of (1.9) from the point of view of the generalized eigenvalue problem well known in the linear algebra. Namely we show that the generalized eigenvalue problem (5.2) for two *arbitrary* tri-diagonal (Jacobi) matrices L and M gives rise to the R_{II} type recurrence relation (1.9).

Note that orthogonality relations of the kind of (1.13) appear quite naturally in the theory of multipoint Padé approximations (MPA) (for details see, e.g., [7, 8, 16]). Hence from our coniderations it follows that MPA leads to the generalized eigenvalue problem for two Jacobi matrices. This result is a "multi-point" analogue of the well-known result that the ordinary (one-point) Padé approximations lead to the ordinary eigenvalue problem for the Jacobi matrix (see, e.g., [5]).

Second, we rewrite the orthogonality relation (1.13) in terms of the *biorthogonal* relation involving two different rational functions constructed from the polynomial solutions of the recurrence relation (1.9). As far as we know such biorthogonal relations are new: previously they were known only for concrete examples (which are related with Wilson's functions and their special and limiting cases [12, 9, 13]).

Finally, we construct spectral transforms of biorthogonal rational functions which are exact analogs of the well known Christoffel and Geronimus transforms for the ordinary orthogonal polynomials. We find also an analogue of the Christoffel–Darboux formula and reproducing kernel for generic biorthogonal rational functions.

In this paper we deal with generic biorthogonal rational functions. Some very special classes of such functions when the linear functional $\mathscr{L}$ has integral realization on the real line or on the unit circle (with additional strong restrictions upon the parameters a_n, b_n, u_n, v_n, ρ_n) were studied, e.g., in [4].

2. GENERALIZED EIGENVALUE PROBLEM FOR FINITE-DIMENSIONAL MATRICES

In this section we recall basic facts and results concerning the generalized eigenvalue problem in linear algebra [18]. We show how restriction of this problem to the tri-diagonal matrices generates just WIMRR.

Let L and M be two $(N+1)\times(N+1)$ matrices with complex elements L_{ik}, M_{ik}, $i, k = 0, 1, ..., N$. By the generalized eigenvalue problem (GEVP) we mean the matrix equation [18]

$$L\psi = \lambda M\psi, \tag{2.1}$$

where λ is an eigenvalue and ψ is corresponding eigenvector with the components $\psi_0, \psi_1, ..., \psi_N$. If the matrix M is invertible then we get the ordinary eigenvalue problem $M^{-1}L\psi = \lambda\psi$. However, sometimes it is convenient to deal directly with Eq. (2.1).

Consider adjoint GEVP

$$L^T\chi = \mu M^T\chi, \tag{2.2}$$

where L^T means transposed matrix. It is easily seen that if $\mu \neq \lambda$ then one has the biorthogonality relation

$$(M\psi, \chi) = (\psi, M^T\chi) = 0, \tag{2.3}$$

where the "scalar product" $(\,,)$ is assumed in the following sense: $(\psi, \chi) = \sum_{i=0}^{N} \psi_i\chi_i$. (Note that this "scalar product" in general does not satisfy positivity requirement in the case of complex values of ψ_n, χ_n.) Indeed, we have the relations

$$(L\psi, \chi) = \lambda(M\psi, \chi) = (\psi, L^T\chi) = \mu(\psi, M^T\chi) = \mu(M\psi, \chi). \tag{2.4}$$

Hence $(\lambda - \mu)(M\psi, \chi) = 0$ and we get (2.3).

The eigenvalues λ (or μ) are obtained as roots of the characteristic $N+1$-order polynomial defined through the determinant

$$D_{N+1}(\lambda) = |L - \lambda M| = 0. \tag{2.5}$$

It is obvious from (2.5) that the sets of eigenvalues λ and μ coincide with one another.

It can be easily shown that if the matrices L, M are real and symmetric and moreover, M is positive definite, then all eigenvalues λ, μ are real [18]. Such situation occurs, e.g., when reducing a pair of quadratic forms (defined by the symmetric matrices L, M) simultaneously to sums of squares [6]. However, we will not restrict ourselves with this (quite special) case.

In what follows we will deal with *tri-diagonal* matrices L and M. In other words, we assume that $L_{ik} = M_{ik} = 0$ if $|i-k| > 1$. Denote by $D_{n+1}(\lambda)$ the determinant obtained from $D_{N+1}(\lambda)$ by deleting all the rows and columns except the first $n+1$ ones (the first two determinants are $D_0 = 1$, $D_1 = L_{00} - \lambda M_{00}$). Then expanding the determinant $D_{n+1}(\lambda)$ about the last column we get the recurrence equation

$$D_{n+1}(\lambda) = \rho_n(v_n - \lambda) D_n(\lambda) - u_n(\lambda - a_n)(\lambda - b_n) D_{n-1}(\lambda) \tag{2.6}$$

with initial conditions $D_0 = 1$, $D_1 = \rho_0(v_0 - \lambda)$. Here we introduce the notations

$$\rho_n = M_{n,n}, \qquad v_n = \frac{L_{n,n}}{M_{n,n}}, \qquad n = 0, 1, ..., N$$

$$u_n = M_{n-1,n} M_{n,n-1}, \qquad b_n = \frac{L_{n-1,n}}{M_{n-1,n}},$$

$$a_n = \frac{L_{n,n-1}}{M_{n,n-1}}, \qquad n = 1, 2, ..., N.$$

Introducing the nth-order polynomials

$$P_n(z) = (-1)^n D_n(z) \tag{2.7}$$

we see that these polynomials satisfy (1.9). Thus the polynomials satisfying R_{II} type recurrence relation (1.9) arise naturally from the characteristic equation for the generalized eigenvalue problem in linear algebra.

Consider now eigenvectors ψ satisfying Eq. (2.1). We have the following set of $N+1$ equations

$$\begin{aligned} &L_{00}\psi_0 + L_{01}\psi_1 \\ &\qquad = \lambda(M_{00}\psi_0 + M_{01}\psi_1), \\ &L_{n,n-1}\psi_{n-1} + L_{nn}\psi_n + L_{n,n+1}\psi_{n+1} \\ &\qquad = \lambda(M_{n,n-1}\psi_{n-1} + M_{nn}\psi_n + M_{n,n+1}\psi_{n+1}) \\ &L_{N,N-1}\psi_{N-1} + L_{NN}\psi_N \\ &\qquad = \lambda(M_{N,N-1}\psi_{N-1} + M_{NN}\psi_N). \end{aligned} \tag{2.8}$$

Choose $\psi_0 = 1$. Then $\psi_1 = (\lambda M_{00} - L_{00})/(L_{01} - \lambda M_{01})$. By induction, it is proved that ψ_n is a rational function in λ being a ratio of two nth-order polynomials.

Quite analogously, for the eigenvector χ we get the system of equations

$$\begin{aligned} L_{00}\chi_0 + L_{10}\chi_1 &= \mu(M_{00}\chi_0 + M_{10}\chi_1), \\ L_{n-1,n}\chi_{n-1} + L_{nn}\chi_n + L_{n+1,n}\chi_{n+1} & \\ &= \mu(M_{n-1,n}\chi_{n-1} + M_{nn}\chi_n + M_{n+1,n}\chi_{n+1}) \\ L_{N-1,N}\chi_{N-1} + L_{NN}\chi_N &= \mu(M_{N-1,N}\chi_{N-1} + M_{NN}\chi_N). \end{aligned} \tag{2.9}$$

Choosing $\chi_0 = 1$ we see that χ_n is also a ratio of two nth order polynomials.

Both ψ_n and χ_n can be expressed in terms of the polynomials $P_n(\lambda)$ defined by (2.7),

$$\psi_n = (-1)^n \frac{P_n(\lambda)}{M_{01} M_{12} \cdots M_{n-1,n} B_n(\lambda)}, \qquad n = 1, 2, ..., N, \tag{2.10}$$

where $A_n(z)$, $B_n(z)$ are defined by (1.11).

Similarly

$$\chi_n = (-1)^n \frac{P_n(\mu)}{M_{10} M_{21} \cdots M_{n,n-1} A_n(\mu)}, \qquad n = 1, 2, ..., N. \tag{2.11}$$

The formulas (2.10) and (2.11) can be verified directly by substituting them into the recurrence relations (2.8) and (2.9).

Thus eigenvectors $\psi_n(\lambda)$ and $\chi_n(\mu)$ are constructed explicitly via the characteristic polynomials $P_n(z)$. Note that all the poles of the rational functions $\psi_n(\lambda)$, $\chi_n(\lambda)$ are known explicitly: they coincide with the points b_j, a_j, $j = 1, 2, ..., n$ (of course it is assumed that any zero of the numerators do not coincide with any zero of the denominators, which is generic situation).

3. BIORTHOGONALITY CONDITIONS: FINITE-DIMENSIONAL CASE

In this section we study biorthogonality conditions for rational functions.

Assume that all the eigenvalues λ_i, $i = 0, 1, ..., N$ of the characteristic equation (2.5) are distinct. Then we can write down the orthogonality condition (2.3) in the form

$$\sum_{k=0}^{N} \psi_k(\lambda_i)\, \eta_k(\lambda_j) = \frac{\delta_{ij}}{w_i}, \tag{3.1}$$

where w_i are some normalization constants and the vector $\eta(\mu)$ is defined by

$$\eta(\mu) = M^T \chi(\mu), \tag{3.2}$$

or, explicitly,

$$\begin{aligned}
&\eta_0(\mu) = M_{00}\chi_0(\mu) + M_{10}\chi_1(\mu), \\
&\eta_n(\mu) = M_{n+1,n}\chi_{n+1}(\mu) + M_{nn}\chi_n(\mu) + M_{n-1,n}\chi_{n-1}(\mu), \quad 0 < n < N, \\
&\eta_N(\mu) = M_{NN}\chi(\mu) + M_{N-1,N}\chi_{N-1}(\mu).
\end{aligned} \tag{3.3}$$

Note that $\eta_n(z)$ is a rational function being a ratio of two polynomials of the orders n and $n+1$. For $\eta_0(z)$ we have from (3.3)

$$\eta_0(z) = \frac{M_{00}(v_0 - a_1)}{z - a_1}. \tag{3.4}$$

It is therefore useful to introduce *new rational functions*

$$\xi_n(z) = \frac{\eta_n(z)}{\eta_0(z)}. \tag{3.5}$$

The functions ξ_n are ratios of two polynomials of nth order, and $\xi_0(z) = 1$. Thus the functions $\xi_n(z)$ have the same functional structure as the functions $\psi_n(z)$ and $\chi_n(z)$.

The relation (3.1) shows that two $(N+1) \times (N+1)$ matrices are inverses of each other. As a consequence, we get the dual orthogonality condition

$$\sum_{s=0}^{N} w_s \psi_i(\lambda_s)\, \eta_k(\lambda_s) = \delta_{ik}. \tag{3.6}$$

The relation (3.6) can be treated as biorthogonality condition for two rational functions $\psi_n(\lambda)$ and $\eta_n(\lambda)$ with respect to some finite discrete measure located at the points λ_s with the masses w_s. Taking into account the relation (3.5) we can rewrite (3.6) in the form

$$\sum_{s=0}^{N} \Omega_s \psi_i(\lambda_s)\, \xi_k(\lambda_s) = \delta_{ik}, \tag{3.7}$$

where the new weights Ω_s are defined as

$$\Omega_s = w_s \eta_0(\lambda_s). \tag{3.8}$$

In order to find explicitly the normalization constants f_s (and hence the weight function w_s) we need some technique connected with the linear functional $\mathscr{L}$ (1.13). This will be done in the next section.

4. BIORTHOGONALITY CONDITIONS: GENERAL CASE

Now we return to general (infinite-dimensional) case of the polynomials $P_n(z)$ satisfying the recurrence relation (1.9).

In what following we will assume that the restrictions (1.12) are fulfilled. Following Ismail and Masson [12] we introduce the linear functional $\mathscr{L}$ providing the orthogonality relation (1.13). We need the following normalization constants

$$q_n = \mathscr{L}\left\{\frac{z^n P_n(z)}{A_n(z)\, B_n(z)}\right\}. \tag{4.1}$$

From the recurrence relation (1.9) we get the relation [12]

$$q_{n+1} - \rho_n q_n + u_n q_{n-1} = 0, \qquad n = 1, 2, \ldots \tag{4.2}$$

whereas initial values q_0 and q_1 can be chosen as arbitrary parameters. As was shown in [12] all entries in the (infinite) "moments" matrix

$$G_{nm} = \mathscr{L}\left\{\frac{1}{A_n(z)\, B_m(z)}\right\} \tag{4.3}$$

are determined uniquely through q_0, q_1 and the recurrence coefficients. For example,

$$G_{01} = \mathscr{L}\left\{\frac{1}{z - b_1}\right\} = \frac{q_0 - q_1/\rho_0}{v_0 - b_1} \tag{4.4}$$

$$G_{10} = \mathscr{L}\left\{\frac{1}{z - a_1}\right\} = \frac{q_0 - q_1/\rho_0}{v_0 - a_1} \tag{4.5}$$

$$G_{11} = \mathscr{L}\left\{\frac{1}{(z - a_1)(z - b_1)}\right\} = \frac{q_0 - q_1/\rho_0}{(v_0 - a_1)(v_0 - b_1)} \tag{4.6}$$

Note also that the leading coefficients of the polynomials $P_n(z) = \kappa_n z^n + O(z^{n-1})$ satisfy the same recurrence relation as (4.2)

$$\kappa_{n+1} - \rho_n \kappa_n + u_n \kappa_{n-1} = 0, \qquad n = 1, 2, \ldots \tag{4.7}$$

with initial conditions $\kappa_0 = 1$, $\kappa_1 = \rho_0$. Comparing (4.2) and (4.7) and taking into account initial conditions we derive the Wronskian relation

$$q_n \kappa_{n-1} - \kappa_n q_{n-1} = u_1 u_2 \cdots u_{n-1} (q_1 - \rho_0 q_0). \tag{4.8}$$

We now define two rational functions $R_n(z)$ and $R_n^{(1)}(z)$ by the formulas

$$R_0(z) = 1, \qquad R_n(z) = \frac{P_n(z)}{B_n(z)} \tag{4.9}$$

$$R_0^{(1)}(z) = 1, \qquad R_n^{(1)}(z) = \frac{P_n(z)}{h_n A_n(z)}, \tag{4.10}$$

where $h_n = u_1 u_2, ..., u_n$.

These functions satisfy the generalized eigenvalue problems:

$$(z - b_{n+1})\, R_{n+1}(z) + \rho_n (v_n - z)\, R_n(z) + u_n (z - a_n)\, R_{n-1}(z) = 0 \tag{4.11}$$

$$u_{n+1}(z - a_{n+1})\, R_{n+1}^{(1)}(z) + \rho_n (v_n - z)\, R_n^{(1)}(z) + (z - b_n)\, R_{n-1}^{(1)}(z) = 0. \tag{4.12}$$

It is clear that the functions $R_n(z)$ and $R_n^{(1)}(z)$ are analogues of the eigenvectors ψ and χ introduced in the Section 2.

We also introduce another set of rational functions $H_n(z)$ by the relation

$$\begin{aligned} H_0(z) &= u_1 R_1^{(1)}(z) - \rho_0 = \rho_0 \frac{a_1 - v_0}{z - a_1} \\ H_n(z) &= R_{n-1}^{(1)}(z) - \rho_n R_n^{(1)}(z) + u_{n+1} R_{n+1}^{(1)}(z), \qquad n > 0. \end{aligned} \tag{4.13}$$

THEOREM 1. *Let $\mathscr{L}$ be linear functional providing orthogonality property* (1.13) *for the polynomials $P_n(z)$. Then we have the following biorthogonality property for the rational functions $\{R_n(z)\}$ and $\{H_m(z)\}$*

$$\mathscr{L}\{R_n(z)\, H_m(z)\} = (q_1 - \rho_0 q_0)\, \delta_{nm}, \qquad n, m = 0, 1, 2, ... \tag{4.14}$$

Proof. Assume first that $m < n - 1$. Then we have

$$\begin{aligned} &\mathscr{L}\{R_n(z)\, H_m(z)\} \\ &\quad = \mathscr{L}\left\{\frac{P_n(z)}{B_n(z)}\left(\frac{P_{m-1}(z)}{h_{m-1} A_{m-1}(z)} - \frac{\rho_m P_m(z)}{h_m A_m(z)} + \frac{u_{m+1} P_{m+1}(z)}{h_{m+1} A_{m+1}(z)}\right)\right\} \\ &\quad = \mathscr{L}\left\{\frac{P_n(z)}{h_m A_n(z)\, B_n(z)}\,(u_m (z - a_m)(z - a_{m+1}) \cdots (z - a_n)\, P_{m-1}(z)\right. \\ &\qquad \left. - \rho_m (z - a_{m+1}) \cdots (z - a_n)\, P_m(z) + (z - a_{m+2}) \cdots (z - a_n))\, P_{m+1}\right\}. \end{aligned} \tag{4.15}$$

Using the relations (1.13) and (4.1) we can rewrite (4.15) in the form

$$\mathscr{L}\{R_n(z)\,H_m(z)\} = q_n(u_m\kappa_{m-1} - \rho_m\kappa_m + \kappa_{m+1})/h_m = 0 \tag{4.16}$$

due to the relation (4.7).

Now assume that $m > n+1$. Then we similarly derive

$$\begin{aligned}\mathscr{L}\{R_n(z)\,H_m(z)\} &= \mathscr{L}\left\{\frac{P_n(z)\,P_{m-1}(z)(z-b_{n+1})\cdots(z-b_{m-1})}{h_{m-1}A_{m-1}(z)\,B_{m-1}(z)}\right\}\\ &\quad - \rho_m\mathscr{L}\left\{\frac{P_n(z)\,P_m(z)(z-b_{n+1})\cdots(z-b_m)}{h_mA_m(z)\,B_m(z)}\right\}\\ &\quad + u_{m+1}\mathscr{L}\left\{\frac{P_n(z)\,P_{m+1}(z)(z-b_{n+1})\cdots(z-b_{m+1})}{h_{m+1}A_{m+1}(z)\,B_{m+1}(z)}\right\}\\ &= \kappa_n(u_mq_{m-1} - \rho_mq_m + q_{m+1})/h_m = 0\end{aligned}$$

due to (4.2). The cases $m = n+1$ and $m = n-1$ are checked analogously.

Finally consider the case $n = m$. Omitting simple technical details we obtain

$$\begin{aligned}\mathscr{L}\{R_n(z)\,H_n(z)\} &= (u_n\kappa_{n-1}q_n - \rho_n\kappa_nq_n + \kappa_nq_{n+1})/h_n\\ &= u_n(\kappa_{n-1}q_n - \kappa_nq_{n-1})/h_n = q_1 - \rho_0q_0\end{aligned}$$

due to the Wronskian formula (4.8). Thus the theorem is proven.

If one introduces modified rational functions

$$T_n(z) = H_n(z)/H_0(z) \tag{4.17}$$

then the biorthogonality relation becomes

$$\mathscr{M}\{R_n(z)\,T_m(z)\} = \delta_{nm}, \tag{4.18}$$

where the modified functional $\mathscr{M}$ is defined on the same space of rational functions $f(z)$ as

$$\mathscr{M}\{f(z)\} = \mathscr{L}\left\{\frac{H_0(z)\,f(z)}{q_1 - \rho_0q_0}\right\}. \tag{4.19}$$

Note that the functions $T_n(z)$ are more convenient for analysis than $H_n(z)$ because $T_n(z)$ are ratio of two nth order polynomials just as the functions $R_n(z)$ and $R_n^{(1)}(z)$.

We also need expression for so-called polynomials of the second kind $Q_n(z)$ satisfying the same recurrence relation (1.9) but with another initial conditions (we follow the choice of [12]): $Q_0=0$, $Q_1=1$. Obviously, the degree of the polynomial $Q_n(z)$ is $n-1$.

First of all, from the recurrence relations (1.9) and initial conditions for $P_n(z)$, $Q_n(z)$ we get the Wronskian identity

$$P_{n+1}(z)\, Q_n(z) - P_n(z)\, Q_{n+1}(z) = -h_n A_n(z)\, B_n(z) \tag{4.20}$$

The polynomials $Q_n(z)$ have the following representation in terms of linear functional of the polynomials $P_n(z)$:

THEOREM 2. *The polynomials of the second kind $Q_n(z)$ have the expression*

$$Q_n(z) = \mathscr{L}_y \left\{ \frac{P_n(y)\, A_n(z)\, B_n(z) - P_n(z)\, A_n(y)\, B_n(y)}{(q_1 - \rho_0 q_0)(z-y)\, A_n(y)\, B_n(y)} \right\}, \tag{4.21}$$

where the functional $\mathscr{L}_y$ acts only on the variable y.

The proof of this theorem is direct: the cases $Q_0(z)$ and $Q_1(z)$ are verified simply from (4.21). Then it is verified that the functions $Q_n(z)$ defined by (4.21) satisfy the recurrence relation (1.9).

The importance of this theorem consists in the possibility of giving explicit expression for the weights w_s in the finite-dimensional case.

Indeed, assume that all $N+1$ roots z_s of the polynomial $P_{N+1}(z)$ are simple. Then we already have the expression for the linear functional

$$\mathscr{L}\{F(z)\} = \sum_{s=0}^{N} w_s F(z_s) \tag{4.22}$$

with some yet unknown weights w_s. Using this expression we find from (4.21)

$$\frac{Q_{N+1}(z)}{P_{N+1}(z)} = -\sum_{s=0}^{N} \frac{w_s}{(q_1 - \rho_0 q_0)(z - z_s)}. \tag{4.23}$$

On the other hand, expanding the left-hand side of (4.23) into partial fractions we find

$$-\frac{w_s}{(q_1 - \rho_0 q_0)} = \frac{Q_{N+1}(z_s)}{P'_{N+1}(z_s)} = \frac{h_N A_N(z_s)\, B_N(z_s)}{P'_{N+1}(z_s)\, P_N(z_s)}, \tag{4.24}$$

where the last equality follows from the Wronskian formula (4.20). Thus if one defines the weights ω_s via the formula

$$\omega_s = -\frac{h_N A_N(z_s)\, B_N(z_s)}{P'_{N+1}(z_s)\, P_N(z_s)}, \qquad s=0, 1, ..., N \tag{4.25}$$

then the finite biorthogonality relation (4.14) becomes

$$\sum_{s=0}^{N} \omega_s R_n(z_s)\, H_m(z_s) = \sum_{s=0}^{N} \Omega_s R_n(z_s)\, T_m(z_s) = \delta_{nm}, \qquad 0 \leqslant n, m \leqslant N, \tag{4.26}$$

where $\Omega_s = \omega_s(\rho_0(a_1 - v_0)/(z_s - a_1))$. We see that the values q_0, q_1 are not needed for construction of the weights ω_s, Ω_s. The formulas (4.25) and (4.26) yield complete solution of the problem of finding finite biorthogonality property for rational functions. As far as we know these formulas are new (in [9, 12, 13] explicit biorthogonality relations like (4.26) were written down only for concrete examples connected with Wilson's rational functions).

5. SPECTRAL TRANSFORMATIONS OF BIORTHOGONAL RATIONAL FUNCTIONS

In this section we introduce two elementary transformations which act on the space of biorthogonal rational functions in analogy with the well known Christoffel and Geronimus transforms for the ordinary orthogonal polynomials (see, e.g., [20]).

Return to the recurrence relation for rational functions (replacing z by λ)

$$b_{n+1} R_{n+1} - \rho_n v_n R_n + a_n u_n R_{n-1} = \lambda(R_{n+1} - \rho_n R_n + u_n R_{n-1}). \tag{5.1}$$

It is convenient to rewrite this relation in terms of generalized eigenvalue problem

$$L\psi = \lambda M \psi, \tag{5.2}$$

where the vector ψ is defined by its components $\psi = \{R_0, R_1, ..., R_n, ...\}$ and the operators L, M act as

$$\begin{aligned} L\, R_n &= b_{n+1} R_{n+1} - \rho_n v_n R_n + a_n u_n R_{n-1} \\ M\, R_n &= R_{n+1} - \rho_n R_n + u_n R_{n-1}. \end{aligned} \tag{5.3}$$

Note that in fact, L and M are *arbitrary* tri-diagonal matrices.

Let D and K be some operators such that the following operator relations hold,

$$\tilde{L}D = KL, \qquad \tilde{M}D = KM, \tag{5.4}$$

where the operators $\tilde{L}$ and $\tilde{M}$ are defined by the same formulae (5.3) but with the replacements $a_n \to \tilde{a}_n$, $u_n \to \tilde{u}_n$, ..., etc. Then it is easily seen that the new vector $\tilde{\psi} = D\psi$ is the eigenvector of the generalized eigenvalue problem

$$\tilde{L}\tilde{\psi} = \lambda \tilde{M}\tilde{\psi} \tag{5.5}$$

with the same eigenvalue λ. The corresponding components $\tilde{R}_n$ can be then interpreted as transformed rational functions.

Assume that the operators D, K are *two-diagonal*

$$\begin{aligned} DR_n &= R_n + \beta_n R_{n-1}, \\ KR_n &= R_n + \gamma_n R_{n-1}, \end{aligned} \tag{5.6}$$

with some coefficients β_n, γ_n to be determined.

Substituting (5.6) into (5.4) and acting to an arbitrary vector with components R_n we arrive at the conditions

$$\begin{aligned} &\tilde{b}_n = b_n, \qquad \tilde{a}_n = a_{n-1}, \qquad \tilde{u}_n = \frac{\gamma_n u_{n-1}}{\beta_{n-1}} \\ &\tilde{\rho}_n = \beta_{n+1} - \gamma_n + \rho_n \\ &\tilde{w}_n = w_n + \beta_{n+1} b_{n+1} - \gamma_n b_n \\ a_{n-1}\tilde{u}_n - \beta_n \tilde{w}_n &= a_n u_n - \gamma_n w_{n-1} \\ \tilde{u}_n - \beta_n \tilde{\rho}_n &= u_n - \rho_{n-1}\gamma_n, \end{aligned} \tag{5.7}$$

where we denote for brevity $w_n = \rho_n v_n$. From (5.7) it is easily found that

$$\begin{aligned} \beta_n &= -\frac{\phi_n}{\phi_{n-1}} \\ \gamma_n &= -\frac{u_n \phi_{n-1} + \phi_{n+1} - \rho_n \phi_n}{u_{n-1}\phi_{n-2} + \phi_n - \rho_{n-1}\phi_{n-1}} = -\frac{M\{\phi_n\}}{M\{\phi_{n-1}\}}, \end{aligned} \tag{5.8}$$

where ϕ_n is a solution of the recurrence equation

$$b_{n+1}\phi_{n+1}+a_n u_n\phi_{n-1}-w_n\phi_n=\mu(\phi_{n+1}+u_n\phi_{n-1}-\rho_n\phi_n) \tag{5.9}$$

and μ is an arbitrary parameter.

In the operator form Eq. (5.9) is written as

$$L\Phi=\mu M\Phi, \tag{5.10}$$

where Φ is the vector $\Phi=\{\phi_0, \phi_1, ..., \phi_n, ...\}$. The transformed vector $\psi=D\{\psi\}$ have the components R_n which can be written as

$$\tilde{R}_n=F(z)(R_n+\beta_n R_{n-1})=F(z)\left(R_n-\frac{\phi_n}{\phi_{n-1}}R_{n-1}\right), \tag{5.11}$$

where $F(z)$ is arbitrary function of z (obviously operator relation (5.5) do not depend on $F(z)$). However, if both $R_n(z)$ and $\tilde{R}_n(z)$ are rational functions in z of the order of $[n/n]$, i.e.,

$$\begin{aligned}
R_0(z)&=\tilde{R}_0(z)=1,\\
R_n(z)&=\frac{P_n(z)}{(z-b_1)(z-b_2)\cdots(z-b_n)},\\
\tilde{R}_n(z)&=\frac{P_n(z)}{(z-\tilde{b}_1)(z-\tilde{b}_2)\cdots(z-\tilde{b}_n)},
\end{aligned} \tag{5.12}$$

where $P_n(z)$ are nth order polynomials satisfying the recurrence relation (1.9), then necessarily $F(z)=1$ and hence

$$\tilde{R}_n=R_n-\frac{\phi_n}{\phi_{n-1}}R_{n-1}. \tag{5.13}$$

The transformation (5.13) formally resembles so-called discrete Darboux transformation for the solutions of the discrete Schrödinger equation (see, e.g., [14, 20]), that is, (5.13) is the exact analogue of the Geronimus transform for the ordinary orthogonal polynomials [20].

Consider now another choice for the operators D, K,

$$\begin{aligned}
D\,R_n&=R_{n+1}+\xi_n R_n,\\
K\,R_n&=R_{n+1}+\eta_n R_n.
\end{aligned} \tag{5.14}$$

In this case we should have analogously

$$\tilde{b}_n = b_{n+1}, \qquad \tilde{a}_n = a_n, \qquad \tilde{u}_n = \frac{\eta_n}{\xi_{n-1}}\, u_n,$$
$$\tilde{\rho}_n = \rho_{n+1} + \xi_{n+1} - \eta_n, \tag{5.15}$$
$$\tilde{w}_n = w_{n+1} + \xi_{n+1} b_{n+2} - \eta_n b_{n+1},$$

where

$$\xi_n = -\frac{\phi_{n+1}}{\phi_n}, \qquad \eta_n = -\frac{\phi_{n+2} - \rho_{n+1}\varphi_{n+1} + u_{n+1}\phi_n}{\phi_{n+1} - \rho_n\phi_n + u_n\phi_{n-1}}, \tag{5.16}$$

and again ϕ_n is some solution of the recurrence equation (5.9).

However, in this case there is restriction upon the possible choice of the solution ϕ_n. Indeed, the transformed components $\tilde{R}_n$ are

$$\tilde{R}_n = F(z)(R_{n+1}(z) + \xi_n R_n(z)). \tag{5.17}$$

From the requirements that both $R_n(z)$ and $\tilde{R}_n(z)$ have the form (5.12) we have that necessarily

$$F(z) = \kappa\, \frac{z - b_1}{z - \mu}, \tag{5.18}$$

where $\kappa = (\mu - b_1)/(\rho_0(v_0 - b_1)$ and

$$\phi_n = R_n(\mu). \tag{5.19}$$

Hence the transformation is written explicitly as

$$\tilde{R}_n(z) = \kappa\, \frac{z - b_1}{z - \mu} \left(R_{n+1}(z) - \frac{R_{n+1}(\mu)}{R_n(\mu)}\, R_n(z) \right). \tag{5.20}$$

The transformation (5.20) is the exact "rational" analogues of the famous Christoffel transformation for orthogonal polynomials (see, e.g., [17, 20]). Note the poles of the function $\tilde{R}_n(z)$ are located at the points $b_2, b_3, ..., b_{n+1}$, whereas the poles of the function $R_n(z)$ are located at the points $b_1, b_2, ..., b_n$.

Let $\mathscr{L}$ be linear functional providing orthogonality condition (1.13) and $\tilde{\mathscr{L}}$ is linear functional providing orthogonality for the functions $\tilde{R}_n(z)$, obtained from $R_n(z)$ by the "Christoffel" transform (5.20), i.e.,

$$\tilde{\mathscr{L}}\left\{\frac{\tilde{R}_n(z)\, z^j}{A_n(z)}\right\}=0, \qquad 0\leqslant j<n. \tag{5.21}$$

THEOREM 3. *The linear functionals $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are connected by the relation*

$$\tilde{\mathscr{L}}=\left(\frac{z-\mu}{z-b_1}\right)\mathscr{L}, \tag{5.22}$$

where as usual by multiplication of the functional to a function $y(z)\,\mathscr{L}$ it is assumed the functional acting on a space of functions $f(z)$ as $\mathscr{L}\{y(z)\, f(z)\}$.

Proof. As the functional $\tilde{\mathscr{L}}$ is uniquely determined through the functions $\tilde{R}_n(z)$ it is sufficient to prove that the functional defined by the relation (5.22) provides the orthogonality relation (5.21). The relationships

$$\tilde{\mathscr{L}}\left\{\frac{\tilde{R}_n(z)\, z^j}{A_n(z)}\right\}=\kappa\mathscr{L}\left\{\frac{(R_{n+1}(z)+\xi_n R_n(z))\, z^j}{A_n(z)}\right\}=0, \qquad j=0, 1, ..., n-1 \tag{5.23}$$

follow from the orthogonality condition (1.13). This proves the theorem.

Note that for the ordinary orthogonal polynomials the functional is multiplied by a *linear* function $z-\mu$ (see, e.g., [5]). In the case of biorthogonal rational functions we have instead the *rational* multiplier $(z-\mu)/(z-b_1)$. In the seminal Wilson's work [19] this observation was crucial for explicit constructions of the rational functions. In fact, Wilson exploited (rather implicitly) the invariance of his rational functions with respect to the Christoffel transformation. Explicitly we provide this program in [15] were some new (not belonging to the Wilson's class) examples of the polynomials of R_{II} type are constructed.

6. THE CHRISTOFFEL–DARBOUX IDENTITY

In this section we derive an analogue of the Christoffel–Darboux identity (CDI) for the rational functions $R_n(z)$ and $T_n(z)$.

First of all from the relations (4.11), (4.12), (4.13), and (4.17) we derive the system of relations

$$\begin{aligned} R^{(1)}_{n+1}(z) &= \kappa^{(1)}_n \frac{z-b_n}{z-a_1} T_n(z) + \kappa^{(2)}_n R^{(1)}_n(z), \\ T_{n+1}(z) &= \kappa^{(3)}_n \frac{z-b_n}{z-a_{n+2}} T_n(z) + \kappa^{(4)}_n \frac{z-a_1}{z-a_{n+2}} R^{(1)}_n(z), \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \kappa^{(1)}_n &= \frac{\rho_0(a_1-v_0)}{u_{n+1}(a_{n+1}-b_n)}, \\ \kappa^{(2)}_n &= \frac{\rho_n(v_n-b_n)}{u_{n+1}(a_{n+1}-b_n)}, \\ \kappa^{(3)}_n &= \frac{\rho_{n+1}(a_{n+2}-v_{n+1})}{u_{n+1}(a_{n+1}-b_n)}, \\ \kappa^{(4)}_n &= \frac{u_{n+1}(a_{n+1}-b_n)(b_{n+1}-a_{n+2})+\rho_n\rho_{n+1}(v_{n+1}-a_{n+2})(b_n-v_n)}{\rho_0 u_{n+1}(a_1-v_0)(a_{n+1}-b_n)}. \end{aligned} \tag{6.2}$$

Note that the parameter b_0 in the coefficients (6.2) may be arbitrary. Then starting from initial values $R^{(1)}_0(z) = T_0(z) = 1$ we can construct recursively all the rational functions $R^{(1)}_n(z)$ and $T_n(z)$ from the system (6.1).

The idea of deriving CDI is the following. Introduce the expression

$$L_n(z, w) = X_n(z, w)\, R^{(1)}_n(z)\, T_n(w) + Y_n(z, w)\, R^{(1)}_n(w)\, T_n(z), \tag{6.3}$$

where $X_n(z, w)$, $Y_n(z, w)$ are some functions in two independent variables z, w. We should choose the functions $X_n(z, w)$, $Y_n(z, w)$ in such a manner that the relations

$$L_{n+1}(z, w) - L_n(z, w) = Z_n(z, w)\, R^{(1)}_n(z)\, T_n(w), \tag{6.4}$$

are satisfied by another set of functions $\{Z_n(z, w)\}$. If $L_0(z, w) = 0$ then summing (6.4) we arrive at the identity

$$L_{n+1}(z, w) = \sum_{k=0}^{n} Z_k(z, w)\, R^{(1)}_k(z)\, T_k(w), \tag{6.5}$$

which is one possible form of CDI.

In order to find $X_n(z, w)$, $Y_n(z, w)$, $Z_n(z, w)$ we express $L_{n+1}(z, w)$ in terms of $R^{(1)}_n$ and T_n using the recurrence formulae (6.1). Then we find that the requirement (6.4) is fulfilled when

$$X_0(z, w) = \frac{w - a_1}{b_0 - a_1},$$

$$X_n(z, w) = \frac{h_n A_n(z)(z - b_n)(w - a_{n+1})}{(z - b_0)\, B_n(z)(b_n - a_{n+1})}, \qquad n = 1, 2, ..., \tag{6.6}$$

$$Y_n(z, w) = -\frac{(z - a_{n+1})(w - a_1)}{(z - a_1)(w - a_{n+1})}\, X_n(z, w), \qquad n = 0, 1, ..., \tag{6.7}$$

$$Z_n(z, w) = \frac{(w - z)(b_n - a_{n+1})}{(z - b_n)(w - a_{n+1})}\, X_n(z, w), \qquad n = 0, 1, ..., \tag{6.8}$$

where $A_n(z)$, $B_n(z)$ are defined by (1.11).

Thus from (6.5) we derive CDI

$$K_{n+1}(z, w) = \sum_{k=0}^{n} R_k(z)\, T_k(w), \tag{6.9}$$

where

$$\begin{aligned} K_{n+1}(z, w) &= \frac{(z - a_{n+1})(z - b_n)}{(w - z)(b_n - a_{n+1})} \\ &\quad \times \left(\frac{w - b_n}{z - b_n} R_n(z)\, T_n(w) - \frac{w - a_1}{z - a_1} \frac{A_n(z)\, B_n(w)}{A_n(w)\, B_n(z)} R_n(w)\, T_n(z) \right) \\ &= \frac{(z - a_{n+2})(z - b_{n+1})}{(w - z)(b_{n+1} - a_{n+2})} \\ &\quad \times \left(\frac{w - a_{n+2}}{z - a_{n+2}} R_{n+1}(z)\, T_{n+1}(w) - \frac{w - a_1}{z - a_1} \right. \\ &\quad \left. \times \frac{A_{n+1}(z)\, B_{n+1}(w)}{A_{n+1}(w)\, B_{n+1}(z)} R_{n+1}(w)\, T_{n+1}(z) \right) \end{aligned} \tag{6.10}$$

is a reproducing kernel on the space of rational functions in the sense explained below.

Assume that

$$F_j(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_j)}{(z - b_1)(z - b_2) \cdots (z - b_j)}$$

is a rational function of the order $[j/j]$ with prescribed poles $b_1, b_2, ..., b_j$ and arbitrary zeroes z_i. Then obviously we can expand $F_j(z)$ in terms of the rational functions $R_k(z)$,

$$F_j(z) = \sum_{k=0}^{j} \tau_k R_k(z) \tag{6.11}$$

with some coefficients τ_k. Then from (4.18), (6.9) and (6.11) we have the identity

$$\mathscr{M}_w\{K_{n+1}(z, w)\, F_j(w)\} = F_j(z), \tag{6.12}$$

where in left-hand side of (6.12) it is assumed that the linear functional $\mathscr{M}_w$ acts only on the variable w. Thus $K_{n+1}(z, w)$ is a reproducing kernel for *any* rational function of the order $[j/j]$ with arbitrary $j \leqslant n$ and prescribed poles $b_1, b_2, ..., b_j$.

Using the CD identity we can easily rediscover biorthogonality relation (4.26). Indeed, assume that z_s, $s = 0, 1, ..., n$ are (simple) zeros of the polynomial $P_{n+1}(z)$. Assume additionally that $z_s \neq a_k,\ b_k,\ k = 1, 2, ..., n+1$. Then obviously z_s are zeros of the rational functions $R_{n+1}(z)$ and $R^{(1)}_{n+1}(z)$.

Putting $z = z_s$, $w = z'_s$ we can rewrite (6.9) in the form

$$\sum_{k=0}^{n} R_k(z_s)\, T_k(z'_s) = \frac{\delta_{ss'}}{\gamma_s}, \tag{6.13}$$

where

$$\gamma_s = \frac{a_{n+2} - b_{n+1}}{(z_s - b_{n+1})(z_s - a_{n+2})\, R'_{n+1}(z_s)\, T_{n+1}(z_s)}. \tag{6.14}$$

Note that $R'_{n+1}(z_s) \neq 0$ because of our assumption that all the zeros of $P_{n+1}(z)$ are simple. Moreover $T_{n+1}(z_s) \neq 0$, otherwise the system (6.1) implies $R_n(z_s) = R_{n-1}(z_s) = \cdots = R_0(z_s) = 0$, which is impossible. Hence γ_s are well defined for all $s = 0, 1, ..., n$.

The set of equations (6.13) can be treated as orthogonality condition for two matrices with entries $R_k(z_s)$ and $\gamma_s T_k(z'_s)$. Hence we established the orthogonality relation

$$\sum_{s=0}^{n} \gamma_s R_k(z_s)\, T_{k'}(z_s) = \delta_{kk'}. \tag{6.15}$$

It is seen that (6.15) is nothing but the finite orthogonality relation (4.26). Indeed, we have

$$R'_{n+1}(z_s) = \frac{P'_{n+1}(z_s)}{B_{n+1}(z_s)} \tag{6.16}$$

$$T_{n+1}(z_s) = \frac{(b_{n+1} - a_{n+2})\, P_n(z_s)}{H_0(z_s)(z_s - a_{n+2})\, h_n A_n(z_s)}, \tag{6.17}$$

where $H_0(z_s) = \rho_0(a_1 - v_0)/(z_s - a_1)$.

Hence we get

$$\gamma_s = -\frac{H_0(z_s)\, h_n A_n(z_s)\, B_n(z_s)}{P_n(z_s)\, P'_{n+1}(z_s)}. \tag{6.18}$$

So γ_s coincides with the weights Ω_s providing discrete orthogonality (4.26) for the rational functions $R_n(z)$ and $T_n(z)$.

7. WHEN BIORTHOGONALITY BECOMES ORTHOGONALITY?

Assume that $\mathscr{L}$ is a linear functional providing biorthogonality relation (4.18). In this relations the rational functions $R_n(z)$ and $T_n(z)$ have degree $[n/n]$ (i.e., are ratios of two polynomials of degree n). In general these functions do not coincide. Hence the term "biorthogonality" is quite appropriate for this situation.

It is natural to ask when biorthogonality becomes pure orthogonality. In other words we demand that the relationship

$$\frac{T_n(z)}{R_n(z)} = \sigma_n \tag{7.1}$$

holds for some constants $\sigma_n \neq 0$ not depending on z. In this case the *orthogonality* relation

$$\mathscr{L}\{R_n(z)\, R_m(z)\} = \frac{\delta_{nm}}{\sigma_n} \tag{7.2}$$

for the rational functions $R_n(z)$ will be satisfied.

We would like to seek restrictions upon the recurrence coefficients $a_n, b_n, u_n, v_n, \rho_n$ under which the relation (7.1) (and hence (7.2)) is valid.

PROPOSITION 1. *The relation (7.1) holds if and only if the parameters* $\{b_n\}$ *satisfy*

$$b_n = a_{n+1}, \qquad n = 1, 2, ..., \tag{7.3}$$

but the other parameters $\{a_n\}$, $\{u_n\}$, $\{v_n\}$, $\{\rho_n\}$ *are arbitrary. In this case*

$$\sigma_n = \frac{\rho_n(v_n - a_{n+1})}{h_n \rho_0(v_0 - a_1)}. \tag{7.4}$$

Proof. The denominator of $R_n(z)$ has zeros at the points $b_1, b_2, ..., b_n$ whereas the denominator of $T_n(z)$ has zeros at the points $a_2, a_3, ..., a_{n+1}$. Whence the relation (7.3) is necessary for (7.1) to hold. Conversely, assume that relation (7.3) takes place. Then obviously $T_0(z) = R_0(z) = 1$ and it is easily verified directly that $T_1(z) = \sigma_1 R_1(z)$. For $n \geqslant 2$ we have from definitions (4.13) and (4.17)

$$\begin{aligned} T_n(z) &= \frac{z - a_1}{\rho_0(a_1 - v_1)} \left(R^{(1)}_{n-1}(z) - \rho_n R^{(1)}_n(z) + u_{n+1} R^{(1)}_{n+1}(z)\right) \\ &= \frac{u_n(z - a_n)\, R_{n-1}(z) - \rho_n(z - a_{n+1})\, R_n(z) + (z - a_{n+2})\, R_{n+1}(z)}{\rho_0(a_1 - v_0)\, h_n}. \end{aligned} \tag{7.5}$$

Now taking into account the recurrence relation (4.11) we arrive at the needed identity (7.1) with σ_n given by (7.4). Thus the proposition is proven.

Note that in this case the recurrence relation for the rational functions $R_n(z)$ is rewritten as

$$(z - a_{n+2})\, R_{n+1}(z) + \rho_n(v_n - z)\, R_n(z) + u_n(z - a_n)\, R_{n-1}(z) = 0. \tag{7.6}$$

The recurrence relations of the type (7.6) for the rational orthogonal functions were considered, e.g., in [2, Problem 4.12].

The simplest example when (7.3) is fulfilled corresponds to the choices

$$a_n = b_n = a = const. \tag{7.7}$$

In this case the recurrence relation (7.6) becomes

$$R_{n+1}(z) - \rho_n R_n(z) + u_n R_{n-1}(z) = \frac{r_n}{z - a} R_n(z), \tag{7.8}$$

where $r_n = \rho_n(a - v_n)$. But the recurrence relation (7.8) is equivalent to the recurrence relation for the *ordinary orthogonal polynomials* $Y_n(x)$

$$Y_{n+1}(x) - \rho_n Y_n(x) + u_n Y_{n-1}(x) = r_n x Y_n(x), \tag{7.9}$$

where $x = 1/(z-a)$. Thus in this case theory of the orthogonal rational functions reduces to theory of the ordinary orthogonal polynomials.

Consider Proposition 1 in the case of Wilson's basic hypergeometric biorthogonal rational functions. These functions depend on 6 parameters a, b, d, e, N, μ and have the following explicit expression [9]

$$R_n(z) = C_n^{(1)}\,{}_{10}W_9(a; be^{-\xi}, \mu be^{\xi}, d, e, aq^{N+1}, sq^{n-1}, q^{-n}; q), \tag{7.10}$$

$$T_n(z) = C_n^{(2)}\,{}_{10}W_9\left(\frac{aq^{1-N}}{de}; be^{-\xi}, \mu be^{\xi}, \frac{q^{1-N}}{d}, \frac{q^{1-N}}{e}, \frac{aq^2}{de}, sq^{n-1}, q^{-n}; q\right), \tag{7.11}$$

where $C_n^{(1,2)}$ are appropriate normalization constants, and the function ${}_{10}W_9(a; b, c, d, e, f, g, h; q)$ stands for short notation of a very-well poised, balanced ${}_{10}\phi_9$ series with the restriction

$$bcdefgh = a^3q^2 \tag{7.12}$$

upon the parameters (for details see [9]). The integer parameter N is the same as in our biorthogonality relation (4.26). The variable ξ stands for parameterization of the argument of rational functions [9]: $z = (e^{\xi} + e^{-\xi}/\mu)/2$. The auxiliary parameter s has the expression $s = a^2q^{2-N}/(\mu b^2 de)$.

The recurrence parameters v_n, u_n, ρ_n have rather complicated expressions, whereas the parameters a_n, b_n have the simple expression [9]

$$\begin{aligned} a_n &= \left(\frac{bsq^{n-2}}{a} + \frac{aq^{2-n}}{bs\mu}\right)\Big/2 \\ b_n &= \left(\frac{aq^n}{b\mu} + \frac{bq^{-n}}{a}\right)\Big/2. \end{aligned} \tag{7.13}$$

Hence the condition (7.3) in this case is reduced to

$$de = q^{1-N}. \tag{7.14}$$

Thus under the condition (7.14) the Wilson biorthogonality becomes pure orthogonality. This is also easily seen if one compares expressions (7.10) and (7.11): under the condition (7.14) both ${}_{10}W_9$ functions coincide.

8. CONFORMAL INVARIANCE OF THE POLYNOMIALS OF R_{II} TYPE

The ordinary orthogonal polynomials $P_n(x)$ possess an obvious property of translational invariance: for arbitrary α the polynomials $P_n(x+\alpha)$ are also ordinary orthogonal polynomials.

For the polynomials $P_n(z)$ of the R_{II} type satisfying the recurrence relation (1.9) there is more general property of conformal invariance with respect to general rational substitution $z \to (\alpha z+\beta)/(\gamma z+\delta)$. Namely, we have

Proposition 2. *Let $P_n(z)$ be polynomials of R_{II} type satisfying the recurrence relation* (1.9). *Then the polynomials*

$$\tilde{P}_n(z) = (\gamma z+\delta)^n \, P_n\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) \tag{8.1}$$

are also polynomials of R_{II} type satisfying the recurrence relation

$$\tilde{P}_{n+1}(z) + \tilde{\rho}_n \, (\tilde{v}_n - z) \, \tilde{P}_n(z) + \tilde{u}_n \, (z-\tilde{a}_n)(z-\tilde{b}_n) \, \tilde{P}_{n-1}(z) = 0 \tag{8.2}$$

with

$$\tilde{a}_n = \frac{\delta a_n - \beta}{\alpha - \gamma a_n}, \qquad \tilde{b}_n = \frac{\delta b_n - \beta}{\alpha - \gamma b_n}, \tag{8.3}$$

$$\tilde{u}_n = u_n \, (\alpha - \gamma \, a_n)(\alpha - \gamma \, b_n),$$

$$\tilde{\rho}_n = \rho_n(\alpha - \gamma v_n),$$

$$\tilde{v}_n = \frac{\delta v_n - \beta}{\alpha - \gamma v_n}. \tag{8.4}$$

The corresponding biorthogonal rational functions are

$$\tilde{R}_n(z) = R_n\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) \tag{8.5}$$

$$\tilde{T}_n(z) = \frac{T_n\left((\alpha z+\beta)/(\gamma z+\delta)\right)}{(\alpha - \gamma b_1)\cdots(\alpha-\gamma b_n)}.$$

The proof of this proposition is elementary.

As a consequence of this proposition we consider the case when one of the parameters a_n, b_n, say b_n, is a constant: $b_n = b$ not depending on n. Then choosing $\alpha = \gamma b$ we arrive at the recurrence relation for the polynomials $\tilde{P}_n(z)$

$$\tilde{P}_{n+1}(z) + \tilde{\rho}_n(\tilde{v}_n - z) \, \tilde{P}_n(z) + \tilde{u}_n(z-\tilde{a}_n) \, \tilde{P}_{n-1}(z) = 0, \tag{8.6}$$

where

$$\tilde{u}_n = \gamma u_n(b-a_n)(\beta - \delta b). \tag{8.7}$$

But the recurrence relation (8.6) describes so-called polynomials of R_I type introduced in [12].

If both parameters are constants $a_n = a$, $b_n = b$ and $a \neq b$ then choosing $\alpha = \gamma b$, $\beta = a\delta$ we arrive at the recurrence relation describing biorthogonal Laurent polynomials (see, e.g., [10])

$$\tilde{P}_{n+1}(z) + \tilde{\rho}_n(\tilde{v}_n - z)\,\tilde{P}_n(z) + \tilde{u}_n z \tilde{P}_{n-1}(z) = 0, \tag{8.8}$$

where $\tilde{u}_n = -u_n \gamma\delta(a-b)^2$.

Finally if $a_n = b_n = a$ we can choose $\alpha = \gamma a$ to obtain the recurrence relation

$$\tilde{P}_{n+1}(z) + \tilde{\rho}_n(v_n - z)\,\tilde{P}_n(z) + \tilde{u}_n \tilde{P}_{n-1}(z) = 0, \tag{8.9}$$

where $\tilde{u}_n = u_n(\beta - a\delta)^2$. But (8.9) is nothing else than the recurrence relation for the ordinary orthogonal polynomials.

We thus have the following

PROPOSITION 3. *If $b_n = b = const$ then the polynomials of type R_{II} reduce to the polynomials of type R_I. If $a_n = a$, $b_n = b$, $a \neq b$ then these polynomials are biorthogonal Laurent polynomials. If $a_n = b_n = a$ then we have the ordinary orthogonal polynomials.*

Consider an example. In [12] the hypergeometric polynomials of R_{II} type

$$P_n(z) = \frac{2^{-n}(1-\eta)_n}{((1+\xi-\eta)/2)_n}\,{}_2F_1\left(\begin{matrix} -n, \eta-\xi-n \\ \eta - n \end{matrix}; 1-z\right), \tag{8.10}$$

where ξ, η are arbitrary parameters where considered. They satisfy the recurrence relation (1.9) with

$$\begin{aligned} a_n &= 0, \qquad b_n = 1, \qquad v_n = \frac{n+\xi}{2n+\xi-\eta+1}, \\ \rho_n &= 1, \qquad u_n = \frac{n(n+\xi-\eta)}{(2n+\xi-\eta-1)(2n+\xi-\eta+1)}. \end{aligned} \tag{8.11}$$

For the corresponding rational functions the biorthogonality relation is written as [12]

$$\int_{1/2-i\infty}^{1/2+i\infty} R_n(z)\,T_m(z)\,g(z)\,dz = 0, \qquad n \neq m, \tag{8.12}$$

i.e., the linear functional is realized on the straight line $(1/2-i\infty, 1/2+i\infty)$.

Consider the polynomials

$$\tilde{P}_n(z)=(z-1)^n\,\frac{2^n((1+\xi-\eta)/2)_n}{(1-\eta)_n}\,P_n(z/(z-1)). \tag{8.13}$$

It is easily verified that the polynomials $\tilde{P}_n(z)$ are biorthogonal Laurent polynomials (BLP) satisfying the recurrence relation

$$\tilde{P}_{n+1}(z)+d_n\tilde{P}_n(z)=z(\tilde{P}_n(z)+b_n\tilde{P}_{n-1}(z)), \tag{8.14}$$

where

$$d_n=-\frac{n+\xi}{n-\eta+1}, \qquad b_n=-\frac{n(n+\xi-\eta)}{(n-\eta)(n-\eta+1)}. \tag{8.15}$$

But the BLP with the recurrence coefficients (8.15) coincide with those introduced by Hendriksen and van Rossum [10]. They are biorthogonal on the unit circle [10]

$$\int_C \tilde{P}_n(z)\,z^{-m}(-z)^{1+\xi}\,(1-z)^{\xi-\eta}\,dz=0, \qquad 0\leqslant m\leqslant n-1. \tag{8.16}$$

The transformation $z\to z/(z-1)$ maps the straight line $(1/2-i\infty, 1/2+i\infty)$ onto the unit circle. Thus the polynomials (8.10) are equivalent to the Hendriksen–Van Rossum BLP. Note that these polynomials were also considered by Askey [1] as concrete examples of rational functions which are biorthogonal on the unit circle. See also [11] where the recurrence relation for the polynomials (8.10) was considered.

9. ABUNDANCE OF BIORTHOGONAL RATIONAL FUNCTIONS

Assume that we have explicitly the polynomials $P_n(z)$ of R_{II} type constructed by the recurrence relation (1.9). Then there is still a freedom in choosing of the orthogonal function $R_n(z)$. Indeed, in the recurrence relation (1.9) the parameters a_n, b_n admit some permutational symmetry which do not change the polynomials $P_n(z)$ (and hence the functional $\mathscr{L}$, providing orthogonality condition (1.13)). Namely, choose another set of these parameters

$$\tilde{a}_n=\varepsilon_n\,a_n+(1-\varepsilon_n)\,b_n, \qquad \tilde{b}_n=\varepsilon_n\,b_n+(1-\varepsilon_n)\,a_n, \tag{9.1}$$

where ε_n can takes (arbitrarily) values 0 or 1 at each n. Then it is clear that the polynomials $P_n(z)$ remain the same under such replacement. Our initial choice corresponds to $\varepsilon_n = 1$ for all n.

Another obvious choice is a simple permutation of the parameters $a_n \leftrightarrow b_n$ which corresponds to $\varepsilon_n = 0$. In this case we get

$$\tilde{R}_n(z) = \frac{P_n(z)}{A_n(z)}, \qquad \tilde{R}_n^{(1)}(z) = \frac{P_n(z)}{h_n\, B_n(z)}. \tag{9.2}$$

That is the new rational function $\tilde{R}_n(z)$ has the same zeros as initial one $R_n(z)$. However, the new biorthogonal partner

$$\tilde{T}_n(z) = \frac{z - b_1}{\rho_0(b_1 - v_1)} \left(\tilde{R}_{n-1}^{(1)}(z) - \rho_n \tilde{R}_n^{(1)}(z) + u_{n+1} \tilde{R}_{n+1}^{(1)}(z)\right), \tag{9.3}$$

will have in general zeros which *do not coincide* with zeros for $T_n(z)$. Hence by this trivial permutation procedure we may obtain some non-trivial new biorthogonal rational functions.

Consider the simple example contained in [12]. Let $\rho_n = 1$, $v_n = -ab$, $u_n = 1/4$, $a_n = a^2$, $b_n = b^2$, where a, b are some constants. Then the polynomials $P_n(z)$ can be written explicitly as [12]

$$P_n(z) = \frac{(z^{1/2} + a)(z^{1/2} + b)^{n+1} - (z^{1/2} + b)(z^{1/2} + a)^{n+1}}{2^{n+1}(a+b)z^{1/2}}. \tag{9.4}$$

Choosing $\varepsilon_n = 1$ we arrive at *different* biorthogonal rational functions $R_n(z)$ and $T_n(z)$ (if $b \neq a$).

However, if one chooses $\varepsilon_n = (1 + (-1)^n)/2$ we get *orthogonal* rational functions

$$R_{2n}(z) = \frac{P_{2n}(z)}{(z-a^2)^n (z-b^2)^n}, \qquad R_{2n+1}(z) = \frac{P_{2n+1}(z)}{(z-a^2)^n\, (z-b^2)^{n+1}}. \tag{9.5}$$

Indeed, in this case $\tilde{b}_n = \tilde{a}_{n+1}$ and hence the condition of the Proposition 1 is fulfilled. Note that the choice (9.5) was exploited in [12] in order to obtain pure orthogonality. We would like to mention also that in this case of constant parameters a_n, b_n we can reduce the problem to the biorthogonal Laurent polynomials due to Proposition 3.

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