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Frobenius–Padé approximants of piecewise analytic functions. II

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Abstract

We prove the convergence of the Frobenius–Padé approximants for a class of piecewise analytic functions which includes $|x|^\alpha$ with $0 < \alpha < 2$. The measures considered for the expansions are symmetric in $[-1, 1]$.

1. Introduction

When a function has discontinuities, as is the case in physical problems with shocks or in image compression, the convergence of a Fourier series or an expansion in a basis of orthogonal polynomials is poor or does not take place. Several methods have been developed to overcome these problems (see, for example, [1] and references therein). One approach is to use Frobenius–Padé approximants (see [2], [4], [5], [6], [8] and [9]). Unfortunately, very little is known about the convergence of these rational functions when there is a singularity of the approximated function in the interior of the interval where the measure used in the expansion is supported (see [3]).

In this note we prove the convergence of the Frobenius–Padé approximants for a class of piecewise analytic functions including all functions of the form $|x|^\alpha$ with $0 < \alpha < 2$. The expansions are given in terms of orthogonal polynomials with respect to a symmetric measure in $[-1, 1]$. As a consequence, we obtain the convergence of Frobenius–Padé approximants for

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$|x|^\alpha$, $-1 < \alpha < 0$, in the spherical metric. This paper is a sequel of [3]. The rational approximation of $|x|^\alpha$ has been a subject of major interest in the recent past (see, for example, [11], [13], [14] and the references therein).

Let $f : [-1, 1] \to \mathbb{R}$ be an even function such that its restriction to $(0, 1]$, $f|_{(0,1]}$, has an analytic extension to $\{z \in \mathbb{C} : \Re(z) > 0\}$, which we also denote by f, that satisfies the following conditions:

(i) For $x \in (-\infty, 0)$ there exist

$$\lim_{z\to x+i0} f(\sqrt{z}) =: f^+(x), \qquad \lim_{z\to x-i0} f(\sqrt{z}) =: f^-(x),$$

when z approaches x from the upper or lower half-plane, respectively. Hereafter, we take the principal branch of the square root in $\mathbb{C}\setminus(-\infty, 0]$. We assume that the previous limits are uniform on each compact subset of $(-\infty, 0)$; that is, for each compact subset $K \subset (-\infty, 0)$ and $\epsilon > 0$ there exists $\delta > 0$ such that for $x \in K, \Im(z) > 0, |z - x| < \delta$ we have

$$|f(\sqrt{z}) - f^+(x)| < \epsilon$$

and the analogous relation for $f^-(x)$.

(ii) If $f^*(x) := (f^+(x) - f^-(x))/i$ for $x \in (-\infty, 0)$, then

$$f^*(x) > 0 \quad \text{and} \quad f^*(x) \in L^1((1 - x)^{-2}dx).$$

(iii) $f(0) = 0$ and

$$\lim_{\epsilon\to 0} \sup_{t\in(-\pi,\pi)} |f(\epsilon\, e^{it/2})| = 0.$$

(iv) We have the following behavior at infinity

$$\lim_{R\to\infty} \inf_{t\in(-\pi,\pi)} |f(R\, e^{it/2})| = \infty, \qquad \lim_{R\to\infty} \sup_{t\in(-\pi,\pi)} \frac{|f(\sqrt{R\, e^{it}})|}{R} = 0.$$

(v) There exist constants α_1, α_2 in $(0, 1)$ such that

$$\alpha_1\pi \le \arg(f^+(x)) \le \alpha_2\pi, \quad -\alpha_2\pi \le \arg(f^-(x)) \le -\alpha_1\pi.$$

for all $x \in (-\infty, 0)$, where the argument function is the main branch in $(-\pi, \pi)$.

Let $\mathcal{A}$ denote the class of all functions f which verify the previous conditions. Observe that if $0 < \alpha < 2$, then $|x|^\alpha \in \mathcal{A}$, and we have

$$\lim_{z \to x \pm i0} z^{\alpha/2} = |x|^{\alpha/2}(\cos \frac{\alpha\pi}{2} \pm i \sin \frac{\alpha\pi}{2}), \quad x \in (-\infty, 0).$$

Notice also that a linear combination with positive coefficients of functions $|x|^\alpha$, $\alpha \in (0, 2)$, is also in $\mathcal{A}$.

Let β be a nontrivial positive Borel measure on $\mathbb{R}$ with finite moments; that is, $x^j \in L^1(\beta)$ for all non negative integer j. Let $\{\varphi_j\}$ denote the sequence of orthonormal polynomials with respect to β. Given $g \in L^1(\beta)$ a *Frobenius-Padé approximant* of order n of g with respect to β is a rational function $\Pi_n = P_n/Q_n$ where P_n and Q_n are polynomials of degree $\leq n$, $Q_n \not\equiv 0$, and

$$\int (Q_n(x)g(x) - P_n(x))\varphi_j(x)\, d\beta(x) = 0, \quad j = 0, 1, \ldots, 2n. \tag{1.1}$$

This means that the Fourier coefficients of $Q_n(x)g(x) - P_n(x)$ with respect to $\{\varphi_j : j \geq 0\}$ are zero for $j \leq 2n$. Thus, $P_n(x)$ is the Fourier partial sum of order n of $Q_n g$ with respect to β and

$$\int Q_n(x)g(x)\varphi_j(x)\, d\beta(x) = 0, \quad j = n+1, n+2, \ldots, 2n,$$

which is a homogeneous system of n linear equations in the coefficients of Q_n. Hence, for each n, a Frobenius-Padé approximant of g always exists. In general it is not unique. However, if for a given n all the solutions of the homogeneous system of equations which determines Q_n have $\deg(Q_n) = n$ then the approximant of this order is unique.

Theorem 1.1. *Let μ be a finite symmetric positive Borel measure on $[-1, 1]$ whose support has an accumulation point different from 0. Given $f \in \mathcal{A}$ let $\Pi_n(f)$ denote the Frobenius-Padé approximant of order n of the function f with respect to μ. Then*

$$\lim_{n \to \infty} \Pi_n(f; z) = \begin{cases} f(z) & \text{if } \Re(z) > 0, \\ f(-z) & \text{if } \Re(z) < 0, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{z : \Re(z) = 0\}$, where we also denote by f the analytic extension of $f|_{(0,1]}$ to $\{z \in \mathbb{C} : \Re(z) > 0\}$.

Corollary 1.2. *Let μ be a measure satisfying the conditions of Theorem 1.1 and let $f \in \mathcal{A}$ be such that $1/f \in L^1(\mu)$. Let $\Pi_n(1/f)$ denote the Frobenius-Padé approximant of order n of the function $1/f$ with respect to the measure μ. Then*

$$\lim_{n\to\infty} \Pi_n(1/f; z) = \begin{cases} 1/f(z) & \text{if } \Re(z) > 0, \\ 1/f(-z) & \text{if } \Re(z) < 0, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{z : \Re(z) = 0\}$.

In Section 3 we prove Theorem 1.1. Its proof is reduced to the study of Frobenius-Padé approximants of a function given on the interval $(0, 1)$ and is based on the orthogonality conditions defining Frobenius-Padé approximation as well as the behavior of the points where $Q_n(z)f(z)-P_n(z)$ equals zero. Section 2 includes several properties of the Frobenius-Padé approximants for functions in class $\mathcal{A}$.

2. Properties of the approximants

Let μ be a nontrivial finite symmetric positive Borel measure on $(-1, 1)$ and let $\mu(\sqrt{x})$ denote the image measure of μ on $(0, 1)$ by the function $\sqrt{x}$.

Lemma 2.1. *Let μ be a nontrivial finite symmetric positive Borel measure on $(-1, 1)$. Let $f \in \mathcal{A}$. The Frobenius-Padé approximant p_n/q_n of order n of the function $f(\sqrt{x})$ with respect to $\mu(\sqrt{x})$ on $(0, 1)$ is unique. The Frobenius-Padé approximant P_{2n}/Q_{2n} of order $2n$ of the function f with respect to μ satisfies $P_{2n}(x) = p_n(x^2)$, $Q_{2n}(x) = q_n(x^2)$. Moreover, there exist exactly $2n + 1$ distinct points $z_1, z_2, \ldots, z_{2n+1}$ in $(0, 1)$ such that*

$$q_n(z_j)f(\sqrt{z_j}) - p_n(z_j) = 0, \quad j = 1, 2, \ldots, 2n + 1. \tag{2.1}$$

(To emphasize the dependance on n sometimes we write $z_{j,n}$ instead of z_j.) Moreover, the polynomial q_n has precisely degree n and if $w_{2n+1}(z) := \prod_j (z - z_j)$, it satisfies the orthogonality relations

$$\int_{-\infty}^{0} q_n(t)t^j \frac{f^*(t)}{w_{2n+1}(t)}\, dt = 0, \quad j = 0, 1, \ldots, n - 1. \tag{2.2}$$

Proof. First we obtain relations (2.1) and (2.2). Let p_n/q_n be a Frobenius-Padé approximant of order n of the function $f(\sqrt{x})$ with respect to the measure $\mu(\sqrt{x})$ on $(0,1)$. For any polynomial h of degree at most $2n$, we have

$$\int_0^1 (q_n(x)f(\sqrt{x}) - p_n(x))h(x)\, d\mu(\sqrt{x}) = 0. \tag{2.3}$$

Then the function $q_n(x)f(\sqrt{x}) - p_n(x)$, which is continuous on the interval $(0,1)$, has at least $2n+1$ sign changes on $(0,1)$. Indeed, if it had at most $2n$ sign changes taking h as the polynomial whose zeros are these points we would have that $(q_n(x)f(\sqrt{x}) - p_n(x))h(x)$ has constant sign on $(0,1)$ and (2.3) could not hold. Therefore, there exist distinct points $z_1, z_2, \ldots, z_{2n+1}$ in $(0,1)$ such that (2.1) takes place. Let $\mathcal{P}_n$ denote this set of $2n+1$ points.

Let $\Gamma_{\epsilon,R}$ be the curves given in Figure 1 with ϵ small enough and R sufficiently large so that all the points in $\mathcal{P}_n$ are in the interior of $\Gamma_{\epsilon,R}$. Since

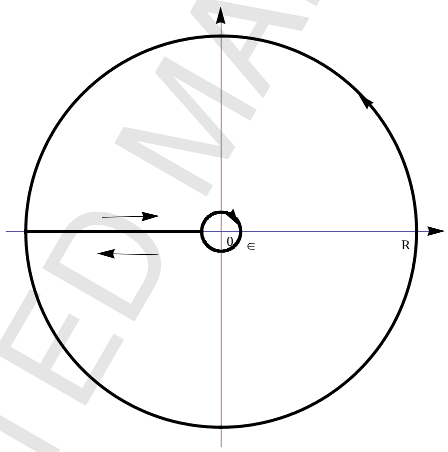

Figure 1: Integration contour $\Gamma_{\epsilon,R}$.

$(q_n(z)f(\sqrt{z}) - p_n(z))/w_{2n+1}(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, 0]$, by the Cauchy theorem we have

$$0 = \int_{\Gamma_{\epsilon,R}} \frac{q_n(z)f(\sqrt{z}) - p_n(z)}{w_{2n+1}(z)} z^j\, dz = \int_{\Gamma_{\epsilon,R}} \frac{q_n(z)f(\sqrt{z})}{w_{2n+1}(z)} z^j\, dz,$$

for all $j = 0, 1, \ldots, n-1$. Here we have deformed the integration contour and used the assumption that the limit for f^+ and f^- are locally uniform on $(-\infty, 0)$. In the last equality we have used the Cauchy theorem for the functions $p_n(z)z^j/w_{2n+1}(z)$, $j = 0, 1, \ldots, n-1$, which are analytic in $\{z \in$

$\mathbb{C} : |z| \geq R\}$ and have a zero of order at least 2 at infinity. As f belongs to the class $\mathcal{A}$ (see conditions (i)–(iv)), we have

$$\lim_{\epsilon\to 0, R\to\infty} \int_{\Gamma_{\epsilon,R}} \frac{q_n(z) f(\sqrt{z})}{w_{2n+1}(z)} z^j \, dz = \int_{-\infty}^{0} q_n(t) t^j \frac{f^*(t)}{w_{2n+1}(t)} \, dt, \quad j = 0, 1, \ldots, n-1.$$

Hence (2.2) follows and q_n turns out to be an orthogonal polynomial of precise degree n. Thus, the Frobenius-Padé approximant of order n of the function $f(\sqrt{x})$ is unique. Moreover, the previous arguments reveal that $q_n(z) f(\sqrt{z}) - p_n(z)$ cannot have more than $2n+1$ zeros in $(0, 1)$. Indeed, if that function had more than $2n+1$ zeros, putting all those points as zeros of what was denoted w_{2n+1} we would find that q_n satisfies more that $n-1$ orthogonality relations implying that $q_n \equiv 0$.

Next we prove the relation between the Frobenius-Padé approximants of $f(x)$ and $f(\sqrt{x})$. Define the even polynomials of degree at most $2n$ given by $q(x^2) := Q_{2n}(x) + Q_{2n}(-x)$ and $p(x^2) := P_{2n}(x) + P_{2n}(-x)$. As f is an even function, we have

$$\int_{-1}^{1} (q(x^2) f(x) - p(x^2)) x^{2j} \, d\mu(x) = 0, \quad j = 0, 1, \ldots, 2n,$$

so

$$\int_{(0,1)} (q(x^2) f(x) - p(x^2)) x^{2j} \, d\mu(x) = 0, \quad j = 0, 1, \ldots, 2n.$$

Making the change of variable $x^2 = t$ we obtain

$$\int_{(0,1)} (q(t) f(\sqrt{t}) - p(t)) t^j \, d\mu(\sqrt{t}) = 0, \quad j = 0, 1, \ldots, 2n.$$

Then p/q is a Frobenius-Padé approximant of $f(\sqrt{x})$ of order n with respect to $\mu(\sqrt{t})$. According to (2.2) q has exact degree n, Q_{2n} degree $2n$. Thus, the Frobenius-Padé approximant of f of order $2n$ is unique and $Q_{2n}(x) = Q_{2n}(-x)$ from which the relation $Q_{2n}(x) = q_n(x^2)$ follows (possibly up to some multiplicative constant). $\square$

The same line of reasoning leads to the following result for Frobenius-Padé approximants of odd degree. The details of the proof are left to the reader.

Lemma 2.2. *Under the hypothesis of the Lemma 2.1, the Frobenius-Padé approximant P_{2n+1}/Q_{2n+1} of order $2n+1$ of the function f with respect to μ satisfies $P_{2n+1}(x) = xp_n(x^2)$, $Q_{2n+1}(x) = xq_n(x^2)$ where p_n/q_n is the Frobenius-Padé approximant of order n of the function $f(\sqrt{x})$ with respect to measure $x\,d\mu(\sqrt{x})$ in $(0,1)$.*

Cauchy's integral formula allows to obtain a convenient integral representation of the remainder.

Lemma 2.3. *Let $\pi_n = p_n/q_n$ be the Frobenius-Padé approximant of order n of the function $f(\sqrt{x})$ with respect to $\mu(\sqrt{x})$ with $x \in (0,1)$. Then*

$$q_n(z)f(\sqrt{z}) - p_n(z) = \frac{w_{2n+1}(z)}{2\pi h_n(z)} \int_{-\infty}^{0} \frac{h_n(x)q_n(x)}{x-z} \frac{f^*(x)\,dx}{w_{2n+1}(x)}, \tag{2.4}$$

where $z \in \mathbb{C} \setminus (-\infty, 0]$, h_n denotes any non null polynomial of degree $\leq n$, and w_{2n+1} is as defined in Lemma 2.1. In particular, if $h_n = q_n$ we have

$$f(\sqrt{z}) - \pi_n(z) = \frac{w_{2n+1}(z)}{2\pi q_n^2(z)} \int_{-\infty}^{0} \frac{q_n^2(x)}{x-z} \frac{f^*(x)\,dx}{w_{2n+1}(x)}. \tag{2.5}$$

Proof. Since $h_n(z)(q_n(z)f(\sqrt{z}) - p_n(z))/w_{2n+1}(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, 0]$, Cauchy's integral formula gives us

$$\frac{h_n(z)(q_n(z)f(\sqrt{z}) - p_n(z))}{w_{2n+1}(z)} = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,R}} \frac{h_n(\zeta)(q_n(\zeta)f(\sqrt{\zeta}) - p_n(\zeta))}{w_{2n+1}(\zeta)(\zeta - z)}\, d\zeta =$$

$$\frac{1}{2\pi i} \int_{\Gamma_{\epsilon,R}} \frac{h_n(\zeta)q_n(\zeta)f(\sqrt{\zeta})}{w_{2n+1}(\zeta)(\zeta - z)}\, d\zeta,$$

where $\Gamma_{\epsilon,R}$ is the same contour described in Figure 1. Now, it remains to make $R \to \infty, \epsilon \to 0$ using the properties of $f \in \mathcal{A}$ and the definition of f^*. □

The rational approximant $\pi_n = p_n/q_n$ given in Lemma 2.1 verifies the following properties.

Lemma 2.4. 1. *The polynomial q_n has n simple zeros in $(-\infty, 0)$ which we denote by $\zeta_j, j = 1, \ldots, n$ and enumerate in the order $\zeta_1 < \zeta_2 < \ldots < \zeta_n$.*

2. *We have*
$$\pi_n(z) = \frac{p_n(z)}{q_n(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z-\zeta_j} + A_n, \tag{2.6}$$
where $\lambda_j < 0$, $j = 1, \ldots, n$, *and* $A_n > 0$.
3. $$f(0) = 0 < \pi_n(0) \quad \text{and} \quad \pi_n(1) < f(1). \tag{2.7}$$
4. *The polynomial* p_n *has exactly* n *real zeros,* $\eta_1, \eta_2, \ldots, \eta_n$, *which alternate with the points* ζ_j, *i.e.*
$$-\infty < \zeta_1 < \eta_1 < \zeta_2 < \ldots < \zeta_{n-1} < \eta_{n-1} < \zeta_n < \eta_n < 0.$$
5. *The zeros of* $f(\sqrt{z}) - \pi_n(z)$ *in* $\mathbb{C} \setminus (-\infty, 0]$ *are precisely the* $2n+1$ *zeros of* w_{2n+1} *and they lie in* $(0,1)$.
6. *The function* π_n *is strictly increasing and strictly concave in* (ζ_n, ∞).

Proof. The first statement is a well known consequence of the orthogonality properties in (2.2). Formula (2.6) follows immediately since the zeros of q_n are simple.

If we take $h_n(z) = \frac{q_n(z)}{z-\zeta_j}$ in (2.4), multiply the resulting equation by $(z-\zeta_j)$, and take limit $z \to \zeta_j$, we obtain
$$-\lambda_j = \frac{w_{2n+1}(\zeta_j)}{2\pi (q_n'(\zeta_j))^2} \int_{-\infty}^{0} \left(\frac{q_n(x)}{x-\zeta_j} \right)^2 \frac{f^*(x)\, dx}{w_{2n+1}(x)}.$$
Since the zeros of w_{2n+1} are in $(0,1)$ and the zeros of q_n are in $(-\infty, 0)$, the previous identity leads to $\lambda_j < 0$, $j = 1, \ldots, n$.

Now we check the inequalities in (2.7) and 4) before proving that $A_n > 0$. Taking $z = 0$ in (2.5), it follows that $\pi_n(0) > 0$. Since $f(\sqrt{z}) - \pi_n(z)$ changes sign at $2n+1$ points on $(0,1)$, it follows that $\pi_k(1) < f(1)$. The inequality $\pi_n(0) > 0$, together with $\lim_{z \to \zeta_\ell^{\pm}} \pi_n(z) = \mp\infty$ allows us to conclude that π_n (or equivalently p_n) has a simple zero in $(\zeta_n, 0)$ and in each interval (ζ_{j-1}, ζ_j) which proves the statements in 4). Thus, $\pi_n(0) = A_n \prod_{j=1}^{n} \frac{\eta_j}{\zeta_j}$, and we get $A_n > 0$.

Next we prove statement 5). We have
$$\Im \left(\int_{-\infty}^{0} \frac{q_n^2(x)}{x-z} \frac{f^*(x)\, dx}{w_{2n+1}(x)} \right) = \Im(z) \int_{-\infty}^{0} \frac{q_n^2(x)}{|x-z|^2} \frac{f^*(x)\, dx}{w_{2n+1}(x)} \neq 0, \quad \text{for } \Im(z) \neq 0,$$

and
$$\int_{-\infty}^{0} \frac{q_n^2(x)}{x-z} \frac{f^*(x)\,dx}{w_{2n+1}(x)} > 0, \qquad \text{for } z \geq 0.$$
Hence, by (2.5) and Lemma 2.1 the zeros of $f(\sqrt{z}) - \pi_n(z)$ in $\mathbb{C} \setminus (-\infty, 0]$ are precisely the $2n+1$ points in $(0,1)$.

Finally, according to the statement 2) the derivative of π_n is positive in $\mathbb{R} \setminus \{\zeta_1, \ldots, \zeta_n\}$, thus we have 6).

□

A direct consequence of part 4 in Lemma 2.4 is

Corollary 2.5. *The poles and the zeros of the Frobenius-Padé approximants of $f \in \mathcal{A}$ are located on the imaginary line $\{z : \Re(z) = 0\}$ and they strictly interlace.*

Now we consider the Newman type-function
$$N_n(z) := \frac{f(\sqrt{z}) - \pi_n(z)}{f(\sqrt{z}) + \pi_n(z)} = \frac{1 - f(\sqrt{z})^{-1}\pi_n(z)}{1 + f(\sqrt{z})^{-1}\pi_n(z)}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad n \geq 1.$$
This is equivalent to
$$\pi_n(z) = f(\sqrt{z}) \left(1 - 2\frac{N_n(z)}{1 + N_n(z)} \right), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.8}$$

Lemma 2.6. *Each function N_n, $n \in \mathbb{N}$, is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and has zeros precisely at $z_1, z_2, \ldots, z_{2n+1}$. The sequences $\{N_n\}$ and $\{\pi_n\}$ are uniformly bounded on each compact subset of $\mathbb{C} \setminus (-\infty, 0]$.*

Proof. We know that all coefficients λ_j, $j = 1, 2, \ldots, n$, in the partial fraction representation (2.6) have identical (negative) signs, thus the value $\pi_n(z)$ runs through the whole extended real line $\overline{\mathbb{R}}$ when z moves along the interval (ζ_j, ζ_{j+1}) with ζ_j and ζ_{j+1} two adjacent poles. Also, we have that
$$\lim_{z \to 0 \pm i0} N_n(z) = -1 = N_n(0) = N_n(\zeta_j), \qquad \lim_{z \to \infty \pm i0} N_n(z) = 1 = N(\eta_j). \tag{2.9}$$

Consider the mapping $\tau_\alpha(x) := \frac{e^{i\alpha\pi} - x}{e^{i\alpha\pi} + x} = \frac{1 - e^{-i\alpha\pi}x}{1 + e^{-i\alpha\pi}x}$ (compare with the definition of $N_n(z)$ above). It satisfies
$$\tau_\alpha(x) + i\cot(\alpha\pi) = i\frac{1 + xe^{i\alpha\pi}}{(e^{i\alpha\pi} + x)\sin(\alpha\pi)} \Rightarrow |\tau_\alpha(x) + i\cot(\alpha\pi)| = \frac{1}{\sin(\pi\alpha)}$$

for $x \in \mathbb{R}$ and $0 < \alpha < 1$. Observe that ± 1 are in the circle $|z + i\cot(\alpha\pi)| = \frac{1}{\sin(\pi\alpha)}$. By condition (v) for the class $\mathcal{A}$ and the definition of the function N_n, it follows that $\arg(N_n(z))$ grows exactly by 2π if z moves from ζ_j to ζ_{j+1} on $\mathbb{R}_- + i0$. Analogously, $\arg(N_n(z))$ grows by 2π if z moves in the opposite direction from ζ_{j+1} to ζ_j on the other bank $\mathbb{R}_- - i0$ of $\mathbb{R}_-$. Because of (2.9) and (v) the same conclusion holds for each one of intervals $(\zeta_n, 0) - i0)$, $((0, \zeta_n) + i0)$ on the lower and upper banks respectively, and on $(-\infty, \zeta_1) + i0 \cup (-\infty, \zeta_1) - i0$ taken as a whole. Thus, $\arg(N_n(z))$ grows by $2\pi(2n+1)$ when z moves once around the boundary of the domain $\mathbb{C} \setminus (-\infty, 0]$ going first along one bank in one direction and returning through the other. From (2.1) we know that $z_1, z_2, \ldots, z_{2n+1}$ are zeros of $N_n(z)$ in $\mathbb{C} \setminus (-\infty, 0]$ and from (2.5) it follows that this function has no other zero in that region. From the argument principle we conclude that the function $N_n(z)$ has no pole and is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

From what was proved above and condition (v), the image of $\mathbb{C} \setminus (-\infty, 0]$ under N_n is contained in the union of disks $\{|z + i\cot(\alpha\pi)| \leq 1/\sin(\alpha\pi) : \alpha \in [\alpha_1, \alpha_2]\}$ and -1 lies on the boundary of the image for each n. On the other hand, condition (iv) leads to $\lim_{z\to\infty} N_n(z) = 1$ in $z \in \mathbb{C} \setminus (-\infty, 0]$. Thus, the image of $\mathbb{C} \setminus (-\infty, 0]$ by N_n is bounded independent of n and the maximum principle entails that the sequence $\{N_n\}$ is a normal family in $\overline{\mathbb{C}} \setminus [-\infty, 0]$ and it can be applied because N_n extends continuously to each banks of $(-\infty, 0]$.

According to (2.8) to prove that $\{\pi_n\}$ is also a normal family it suffices to show that $\{N_n\}$ remains bounded away from -1 in $\mathbb{C} \setminus (-\infty, 0]$. Suppose there is a point $\zeta \in \mathbb{C} \setminus (-\infty, 0]$ and subindices Λ such that $\lim_{n\in\Lambda} N_n(\zeta) = -1$. We can assume that Λ is such that $\lim_{n\in\Lambda} N_n = N$ exists uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ and $N(\zeta) = -1$. As -1 is on the boundary of the image of $\mathbb{C} \setminus (-\infty, 0]$ for each N_n, the same happens with N. By the open mapping theorem, we have $N \equiv -1$ in $\mathbb{C} \setminus (-\infty, 0]$ but this is impossible since $N_n(1) > 0$ for all n. □

3. Proof of Theorem 1.1

Due to the connections given in Lemmas 2.1 and 2.2 between the Frobenius-Padé approximants of $f \in \mathcal{A}$ and the restriction of $f(\sqrt{x})$ to the interval $(0, 1)$, in order to obtain Theorem 1.1 it is sufficient to prove the following result.

Theorem 3.1. *Let γ be a finite positive Borel measure on $[0,1]$ whose support,* $\operatorname{supp}(\gamma)$*, has an accumulation point different from 0. Let $f \in \mathcal{A}$ and denote by π_n the Frobenius-Padé approximants of order n of the function $f(\sqrt{x})$ with respect to γ. Then*

$$\lim_{n\to\infty} \pi_n(z) = f(\sqrt{z}),$$

uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$.

Proof. According to Lemma 2.6, the sequence $\{\pi_n\}_{n\in\mathbb{N}}$ is a normal family in $\mathbb{C}\setminus(-\infty,0]$. By Montel's theorem it is sufficient to prove that each convergent subsequence of $\{\pi_n\}_{n\in\mathbb{N}}$ converges to $f(\sqrt{z})$. We divide the proof in two cases depending on whether the condition (3.1) given below is verified or not. In what follows $\{\pi_n\}_{n\in\Lambda}$ denotes a convergent subsequence.

Define the functions

$$\phi(z) := \frac{\sqrt{z}-1}{\sqrt{z}+1}, \quad \phi_n(z) := \prod_{j=1}^{2n+1} \frac{\phi(z)-\phi(z_{j,n})}{1-\phi(z)\phi(z_{j,n})},$$

$z \in \mathbb{C} \setminus (-\infty, 0]$. Observe that ϕ is a conformal mapping of $\mathbb{C} \setminus (-\infty, 0]$ onto the open unit disk which transforms the interval $(0,1)$ onto the interval $(-1,0)$. It is well known that

$$\lim_{n\in\Lambda} \sum_{j=1}^{2n+1} (1 - |\phi(z_{j,n})|) = \infty \tag{3.1}$$

implies that

$$\lim_{n\in\Lambda} \phi_n(z) = 0$$

uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ (see, for example, [7] or [15], p. 281).

According to Lemma 2.6, the sequence $\{N_n\}_{n\in\Lambda}$ is a normal family in $\mathbb{C} \setminus (-\infty, 0]$. The points in $\mathcal{P}_n$ are zeros of N_n. Hence, the sequence $\{N_n/\phi_n\}_{n\in\Lambda}$ is also a normal family of analytic functions in $\mathbb{C} \setminus (-\infty, 0]$ and given a compact set $K \subset \mathbb{C} \setminus (-\infty, 0]$, there exists a constant $C > 0$ such that

$$|N_n(z)| \le C|\phi_n(z)|, \quad n \in \Lambda, \quad z \in K.$$

Therefore, if (3.1) holds, then $\lim_{n\in\Lambda} N_n(z) = 0$ uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ and by (2.8) the statement of the theorem follows.

Now, assume that the interpolation points $\{\mathcal{P}_n\}_{n\in\Lambda}$ do not satisfy (3.1). Then, there exists a subsequence of indexes in Λ, which we also denote Λ, such that for all $a \in (0,1]$ the number of interpolations points in $[a,1]$ is bounded as a function of $n \in \Lambda$. From now on we only consider such values of $n \in \Lambda$. For the proof of the statement we will show that the approximants converge to $f(\sqrt{x})$ on a subinterval of $(0,1)$ in an L_1 sense. The proof relies on some chain of inequalities and we enumerate the interpolation points in $\mathcal{P}_n$, $n \in \Lambda$, so that

$$z_{1,n} < z_{2,n} < \ldots < z_{2n+1,n}.$$

Let a, b, c be real numbers in $(0,1)$ such that $\operatorname{supp}(\gamma) \cap [a,1]$ has infinitely many points, $b < a$, $c < a - b$. For $t \in (0,1)$, let $\sharp\{z_{j,n} < t\}$ denote the number of points in $\mathcal{P}_n \cap (0,t)$. Without loss of generality we can assume that $q_n(a) = 1$.

Let $d_n(z) := q_n(z) f(\sqrt{z}) - p_n(z)$. As q_n is a positive strictly increasing function on $[0,1]$, the condition $q_n(a) = 1$ implies

$$\int_a^1 |f(\sqrt{x}) - \pi_n(x)| \prod_{z_{j,n} \geq b} |x - z_{j,n}|\, d\gamma \leq \int_a^1 |d_n(x)| \prod_{z_{j,n} \geq b} |x - z_{j,n}|\, d\gamma, \quad (3.2)$$

and $(z_{1,n} < a)$

$$\int_0^{z_{1,n}} \left| f(\sqrt{x}) - \pi_n(x) \right| \prod_{z_{j,n} \geq c} |x - z_{j,n}|\, d\gamma \geq \int_0^{z_{1,n}} |d_n(x)| \prod_{z_{j,n} \geq c} |x - z_{j,n}|\, d\gamma. \tag{3.3}$$

From the definition of Frobenius-Padé approximants we have

$$\begin{aligned}
0 &= \left| \int_0^1 d_n(x) \prod_{j=2}^{2n+1} (x - z_{j,n})\, d\gamma \right| \\
&= \left| \int_0^{z_{1,n}} d_n(x) \prod_{j=2}^{2n+1} (x - z_{j,n})\, d\gamma + \int_{z_{1,n}}^1 d_n(x) \prod_{j=2}^{2n+1} (x - z_{j,n})\, d\gamma \right| \\
&\geq \left| \int_{z_{1,n}}^1 d_n(x) \prod_{j=2}^{2n+1} (x - z_{j,n})\, d\gamma \right| - \int_0^{z_{1,n}} |d_n(x)| \prod_{j=2}^{2n+1} |x - z_{j,n}|\, d\gamma \\
&= \int_{z_{1,n}}^1 |d_n(x)| \prod_{j=2}^{2n+1} |x - z_{j,n}|\, d\gamma - \int_0^{z_{1,n}} |d_n(x)| \prod_{j=2}^{2n+1} |x - z_{j,n}|\, d\gamma.
\end{aligned}$$

In the last equality we have used that the function $d_n(x)\prod_{j=2}^{2n+1}(x-z_{j,n})$ has constant sign in $[z_{1,n},1]$ (recall that according to part 5 in Lemma 2.4 the only zeros of d_n are at the points in $\mathcal{P}_n$). Hereafter, all the products exclude the term $z_{1,n}$. We have shown that

$$\int_{z_{1,n}}^{1} |d_n(x)| \prod_{j=2}^{2n+1} |x-z_{j,n}|\, d\gamma \le \int_0^{z_{1,n}} |d_n(x)| \prod_{j=2}^{2n+1} |x-z_{j,n}|\, d\gamma.$$

Bounding the left-hand from below and the right-hand from above, it follows that for n large enough

$$\begin{aligned}\min_{x\in[a,1]} \prod_{z_{j,n}<b} |x-z_{j,n}| \int_a^1 |d_n(x)| \prod_{z_{j,n}\ge b} |x-z_{j,n}|\, d\gamma \\ \le \max_{x\in[0,z_{1,n}]} \prod_{z_{j,n}<c} |x-z_{j,n}| \int_0^{z_{1,n}} |d_n(x)| \prod_{z_{j,n}\ge c} |x-z_{j,n}|\, d\gamma. \quad (3.4)\end{aligned}$$

Since the number of elements of $\mathcal{P}_n$ in $[b,1]$ and $[c,1]$ is bounded as a function of $n\in\Lambda$, we have

$$\sharp\{z_{j,n}<b\} \sim 2n, \quad \sharp\{z_{j,n}<c\} \sim 2n \quad \text{as } n\to\infty. \quad (3.5)$$

Therefore,

$$\max_{x\in[0,z_{1,n}]} \prod_{z_{j,n}<c} |x-z_{j,n}| \le c^{\sharp\{z_{j,n}<c\}}, \quad \min_{x\in[a,1]} \prod_{z_{j,n}<b} |x-z_{j,n}| \ge (a-b)^{\sharp\{z_{j,n}<b\}}. \quad (3.6)$$

From the inequalities in (2.7), we also have

$$|f(\sqrt{x})-\pi_n(x)| = \pi_n(x) - f(\sqrt{x}) < f(1) - f(\sqrt{x}), \quad x\in(0,z_{1,n}). \quad (3.7)$$

Combining (3.2)–(3.7), we get

$$\begin{aligned}&\int_a^1 \left|f(\sqrt{x})-\pi_n(x)\right| \prod_{z_{j,n}\ge b} |x-z_{j,n}|\, d\gamma \\ &\le \left(\frac{c}{a-b}\right)^{2n+o(n)} \int_0^{z_{1,n}} \left|f(\sqrt{x})-\pi_n(x)\right| \prod_{z_{j,n}\ge c} |x-z_{j,n}|\, d\gamma \\ &\le \left(\frac{c}{a-b}\right)^{2n+o(n)} \int_0^{z_{1,n}} (f(1)-f(\sqrt{x})) \prod_{z_{j,n}\ge c} |x-z_{j,n}|\, d\gamma.\end{aligned}$$

This implies that $(f(\sqrt{x})-\pi_n(x))\prod_{z_{j,n}\geq b}|x-z_{j,n}|, n\in\Lambda$, converges to 0 in $L^1(\gamma|_{[a,1]})$. Hence this sequence has a subsequence which converges to 0 γ–a.e. on $[a,1]$ (see Theorem 3.12 in [12]). For these indexes, as the number of interpolations points in $[b,1]$ is bounded as a function of $n\in\Lambda$, there exists the limit of $\prod_{z_{j,n}\geq b}|x-z_{j,n}|$ (at least of a subsequence). Since the support of γ has an accumulation point in $[a,1]$, from the uniqueness principle of analytic functions the limit function of $\{\pi_n\}_{n\in\Lambda}$ must be equal to $f(\sqrt{z})$ in $\mathbb{C}\setminus(-\infty,0]$. $\square$

For a fixed n, $\pi_n(x)$ is an increasing function in $[0,\infty)$. Thus we have the same property for the function $f(\sqrt{x})$. We also know that the functions in class $\mathcal{A}$ have no zero outside the imaginary line. We can summarize some properties of the functions in class $\mathcal{A}$ in the following result.

Corollary 3.2. *If $f\in\mathcal{A}$, then f is a strictly increasing function in $[0,\infty)$ with no zero outside the imaginary line.*

Since

$$\int_0^1 (q(t)\frac{1}{f(\sqrt{t})}-p(t))t^j\,d\mu(\sqrt{t}) = \int_0^1 (q(t)-p(t)f(\sqrt{t}))t^j\,\frac{d\mu(\sqrt{t})}{f(\sqrt{t})},$$

Corollary 1.2 holds. Written in terms of functions on $[0,1]$ we have:

Corollary 3.3. *Under the assumptions of Theorem 3.1 assume that $1/f(\sqrt{x})\in L^1(\gamma)$, and let π_n^* denote the Frobenius-Padé approximant of order n of the function $1/f(\sqrt{x})$ with respect to γ. Then*

$$\lim_{n\to\infty}\pi_n^*(z)=1/f(\sqrt{z}),$$

uniformly on compact subsets of $\mathbb{C}\setminus(-\infty,0]$.

Example 3.4. If $-1<\alpha<2$, $d\mu(x):=(1-x^2)^\lambda\,dx$, $x\in[-1,1]$, $\lambda>-1$, and Π_n denotes the Frobenius-Padé approximants of order n of $|x|^\alpha$ with respect to μ, then

$$\lim_{n\to\infty}\Pi_n(z)=\begin{cases} z^\alpha, & \text{if } \Re(z)>0,\\ (-z)^\alpha, & \text{if } \Re(z)<0,\end{cases}$$

uniformly on compact subsets of $\mathbb{C}\setminus\{z:\Re(z)=0\}$. Moreover, when $-1<\alpha<0$, since each Frobenius-Padé approximant of $x^{-\alpha/2}$ is a positive increasing function in $[0,1]$, if we define z^α as $-\infty$ at zero and consider the spherical metric, then the above limit is true on compact subsets of $(\mathbb{C}\setminus\{z:\Re(z)=0\})\cup\{0\}$.

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References

[1] **B. Adcock, A. C. Hansen**, Generalized sampling and the stable and accurate reconstruction of piecewise analytic functions from their Fourier coefficients. Math. Comp. 84 (2015), no. 291, 237–270.

[2] **B. Beckermann, A. C. Matos , F. Wielonsky**, Reduction of the Gibbs phenomenon for smooth functions with jumps by the ϵ–algorithm. J. Comput. Appl. Math. 219 (2008), no. 2, 329–349.

[3] **M. Bello Hernández, C. Santos Touza**, Frobenius-Padé approximants of $|x|$. J. Approx. Theory 211 (2016), 1–15.

[4] **M. Bello Hernández, J. Mínguez Ceniceros**, Convergence of Fourier-Padé approximants for Stieltjes functions. Canad. J. Math. 58 (2006), no. 2, 249–261.

[5] **M. Bello Hernández, G. López Lagomasino, J. Mínguez Ceniceros**, Fourier–Padé approximants for Angelesco systems. Constr. Approx. 26 (2007), 339–359.

[6] **A. A. Gonchar, E. A. Rakhmanov, S. P. Suetin**, On the rate of convergence of Padé approximants of orthogonal expansions. Progress in approximation theory (Tampa, FL, 1990), 169–190, Springer Ser. Comput. Math., 19, Springer, New York, 1992.

[7] **G. López Lagomasino**, Asymptotics of polynomials orthogonal with respect to varying measures. Const. Approx. 5 (1989), 199–219.

[8] **D. S. Lubinsky**, **A. Sidi**, Convergence of linear and nonlinear Padé approximants from series of orthogonal polynomials. Trans. Amer. Math. Soc. 278 (1983), no. 1, 333–345.

[9] **G. Németh, G. Paris**, The Gibbs phenomenon in generalized Padé approximation. J. Math. Phys. 26 (1985), no. 6, 1175–1178.

[10] **E. M. Nikishin, V. N. Sorokin**, Rational Approximation and Orthogonality. Trans. Mathematical Monograph, Vol. 92, American Mathematical Society, Providence, R.I., 1991.

[11] **A. A. Pekarskii**, Approximation by rational functions with free poles. East J. Approx. 13 (2007), 227–319.

[12] **W. Rudin**, Real and Complex Analysis. McGraw-Hill International Edition, 1987.

[13] **H. R. Stahl**, Best uniform rational approximation of x^α on $[0,1]$. Acta Math. 190 (2003), no. 2, 241–306.

[14] **N. S. Vjačeslavov**, Approximation of x^α by rational functions. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 1, 92–109, 239.

[15] **J. Walsh**, Interpolation and Approximation by Rational Functions in the Complex Domain. Colloquium Publications, Vol. XX, American Mathematical Society, Providence, R. I. 1969.