

Full length article

Positivity and Fourier integrals over regular hexagon

Yuan Xu

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, United States

Received 30 August 2015; received in revised form 8 February 2016; accepted 23 February 2016

Available online 3 March 2016

Communicated by Feng Dai

Abstract

Let $f \in L^1(\mathbb{R}^2)$ and let \widehat{f} be its Fourier integral. We study summability of the partial integral $S_{\rho, \mathbb{H}}(x) = \int_{\{\|y\|_{\mathbb{H}} \leq \rho\}} e^{ix \cdot y} \widehat{f}(y) dy$, where $\|y\|_{\mathbb{H}}$ denotes the uniform norm taken over the regular hexagonal domain. We prove that the Riesz (R, δ) means of the inverse Fourier integrals are nonnegative if and only if $\delta \geq 2$. Moreover, we describe a class of $\|\cdot\|_{\mathbb{H}}$ -radial functions that are positive definite on \mathbb{R}^2 .
© 2016 Elsevier Inc. All rights reserved.

MSC: 42B08; 41A25; 41A63

Keywords: Fourier integral; Hexagon; Positivity Bochner–Riesz means; Positive definite function

1. Introduction

The classical Bochner–Riesz means of the Fourier integral have kernels that are radial functions, or the $\|\cdot\|_2$ -radial functions, where $\|\cdot\|_2$ denotes the usual Euclidean norm. We study their analogues that have kernels being $\|\cdot\|_{\mathbb{H}}$ -radial functions, where $\|\cdot\|_{\mathbb{H}}$ denotes the uniform norm of the regular hexagonal domain of \mathbb{R}^2 , and $\|\cdot\|_{\mathbb{H}}$ -radial functions that are positive definite functions on \mathbb{R}^2 .

Let f be a function in $L^1(\mathbb{R}^d)$. The Fourier transform \widehat{f} and its inverse are defined by

$$\widehat{f}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot y} f(x) dx \quad \text{and} \quad f(x) = \int_{\mathbb{R}^d} e^{iy \cdot x} \widehat{f}(y) dy, \quad (1.1)$$

E-mail address: yuan@uoregon.edu.

where the latter integral need not exist for an arbitrary function $f \in L^1(\mathbb{R}^d)$, a fact that motivates the study of summability methods. The classical Bochner–Riesz means (cf. [6]) of the inverse Fourier transform are defined by

$$S_{R,\delta}^{(2)} f(x) = \int_{\|y\|_2 \leq R} \left(1 - \frac{\|y\|_2^2}{R^2}\right)^\delta e^{iy \cdot x} \widehat{f}(y) dy. \quad (1.2)$$

The convergence of these means has been studied extensively. If $\|y\|_2$ is replaced by the ℓ_1 norm $|y|_1 := |y_1| + \cdots + |y_d|$ in (1.2), we denote the new means by $S_{R,\delta}^{(1)} f$ and call them ℓ_1 -Riesz (R, δ) means. It was proved in [2] that, in ℓ_1 summability, the (R, δ) means $S_{R,\delta}^{(1)} f$ define positive linear transformations on $L^1(\mathbb{R}^d)$ exactly when $\delta \geq 2d - 1$. In contrast, in ℓ_2 summability, the Bochner–Riesz means do not define positive transformations for any $\delta > 0$ [4].

In the present paper we study the case when $\|\cdot\|_2$ in (1.2) is replaced by the uniform norm $\|\cdot\|_H$ over the regular hexagonal domain in \mathbb{R}^2 . In this case it is more convenient to work in homogeneous coordinates of

$$\mathbb{R}_H^3 := \{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\},$$

for which the regular hexagonal domain is equivalent to $\{\mathbf{t} \in \mathbb{R}_H^3 : \|\mathbf{t}\|_H \leq 1\}$, where

$$\|\mathbf{t}\|_H := \max_{1 \leq i \leq 3} |t_i|.$$

In \mathbb{R}_H^3 the Fourier transform and its inverse can be defined by

$$\widehat{f}(\mathbf{s}) = \frac{1}{3\pi^2} \int_{\mathbb{R}_H^3} e^{-\frac{2i}{3}\mathbf{t} \cdot \mathbf{s}} f(\mathbf{t}) d\mathbf{t} \quad \text{and} \quad f(\mathbf{t}) = \int_{\mathbb{R}_H^3} e^{\frac{2i}{3}\mathbf{t} \cdot \mathbf{s}} \widehat{f}(\mathbf{s}) d\mathbf{s}, \quad (1.3)$$

as we shall see in the next section. The Riesz (R, δ) means then become

$$S_{R,\delta} f(\mathbf{t}) := \int_{\|\mathbf{s}\|_H \leq R} \left(1 - \frac{\|\mathbf{s}\|_H}{R}\right)^\delta e^{is \cdot \mathbf{t}} \widehat{f}(\mathbf{s}) d\mathbf{s}. \quad (1.4)$$

The symmetry of the regular hexagonal domain makes it possible to derive a close form for the Dirichlet kernel,

$$D_R(\mathbf{t}) = \int_{\|\mathbf{s}\|_H \leq R} e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}_H^3,$$

which can be used to establish the following theorem.

Theorem 1.1. *For $\delta \geq 2$ the Riesz (R, δ) means of the hexagonal partial integral (1.4) define positive linear transformations on $L^1(\mathbb{R}_H^3)$; the order of the summability to assure positivity is best possible.*

We note that the minimal order of the summability to assure positivity of the Riesz (R, δ) means for ℓ_1 summability is $\delta \geq 3$ when $d = 2$.

A function $\phi : \mathbb{R}_H^3 \mapsto \mathbb{R}$ is called $\|\cdot\|_H$ invariant, or $\|\cdot\|_H$ -radial, if $\phi(\mathbf{t}) = \phi_0(\|\mathbf{t}\|_H)$ for some $\phi_0 : \mathbb{R}_+ := [0, \infty) \mapsto \mathbb{R}$. Although the Dirichlet kernel is not $\|\cdot\|_H$ radial, it has additional structure that allows us to characterize positive definiteness of such functions.

A function ϕ on \mathbb{R}^d is said to be *positive definite* if for any set of points x_1, \dots, x_N in \mathbb{R}^d and scalars c_1, \dots, c_N in \mathbb{C} , $N \in \mathbb{N}$,

$$\sum_{j=1}^N \sum_{k=1}^N c_j \bar{c}_k \phi(x_j - x_k) \geq 0;$$

that is, the matrix $[\phi(x_j - x_k)]_{j,k=1}^N$ is positive semi-definite for all $\{x_j\}$, $\{c_j\}$ and N . Bochner proved in [3] that a continuous function ϕ on \mathbb{R}^d is positive definite exactly when it is the Fourier integral of a finite positive measure on \mathbb{R}^d . Shoenberg specialized Bochner's theorem for ℓ_2 radial functions and proved that $\phi(\|x\|_2)$ is positive definite exactly when

$$\phi(t) = \int_0^\infty \Omega_d(tu) d\alpha(u) \quad \text{with } \Omega_d(r) := \left(\frac{2}{r}\right)^{\frac{d-2}{2}} J_{\frac{d-2}{2}}(r) \quad (1.5)$$

and $\alpha(u)$ is non-decreasing and bounded for $u \geq 0$, where J_γ is the Bessel function. An analogue result was proved in [2] for ℓ_1 -radial functions $\phi(\|x\|_1)$, for which ϕ is characterized by (1.5) with Ω_d replaced by m_d , where m_d is a function that can be defined either recursively or as the Fourier transform of the B-spline function $x \mapsto M(\cdot|x_1^2, \dots, x_d^2)$ on \mathbb{R}^d . In particular, m_2 is given by

$$m_2(\xi) = \int_\xi^\infty \frac{\sin u}{u} du.$$

In the case of $d = 2$, under the change of variables $x = s + t$ and $y = s - t$, the ℓ_1 ball $\{x \in \mathbb{R}^2 : |x|_1 \leq 1\}$ becomes the square $[-1, 1]^2$ and the ℓ_1 -radial functions become $\|\cdot\|_\infty$, or ℓ_∞ -radial functions. Hence, the result in [2] also gives a characterization of ℓ_∞ radial functions.

Our second result shows that the class of positive definite $\|\cdot\|_H$ radial functions is at least as big as the class of positive definite ℓ_1 or ℓ_∞ radial functions.

Theorem 1.2. *Let $\phi \in C_b(\mathbb{R}_+)$. The function $\phi(\|\mathbf{t}\|_H)$, $\mathbf{t} \in \mathbb{R}_H^3$, is positive definite on \mathbb{R}_H^3 if there exists an increasing bounded function α on \mathbb{R}_+ such that*

$$\phi(t) = \int_0^\infty m_2(tu) d\alpha(u).$$

We do not know if the converse of Theorem 1.2 holds; that is, whether $\phi(\|\mathbf{t}\|_H)$ is positive definite *only if* ϕ is of the form given in the theorem. In [2], the inverse in the ℓ_1 case was established by writing the Dirichlet kernel as an integral against a B-spline function whose knots are the variables, and studying the Fourier transform of the B-spline function. The kernel in the hexagonal setting can also be written as an integral against a B-spline function, but the resulted B-spline function is no longer integrable, which prevents us from following the same approach.

The paper is organized as follows. The set-up of the Fourier integral on the hexagonal domain and the analysis, based on the closed formula for the Dirichlet kernel, that leads to the proof of Theorem 1.1 are given in Section 2. The positive definiteness of $\|\cdot\|_H$ -radial functions is discussed in Section 3.

2. Fourier integral on the hexagonal domain

The regular hexagonal domain on \mathbb{R}^2 is given by

$$H = \left\{ (x_1, x_2) : -1 \leq x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1 \right\}.$$

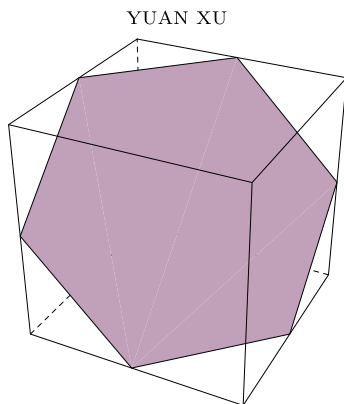


Fig. 1. Regular hexagon in \mathbb{R}_H^3 .

We consider the Fourier integral $\int_H \widehat{f}(y) e^{-ix \cdot y} dy$. Such integrals have been studied, say, in [1], where the Turán's problem on positive definite functions on the hexagonal domain is studied. For our purpose, it is more convenient to use homogeneous coordinates $\mathbf{t} = (t_1, t_2, t_3)$ in the space

$$\mathbb{R}_H^3 := \{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\},$$

for which the hexagonal domain H becomes

$$H := \{\mathbf{t} \in \mathbb{R}_H^3 : -1 \leq t_1, t_2, t_3 \leq 1\} = \{\mathbf{t} \in \mathbb{R}_H^3 : \|\mathbf{t}\|_H \leq 1\},$$

where $\|\mathbf{t}\|_H = \max_{1 \leq i \leq 3} |t_i|$. Geometrically H is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$ as shown in Fig. 1. Such a convenience is used in [5,7] where the Fourier series associated with the hexagonal lattice and their applications in discrete Fourier transform are studied.

For convenience, we adopt the convention of using bold letters, such as \mathbf{t} , to denote points in homogeneous coordinates of \mathbb{R}_H^3 throughout this paper. The relation between $(x_1, x_2) \in H$ and $\mathbf{t} \in H$ is given by

$$t_1 = -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2}, \quad t_2 = x_2, \quad t_3 = -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2},$$

or, equivalently, using the fact that $t_1 + t_2 + t_3 = 0$,

$$x_1 = \frac{1}{3}(t_1 - t_3) = \frac{1}{3}(2t_1 + t_2), \quad x_3 = \frac{1}{3}(t_2 - t_3) = \frac{1}{3}(t_1 + 2t_2). \quad (2.1)$$

Let $d\mathbf{t}$ be the Lebesgue measure on \mathbb{R}_H^3 . Then $d\mathbf{t} = \sqrt{1 + |\nabla g(t_1, t_2)|^2} dt_1 dt_2 = \sqrt{3} dt_1 dt_2$ with $g(t_1, t_2) = -t_1 - t_2$. Moreover, computing the Jacobian of the change of variables shows that $dt_1 dt_2 = \frac{\sqrt{3}}{2} dx_1 dx_2$. Consequently, $dx_1 dx_2 = \frac{2}{3} d\mathbf{t}$. With x and y associated to \mathbf{t} and \mathbf{s} , respectively, through (2.1), it is easy to see that $x \cdot y = \frac{2}{3} \mathbf{s} \cdot \mathbf{t}$. If we identify the function $f(x)$ on \mathbb{R}^2 with $f(\mathbf{t})$ on \mathbb{R}_H^3 , then the Fourier transform and its inversion (1.1) translate to

$$\widehat{f}(x) \mapsto \frac{1}{6\pi^2} \int_{\mathbb{R}_H^3} f(\mathbf{t}) e^{-\frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} d\mathbf{t} \quad \text{and} \quad f(x) \mapsto \frac{2}{3} \int_{\mathbb{R}_H^3} \widehat{f}(\mathbf{s}) e^{\frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} d\mathbf{s}.$$

For convenience we renormalize them and defined the Fourier transform and its inverse on \mathbb{R}_H^3 as in (1.3), that is,

$$\widehat{f}(\mathbf{s}) = \frac{1}{3\pi^2} \int_{\mathbb{R}_H^3} e^{-\frac{2i}{3}\mathbf{t}\cdot\mathbf{s}} f(\mathbf{t}) d\mathbf{t} \quad \text{and} \quad f(\mathbf{t}) = \int_{\mathbb{R}_H^3} e^{\frac{2i}{3}\mathbf{t}\cdot\mathbf{s}} \widehat{f}(\mathbf{s}) d\mathbf{s}.$$

The usual definition for the convolution $f * g$ extends to $f, g \in L^1(\mathbb{R}_H^3)$ by

$$f * g(\mathbf{t}) = \int_{\mathbb{R}_H^3} f(\mathbf{t} - \mathbf{s}) g(\mathbf{s}) d\mathbf{s}, \quad f, g \in L^1(\mathbb{R}_H^3).$$

For $\rho > 0$ we first consider the hexagonal summability of the inverse Fourier integral

$$\sigma_\rho(f; \mathbf{t}) = \int_{\|\mathbf{s}\|_H \leq \rho} \widehat{f}(\mathbf{s}) e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} d\mathbf{s} = (f * D_\rho)(\mathbf{t}), \quad (2.2)$$

where D_ρ is the Dirichlet kernel for the regular hexagonal domain,

$$D_\rho(\mathbf{t}) := \int_{\|\mathbf{t}\|_H \leq \rho} e^{-\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}_H^3.$$

Our first result is a closed form for the Dirichlet kernel for the hexagonal domain.

Proposition 2.1. For $\rho > 0$,

$$D_\rho(\mathbf{t}) = -\frac{9}{2} \left[\frac{\cos \left[\frac{2}{3}\rho(t_1 - t_2) \right]}{(t_2 - t_3)(t_3 - t_1)} + \frac{\cos \left[\frac{2}{3}\rho(t_2 - t_3) \right]}{(t_3 - t_1)(t_1 - t_2)} + \frac{\cos \left[\frac{2}{3}\rho(t_3 - t_1) \right]}{(t_1 - t_2)(t_2 - t_3)} \right]. \quad (2.3)$$

Proof. Let $D(\mathbf{t}) := D_1(\mathbf{t})$. A simple change of variable shows that

$$D_\rho(\mathbf{t}) = \rho^2 D(\rho\mathbf{t}).$$

Hence, we only need to work with the case $\rho = 1$. The hexagonal domain can be partitioned into three parallelograms, as shown in Fig. 2, which leads to

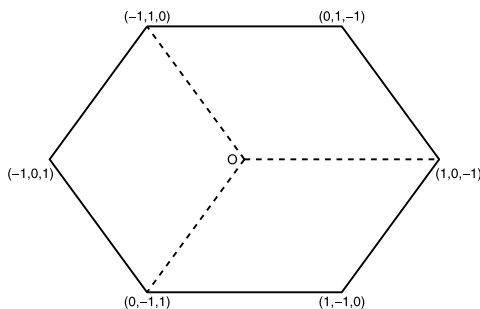
$$D(\mathbf{t}) = \int_0^1 \int_{-1}^0 e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} ds_1 ds_2 + \int_0^1 \int_{-1}^0 e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} ds_2 ds_3 + \int_0^1 \int_{-1}^0 e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} ds_3 ds_1.$$

For $\mathbf{s}, \mathbf{t} \in \mathbb{R}_H^3$, we can write $\mathbf{s} \cdot \mathbf{t} = (t_1 - t_3)s_1 + (t_2 - t_3)s_2$ by using $s_1 + s_2 + s_3 = 0$, so that the first integral can be easily evaluated as

$$I(1, 2) := \int_0^1 \int_{-1}^0 e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} ds_1 ds_2 = \left(\frac{3}{2}\right)^2 \frac{\left(1 - e^{\frac{2i}{3}(t_3 - t_1)}\right) \left(1 - e^{\frac{2i}{3}(t_2 - t_3)}\right)}{(t_1 - t_3)(t_2 - t_3)}.$$

The other two integrals are evidently just $I(2, 3)$ and $I(3, 1)$. The nominator of the last fraction in the right hand side of $I(1, 2)$ can be written as

$$1 - e^{-\frac{2i}{3}(t_3 - t_1)} - e^{-\frac{2i}{3}(t_1 - t_2)} - e^{-\frac{2i}{3}(t_2 - t_3)} + 2 \cos \left[\frac{2}{3}(t_1 - t_2) \right].$$

Fig. 2. Hexagon and its partition in \mathbb{R}_H^3 .

Without the cosine term, the above expression is invariant under the permutation. Hence, using the fact that $(t_1 - t_2) + (t_2 - t_3) + (t_3 - t_1) = 0$, summing up $I(1, 2) + I(2, 3) + I(3, 1)$ shows that

$$D(\mathbf{t}) = -\frac{9}{2} \left[\frac{\cos \left[\frac{2}{3}(t_1 - t_2) \right]}{(t_2 - t_3)(t_3 - t_1)} + \frac{\cos \left[\frac{2}{3}(t_2 - t_3) \right]}{(t_3 - t_1)(t_1 - t_2)} + \frac{\cos \left[\frac{2}{3}(t_3 - t_1) \right]}{(t_1 - t_2)(t_2 - t_3)} \right],$$

which is the desired formula for $\rho = 1$. This completes the proof. \square

Along the same line of the proof, we have the following proposition on the Fourier transform of $\|\cdot\|_H$ radial functions.

Proposition 2.2. Let $\phi \in C_0(\mathbb{R}_+)$. For any $r > 0$ and $\mathbf{t} \in \mathbb{R}_H^3$,

$$\int_{\|\mathbf{s}\|_H \leq r} e^{\pm \frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} \phi(\|\mathbf{s}\|_H) d\mathbf{s} = \int_0^r E_\rho(\mathbf{t}) \phi(\rho) d\rho \quad (2.4)$$

for almost all \mathbf{t} , where

$$E_\rho(\mathbf{t}) = \frac{d}{d\rho} D_\rho(\mathbf{t}).$$

Proof. As in the proof of Proposition 2.1, we can split the integral into three pieces,

$$\int_{\|\mathbf{s}\|_H \leq r} e^{\frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} \phi(\|\mathbf{s}\|_H) d\mathbf{s} = J(1, 2) + J(2, 3) + J(3, 1),$$

where, writing $\mathbf{s} \cdot \mathbf{t} = s_1(t_1 - t_3) + s_2(t_2 - t_3)$ for $\mathbf{t} \in \mathbb{R}_H^3$,

$$\begin{aligned} J(1, 2) &:= \int_0^r \int_{-r}^0 e^{\frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} \phi(\|\mathbf{s}\|_H) ds_1 ds_2 \\ &= \int_0^r \int_0^r e^{\frac{2i}{3} (s_1(t_1 - t_3) - s_2(t_2 - t_3))} \phi(\|\mathbf{s}\|_H) ds_1 ds_2, \end{aligned}$$

and $J(2, 3)$ and $J(3, 1)$ are permutations of $J(1, 2)$. For $s_1 < 0$ and $s_2 > 0$, $\|\mathbf{s}\|_H = \max\{s_1, s_2\}$. Hence, it follows that

$$\begin{aligned} J(1, 2) &= \int_0^r e^{\frac{2i}{3}s_1(t_1-t_3)} \phi(s_1) ds_1 \int_0^{s_1} e^{-\frac{2i}{3}s_2(t_2-t_3)} ds_2 \\ &\quad + \int_0^r e^{-\frac{2i}{3}s_2(t_2-t_3)} \phi(s_2) ds_2 \int_0^{s_2} e^{\frac{2i}{3}s_1(t_1-t_3)} ds_1 \\ &= \frac{3}{2} \int_0^r \phi(\rho) \left[\frac{e^{\frac{2i}{3}\rho(t_1-t_3)} - e^{\frac{2i}{3}\rho(t_1-t_2)}}{i(t_2-t_3)} - \frac{e^{\frac{2i}{3}\rho(t_3-t_2)} - e^{\frac{2i}{3}\rho(t_1-t_2)}}{i(t_1-t_3)} \right] d\rho. \end{aligned}$$

The terms inside the bracket can be rewritten as

$$\frac{(t_1-t_3)e^{\frac{2i}{3}\rho(t_1-t_3)} + (t_3-t_2)e^{\frac{2i}{3}\rho(t_3-t_2)} - (t_1-t_2)e^{\frac{2i}{3}\rho(t_1-t_2)}}{i(t_2-t_3)(t_1-t_3)}$$

and the nominator of this expression can be further rewritten as

$$\begin{aligned} &(t_1-t_3)e^{\frac{2i}{3}\rho(t_1-t_3)} + (t_3-t_2)e^{\frac{2i}{3}\rho(t_3-t_2)} + (t_2-t_1)e^{\frac{2i}{3}\rho(t_2-t_1)} \\ &\quad - 2i(t_1-t_2) \sin \left[\frac{2}{3}\rho(t_1-t_2) \right], \end{aligned}$$

in which the sum in the first line is invariant under permutation. Consequently, adding $J(1, 2)$, $J(2, 3)$ and $J(3, 1)$ gives

$$\int_{\|\mathbf{t}\|_H \leq r} e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} \phi(\|\mathbf{s}\|_H) d\mathbf{s} = \int_0^r \phi(\rho) E_\rho(\mathbf{t}) d\rho$$

where

$$\begin{aligned} E_\rho(\mathbf{t}) &= 3 \left[\frac{(t_1-t_2) \sin \left[\frac{2}{3}\rho(t_1-t_2) \right]}{(t_2-t_3)(t_3-t_1)} \right. \\ &\quad \left. + \frac{(t_2-t_3) \sin \left[\frac{2}{3}\rho(t_2-t_3) \right]}{(t_3-t_1)(t_1-t_2)} + \frac{(t_3-t_1) \sin \left[\frac{2}{3}\rho(t_3-t_1) \right]}{(t_1-t_2)(t_2-t_3)} \right]. \end{aligned} \quad (2.5)$$

It is now easy to see that $\frac{d}{d\rho} D_\rho(\mathbf{t}) = E_\rho(\mathbf{t})$ and (2.4) follows. \square

The computation of the proof also shows that

$$\int_0^r \phi(\|\mathbf{t}\|_H) d\mathbf{t} = 6 \int_0^r \rho \phi(\rho) d\rho.$$

This proposition allows us to compute the Fourier transform of $\|\cdot\|_H$ radial functions.

Example 2.3. For $a > 0$ and $\mathbf{t} \in \mathbb{R}_H^3$, let $\phi(\mathbf{t}) = e^{-\frac{2a}{3}\|\mathbf{t}\|_H}$. Then

$$\widehat{\phi}(\mathbf{t}) = \frac{27a^2(2a^2 + t_1^2 + t_2^2 + t_3^2)}{4(a^2 + (t_1-t_2)^2)(a^2 + (t_2-t_3)^2)(a^2 + (t_3-t_1)^2)}.$$

Proof. We use the explicit formula of E_ρ in (2.5) and the elementary integral

$$\int_0^\infty e^{-a\rho} \sin(b\rho) d\rho = \frac{b}{a^2 + b^2}, \quad a > 0,$$

then simplify the computation using $t_1 t_2 + t_2 t_3 + t_3 t_1 = -(t_1^2 + t_2^2 + t_3^2)/2$, which comes from $(t_1 + t_2 + t_3)^2 = 0$. \square

Another example is the Gaussian $\phi(\mathbf{t}) = e^{-\|\mathbf{t}\|_H^2}$. The Fourier transform of this function, however, does not have a closed form, since

$$\int_0^\infty e^{-\rho^2} \sin(b\rho) d\rho = F\left(\frac{b}{2}\right), \quad \text{with } F(t) := e^{-t^2} \int_0^t e^{s^2} ds.$$

For $\delta > 0$, the Cesàro (C, δ) means of a function $s : \mathbb{R}_+ \mapsto \mathbb{C}$ are defined by

$$s^\delta(\rho) = \frac{\delta}{\rho^\delta} \int_0^\rho (\rho - u)^{\delta-1} s(u) du, \quad \rho > 0.$$

Because of the convolution structure of the partial integral (2.2), its (C, δ) means can be viewed as convolving the function f with the (C, δ) means of the Dirichlet kernel; that is, define

$$D_R^\delta(\mathbf{t}) := \frac{\delta}{R^\delta} \int_0^R (R - \rho)^{\delta-1} D_\rho(\mathbf{t}) d\rho,$$

then the (C, δ) means of the integral in (2.2) is defined by

$$\sigma_R^\delta(f; \mathbf{t}) = (f * D_R^\delta)(\mathbf{t}), \quad R > 0, \mathbf{t} \in \mathbb{R}_H^3.$$

Our next result shows that the Cesàro (C, δ) means and the Reize (R, δ) means, defined in (1.4), of hexagonal summability of the Fourier integral are identical.

Corollary 2.4. *Let $f \in L^1(\mathbb{R}_H^2)$ and $\delta > 0$. Then for any $r > 0$,*

$$\begin{aligned} S_{R,\delta} f(\mathbf{t}) &= \int_{\|\mathbf{s}\|_H \leq 1} \left(1 - \frac{\|\mathbf{s}\|_H}{R}\right)^\delta e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} \widehat{f}(\mathbf{s}) d\mathbf{s} \\ &= \frac{\delta}{R^\delta} \int_0^R (R - \rho)^{\delta-1} \sigma_\rho(f; \mathbf{t}) d\rho = (f * D_R^\delta)(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}_H^3. \end{aligned}$$

Proof. Let $\phi \in C_0(\mathbb{R}_+)$ be locally absolutely continuous. Integration by parts in the right hand side of (2.4) shows that

$$\int_{\|\mathbf{s}\|_H \leq R} \phi(\|\mathbf{s}\|_H) e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} d\mathbf{s} = \phi(R) D_R(\mathbf{t}) - \int_0^R \phi'(\rho) D_\rho(\mathbf{t}) d\rho.$$

Setting $\phi(t) = (1 - t/R)_+^\delta$ for $t > 0$ and $\delta > 0$, this identity becomes

$$\int_{\|\mathbf{s}\|_H \leq R} \left(1 - \frac{\|\mathbf{s}\|_H}{R}\right)^\delta e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} d\mathbf{s} = \frac{\delta}{R^\delta} \int_0^R (R - \rho)^{\delta-1} D_\rho(\mathbf{t}) d\rho.$$

Taking the convolution with f proves the corollary. \square

By the close form of the Dirichlet kernel in (2.3), we immediately conclude that

$$D_R^\delta(\mathbf{t}) = -\frac{9}{2} \left[\frac{F_\delta\left(\frac{2}{3}R(t_1 - t_2)\right)}{(t_2 - t_3)(t_3 - t_1)} + \frac{F_\delta\left(\frac{2}{3}R(t_2 - t_3)\right)}{(t_3 - t_1)(t_1 - t_2)} + \frac{F_\delta\left(\frac{2}{3}R(t_3 - t_1)\right)}{(t_1 - t_2)(t_2 - t_3)} \right], \quad (2.6)$$

where

$$F_\delta(u) := \delta \int_0^1 \cos(\rho u) (1 - \rho)^{\delta-1} d\rho, \quad u > 0.$$

Elementary computation shows that F_δ can be written as a ${}_1F_2$ series

$$F_\delta(t) = {}_1F_2\left(1; \frac{\delta+1}{2}, \frac{\delta+2}{2}; -\frac{t^2}{4}\right).$$

For some special values of δ , F_δ enjoys compact expression. For example,

$$F_1(t) = \frac{\sin t}{t} \quad \text{and} \quad F_2(t) = \frac{2(1 - \cos t)}{t^2}. \quad (2.7)$$

The kernel D_R^δ turns out to be positive for $\delta \geq 2$. More precisely, we prove the following theorem.

Theorem 2.5. *The kernel $D_R^\delta(\mathbf{t})$ is nonnegative on \mathbb{R}_H^3 if, and only if, $\delta \geq 2$.*

Proof. First we prove that $D_R^\delta(\mathbf{t}) \geq 0$ if $\delta \geq 2$. For $\delta, \mu > 0$ it is easy to verify that

$$D_R^{\delta+\mu}(\mathbf{t}) = \frac{\Gamma(\delta + \mu + 1)}{\Gamma(\delta + 1)\Gamma(\mu)} \frac{1}{R^{\delta+\mu}} \int_0^R (R - \rho)^{\mu-1} \rho^\delta D_\rho^\delta(\mathbf{t}) d\rho, \quad R > 0.$$

Thus, it follows that $D_R^{\delta+\mu}(\mathbf{t})$ is nonnegative if $D_R^\delta(\mathbf{t})$ is for all R . Hence, it suffices to show that $D_R^2(\mathbf{t})$ is nonnegative for $\mathbf{t} \in \mathbb{R}_H^3$ and $R > 0$. Using the close form (2.6) of D_R^δ and the explicit formula of F_2 in (2.7), it follows readily that $D_R^2(\mathbf{t}) \geq 0$ if

$$\begin{aligned} G_R(\mathbf{t}) := & -(t_2 - t_3)(t_3 - t_1) \left(1 - \cos\left[\frac{2}{3}R(t_1 - t_2)\right]\right) \\ & - (t_3 - t_1)(t_1 - t_2) \left(1 - \cos\left[\frac{2}{3}R(t_2 - t_3)\right]\right) \\ & - (t_1 - t_2)(t_2 - t_3) \left(1 - \cos\left[\frac{2}{3}R(t_3 - t_1)\right]\right) \end{aligned}$$

is nonnegative. Evidently, it suffices to establish the non-negativity of $G_R(\mathbf{t})$ when $\frac{2}{3}R = 1$, which we denote by $G(\mathbf{t})$. It turns out that $G(\mathbf{t})$ can be written as a sum of squares, from which the nonnegativity of $G(\mathbf{t})$ follows immediately. Indeed, the following identity holds,

$$\begin{aligned} G(\mathbf{t}) = & \frac{1}{2} [(t_1 - t_2) \cos t_3 + (t_2 - t_3) \cos t_1 + (t_3 - t_1) \cos t_2]^2 \\ & + \frac{1}{2} [(t_1 - t_2) \sin t_3 + (t_2 - t_3) \sin t_1 + (t_3 - t_1) \sin t_2]^2. \end{aligned}$$

The difficulty lies in identifying the formula. The verification is tedious but straightforward, and it can be checked by a computer algebra system. This proves the positivity of $D_R^2(\mathbf{t})$.

Next we prove that $D_R^\delta(\mathbf{t})$ is not nonnegative when $0 < \delta < 2$. We only need to consider the case $1 < \delta < 2$, since if $D_R^\delta(\mathbf{t})$ is nonnegative for some δ that satisfies $0 < \delta \leq 1$, then it has to

be nonnegative for $1 < \delta < 2$. Assume $1 < \delta < 2$. It suffices to show that $D_R^\delta(\mathbf{t})$ is negative for some $\mathbf{t} \in \mathbb{R}_H^3$. Using the explicit formula of (2.6), it is easy to see that

$$D_R^\delta\left(\frac{3\pi}{R}, -\frac{3\pi}{R}, 0\right) = \frac{R^2}{2\pi^2} [F_\delta(2\pi) - F_\delta(4\pi)].$$

Integrating by parts shows that

$$\begin{aligned} F_\delta(2\pi) - F_\delta(4\pi) &= \frac{\delta(\delta-1)}{4\pi} \int_0^1 (1-s)^{\delta-2} [2\sin(2\pi s) - \sin(4\pi s)] ds \\ &= \frac{\delta(\delta-1)}{2\pi} \int_0^1 (1-s)^{\delta-2} \sin(2\pi s) [1 - \cos(2\pi s)] ds. \end{aligned}$$

Splitting the last integral as two, one over $[0, 1/2]$ and the other over $[1/2, 1]$, and changing variable $t \mapsto 1-s$ in the second integral, we see that

$$F_\delta(2\pi) - F_\delta(4\pi) = \frac{\delta(\delta-1)}{2\pi} \int_0^{1/2} \left[(1-s)^{\delta-2} - s^{\delta-2} \right] \sin(2\pi s) [1 - \cos(2\pi s)] ds.$$

Since for $0 < s < 1/2$, $\sin(2\pi s)[1 - \cos(2\pi s)] \geq 0$ and $(1-s)^{\delta-2} - s^{\delta-2} < 0$ as $\delta-2 < 0$, we conclude that $F_\delta(2\pi) - F_\delta(4\pi) < 0$ for $1 < \delta < 2$. Consequently $D_R^\delta\left(\frac{3\pi}{R}, -\frac{3\pi}{R}, 0\right)$ is negative for $1 < \delta < 2$. \square

Corollary 2.6. For $\delta \geq 2$, the Cesàro (C, δ) means $\sigma_R^\delta(f)$ define positive linear transformations on $L^1(\mathbb{R}^2)$; the order of summability to assure positivity is best possible.

By Corollary 2.4, this also proves Theorem 1.1. As in the case of ℓ_1 , the positivity of the kernel and its proof are motivated by the corresponding result on the summability of the Fourier series. Indeed, it was proved in [7] that the Cesàro (C, δ) means of the Fourier series associated with the hexagonal lattice are nonnegative if $\delta \geq 2$.

3. Positive definite hexagonal radial functions

In this section we consider hexagonal invariant functions that depend only on $\|\cdot\|_H$, which we call $\|\cdot\|_H$ radial functions. We start with the proof of Theorem 1.2, which we restate below, that gives a sufficient condition for a $\|\cdot\|_H$ radial function to be positive definite.

Theorem 3.1. Let $\phi \in C_b(\mathbb{R}_+)$. The function $\phi(\|\mathbf{t}\|_H)$, $\mathbf{t} \in \mathbb{R}_H^3$, is positive definite on \mathbb{R}_H^3 if there exists an increasing bound function α on \mathbb{R}_+ such that

$$\phi(t) = \int_0^\infty m_2(tu) d\alpha(u) \quad \text{with} \quad m_2(\xi) = \int_\xi^\infty \frac{\sin(u)}{u} du. \quad (3.1)$$

Proof. By Bochner's theorem, $\phi(\|\mathbf{t}\|_H)$ is positive definite exactly when it is the Fourier transform of a nonnegative, integrable function, say Φ , on \mathbb{R}_H^3 . By (2.4), we need to show that

$$\Phi(\mathbf{t}) = \int_0^\infty \phi(\rho) E_\rho(\mathbf{t}) d\rho$$

is nonnegative. For m_2 given in (3.1), it is sufficient to write

$$\Phi(\mathbf{t}) = \int_0^\infty \left[\int_0^\infty E_\rho(\mathbf{t}) m_2(\rho u) d\rho \right] d\alpha(u)$$

and prove that the inner integral is nonnegative. Using the fact that $E_{\rho/u}(\mathbf{t}) = u^{-1}E_{\rho}(\mathbf{t}/u)$, which follows immediately from (2.5), it is enough to prove that

$$J(\mathbf{t}) := \int_0^\infty E_{\frac{3}{2}\rho}(\mathbf{t})m_2(\rho)d\rho \geq 0, \quad \mathbf{t} \in \mathbb{R}_H^3.$$

Let χ_E be the characteristic function of the set $E \subset \mathbb{R}_H^3$. We need the evaluation

$$\begin{aligned} \int_0^\infty \sin(u\rho) \int_\rho^\infty \frac{\sin s}{s} ds d\rho &= \int_0^\infty \frac{\sin s}{s} \int_0^s \sin(u\rho) d\rho ds \\ &= \frac{1}{u} \int_0^\infty \sin s (1 - \cos(su)) \frac{ds}{s} = \frac{\pi}{4u} (1 - \operatorname{sign}(1 - |u|)). \end{aligned}$$

Together with the fact that $(t_1 - t_2) + (t_2 - t_3) + (t_3 - t_1) = 0$ and $1 + \operatorname{sign}(1 - |t_1 - t_2|) = 2\chi_{\{|t_1 - t_2| \leq 1\}}(\mathbf{t})$, it follows from (2.5) that

$$J(\mathbf{t}) = -\frac{3\pi}{2} \left[\frac{\chi_{\{|t_1 - t_2| \leq 1\}}(\mathbf{t})}{(t_2 - t_3)(t_3 - t_1)} + \frac{\chi_{\{|t_2 - t_3| \leq 1\}}(\mathbf{t})}{(t_3 - t_1)(t_1 - t_2)} + \frac{\chi_{\{|t_3 - t_1| \leq 1\}}(\mathbf{t})}{(t_1 - t_2)(t_2 - t_3)} \right].$$

The value of $J(\mathbf{t})$ depends on regions of $\mathbf{t} \in \mathbb{R}_H^3$ determined by the support sets of the three characteristic functions. In the region $E_{---} := \{\mathbf{t} : |t_1 - t_2| < 1, |t_2 - t_3| < 1, |t_3 - t_1| < 1\}$, $J(\mathbf{t}) = 0$ since $(t_1 - t_2) + (t_2 - t_3) + (t_3 - t_1) = 0$. We now consider the region $E_{--+} := \{\mathbf{t} : |t_1 - t_2| < 1, |t_2 - t_3| < 1, |t_3 - t_1| > 1\}$. In this case,

$$J(\mathbf{t}) = -\frac{3\pi}{4} \left[\frac{1}{(t_2 - t_3)(t_3 - t_1)} + \frac{1}{(t_3 - t_1)(t_1 - t_2)} \right] = \frac{3\pi}{4} \frac{1}{(t_1 - t_2)(t_2 - t_3)},$$

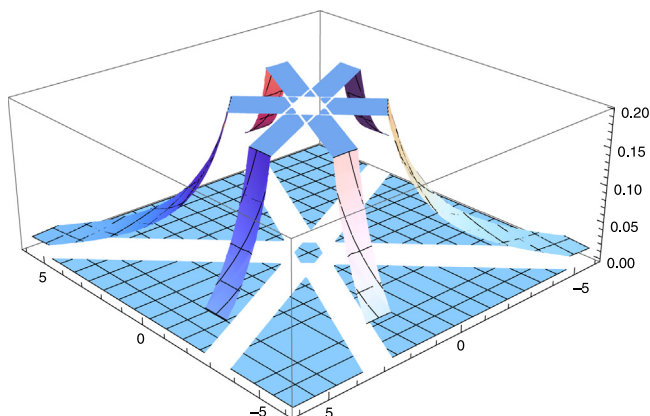
where the second equality follows from $t_1 + t_2 + t_3 = 0$, which is positive if $t_1 - t_2$ and $t_2 - t_3$ have the same sign. Assume these two factors have different sign, say $0 < t_1 - t_2$ and $t_2 - t_3 < 0$. Then $\mathbf{t} \in E_{--+}$ satisfies

$$0 < t_1 - t_2 < 1, \quad -1 < t_2 - t_3 < 0, \quad |t_3 - t_1| > 1.$$

The third inequality has two possibilities. In the case of $t_3 - t_1 > 1$, the second and the third inequality imply that $t_1 < t_2$, which contradicts the first inequality. In the case of $t_3 - t_1 < -1$, the first and the third inequality imply that $t_3 < t_2$, which contradicts the second inequality. Consequently, the set E_{--+} does not contain elements for which $0 < t_1 - t_2$ and $t_2 - t_3 < 0$, nor does it contain elements for which $t_1 - t_2 < 0$ and $0 < t_2 - t_3$. As a result, $J(\mathbf{t})$ is nonnegative for $\mathbf{t} \in E_{--+}$. By symmetry, this holds for permutations of E_{--+} . Next we consider the region $E_{-++} := \{\mathbf{t} : |t_1 - t_2| < 1, |t_2 - t_3| > 1, |t_3 - t_1| > 1\}$, for which

$$J(\mathbf{t}) = -\frac{3\pi}{4} \frac{1}{(t_2 - t_3)(t_3 - t_1)}$$

is nonnegative if $t_2 - t_3$ and $t_3 - t_1$ have the opposite sign. Assume those two factors have the same sign, say, $t_2 - t_3 > 0$ and $t_3 - t_1 > 0$, then $t_2 - t_1 = t_2 - t_3 + t_3 - t_1 > 0$, so that $\mathbf{t} \in E_{-++}$ satisfies $t_2 - t_3 > 1$, $t_3 - t_1 > 1$ and $t_2 - t_1 < 1$. However, the first two inequalities imply that $t_2 > t_1 + 2$, which contradicts the third inequality. Hence, the set E_{-++} does not contain \mathbf{t} for which $t_2 - t_3$ and $t_3 - t_1$ have the same sign. Consequently, $J(\mathbf{t})$ is nonnegative on E_{-++} . By symmetry, this holds for permutations of E_{-++} . Finally, it is evident that $J(\mathbf{t}) = 0$ on E_{+++} , the definition of which should be obvious by now. Thus, we have proved that $J(\mathbf{t}) \geq 0$ for all $\mathbf{t} \in \mathbb{R}_H^3$, which completes the proof of the theorem. \square

Fig. 3. The function $J(\mathbf{t})$.

As shown in the proof, the support set of the function $J(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}_H^3$, is relatively small. The graph of the function looks like a hexagonal spider (that has six legs), which is depicted in Fig. 3.

We do not know if the sufficient condition in the above theorem is necessary. The theorem shows that the class of positive definite $\|\cdot\|_H$ radial functions is at least as large as the class of positive definite ℓ_1 radial functions. In the latter case, the condition is necessary and it is established by writing the corresponding kernel in terms of B-spline functions. An analogue representation can be given in the hexagonal setting, which we discuss below.

Recall that the first divided difference $[a, b]f$ is defined by

$$[a, b]f := \frac{f(b) - f(a)}{b - a}, \quad a, b \in \mathbb{R}, \quad a \neq b,$$

which can also be written as

$$[a, b]f = \int_{\mathbb{R}} f'(u) B(u|a, b) du \quad \text{with} \quad B(u|a, b) := \begin{cases} \frac{1}{b-a} & \text{if } b > u > a \\ \frac{1}{a-b} & \text{if } a > u > b \\ 0 & \text{otherwise.} \end{cases}$$

The function $B(\cdot|a, b)$ is the simplest example of B-spline functions and it evidently satisfies

$$B(u|a, b) \geq 0, \quad u \in \mathbb{R}, \quad \text{and} \quad \int_{\mathbb{R}} B(u|a, b) du = 1.$$

Proposition 3.2. For $\mathbf{t} \in \mathbb{R}_H^3$ and $u \in \mathbb{R}$, define

$$\begin{aligned} M_1(u|\mathbf{t}) &:= B(u|t_1 - t_3, t_2 - t_3) + B(u|t_2 - t_1, t_3 - t_1) + B(u|t_3 - t_2, t_1 - t_2). \\ M(u|\mathbf{t}) &:= \frac{1}{2} [M_1(u|\mathbf{t}) + M_1(u|-\mathbf{t})]. \end{aligned}$$

Then, for $\rho > 0$,

$$E_\rho(\mathbf{t}) = 2\rho \int_0^\infty \cos\left(\frac{2}{3}\rho u\right) M(u|\mathbf{t}) du. \quad (3.2)$$

Proof. In the expression of E_ρ in (2.5), we apply the partial fraction

$$-\frac{t_1 - t_2}{(t_2 - t_3)(t_3 - t_1)} = \frac{1}{t_2 - t_3} + \frac{1}{t_3 - t_1}$$

to the first term in the right hand side and two analogues partial fractions to the other two terms. Rearranging the terms leads to

$$E_\rho(\mathbf{t}) = 3 \left([t_1 - t_3, t_2 - t_3] \sin \left[\frac{2}{3} \rho \{\cdot\} \right] + [t_2 - t_1, t_3 - t_1] \sin \left[\frac{2}{3} \rho \{\cdot\} \right] + [t_3 - t_2, t_1 - t_2] \sin \left[\frac{2}{3} \rho \{\cdot\} \right] \right).$$

Writing the divided differences in terms of B-spline functions lead immediately to

$$E_\rho(\mathbf{t}) = 2\rho \int_{-\infty}^{\infty} \cos \left(\frac{2}{3} \rho u \right) M_1(u|\mathbf{t}) du.$$

Directly from the definition, it is easy to verify that $B(u|a, b)$ satisfies $B(-u|a, b) = B(u| -a, -b)$, from which follows $M_1(-u|\mathbf{t}) = M_1(u| -\mathbf{t})$. Consequently, with our definition of $M(u|\mathbf{t})$, we can write the integral expression of $E_\rho(\mathbf{t})$ over $\mathbf{t} \in \mathbb{R}_H^3$ as the integral over \mathbb{R}_+ . \square

As a consequence of Propositions 2.2 and 3.2, we immediately deduce the following result on the inverse Fourier transform of hexagonal invariant functions.

Proposition 3.3. Let $\phi \in C_0(\mathbb{R}_+)$ such that the function $u \mapsto u\phi(u)$ is in $L^1(\mathbb{R}_+)$. Then

$$\int_{\mathbb{R}_H^3} \phi(\|\mathbf{s}\|_H) e^{\frac{2i}{3}\mathbf{s} \cdot \mathbf{t}} d\mathbf{s} = \int_0^\infty \psi(u) M(u|\mathbf{t}) du \quad (3.3)$$

where

$$\psi(u) := 4 \int_0^\infty \rho \cos \left(\frac{2}{3} \rho u \right) \phi(\rho) d\rho.$$

It is easy to see that the function $M(u|\mathbf{t})$ satisfies

$$M(u|\mathbf{t}) \geq 0, \quad u \in \mathbb{R}, \quad \mathbf{t} \in \mathbb{R}_H^3, \quad \text{and} \quad \int_{\mathbb{R}} M(u|\mathbf{t}) du = 3.$$

Furthermore, it also satisfies

$$M(u|u\mathbf{t}) = \frac{1}{u} M(1|\mathbf{t}), \quad u > 0.$$

Thus, we only need to consider the $M(1|\mathbf{t})$. In Fig. 4, we depict this function in the regular rectangle coordinates.

The formula (3.3) suggests that we consider the Fourier transform of $M(u|\mathbf{t})$, as an analogue of the study in the ℓ_1 case, for which the corresponding B-spline is $B(u|x_1^2, \dots, x_d^2)$ for $x \in \mathbb{R}^d$. However, the function $x \mapsto B(u|x_1^2, \dots, x_d^2)$ is integrable on \mathbb{R}^d , which warrants the existence of its Fourier transform, whereas the function $\mathbf{t} \mapsto M(u|\mathbf{t})$ is not integrable on \mathbb{R}_H^3 . The latter can be seen, for example, from the formula

$$M_1(u|\mathbf{t}) = \frac{1}{t_2 - t_1} + \frac{1}{t_2 - t_3} = \frac{3t_2}{(t_2 - t_1)(t_2 - t_3)}, \quad \mathbf{t} \in \Omega,$$

where $\Omega = \{\mathbf{t} \in \mathbb{R}_H^3 : t_2 - t_3 > 1, t_2 - t_1 > 1, t_3 - t_1 > -1, t_1 - t_3 > -1\}$.

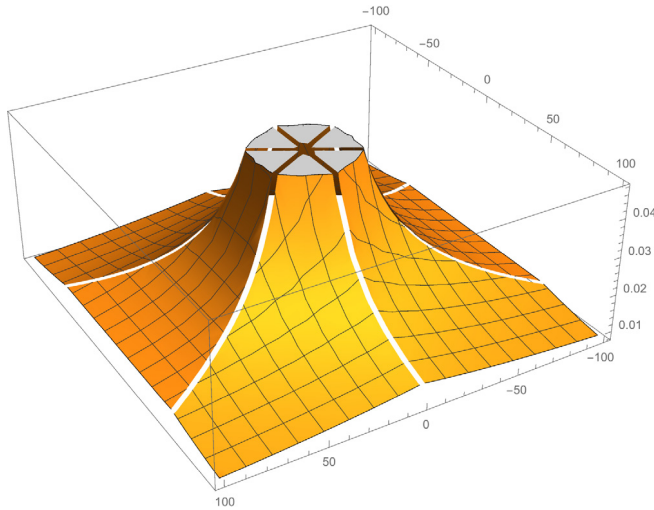


Fig. 4. The function $M(1|\cdot)$ on the usual Cartesian coordinates.

Let us also point out that, since $M(u|\mathbf{t}) \geq 0$, if ψ is nonnegative on \mathbb{R}_+ then, by (3.3), the Fourier integral of $\phi(\|\cdot\|_{\mathbb{H}})$ is nonnegative. This is, however, not necessary. Indeed, if $\phi(u) = e^{-u}$, then $\Phi(\mathbf{t}) = \int_{\mathbb{R}_{\mathbb{H}}^3} \phi(s) e^{\frac{2i}{3} \mathbf{s} \cdot \mathbf{t}} ds \geq 0$ by Example 2.3. However, in this case

$$\psi(u) = 4 \int_0^\infty \rho \cos(u\rho) e^{-\rho} d\rho = \frac{4(1-u^2)}{(1+u^2)^2},$$

which is not nonnegative if $|u| > 1$. Hence, the expression (3.3) is far less useful than its counterpart in the ℓ_1 case.

Acknowledgment

The work was supported in part by NSF Grant DMS-1510296.

References

- [1] V.V. Arestov, E. Berdysheva, Turán's problem for positive definite functions with supports in a hexagon, Proc. Steklov Inst. Math. (suppl. 1) (2001) S20–S29. Approximation Theory. Asymptotical Expansions.
- [2] H. Berens, Y. Xu, ℓ_1 -summability for multivariate Fourier integrals and positivity, Math. Proc. Cambridge Philos. Soc. 122 (1997) 149–172.
- [3] S. Bochner, Monotone Funktionen, Stieltjes integrale und harmonische analyse, Math. Ann. 108 (1933) 378–410.
- [4] B.I. Golubov, On Abel–Poisson type and Riesz means, Anal. Math. 7 (1981) 161–184.
- [5] H. Li, J. Sun, Y. Xu, Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle, SIAM J. Numer. Anal. 46 (2008) 1653–1681.
- [6] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, 1971.
- [7] Y. Xu, Fourier series and approximation on hexagonal and triangular domains, Constr. Approx. 31 (2010) 115–138.

Further reading

- [1] I.J. Schoenberg, Metric spaces and completely monotone functions, Ann. of Math. 39 (1938) 811–841.