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On approximations for functions in the space of uniformly convergent Fourier series[☆]

Holger Boche[a,b], Volker Pohl[a,*]

[a] Technical University of Munich, Institute of Theoretical Information Technology, Arcisstraße 21, 80333 München, Germany

[b] Munich Center for Quantum Science and Technology (MCQST), 80799 München, Germany

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Abstract

This paper studies the possibility of approximating functions in the space of all uniformly convergent symmetric and non-symmetric Fourier series from finitely many samples of the given function. It is shown that no matter what approximation method is chosen, there always exists a residual subset such that the approximation method diverges for all functions from this subset. This general result implies that there exists no method to effectively calculate the Fourier series expansion on a digital computer for all functions from the space of uniformly convergent Fourier series. In particular, there exists no Turing computable approximation method in these spaces.

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* Corresponding author.
E-mail addresses: boche@tum.de (H. Boche), volker.pohl@tum.de (V. Pohl).

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1. Introduction and motivation

The Fourier series of a 2π periodic function f represents this function as an (infinite) sum of pure frequencies

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt}\ , \qquad t \in [-\pi, \pi) \tag{1}$$

with its so called *Fourier coefficients*

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-\mathrm{i}n\tau}\, \mathrm{d}\tau\ , \qquad n \in \mathbb{Z}\ . \tag{2}$$

This decomposition of a signal into its frequency components is a fundamental tool in applied mathematics, engineering, and physics. Basically all areas of applied mathematics are penetrated by Fourier analysis techniques to an extend that it seems almost impossible to work in these areas without using Fourier analysis. Many physical phenomena, for example, are much more simpler to describe and to analyze in the Fourier domain, i.e. in terms of the Fourier coefficients [7,13]. Moreover, signal- and system theory as well as the design and the implementation of filters in engineering and signal processing rely heavily on Fourier analysis techniques [21–24,26].

Whether these Fourier analysis techniques are justifiable, depends on the question whether the Fourier coefficients determine uniquely the function f and whether it is possible to reconstruct f from its Fourier coefficients. So given the Fourier coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$ of an $f \in \mathcal{B}$ in a Banach space $\mathcal{B}$, the question is whether the sum in (1) exists and converges to f in $\mathcal{B}$. To investigate this, one considers usually the partial *symmetric Fourier series*

$$\big(\mathrm{S}_N f\big)(t) = \sum_{n=-N}^{N} c_n(f)\, \mathrm{e}^{\mathrm{i}nt}\ , \qquad t \in \mathbb{T} := [-\pi, \pi) \tag{3}$$

and asks whether $\lim_{N\to\infty} \|f - \mathrm{S}_N f\|_{\mathcal{B}} = 0$ for all $f \in \mathcal{B}$. Whether or not this is true, depends on the Banach space $\mathcal{B}$. Since its first application as an approximation method for functions, numerous investigations on the convergence behavior of Fourier series on different Banach spaces appeared. Examples include the construction of *Kolmogoroff* [17,18] showing that there exist functions in $L^1(\mathbb{T})$ whose Fourier series diverges at every point in $\mathbb{T}$. *Carleson's theorem* [6], on the other hand, shows that the Fourier series of an $L^2(\mathbb{T})$-function converges almost everywhere on $\mathbb{T}$ and *Hunt* [16] extended this result to all spaces $L^p(\mathbb{T})$ with $1 < p < +\infty$. Moreover, by celebrated classical results due to *du Bois-Reymond*, *Lebesgue*, and *Fejér* it is well known that the Fourier series of a continuous function may diverge at some points in $\mathbb{T}$ [8,11,20], and Carleson's theorem implies that the set of all divergence points has Lebesgue measure zero for every $f \in \mathcal{C}(\mathbb{T})$. Nevertheless, the set of all $f \in \mathcal{C}(\mathbb{T})$ with a pointwise divergent Fourier series is a residual set in $\mathcal{C}(\mathbb{T})$. Apart from these classical results there are many more elaborated investigations on the convergence of the Fourier series [10,28] or of other summation methods [12,19,34] in several function spaces, and we refer to books like [9,14,29,35] for an overview on the extensive theory of Fourier series.

In the engineering literature (see, e.g., [21,23]), it is often assumed that for continuous functions f, the corresponding partial Fourier series $\mathrm{S}_N f$ converges to f as $N \to \infty$. However, as mentioned above, it is a classical result that this is generally not true. Then, there are basically two ways to resolve this issue. On the one hand, one may apply alternative summation methods (e.g. arithmetic means of the partial Fourier series) to achieve uniform convergence for all $f \in \mathcal{C}(\mathbb{T})$. On the other hand, one may restrict the function space to an appropriate subset of $\mathcal{C}(\mathbb{T})$. A particular appealing space might be the set $\mathcal{U}_{\mathrm{s}}$ of all continuous, 2π-periodic function for which the partial Fourier series (3) converges uniformly on $\mathbb{T}$. Equipped with an appropriate norm, $\mathcal{U}_{\mathrm{s}}$ becomes a Banach space, namely the largest space of continuous functions on which the Fourier series converges uniformly on $\mathbb{T}$ (and in norm) for every $f \in \mathcal{U}_{\mathrm{s}}$. In particular, the Fourier series (3) allows us to determine to every $f \in \mathcal{U}_{\mathrm{s}}$ an approximation $f_N = \mathrm{S}_N f$ in such a way, that $\|f - f_N\|_{\mathcal{U}_{\mathrm{s}}}$ gets smaller than any given bound $\epsilon > 0$ provided $N \in \mathbb{N}$ is sufficiently large. So $\mathcal{U}_{\mathrm{s}}$ seems to be an appropriate function space for working with Fourier series.

However, in practice, the function f is usually not given at all points $t \in \mathbb{T}$ but only on a discrete subset $\mathcal{Z}_N \subset \mathbb{T}$ of finite cardinality $|\mathcal{Z}_N| = Z_N \in \mathbb{N}$. Therefore, it will generally be impossible to calculate the integral (2) exactly from the known samples $\{f(\tau) : \tau \in \mathcal{Z}_N\}$ of f. Nevertheless, it is possible to find numerical integration methods which determine approximations $c_{N,n}(f)$ of the true Fourier coefficient $c_n(f)$ based on the samples of f on $\mathcal{Z}_N$ such that

$$\lim_{N\to\infty} c_{N,n}(f) = c_n(f) \qquad \text{for all } n \in \mathbb{Z}\,.$$

If the exact Fourier coefficients $c_n(f)$ in (3) are replaced by the approximations $c_{N,n}(f)$, the question arises whether the series

$$\big(\widetilde{\mathrm{S}}_N f\big)(t) = \textstyle\sum_{n=-N}^{N} c_{N,n}(f)\,\mathrm{e}^{\mathrm{i}nt} \tag{4}$$

still converges to f, in the norm of $\mathcal{U}_{\mathrm{s}}$, for every $f \in \mathcal{U}_{\mathrm{s}}$? More formally, we may ask

Question 1. *Is it possible to find a family $\{\mathcal{Z}_N\}_{N\in\mathbb{N}}$ of discrete sampling sets $\mathcal{Z}_N \subset \mathbb{T}$ and a method to determine approximations $c_{N,n}(f)$ of the Fourier coefficients $c_n(f)$ from the samples $\{f(\tau) : \tau \in \mathcal{Z}_N\}$ such that the operators defined in (4) satisfy*

$$\lim_{N\to\infty} \big\|f - \widetilde{\mathrm{S}}_N(f)\big\|_{\mathcal{U}_{\mathrm{s}}} = 0 \qquad \textit{for all } f \in \mathcal{U}_{\mathrm{s}}\ ?$$

Approximating f by the series (4) might even be too specific. So we may ask the more general question.

Question 2. *Is it possible to find a family $\{\mathcal{Z}_N\}_{N\in\mathbb{N}}$ of discrete sampling sets $\mathcal{Z}_N \subset \mathbb{T}$ and a family $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ of approximation operators $\mathrm{A}_N : \mathcal{U}_{\mathrm{s}} \to \mathcal{U}_{\mathrm{s}}$*

$$\mathrm{A}_N \ :\ \{f(\tau)\ :\ \tau \in \mathcal{Z}_N\} \mapsto \widetilde{f}_N$$

such that

$$\lim_{N\to\infty} \big\|f - \widetilde{f}_N\big\|_{\mathcal{U}_{\mathrm{s}}} = \lim_{N\to\infty} \big\|f - \mathrm{A}_N(f)\big\|_{\mathcal{U}_{\mathrm{s}}} = 0 \qquad \textit{for all } f \in \mathcal{U}_{\mathrm{s}}\ ?$$

Clearly, the approximation method (4) in Question 1, is just a special case of the more general setting in Question 2. Already at this point, we want to emphasize that Questions 1 and 2 make no assumption on the linearity of the approximation methods which determine the approximate Fourier coefficients $c_{N,n}(f)$ or on the linearity of the approximation operators A_N, respectively. In both cases, these approximation operations might be non-linear.

This paper is going to show that both questions have a negative answer. So on the space $\mathcal{U}_{\mathrm{s}}$ of all uniformly convergent Fourier series, there exists no method which is able to approximate every $f \in \mathcal{U}_{\mathrm{s}}$ arbitrarily well from discrete samples of f. In particular, it follows that there exists no method to determine an approximate Fourier series (4) which converges for every $f \in \mathcal{U}_{\mathrm{s}}$.

The main restriction on the approximation methods, considered in this paper, is the assumption that only finitely many samples of the given function f can be processed. This is a necessary condition for any algorithm which should be implemented on a digital computer. Otherwise, an infinitely large memory and infinite processing time would be needed to calculate the result. So the statements of this paper will show that there exists no numerical method which is able to determine the Fourier approximation for all functions in $\mathcal{U}_{\mathrm{s}}$. This statement can be made even more descriptive by formulating it in the framework of Turing computable functions [1,30,31], which will be done at the end of this paper. So even although $\mathcal{U}_{\mathrm{s}}$ is, by construction, the space on which the Fourier series converges uniformly, it is a poor space for actually calculating Fourier series approximations of functions from $\mathcal{U}_{\mathrm{s}}$ in the above sense.

The outline of this paper is as follows. Section 2 introduces our notations and gives some basic definitions. In particular, the space $\mathcal{U}_{\mathrm{s}}$ of all uniformly convergent symmetric Fourier series will be introduced. To obtain more general results, Section 2 introduces also the space $\mathcal{U}$ of uniformly convergent *non-symmetric Fourier series* $\big(\mathrm{S}_{N,M} f\big)(t) = \sum_{n=-M}^{N} c_n(f)\,\mathrm{e}^{\mathrm{i}nt}$, where $N, M \in \mathbb{N}_0$ are arbitrary and not necessarily equal, and discusses some of its properties and its relation to $\mathcal{U}_{\mathrm{s}}$. Then Section 3 states our main results. To this end, an axiomatic characterization of general sampling-based approximation methods $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ on $\mathcal{U}_{\mathrm{s}}$ and $\mathcal{U}$ is introduced. Then it is proved that no such approximation method exists on $\mathcal{U}_{\mathrm{s}}$ and $\mathcal{U}$. As a particular case, the sampling-based Fourier series approximation on $\mathcal{U}_{\mathrm{s}}$ and $\mathcal{U}$ is discussed in some detail in Section 4 whereas Section 5 will demonstrate that there exist no computational bases in $\mathcal{U}_{\mathrm{s}}$ and $\mathcal{U}$. Finally, Section 6 shows that there exist no Turing computable approximation methods in $\mathcal{U}_{\mathrm{s}}$ and $\mathcal{U}$. The proofs of our main results are given in Section 8. Before that some auxiliary results, necessary for the poofs in Section 8, are given in Section 7.

2. Notation, basic definitions and properties

The first subsection introduces our notation whereas the second subsection defines the signal spaces $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ on which the approximation operators are investigated in later sections.

2.1. General notation

Throughout this paper, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ stands for the additive quotient group of real numbers modulo 2π. Then $\mathcal{C}(\mathbb{T})$ is the Banach space of continuous function on $\mathbb{T}$ equipped with the maximum norm $\|f\|_\infty = \max_{t\in\mathbb{T}} |f(t)|$. The subset of all *trigonometric polynomial* is denoted by $\mathcal{P}$, i.e. the set of all $f(t) = \sum_{n=-M}^{N} c_n\,\mathrm{e}^{\mathrm{i}nt}$ with non-negative integers $N, M \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and with coefficients $c_n \in \mathbb{C}$. The *degree* of $f \in \mathcal{P}$, denoted by $\deg(f)$, is the maximum of N and M. To simplify notation, we often write

$$e_n(t) = \mathrm{e}^{\mathrm{i}nt}\,, \qquad n \in \mathbb{Z} \tag{5}$$

for the monomials in $\mathcal{P}$. For any $a \in \mathbb{T}$, the *translation operator* $\mathrm{T}_a : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ is defined by

$$\big(\mathrm{T}_a f\big)(t) = f(t-a)\,, \qquad t \in \mathbb{T}\,. \tag{6}$$

It is well known that every $f \in \mathcal{C}(\mathbb{T})$ is uniquely determined by the set $\{c_n(f)\}_{n\in\mathbb{Z}}$ of its *Fourier coefficients* (2). Then the *non-symmetric partial Fourier series* $\mathrm{S}_{N,M} : \mathcal{C}(\mathbb{T}) \to \mathcal{P}$ is defined for arbitrary $N, M \in \mathbb{N}_0$ by

$$\big(\mathrm{S}_{N,M} f\big)(t) = \sum_{n=-M}^{N} c_n(f)\,\mathrm{e}^{\mathrm{i}nt}\,, \qquad t \in \mathbb{T}\,. \tag{7}$$

We simply write S_N for $\mathrm{S}_{N,N}$, thus $\mathrm{S}_N f$ is the usual (symmetric) partial Fourier series (3) of f. Inserting the Fourier coefficients (2) into (7), one obtains the integral representation of $\mathrm{S}_{N,M}$

$$\begin{aligned}\big(\mathrm{S}_{N,M} f\big)(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,\mathcal{D}_{N,M}(t-\tau)\,\mathrm{d}\tau \qquad \text{with} \\ \mathcal{D}_{N,M}(\tau) &= \frac{\mathrm{e}^{\mathrm{i}\left(N+\frac{1}{2}\right)t} - \mathrm{e}^{\mathrm{i}\left(M+\frac{1}{2}\right)t}}{2\mathrm{i}\,\sin(\tau/2)}\ ,\end{aligned} \tag{8}$$

and we notice that $\mathrm{S}_{N,M}$ commutes with the translation operator T_a for every $a \in \mathbb{R}$, i.e.

$$\mathrm{S}_{N,M}\mathrm{T}_a = \mathrm{T}_a\mathrm{S}_{N,M} \qquad \text{for all } N, M \in \mathbb{N}_0\ . \tag{9}$$

We frequently need the notation of an open ball

$$B_\delta(f_0, \mathcal{B}) = \left\{ f \in \mathcal{B} \ :\ \|f - f_0\|_{\mathcal{B}} < \delta \right\}\ . \tag{10}$$

in a Banach space $\mathcal{B}$ with center $f_0 \in \mathcal{B}$ and radius $\delta > 0$. Finally, we recall that a subset $\mathcal{N}$ of a topological space $\mathcal{B}$ is said to be *nowhere dense* in $\mathcal{B}$ if its closure contains no nonempty open set of $\mathcal{B}$. $\mathcal{N}$ is said to be *meager* (of first category) if it is the countable union of nowhere dense sets, and it is said to be *nonmeager* (of second category) if it is not meager. The complement of a meager set in a complete metric space is called a *residual set* and Baire's theorem implies that any residual set is nonmeager and dense. Conversely, any open and dense subset is a residual set.

2.2. *The spaces $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ of uniformly convergent fourier series*

It is well known that the symmetric and non-symmetric Fourier series $\mathrm{S}_N f$ and $\mathrm{S}_{N,M} f$, respectively, converges to f in several Banach spaces of functions on $\mathbb{T}$, e.g. in $L^2(\mathbb{T})$ or in the Wiener algebra $\mathcal{W}$. Nevertheless, the series $\mathrm{S}_{N,M} f$ and $\mathrm{S}_N f$ do usually not converge to $f \in \mathcal{C}(\mathbb{T})$ as $N, M \to \infty$ in the uniform norm of $\mathcal{C}(\mathbb{T})$. For this reason, we introduce $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ as the largest subset of $\mathcal{C}(\mathbb{T})$ for which $\mathrm{S}_{N,M} f$ and $\mathrm{S}_N f$ converge uniformly to f, respectively. The precise definition of these spaces is given next.

Definition 2.1. We write U for the set of all $f \in \mathcal{C}(\mathbb{T})$ for which

$$\|f\|_U := \sup_{N,M\in\mathbb{N}_0} \left\|\mathrm{S}_{N,M} f\right\|_\infty < +\infty\ , \tag{11}$$

and we write U_{s} for the set of all $f \in \mathcal{C}(\mathbb{T})$ for which

$$\|f\|_{U_{\mathrm{s}}} := \sup_{N\in\mathbb{N}_0} \|\mathrm{S}_N f\|_\infty < +\infty\ . \tag{12}$$

Moreover, $\mathcal{U} \subset U$ and $\mathcal{U}_{\mathrm{s}} \subset U_{\mathrm{s}}$ stand for the closed linear span of the polynomials $\mathcal{P}$ with respect to the norm $\|\cdot\|_U$ and $\|\cdot\|_{U_{\mathrm{s}}}$, respectively, i.e.

$$\mathcal{U} := \overline{\mathrm{span}\,\{f \in \mathcal{P}\}}^{\|\cdot\|_U} \qquad \text{and} \qquad \mathcal{U}_{\mathrm{s}} := \overline{\mathrm{span}\,\{f \in \mathcal{P}\}}^{\|\cdot\|_{U_{\mathrm{s}}}}\ . \tag{13}$$

Definition 2.1 introduces two pairs of Banach spaces U, U_{s} and $\mathcal{U}$, $\mathcal{U}_{\mathrm{s}}$. Even though we primarily work with $\mathcal{U}$, $\mathcal{U}_{\mathrm{s}}$, some properties of all four spaces and their relation to each other are shortly discussed. In particular, we emphasize the fundamental difference between U and U_{s} on the one hand and $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ on the other hand, and we give a motivation for introducing of $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$. Especially Corollary 2.5 and Lemma 2.6 will show again that $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ are very well suited for applications in signal processing which rely on Fourier series techniques. The spaces $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ and their properties were already discussed elsewhere [3–5]. Nevertheless, for completeness and to being self-contained, the proofs of the following statements are given in the Appendix.

First, we notice that U and U_{s}, equipped with the norm (11) and (12), respectively, are Banach spaces which are continuously embedded in $\mathcal{C}(\mathbb{T})$.

Theorem 2.2. *The sets U and U_{s}, introduced in Definition 2.1 and equipped with the norm* (11) and (12)*, respectively, have the following properties*

1. *U and U_{s} are continuously embedded in $\mathcal{C}(\mathbb{T})$ with*

$$\|f\|_{\infty} \leq \|f\|_{U_{\mathrm{s}}} \leq \|f\|_{U} \qquad \text{for all } f \in U\ . \tag{14}$$

2. *U and U_{s} are Banach spaces.*

Remark 2.1. The first statement implies obviously $U_{\mathrm{s}} \subset U$ and $\|f\|_{\infty} \leq \|f\|_{U_{\mathrm{s}}}$ for all $f \in U_{\mathrm{s}}$.

The space U is defined so that the partial Fourier series $\mathrm{S}_{N,M} f$ is uniformly bounded for every $f \in U$. Similarly, U_{s} is the set of all $f \in \mathcal{C}(\mathbb{T})$ for which the symmetric Fourier series $\mathrm{S}_N f$ is uniformly bounded. Nevertheless, these properties do not imply that $\mathrm{S}_{N,M} f$ and $\mathrm{S}_N f$ converge to f for every $f \in U$ and $f \in U_{\mathrm{s}}$, respectively. This observation follows from the following lemma.

Lemma 2.3. *There exists an $f \in U_{\mathrm{s}}$ such that*

$$\liminf_{N\to\infty} \|\mathrm{S}_N f\|_{\infty} < \limsup_{N\to\infty} \|\mathrm{S}_N f\|_{\infty} < +\infty\ ,$$

and there exists an $f \in U$ such that

$$\liminf_{N,M\to\infty} \left\|\mathrm{S}_{N,M} f\right\|_{\infty} < \limsup_{N,M\to\infty} \left\|\mathrm{S}_{N,M} f\right\|_{\infty} < +\infty\ .$$

The statement of Lemma 2.3, that $\mathrm{S}_{N,M} f$ and $\mathrm{S}_N f$ do not converge uniformly for all $f \in U$ and $f \in U_{\mathrm{s}}$, respectively, is the reason for introducing $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$. In these spaces, $\mathrm{S}_{N,M} f$ and $\mathrm{S}_N f$ converge in norm and uniformly for every $f \in \mathcal{U}$ and every $f \in \mathcal{U}_{\mathrm{s}}$, respectively.

Theorem 2.4. *The spaces $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$, given in Definition 2.1, are Banach spaces and it holds*

$$\text{(1)} \quad \lim_{N\to\infty} \|f - \mathrm{S}_N f\|_{\mathcal{U}_{\mathrm{s}}} = 0 \qquad \text{for all } f \in \mathcal{U}_{\mathrm{s}} \tag{15}$$

$$\text{(2)} \quad \lim_{N,M\to\infty} \left\|f - \mathrm{S}_{N,M} f\right\|_{\mathcal{U}} = 0 \qquad \text{for all } f \in \mathcal{U}$$

$$\text{(3)} \quad \mathcal{U}_{\mathrm{s}} = \left\{ f \in U_{\mathrm{s}} \ :\ \lim_{N\to\infty} \|f - \mathrm{S}_N f\|_{\infty} = 0 \right\} \subsetneq U_{\mathrm{s}}$$

$$\text{(4)} \quad \mathcal{U} = \left\{ f \in U \ :\ \lim_{M,N\to\infty} \left\|f - \mathrm{S}_{M,N} f\right\|_{\infty} = 0 \right\} \subsetneq U\ .$$

So $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ and are closed subspaces of U and U_{s} containing all $f \in \mathcal{C}(\mathbb{T})$ with a uniformly convergent Fourier series. In this respect $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ are the largest spaces of continuous functions for which the non-symmetric and the symmetric Fourier series, respectively, converges uniformly.

The first statement of Theorem 2.4 is based on the fact that $\|S_N\|_{\mathcal{U}_s\to\mathcal{U}_s} = 1$ for all $N \in \mathbb{N}$ (cf. the proof of Theorem 2.4 in Appendix A.2). This uniform boundedness of the operator norms implies also that the monomials (5) form a basis for $\mathcal{U}_s$, provided they are ordered appropriately.

Corollary 2.5. *The sequence* $\widetilde{\boldsymbol{e}} = \{e_0, e_1, e_{-1}, e_2, e_{-2}, e_3, \ldots\}$ *forms a Schauder basis for* $\mathcal{U}_s$.

It is obvious from Definition 2.1 that $\mathcal{U} \subset \mathcal{U}_s$ and since the polynomials are dense in $\mathcal{U}$, the space $\mathcal{U}_s$ may be considered as the closure of $\mathcal{U}$ with respect to U_s-norm, i.e. $\mathcal{U}_s = \overline{\mathcal{U}}^{\|\cdot\|_{U_s}}$. On the other side, there exist functions $f_* \in \mathcal{U}_s$ which do not belong to $\mathcal{U}$, i.e. for which

$$\limsup_{N,M\to\infty} \left\|S_{N,M} f_*\right\|_\infty = +\infty\,. \tag{16}$$

To see this, we define for any $K \in \mathbb{N}$ the trigonometric polynomial

$$f_K(t) = C_0 \sum_{k=1}^{K} \frac{\sin(kt)}{k} = \frac{C_0}{2\mathrm{i}} \sum_{\substack{k=-K\\k\neq 0}}^{K} \frac{1}{k}\,\mathrm{e}^{\mathrm{i}kt}\,, \qquad t \in \mathbb{T}\,,$$

wherein $C_0 > 0$ is a constant, independent of K, which can be chosen such that $\|f_K\|_\infty \leq 1$ for all $K \in \mathbb{N}$ (cf. [35, Chapter II.9]). Then it is clear from the definition that $S_N f_K = f_N$ for all $N \leq K$ and that $S_N f_K = f_K$ for all $N > K$. So $\|f_K\|_{U_s} = \sup_{N\in\mathbb{N}} \|S_N f_K\|_\infty \leq 1$ showing that $f_K \in \mathcal{U}_s$ for all $K \in \mathbb{N}$. Nevertheless, for the non-symmetric Fourier series of f_K, one obtains

$$\sup_{N,M\in\mathbb{N}} \left\|S_{N,M} f_K\right\|_\infty \geq \left\|S_{K,0} f_K\right\|_\infty \geq \left|\left(S_{K,0} f_K\right)(0)\right| = \tfrac{C_0}{2} \textstyle\sum_{k=1}^{K} \tfrac{1}{k} \geq \tfrac{C_0}{2} \log(K+1)$$

showing that the norms of the operators $S_{N,M} : \mathcal{U}_s \to \mathcal{C}(\mathbb{T})$

$$\left\|S_{N,M}\right\|_{\mathcal{U}_s\to\mathcal{C}(\mathbb{T})} = \sup_{f\in\mathcal{U}_s,\ \|f\|_{U_s}\leq 1} \left\|S_{N,M} f\right\|_\infty \geq \left\|S_{N,M} f_K\right\|_\infty \geq \tfrac{C_0}{2} \textstyle\sum_{k=1}^{K} \tfrac{1}{k}$$

are not uniformly bounded. Then the uniform boundedness principle implies that there exists an $f_* \in \mathcal{U}_s$ for which (16) holds.

Let $\mathcal{S}$ be a subspace of $\mathcal{C}(\mathbb{T})$. We say that $\mathcal{S}$ is *shift-invariant* if $f \in \mathcal{S}$ implies that $T_a f = f(\cdot - a) \in \mathcal{S}$ for every $a \in \mathbb{T}$. Such spaces play an important role in practical application because there the question whether f belongs to a certain signal space $\mathcal{S}$ should often not depend on a shift of the signal. If in such a shift-invariant subspace $\mathcal{S}$, the Fourier series converges at least at one point $t_0 \in \mathbb{T}$, it converges on all points in $\mathbb{T}$ and $\mathcal{S}$ is continuously embedded in $\mathcal{U}_s$.

Lemma 2.6. *Let* $\mathcal{S} \subset \mathcal{C}(\mathbb{T})$ *be a shift-invariant Banach space of continuous functions on* $\mathbb{T}$ *such that* $\lim_{N\to\infty} (S_N f)(t_0) = f(t_0)$ *for some* $t_0 \in \mathbb{T}$ *and for all* $f \in \mathcal{S}$. *Then there exists a constant* $C(\mathcal{S})$ *such that* $\|f\|_{\mathcal{U}_s} \leq C(\mathcal{S})\,\|f\|_{\mathcal{S}}$ *for all* $f \in \mathcal{S}$.

Remark 2.2. It is easy to see that a similar statement holds for $\mathcal{U}$ and shift-invariant subspaces $\mathcal{S}$ in which $\lim_{N,M\to\infty} \left(S_{N,M} f\right)(t_0) = f(t_0)$ for some $t_0 \in \mathbb{T}$ and all $f \in \mathcal{S}$.

Corollary 2.5 and Lemma 2.6 show again that $\mathcal{U}_s$ and $\mathcal{U}$ are very natural spaces for typical applications in signal processing which are often based on Fourier series techniques and in which the monomials (5) are used as a natural basis.

3. General approximations in $\mathcal{U}$ and $\mathcal{U}_s$

We are going to investigate Question 2 from the introduction, namely the problem of approximating functions f in $\mathcal{U}$ and $\mathcal{U}_s$ based on finitely many samples of f. To this end, we consider sequences $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ of approximation operators on $\mathcal{U}$ or $\mathcal{U}_s$ which should have the property that $A_N(f)$ converges to f in the norm of $\mathcal{U}$ and $\mathcal{U}_s$ for all $f \in \mathcal{U}$ and $f \in \mathcal{U}_s$, respectively. One basic requirement on these operators is the assumption that they are lower semicontinuous.

Definition 3.1 (*Lower Semicontinuous Operators*)**.** Let $A : \mathcal{B}_1 \to \mathcal{B}_2$ be a mapping from a Banach space $\mathcal{B}_1$ into a Banach space $\mathcal{B}_2$. We say that A is lower semicontinuous if for every $\lambda \geq 0$ the set $\left\{f \in \mathcal{B}_1 \ : \ \|A(f)\|_{\mathcal{B}_2} \leq \lambda\right\}$ is closed.

Apart from this lower semicontinuous property, we require that the sequences $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ of approximation operators satisfy two very simple and natural properties, namely

(A) Each A_N should be concentrated on a finite sampling set, so that the calculation of $(A_N f)(t)$ can be implemented on a digital computer.
(B) $A_N(f)$ should converge to f at least for all f from a dense subset of the function space $\mathcal{U}$ or $\mathcal{U}_s$. This is a very basic necessary requirement for any approximation method $\boldsymbol{A}$ which is expected to converge for all f in $\mathcal{U}$ or $\mathcal{U}_s$.

All of our requirements on the sequence $\boldsymbol{A}$ of approximation operators are formalized by the following definition.

Definition 3.2 (*Sampling-based Approximation Method*)**.** Let $\mathcal{B}$ be a Banach space of continuous functions on $\mathbb{T}$ and let $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ be a sequence of lower semicontinuous operators $A_N : \mathcal{B} \to \mathcal{B}$. We say that $\boldsymbol{A}$ is a *sampling-based approximation method for* $\mathcal{B}$, if its satisfies the following two properties:

(A) To every $N \in \mathbb{N}$ there exists a finite set $\mathcal{Z}_N \subset \mathbb{T}$ of cardinality $|\mathcal{Z}_N| = Z_N \in \mathbb{N}$ such that for arbitrary $f_1, f_2 \in \mathcal{B}$

$$f_1(\tau) = f_2(\tau) \quad \text{for all } \tau \in \mathcal{Z}_N$$
$$\text{implies} \quad [A_N(f_1)](t) = [A_N(f_2)](t) \quad \text{for all } t \in \mathbb{T}\,.$$

(B) There exists a dense subset $\mathcal{M} \subset \mathcal{B}$ such that

$$\lim_{N\to\infty} \|f - A_N(f)\|_{\mathcal{B}} = 0 \qquad \text{for all } f \in \mathcal{M}\,.$$

Remark 3.1. Definition 3.2 is formulated for general Banach spaces $\mathcal{B}$ of continuous functions on $\mathbb{T}$. Nevertheless, this paper considers solely the two particular cases $\mathcal{B} = \mathcal{U}$ and $\mathcal{B} = \mathcal{U}_s$.

Remark 3.2. An operator $A_N : \mathcal{B} \to \mathcal{B}$ which satisfies (A) is said to be *concentrated* $\mathcal{Z}_N$.

Remark 3.3. It is emphasized that we do *not* require that the operators $A_N : \mathcal{B} \to \mathcal{B}$ are linear. Also the dense subset $\mathcal{M}$, appearing in Property (B), is *not* assumed to have any *linear structure*.

The two properties of Definition 3.2 imply no serious restriction on the approximation methods $\boldsymbol{A}$. They only require that the calculation of $A_N(f)$ is based on finitely many samples of f. This is a necessary condition for implementing such a method on a digital computer. Secondly, they require that the algorithm converges at least for a dense subset $\mathcal{M}$ of $\mathcal{B}$. This is a very weak necessary condition for $\boldsymbol{A}$ to be able to approximate every function in $\mathcal{B}$.

Despite these weak assumptions on the approximation methods, one obtains that on $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ there exists no approximation method $\boldsymbol{A}$ satisfying the requirements of Definition 3.2 and which converges for all f in $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$.

Theorem 3.3. *Let $\mathcal{B}$ be either $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$ and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be an arbitrary sampling-based approximation method for $\mathcal{B}$ according to Definition 3.2. Then*

$$\mathcal{R}(\boldsymbol{A}) = \left\{ f \in \mathcal{B} \ : \ \limsup_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathcal{B}} = +\infty \right\}$$

is a residual set in $\mathcal{B}$.

So to any approximation method $\boldsymbol{A}$ satisfying the conditions of Definition 3.2 there exists a dense and nonmeager subset $\mathcal{R}(\boldsymbol{A}) \subset \mathcal{B}$ such that

$$\limsup_{N\to\infty} \|f - \mathrm{A}_N(f)\|_{\mathcal{B}} = +\infty \qquad \text{for all } f \in \mathcal{R}(\boldsymbol{A}) \,.$$

To prove Theorem 3.3 some preliminary results are necessary which will be presented and proved in Section 7. The proof of Theorem 3.3 is then given in Section 8. Next, the following three sections will discuss some consequences and applications of Theorem 3.3.

4. Application 1: Sampling-based fourier approximations in $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$

Section 3 considered general sampling-based methods to approximate functions in $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ from finitely many samples of the given function. To illustrate this result, this section studies a particular example of such approximation methods, namely the approximation of f by its truncated symmetric Fourier series $\mathrm{S}_N f$. Thus, we consider Question 1 from the introduction.

As in the previous section, $\mathcal{B}$ stands always for either $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$. According to Theorem 2.4, the symmetric Fourier series $\mathrm{S}_N f$ converges to f for every $f \in \mathcal{B}$, uniformly on $\mathbb{T}$ and in the norm of $\mathcal{B}$. So it seems to be natural to approximate $f \in \mathcal{B}$ by the partial sum $\mathrm{S}_N f$. But to calculate (3), the exact Fourier coefficients $\{c_n(f)\}_{n=-N}^{N}$ are needed. However, since these coefficients are given by an integral (2) over f, it is clear that $c_n(f)$ can generally only be determined exactly if $f(t)$ is known at almost all points $t \in \mathbb{T}$. Nevertheless, on a digital computer only finitely many values of $f(t)$ can be processed. So in practice, one will replace the exact coefficients $c_n(f)$ by some "good approximation" $c_{N,n}(f)$ of $c_n(f)$ obtained via a numerical integration in (2), based on finitely many known samples $\{f(\tau_n)\}_{n=1}^{Z(N)}$ of f. So instead of (3), one determines for $N = 0, 1, 2, \ldots$ the approximations

$$\widetilde{f}_N(t) = \big[\mathrm{E}_N(f)\big](t) = \textstyle\sum_{n=-N}^{N} c_{N,n}(f)\, \mathrm{e}^{\mathrm{i}nt} \,, \qquad t \in \mathbb{T} \,, \tag{17}$$

with certain approximations $c_{N,n}(f)$ of the exact Fourier coefficients (2). Then, we say that the sequence $\boldsymbol{E} = \{\mathrm{E}_N\}_{N\in\mathbb{N}}$ of operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ is an *effective approximation method* for $\mathcal{B}$ if

$$\lim_{N\to\infty} \big\| f - \widetilde{f}_N \big\|_{\mathcal{B}} = \lim_{N\to\infty} \big\| f - \mathrm{E}_N(f) \big\|_{\mathcal{B}} = 0 \qquad \text{for all } f \in \mathcal{B} \,.$$

We ask whether it is possible to find an effective approximation method for $\mathcal{B} = \mathcal{U}$ or $\mathcal{B} = \mathcal{U}_{\mathrm{s}}$ and how we have to calculate the approximate Fourier coefficients $c_{N,n}(f)$ such that the corresponding sequence $\boldsymbol{E}$ becomes effective. To formalize our approach, we characterize again the approximation methods by two simple properties. In the present situation our approximation methods E_N have already slightly more structure than in the general case, discussed in Section 3.

Definition 4.1 (*Sampling-based Fourier Approximation*)**.** Let $\mathcal{B}$ stand for $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$ and let $\boldsymbol{E} = \{\mathrm{E}_N\}_{N\in\mathbb{N}}$ be a sequence of operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ of the form (17) with continuous functionals $c_{N,n} : \mathcal{B} \to \mathbb{C}$. We say that $\boldsymbol{E}$ is a *sampling-based Fourier approximation*, if it satisfies the following two properties:

(a) To every pair $(N, n) \in \mathbb{N} \times [-N, \dots, -1, 0, 1 \dots, N]$ there exists a finite set $\mathcal{Z}_{N,n} \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathcal{B}$

$$f_1(\tau) = f_2(\tau) \quad \text{for all } \tau \in \mathcal{Z}_{N,n} \qquad \text{implies} \qquad c_{N,n}(f_1) = c_{N,n}(f_2)\,.$$

(b) The functionals $c_{N,n} : \mathcal{B} \to \mathbb{C}$ satisfy for every $n \in \mathbb{Z}$

$$\lim_{N\to\infty} c_{N,n}(f) = c_n(f) \qquad \text{for all } f \in \mathcal{B}\,.$$

Property (a) requires that the approximate Fourier coefficients $c_{N,n}(f)$ are uniquely determined by the values of f on a finite sampling set $\mathcal{Z}_{N,n}$. If two functions f_1 and f_2 coincide on this sampling set then the corresponding approximate Fourier coefficients $c_{N,n}(f_1)$ and $c_{N,n}(f_2)$ have to be equal. Property (b) requires that for every $n \in \mathbb{Z}$ the number $c_{N,n}(f)$ is a good approximation of the true Fourier coefficient $c_n(f)$ in the sense that $c_{N,n}(f)$ converges to $c_n(f)$ as N goes to infinity. The intuition behind this assumption is that the cardinality $\left|\mathcal{Z}_{N,n}\right|$ of the sampling sets increases as N increases. So to satisfy Property (b), one has to choose a proper numerical integration method for the determination of $c_{N,n}(f)$ which converges to $c_n(f)$ if the number of sampling points goes to infinity. Such a method is easy to find because the integrand in (2) is a continuous function.

If $\boldsymbol{E} = \{\mathrm{E}_N\}_{N\in\mathbb{N}}$ is a sampling-based Fourier approximation with Properties (a) and (b) then the calculation of $\mathrm{E}_N(f)$ is based on the values of f on the finite sampling set $\mathcal{Z}_N = \bigcup_{n=-N}^{N} \mathcal{Z}_{N,n}$. In many concrete situations (cf. Example 1 below), one will choose the sampling sets $\mathcal{Z}_{N,n}$ to be equal for all $n = 0, \pm 1, \pm 2, \dots, \pm N$. Nevertheless, our approach allows for the general situation where all sampling sets might be different. We emphasis also that the sampling sets $\left\{\mathcal{Z}_{N,n}\right\}_{n=-N}^{N}$ can be completely different for different N, i.e. we do not require that $\mathcal{Z}_{N,n} \subset \mathcal{Z}_{N+k,n}$ for any $k \in \mathbb{N}$. To illustrate the approximation sequences $\{\mathrm{E}_N\}_{N\in\mathbb{N}}$ characterized by Definition 4.1, we give a concrete and simple example.

Example 1. For each $N \in \mathbb{N}$, choose the sampling sets

$$\mathcal{Z}_{N,n} = \left\{t_{N,k} = (k-N)\frac{\pi}{N} \;:\; k = 0, 1, 2, \dots, 2N-1\right\}$$
$$\text{for every } n = 0, \pm 1, \pm 2, \dots, \pm N\,.$$

Then we approximate the integral in (2) by its Riemann sum with nodes $\mathcal{Z}_{N,n}$

$$c_{N,n}(f) = \frac{\pi}{N}\frac{1}{2\pi}\sum_{k=0}^{2N-1} f(t_{N,k})\,\overline{e_n(t_{N,k})} = \frac{1}{2N}\sum_{k=0}^{2N-1} f([k-N]\tfrac{\pi}{N})\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{N}n(k-N)}\,,$$

and consider the operators $\mathrm{E}_N(f) = \sum_{n=-N}^{N} c_{N,n}(f)\,\mathrm{e}^{\mathrm{i}nt}$. It is clear that the so defined functionals $c_{N,n} : \mathcal{B} \to \mathbb{C}$ have Property (a) and that they are continuous. Moreover, by well known properties of the Riemann integral, they also satisfy Property (b).

Even though the functionals $c_{N,n} : \mathcal{B} \to \mathbb{C}$ in the previous example were linear, we emphasis that the two properties of Definition 4.1 do not require that any $c_{N,n}$ is linear. So these functionals and in turn the operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ can be *non-linear*, in general.

Similarly as in Section 3, one can show now that every sampling-based Fourier approximation satisfying Properties (a) and (b) of Definition 4.1 diverges on the spaces $\mathcal{U}$ and $\mathcal{U}_s$.

Theorem 4.2. *Let $\mathcal{B}$ stand for $\mathcal{U}$ or $\mathcal{U}_s$ and let $\boldsymbol{E} = \{\mathrm{E}_N\}_{N\in\mathbb{N}}$ be a sampling-based Fourier approximation with operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ of the form (17) and having Properties (a) and (b) of Definition 4.1. Then*

$$\mathcal{R}(\boldsymbol{E}) = \left\{ f \in \mathcal{B} \;:\; \limsup_{N\to\infty} \|\mathrm{E}_N(f)\|_{\mathcal{B}} = +\infty \right\}$$

is a residual set in $\mathcal{B}$.

This theorem is an immediate consequence of Theorem 3.3 since every sampling-based basis expansion $\boldsymbol{E}$, as defined in Definition 4.1, satisfies also the conditions of Definition 3.2. Nevertheless, a detailed verification of this statement is given in Section 8.

5. Application 2: Computational bases in $\mathcal{U}$ or $\mathcal{U}_s$

As before, $\mathcal{B}$ stands always for either $\mathcal{U}$ or $\mathcal{U}_s$. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a basis for $\mathcal{B}$. Then to every $f \in \mathcal{B}$ there exists a unique sequence $\{a_n(f)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $f = \sum_{n\in\mathbb{N}} a_n(f)\,\varphi_n$ and where the sum converges in the norm of $\mathcal{B}$. Similarly as in the previous section, we may try to approximate f by the partial sum

$$\Phi_N f = \textstyle\sum_{n=1}^{N} a_n(f)\,\varphi_n\ . \tag{18}$$

Since $\{\varphi_n\}_{n=1}^{\infty}$ is a basis, $\Phi_N f$ converges to f as $N \to \infty$. Nevertheless, working on a digital computer, it might not be possible to determine the coefficients $a_n(f)$ in (18) exactly from only finitely many samples of f. Then, similar as in Section 4, one has to replace the exact coefficients $a_n(f)$ by certain approximations $a_{N,n}(f)$ which yields approximation operators of the form

$$E_N(f) = \textstyle\sum_{n=1}^{N} a_{N,n}(f)\,\varphi_n\ , \qquad N \in \mathbb{N}\ .$$

By the same arguments as in Section 4, it follows that there exists a residual set $\mathcal{R} \subset \mathcal{B}$ such that

$$\limsup_{N\to\infty} \|\mathrm{E}_N(f)\|_{\mathcal{B}} = +\infty \qquad \text{for all } f \in \mathcal{R}\ ,$$

irrespectively of how the approximations $a_{N,n}(f)$ of the true coefficients $a_n(f)$ are chosen.

However, it is known that there exist Banach spaces which posses a basis $\{\varphi_n\}_{n\in\mathbb{N}}$ such that each of the corresponding coefficient functionals $\{a_n(f)\}$ is uniquely determined by only finitely many samples of f. Such bases are said to be *computable*.

Definition 5.1 (*Computational Basis*)**.** Let $\mathcal{B}$ be a separable Banach space of continuous functions on $\mathbb{T}$. A basis $\boldsymbol{\varphi} = \{\varphi_n\}_{n\in\mathbb{N}}$ of $\mathcal{B}$ is said to be *computational* if the corresponding coefficient functionals $\{a_n(f)\}_{n\in\mathbb{N}}$ of $\boldsymbol{\varphi}$ have the following property: To every $n \in \mathbb{Z}$ there exist an $K = K(n) \in \mathbb{N}$ and distinct numbers $\tau_{1,n}, \ldots, \tau_{K,n} \in \mathbb{T}$ such that the value $a_n(f)$ does only depend on the values $f(\tau_{k,n})$, $1 \le k \le K(n)$ for every $f \in \mathcal{B}$.

Remark 5.1. In other words, $\boldsymbol{\varphi}$ is a computational basis if and only if for all functions $f, g \in \mathcal{B}$ with $f_1(\tau_{k,n}) = f_2(\tau_{k,n})$ for all $k = 1, \ldots, K(n)$ one has $a_n(f_1) = a_n(f_2)$ for all $n = 1, 2, 3, \ldots$.

If $\boldsymbol{\varphi}$ is a computational basis then to every $n \in \mathbb{Z}$ there exists a finite sampling set $\mathcal{Z}_n = \{\tau_{1,n}, \ldots, \tau_{K(n),n}\} \subset \mathbb{T}$ which allows to calculate $a_n(f)$ exactly based on the values $\{f(\tau)\}_{\tau\in\mathcal{Z}_n}$. So a computational basis allows to determine the partial sum operators $\Phi_N : \mathcal{B} \to \mathcal{B}$ in (18) exactly for all $f \in \mathcal{B}$ from only finitely many samples of f. For a large number of Banach spaces of continuous functions on $\mathbb{T}$, computational bases are known [27]. One example is the *spline basis* for $C(\mathbb{T})$. Nevertheless, Theorem 3.3 implies the following statement.

Corollary 5.2. *The two spaces $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$ do not posses a computational basis.*

This corollary follows from the observation that the partial sum operators (18) associated with any computational basis $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies the properties of Definition 3.2. Then Corollary 5.2 follows form Theorem 3.3 in a similar way as Theorem 4.2 follows from Theorem 3.3.

6. Application 3: Approximations by turing computable functions

Theorem 3.3 showed that there exists no sampling-based method which is able to determine an approximation for all f in $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$. This section is going to translate this statement into the language of *Turing computability*. Generally speaking, a method or a function is said to be *computable* if there exists an algorithm on an abstract machine which can emulate the method or function. Thus, given a (countable) sequence of input symbols, the machine will be able to return the corresponding output symbols as prescribed by the given method or function. The concrete notion of computability is characterized by the model of the abstract machine on which the corresponding algorithm is assumed to be executed. In this paper, we consider only the model of a so called *Turing machine*. This particular computational model describes a natural limit on algorithms which can be implemented on digital computers [1,30,31].

Before we can reformulate our main result in terms of Turing computable functions, we have to relate the approximation operators A_N to certain functions which can then be analyzed using the notion of Turing computability. These functions are introduced in the following definition and related to our approximation sequence $\boldsymbol{A}$ by the subsequent lemma.

Definition 6.1. Let $Z \in \mathbb{N}$ be arbitrary and let $\mathcal{B}$ be a separable Banach space of continuous functions on $\mathbb{T}$. Then we write $\mathcal{T}(Z, \mathcal{B})$ for the set of all functions $F : \mathbb{R}^Z \times \mathbb{T} \to \mathbb{R}$ satisfying

1. $F(\mathbf{x}; \cdot) \in \mathcal{B}$ for every $\mathbf{x} \in \mathbb{R}^Z$.
2. the mapping $\mathrm{F} : \mathbb{R}^Z \to \mathcal{B}$, given by $\mathrm{F} : \mathbf{x} \mapsto F(\mathbf{x}; \cdot)$, is lower semicontinuous.

Remark 6.1. As before and as in the remainder of this section, the Banach space $\mathcal{B}$, appearing in this definition, will always be either $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$.

Now we consider again approximation methods $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ described by Definition 3.2. The following lemma shows that every $\boldsymbol{A}$, satisfying the first property of Definition 3.2, can be associated with a sequence $\{F_N\}_{N\in\mathbb{N}}$ of functions $F_N = \mathcal{T}(Z_N, \mathcal{B})$, wherein $Z_N = |\mathcal{Z}_N|$ denotes the cardinality of the sampling set $\mathcal{Z}_N$ on which A_N is concentrated.

Lemma 6.2. *Let $\mathcal{B}$ stand for either $\mathcal{U}$ or $\mathcal{U}_s$. A sequence $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ of lower semicontinuous mappings on $\mathcal{B}$ has Property (A) of Definition 3.2 if and only if to every $N \in \mathbb{N}$ there exist a finite set $\mathcal{Z}_N = \{\tau_1, \tau_2, \ldots, \tau_{Z_N}\} \subset \mathbb{T}$ and a function $F_N \in \mathcal{T}(Z_N, \mathcal{B})$ such that for every $f \in \mathcal{B}$*

$$\big[\mathrm{A}_N(f)\big](t) = F_N\big(f(\tau_1), f(\tau_2), \ldots, f(\tau_{Z_N})\ ;\ t\big) \qquad \text{for all } t \in \mathbb{T}\ .$$

Proof. Let $N \in \mathbb{N}$ be arbitrary and let $\mathcal{Z}_N = \{\tau_1, \tau_2, \ldots, \tau_{Z_N}\} \subset \mathbb{T}$ be an arbitrary sampling set of cardinality Z_N. Assume $F_N \in \mathcal{T}(Z_N, \mathcal{B})$ is a function according to Definition 6.1. Therewith, we define the mapping $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$ by

$$[\mathrm{A}_N(f)]\,(t) = F_N\big(\ f(\tau_1), f(\tau_2), \ldots, f(\tau_{Z_N})\ ;\ t\ \big)\ , \qquad t \in \mathbb{T}\ .$$

It is obvious that A_N satisfies Property (A) of Definition 3.2. Moreover, since $\mathrm{F}_N\ :\ \mathbf{x} \mapsto F_N(\mathbf{x};\cdot)$ is lower semicontinuous, it follows, using (14), that also the corresponding $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$ is lower semicontinuous.

Conversely, assume $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ is a sequence of lower semicontinuous mappings with Property (A), and let $\mathcal{Z}_N = \{\tau_1, \tau_2, \ldots, \tau_{Z_N}\} \subset \mathbb{T}$ be the associated sampling sets. Fix an arbitrary $N \in \mathbb{N}$. Because $\boldsymbol{A}$ satisfies Property (A), i.e. because A_N is concentrated on the sampling set $\mathcal{Z}_N$, the function $\mathrm{A}_N f \in \mathcal{B}$ depends only on the values $\{f(\tau_n)\}_{n=1}^{Z_N} \in \mathbb{R}^{Z_N}$. Let $\mathbf{x} = (x_1, \ldots, x_{Z_N})^{\mathrm{T}} \in \mathbb{R}^{Z_N}$ be arbitrary. Then there always exists an $g_{\mathbf{x}} \in \mathcal{C}(\mathbb{T})$ such that

$$g_{\mathbf{x}}(\tau_n) = x_n \qquad \text{for all } n = 1, 2, \ldots, Z_N\ ,$$

and such that $\|g_{\mathbf{x}}\|_\infty = \|\mathbf{x}\|_\infty = \max_{n\in[1,\ldots,Z_N]} |x_n|$. Applying Lemma 7.1, one can always find an $f_{\mathbf{x}} \in \mathcal{B}$ with $\|f_{\mathbf{x}}\|_{\mathcal{B}} \le 2\,\|g_{\mathbf{x}}\|_\infty = 2\,\|\mathbf{x}\|_\infty$ and such that

$$f_N(\tau_n) = g_N(\tau_n) = x_n \qquad \text{for all } n = 1, 2, \ldots, Z_N\ .$$

Finally, we define the function $F_N : \mathbb{R}^{Z_N} \times \mathbb{T}$ by $F_N(\mathbf{x}; t) := [\mathrm{A}_N(f_{\mathbf{x}})]\,(t)$ for $t \in \mathbb{T}$. Since $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$, it is clear that $F_N(\mathbf{x};\cdot) \in \mathcal{B}$ for every arbitrary $\mathbf{x} \in \mathbb{R}^{Z_N}$. Moreover, since A_N is assumed to be lower semicontinuous, it follows easily that the corresponding mapping $\mathrm{F}_N : \mathbb{R}^{Z_N} \to \mathcal{B}$, given by $\mathrm{F}_N\ :\ \mathbf{x} \mapsto F_N(\mathbf{x};\cdot) = \mathrm{A}_N(f_{\mathbf{x}})$ is lower semicontinuous, showing that $F_N \in \mathcal{T}(Z_N, \mathcal{B})$. ■

So every approximation operator A_N is associated with a function F_N in the class $\mathcal{T}(Z_N, \mathcal{B})$. This allows us to apply techniques from computability theory to analyze the computability of approximation methods having the Properties (A) and (B) of Definition 3.2. This is necessary, because to compute the values of $[\mathrm{A}_N(f)]\,(t) = F_N(f(\tau_1), \ldots, f(\tau_{Z_N}); t)$, $t \in \mathbb{T}$ on a computer, the function F_N has to satisfy some reasonable computability conditions. Here, we only discuss so called Turing computable functions, and we shortly review the necessary definitions.

Definition 6.3 (*Computable Vectors*). Let $\mathbf{x} \in \mathbb{R}^M$ be an M-dimensional real vector.

1. A sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}^M$ of rational vectors is said to be a *rapidly converging Cauchy name* of $\mathbf{x}$, if $\mathbf{x}_n$ converges rapidly to $\mathbf{x}$ in the following sense: For all $n, m \in \mathbb{N}$ with $m > n$, one has $\|\mathbf{x}_m - \mathbf{x}_n\|_{\mathbb{R}^M} \le 2^{-n}$.
2. A vector $\mathbf{x} \in \mathbb{R}^M$ is said to be *computable*, if there exists a rapidly converging Cauchy name of $\mathbf{x}$.

Definition 6.4 (*Computable Function*)**.** We call $F : \mathbb{R}^M \to \mathbb{R}$ a *computable function* if there is an algorithm that transforms each rapidly converging Cauchy name of an arbitrary $\mathbf{x} \in \mathbb{R}^M$ into a rapidly converging Cauchy name of $F(\mathbf{x})$.

Remark 6.2. Following usual conventions, the notion "algorithm" in Definition 6.4 means that there exists a Turing machine which determines the corresponding mapping.

Remark 6.3. In his early works Turing introduced the notion of computable numbers in $\mathbb{R}$ [30,31]. These numbers have to be approximable by rational numbers using a Turing machine. Since the number of Turing machines is countable also the number of computable numbers is countable. Later, Turing introduced machines with an oracle [32]. By considering the inputs as given by an oracle from outside and not being itself calculated on a Turing machine, these machines can handle functions with arbitrary real inputs (see, e.g., [1] for further discussions). This paper considers only approximation methods computable on Turing machines with oracle. So here "Turing computable" always means "computable on a Turing machine with oracle".

Definition 6.5 (*Turing Computable Approximation Method*)**.** Let $\mathcal{B}$ stand for $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$. A sequence $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ of mappings $\mathcal{B} \to \mathcal{B}$ is said to be a *Turing computable approximation method* on $\mathcal{B}$ if to every $N \in \mathbb{N}$ there exist an $Z_N \in \mathbb{N}$, a finite sampling set $\mathcal{Z}_N = \{\tau_{N,1}, \tau_{N,2}, \dots, \tau_{N,Z_N}\} \subset \mathbb{T}$, and a computable function $F_N \in \mathcal{T}(Z_N, \mathcal{B})$ so that for every $f \in \mathcal{B}$

$$\big[A_N(f)\big](t) = F_N\big(f(t_{N,1}), f(t_{N,2}), \dots, f(t_{N,Z_N})\ ;\ t\big) \qquad \text{for all } t \in \mathbb{T}\,. \tag{19}$$

Remark 6.4. Definition 6.5 is basically a reformulation of Property (A) of Definition 3.2 in the context of Turing computability where the lower-semicontinuity of the operators A_N follows from the fact that each computable function F_N is continuous [33].

After these preparations, we can recast Theorem 3.3 in the framework of Turing computability.

Theorem 6.6. *Let $\mathcal{B}$ stand for $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$, and let $\boldsymbol{A} = \{A_N\}_{N\in\mathbb{N}}$ be a Turing computable approximation method on $\mathcal{B}$ such that there exists a dense subset $\mathcal{M} \subset \mathcal{B}$ so that*

$$\lim_{N\to\infty} \big\| f - A_N(f)\big\|_{\mathcal{B}} = 0 \qquad \textit{for all } f \in \mathcal{M}\,. \tag{20}$$

Then

$$\Big\{ f \in \mathcal{B}\ :\ \limsup_{N\to\infty} \|A_N(f)\|_{\mathcal{B}} = +\infty \Big\} \tag{21}$$

is a residual set in $\mathcal{B}$.

Remark 6.5. Condition (20) is equivalent to Property (B) in Definition 3.2 and requires that the method $\boldsymbol{A}$ converges to the desired function f at least for all f from a dense subset of $\mathcal{B}$.

Proof. The sequence $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ is a Turing computable approximation method on $\mathcal{B}$. So to every $N \in \mathbb{N}$ there exists a sampling set $\mathcal{Z}_N \subset \mathbb{T}$ of cardinality $Z_N = |\mathcal{Z}_N|$ and an $F_N \in \mathcal{T}(Z_N, \mathcal{B})$ such that (19) holds for every $f \in \mathcal{B}$. Moreover, every Turing computable function is continuous [33] and so Lemma 6.2 implies that $\boldsymbol{A}$ has Property (A) of Definition 3.2. Furthermore, Requirement (20) implies that $\boldsymbol{A}$ has Property (B) of Definition 3.2. So Theorem 3.3 is applicable, showing that (21) is a residual set in $\mathcal{B}$. ■

According to Theorem 6.6, there exists no method $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ which can be implemented on an abstract Turing machine and which is able to calculate arbitrary values $f(t)$, $t \in \mathbb{T}$ for all functions f from the spaces $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$. In particular, it is impossible to calculate a partial Fourier series (3) or (7) on a Turing machine for all $f \in \mathcal{U}_{\mathrm{s}}$ or $f \in \mathcal{U}$, respectively.

7. Auxiliary results

This section presents two auxiliary results needed to prove Theorem 3.3 in Section 8. Nevertheless, both results may be of some interest by themselves. Again, $\mathcal{B}$ stands always for $\mathcal{U}$ or $\mathcal{U}_{\mathrm{s}}$.

7.1. Interpolating continuous functions by functions in $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$

The following interpolation lemma will be of fundamental importance for the proof of our main results. Let $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$ be an arbitrary sampling-based approximation operator concentrated on a finite sampling set $\mathcal{Z}_N$, and let $\{f(\tau_n) : \tau_n \in \mathcal{Z}_N\}$ be the samples of $f \in \mathcal{B}$ on which the calculation of $\mathrm{A}_N(f)$ is based. Then the following interpolation lemma will show that the operator A_N cannot decide from the given samples $\{f(\tau_n) : \tau_n \in \mathcal{Z}_N\}$ whether f belongs to $\mathcal{B}$ or $\mathcal{C}(\mathbb{T})$, because to every $f \in \mathcal{C}(\mathbb{T})$ there exists a function $g \in \mathcal{B}$ which coincides with f on the sampling set $\mathcal{Z}_N$. Then, in connection with Property (B) of Definition 3.2, every approximation method which shows a bad convergence behavior on $\mathcal{C}(\mathbb{T})$ will show a similar bad convergence behavior on the subset $\mathcal{B} \subset \mathcal{C}(\mathbb{T})$. Exactly in this way, the following approximation lemma will be applied in the proof of Theorem 3.3.

Lemma 7.1 (*Interpolation Lemma*). *Let $\mathcal{Z} \subset \mathbb{T}$ be an arbitrary discrete subset of $\mathbb{T}$. To every $f \in \mathcal{C}(\mathbb{T})$ there exists an $f_1 \in \mathcal{U}$ with $\|f_1\|_U \leq 2\,\|f\|_\infty$ and such that $f_1(\tau) = f(\tau)$ for all $\tau \in \mathcal{Z}$.*

Remark 7.1. Since $\mathcal{U} \subset \mathcal{U}_{\mathrm{s}}$ and $\|f_1\|_{U_{\mathrm{s}}} \leq \|f_1\|_U$ one has also $f_1 \in \mathcal{U}_{\mathrm{s}}$ with $\|f_1\|_{U_{\mathrm{s}}} \leq 2\,\|f\|_\infty$.

Remark 7.2. A similar interpolation lemma was used in [2] but for a different Banach space.

Proof. (1) For $\delta \in (0,1)$, we consider the function $\Delta_\delta \in \mathcal{C}(\mathbb{T})$ defined by $\Delta_\delta(t) = \max\left(0, 1 - \frac{|t|}{\delta}\right)$ for $t \in \mathbb{T}$, and which can be written as a Fourier series

$$\begin{aligned} &\Delta_\delta(t) = \textstyle\sum_{n=-\infty}^{\infty} c_n(\Delta_\delta)\,\mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad c_0(\Delta_\delta) = \frac{\delta}{2\pi} \quad \text{and} \\ &c_n(\Delta_\delta) = \frac{[1-\cos(n\delta)]}{\delta\,\pi\,n^2}\ ,\ n \geq 1\ . \end{aligned} \tag{22}$$

All Fourier coefficients $c_n(\Delta_\delta)$ are non-negative, so $\Delta_\delta(0) = \sum_{n=0}^{\infty} c_n(\Delta_\delta) = \|\boldsymbol{c}\|_{\ell^1} = \|\Delta_\delta\|_{\mathcal{W}} = 1$, showing that (22) converges absolutely and satisfies

$$\|\Delta_\delta\|_U = \sup_{N,M\in\mathbb{N}_0} \left\|\mathrm{S}_{N,M}\Delta_\delta\right\|_\infty \leq \|\Delta_\delta\|_{\mathcal{W}} = 1 \tag{23}$$

using that $\left\|\mathrm{S}_{N,M}\Delta_\delta\right\|_\infty = \left\|\sum_{n=-M}^{N} c_n(\Delta_\delta)\,e_n\right\|_\infty \leq \sum_{n=-M}^{N} c_n(\Delta_\delta) \leq \|\Delta_\delta\|_{\mathcal{W}}$ for all $N, M \in \mathbb{N}_0$, and where $\|\Delta_\delta\|_{\mathcal{W}}$ stands for the usual norm of Δ_δ in the Wiener algebra $\mathcal{W}$.

(2) Next, we show that for every $\mu \in (0, 1)$ and every $\epsilon > 0$ there exists a $\delta_0 = \delta_0(\mu, \epsilon)$ with $0 < \delta_0 < \mu$ such that for every $\delta \in (0, \delta_0)$ and for all $N, M \in \mathbb{N}_0$

$$\left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t)\right| < \epsilon \qquad \text{for all } t \in \mathbb{T}\backslash[-\mu, \mu]\ .$$

Indeed, using the integral representation (8) of $\mathrm{S}_{N,M}$, we obtain for all $\delta \in (0, 1)$ and all $t \in \mathbb{T}$

$$\left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t)\right| \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |\Delta_\delta(\tau)| \left|\mathcal{D}_{N,M}(t-\tau)\right| \mathrm{d}\tau \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{1}{|\sin([t-\tau]/2)|}\,\mathrm{d}\tau\ ,$$

because $\|\Delta_\delta\|_\infty \leq 1$. Then, choosing an arbitrary μ satisfying $1 > \mu > \delta$, one obtains

$$\left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t)\right| \leq \frac{\delta}{\pi}\frac{1}{|\sin([\mu-\delta]/2)|} \qquad \text{for all } t \in \mathbb{T}\backslash[-\mu, \mu]\ .$$

So to every $\epsilon > 0$, we can choose $\delta \in (0, \mu)$ so that $\left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t)\right| < \epsilon$ for all $t \in \mathbb{T}\backslash[-\mu, \mu]$.

(3) Let $\mathcal{Z} = \{\tau_n\}_{n=1}^{Z} \subset \mathbb{T}$ be an arbitrary sampling set of cardinality Z. Then we set

$$\rho = \min\left\{|\tau_n - \tau_m| \ :\ \tau_n, \tau_m \in \mathcal{Z} \text{ with } n \neq m\right\} \qquad \text{and} \qquad \mu = \rho/3\ .$$

Using the previous step of this proof, we choose $\delta < \mu$ such that for all $N, M \in \mathbb{N}_0$ always

$$\left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t)\right| \leq \tfrac{1}{Z} \qquad \text{for all } t \in \mathbb{T}\backslash[-\mu, \mu]\ . \tag{24}$$

(4) Let $f \in \mathcal{C}(\mathbb{T})$ be arbitrary. With f we associate the function

$$f_\delta(t) = \textstyle\sum_{n=1}^{Z} f(\tau_n)\,\Delta_\delta(t - \tau_n) = \sum_{n=1}^{Z} f(\tau_n)\left(\mathrm{T}_{\tau_n}\Delta_\delta\right)(t) \tag{25}$$

with δ as chosen in the previous step and where T_{τ_n} stands for the translation operators (6). Then we fix arbitrary numbers $N, M \in \mathbb{N}_0$ and apply $\mathrm{S}_{N,M}$ to f_δ. Using (9), this yield

$$\begin{aligned}\left(\mathrm{S}_{N,M} f_\delta\right)(t) &= \textstyle\sum_{n=1}^{Z} f(\tau_n)\left(\mathrm{S}_{N,M}\mathrm{T}_{\tau_n}\Delta_\delta\right)(t) = \sum_{n=1}^{Z} f(\tau_n)\left(\mathrm{T}_{\tau_n}\mathrm{S}_{N,M}\Delta_\delta\right)(t) \\ &= \textstyle\sum_{n=1}^{Z} f(\tau_n)\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t - \tau_n)\ .\end{aligned} \tag{26}$$

(5) Now we fix an arbitrary $t \in \mathbb{T}$. By the definition of ρ and μ, there exists at most one index $\hat{n} \in \{1, 2, \ldots, Z\}$ such that $t - \tau_{\hat{n}} \in [-\mu, \mu]$. Consequently, we have

$$\left|\left(\mathrm{S}_{N,M}\mathrm{T}_{\tau_n}\Delta_\delta\right)(t)\right| = \left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t - \tau_n)\right| \leq \begin{cases} 1 & \text{if} \quad n = \hat{n} \\ 1/Z & \text{if} \quad n \neq \hat{n} \end{cases}\ ,$$

where the first line follows from (23) and where the second line is a consequence of (24). Using these upper bounds and applying the triangle inequality to the absolute value of (26) yields

$$\begin{aligned}\left|\left(\mathrm{S}_{N,M} f_\delta\right)(t)\right| &\leq \left|f(\tau_m)\right| \left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t - \tau_{\hat{n}})\right| + \sum_{n=1, n\neq m}^{Z} \left|f(\tau_n)\right| \left|\left(\mathrm{S}_{N,M}\Delta_\delta\right)(t - \tau_n)\right| \\ &\leq \left|f(\tau_m)\right| + \frac{1}{Z}\sum_{n=1, n\neq m}^{Z} \left|f(\tau_n)\right| \leq \|f\|_\infty + \tfrac{Z-1}{Z}\,\|f\|_\infty \leq 2\,\|f\|_\infty\ ,\end{aligned}$$

and since $t \in \mathbb{T}$ was arbitrary, we have $\left\|\mathrm{S}_{N,M} f_\delta\right\|_\infty \leq 2\,\|f\|_\infty$ for all $N, M \in \mathbb{N}$ and therefore

$$\|f_\delta\|_U = \sup_{N,M\in\mathbb{N}_0} \left\|\mathrm{S}_{N,M} f_\delta\right\| \leq 2\,\|f\|_\infty\ .$$

(6) Finally, the definition of f_δ in (25) shows that $f_\delta(\tau_n) = f(\tau_n)$ for all $\tau_n \in \mathcal{Z}$ because $\delta < \rho/3$ was chosen such that the support sets of the functions $\left\{\mathrm{T}_{\tau_n}\Delta_\delta\right\}_{n=1}^{Z}$ are mutually disjoint. So the function $f_1 = f_\delta \in \mathcal{U}$, defined in (25), has all the properties claimed by the lemma. ■

7.2. A generalized uniform boundedness principle for non-linear operators

Since Theorem 3.3 allows for approximation operators $\mathrm{A}_N : \mathcal{B} \to \mathcal{B}$ which are not necessarily linear, we will need a generalization of the well known uniform boundedness principle for linear operators. This principle can be formulated as follows (see, e.g., [25, Chapter 5]). Let $\boldsymbol{\Phi} = \{\Phi_N\}_{N\in\mathbb{N}}$ be a family of bounded *linear* functionals on a Banach space $\mathcal{B}$ and assume that there exists a residual set $\mathcal{K} \subset \mathcal{B}$ such that

$$\sup_{N\in\mathbb{N}} |\Phi_N(f)| < +\infty \qquad \text{for all } f \in \mathcal{K}\,.$$

Then the functionals Φ_N are uniformly bounded by a constant $M < \infty$, i.e. $\|\Phi_N\| \leq M$ for all $N \in \mathbb{N}$. By the linearity of Φ_N, this conclusion may be stated as follows: To every arbitrary ball $B_\delta(f, \mathcal{B}) \subset \mathcal{B}$ with radius $\delta > 0$ and center $f \in \mathcal{B}$ there exists a constant $M = M(\delta, f)$ such that

$$|\Phi_N(f)| \leq M \qquad \text{for every } N \in \mathbb{N} \text{ and for all } f \in B_\delta(f, \mathcal{B})\,. \tag{27}$$

This uniform boundedness principle (also known as Banach–Steinhaus theorem) can be generalized to non-linear (lower semi-) continuous functionals on $\mathcal{B}$ [15, Satz 4.4]. However, then Conclusion (27) does no longer hold for arbitrary balls $B_\delta(f, \mathcal{B}) \subset \mathcal{B}$, but there exists only one fixed ball $B_{\delta_0}(f_0, \mathcal{B}) \subset \mathcal{B}$, determined by the family of functionals $\boldsymbol{\Phi}$, such that (27) holds only for all $f \in B_{\delta_0}(f_0, \mathcal{B})$. For completeness, we state this generalized uniform boundedness principle in the form as it will be needed later, together with its short proof.

Lemma 7.2 (*Generalized Uniform Boundedness Principle*)**.** *Let $\mathcal{B}$ be a Banach space, let $\boldsymbol{\Phi} = \{\Phi_N\}_{N\in\mathbb{N}}$ be a family of lower semicontinuous functionals $\Phi_N : \mathcal{B} \to \mathbb{R}_+$ and assume there exists a residual set $\mathcal{K} \subset \mathcal{B}$ so that*

$$\sup_{N\in\mathbb{N}} \Phi_N(f) = C(f) < +\infty \qquad \text{for all } f \in \mathcal{K}\,.$$

Then there exist a constant $C_{\boldsymbol{\Phi}} < \infty$, an $f_0 \in \mathcal{B}$, and a $\delta > 0$ so that for every $f \in B_\delta(f_0, \mathcal{B})$ always $\Phi_N(f) \leq C_{\boldsymbol{\Phi}}$ for all $N \in \mathbb{N}$.

Proof. For any $\lambda \geq 0$, we define the sets

$$L_N(\lambda) = \left\{ f \in \mathcal{B} \;:\; \Phi_N(f) \leq \lambda \right\} \qquad \text{and} \qquad L_{\boldsymbol{\Phi}}(\lambda) = \textstyle\bigcap_{N\in\mathbb{N}} L_N(\lambda)\,.$$

Note that for every $\lambda \geq 0$ the sets $L_N(\lambda)$ are either empty or closed because any Φ_N is lower semicontinuous. Consequently, also $L_{\boldsymbol{\Phi}}(\lambda)$ is (if it is not empty) a closed subset of $\mathcal{B}$, because the intersection of closed sets is closed.

Let $f \in \mathcal{K}$ be arbitrary. Then $\sup_{N\in\mathbb{N}} \Phi_N(f) \leq C(f) < \infty$ by the assumption of the lemma. So if $\lambda > C(f)$ then $f \in L_{\boldsymbol{\Phi}}(\lambda)$ and so $\mathcal{K} \subset \bigcup_{\lambda\in\mathbb{N}} L_{\boldsymbol{\Phi}}(\lambda)$. In other words, $\mathcal{K}$ is contained in a countable union of closed sets and since $\mathcal{K}$ is assumed to be of second category there exists a $\lambda_0 \in \mathbb{N}$ such that $L_{\boldsymbol{\Phi}}(\lambda_0)$ is not nowhere dense. So there exist a $\delta_0 > 0$ and an $f_0 \in \mathcal{B}$ such that $L_{\boldsymbol{\Phi}}(\lambda_0) \cap B_{\delta_0}(f_0, \mathcal{B})$ is dense in $B_{\delta_0}(f_0, \mathcal{B})$. However, since $L_{\boldsymbol{\Phi}}(\lambda_0)$ is a closed set, we have $B_{\delta_0}(f_0, \mathcal{B}) \subset L_{\boldsymbol{\Phi}}(\lambda_0)$. ■

8. Proof of the divergence results

This section proves the main divergence results from Sections 3 and 4. First, Theorem 3.3 is proved, showing the divergence of general approximation methods in $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$. Afterward, the proof of Theorem 4.2 is presented. The last subsection contains two auxiliary lemmas which are needed for the proof of Theorem 3.3. Recall from (10) that $B_\delta(f_0, \mathcal{B})$ denotes the open ball in the Banach space $\mathcal{B}$ with center $f_0 \in \mathcal{B}$ and radius $\delta > 0$.

8.1. *Proof of Theorem 3.3*

We prove the statement of the theorem for $\mathcal{U}_\mathrm{s}$ by contradiction. So assume the statement of the theorem is wrong, i.e. assume the set

$$G(\boldsymbol{A}) = \left\{ f \in \mathcal{U}_\mathrm{s} \ : \ \sup_{N\in\mathbb{N}} \|\mathrm{A}_N(f)\|_{U_\mathrm{s}} =: C(f) < +\infty \right\} \tag{28}$$

is of second category in $\mathcal{U}_\mathrm{s}$. Based on this assumption we deduct a contraction in several steps.

(1) The assumption that (28) is of second category implies that the family $\{\Phi_N\}_{N\in\mathbb{N}}$ of functionals $\Phi_N : \mathcal{U}_\mathrm{s} \to \mathbb{R}_+$ given by $\Phi_N(f) = \|\mathrm{A}_N(f)\|_{U_\mathrm{s}}$ satisfies the conditions of the generalized uniform boundedness principle (Lemma 7.2). So there exists a constant $C_{\boldsymbol{\Phi}} < \infty$, a function $f_0 \in \mathcal{U}_\mathrm{s}$, and a $\delta_0 > 0$ such that for all $f \in B_{\delta_0}(f_0, \mathcal{U}_\mathrm{s})$ always

$$\|\mathrm{A}_N(f)\|_{U_\mathrm{s}} \le C_{\boldsymbol{\Phi}} \qquad \text{for all } N \in \mathbb{N}\ .$$

(2) Let $\mathcal{M} \subset \mathcal{U}_\mathrm{s}$ be the dense subset of Property (B) in Definition 3.2 for the operator sequence $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$. Then there exists an $f_1 \in \mathcal{M}$ such that $\|f_1 - f_0\|_{U_\mathrm{s}} < \delta_0/2$ and we certainly have $B_{\delta_0/2}(f_1, \mathcal{U}_\mathrm{s}) \subset B_{\delta_0}(f_0, \mathcal{U}_\mathrm{s})$ which implies for every $N \in \mathbb{N}$

$$\|\mathrm{A}_N(f)\|_{U_\mathrm{s}} \le C_{\boldsymbol{\Phi}} \qquad \text{for every } f \in B_{\delta_0/2}(f_1, \mathcal{U}_\mathrm{s})\ . \tag{29}$$

(3) Let $N \in \mathbb{N}$ be arbitrary and let $\mathcal{Z}_N \subset \mathbb{T}$ be the sampling set associated with A_N. We choose an arbitrary $f \in B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))$ and consider the function $g := f - f_0 \in \mathcal{C}(\mathbb{T})$ with norm $\|g\|_\infty < \delta_0/4$. According to Lemma 7.1 there exists a $g_N \in \mathcal{U}_\mathrm{s}$ such that

$$g_N(\tau) = g(\tau) \quad \text{for all } \tau \in \mathcal{Z}_N \qquad \text{and} \qquad \|g_N\|_{U_\mathrm{s}} \le 2\, \|g\|_\infty < \delta_0/2\ .$$

Then we consider the function $q_N := f_1 + g_N$ which the two properties

$$\begin{aligned} &q_N \in B_{\delta_0/2}(f_1, \mathcal{U}_\mathrm{s}) \\ &q_N(\tau) = f_1(\tau) + g_N(\tau) = f_1(\tau) + g(\tau) = f(\tau)\ , \qquad \text{for all } \tau \in \mathcal{Z}_N\ . \end{aligned}$$

So by Property (A) of the sequence $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$, one has $\mathrm{A}_N(q_N) = \mathrm{A}_N(f)$ and (29) shows that $\|\mathrm{A}_N(f)\|_{U_\mathrm{s}} = \|\mathrm{A}_N(q_N)\|_{U_\mathrm{s}} \le C_{\boldsymbol{\Phi}}$. Since $f \in B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))$ and $N \in \mathbb{N}$ was arbitrary, we thus get

$$\|\mathrm{A}_N(f)\|_{U_\mathrm{s}} \le C_{\boldsymbol{\Phi}} \qquad \text{for all } f \in B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))\ . \tag{30}$$

(4) Now, we define two subsets of $\mathcal{C}(\mathbb{T})$ as follows

$$\begin{aligned} &\mathcal{M}_0 := \left\{ g \in B_{\delta_0/4}(0, \mathcal{C}(\mathbb{T})) \ : \ \exists g_1, g_2 \in \mathcal{M} \cap B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T})) \text{ so that } g = g_1 - g_2 \right\} \\ &\mathcal{M}_1 = \lambda \cdot \mathcal{M}_0 := \{ f = \lambda f_0 \ : \ \lambda \in \mathbb{R},\, f_0 \in \mathcal{M}_0 \}\ , \end{aligned}$$

and we notice that $\mathcal{M}_0$ is dense in $B_{\delta_0/4}(0, \mathcal{C}(\mathbb{T}))$ and that $\mathcal{M}_1$ is dense in $\mathcal{C}(\mathbb{T})$. These properties of $\mathcal{M}_0$ and $\mathcal{M}_1$ are verified by Lemmas 8.1 and 8.2 in Section 8.3. Let $g \in \mathcal{M} \cap B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))$ be arbitrary. Since $g \in \mathcal{M} \subset \mathcal{U}_\mathrm{s}$, we have $\|g\|_{U_\mathrm{s}} < +\infty$ and the triangle inequality and (30) yields

$$\|g\|_{U_\mathrm{s}} \le \|g - \mathrm{A}_N(g)\|_{U_\mathrm{s}} + \|\mathrm{A}_N(g)\|_{U_\mathrm{s}} \le \|g - \mathrm{A}_N(g)\|_{U_\mathrm{s}} + C_{\boldsymbol{\Phi}}\ .$$

Moreover, using Property (B) of the sequence $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$, one gets

$$\|g\|_{U_\mathrm{s}} \leq \limsup_{N\to\infty} \|g - \mathrm{A}_N(g)\|_{U_\mathrm{s}} + C_{\boldsymbol{\Phi}} = C_{\boldsymbol{\Phi}} \ ,$$

showing that $\|g\|_{U_\mathrm{s}} \leq C_{\boldsymbol{\Phi}}$ for all $g \in \mathcal{M} \cap B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))$.

Then we consider arbitrary functions $g_1, g_2 \in \mathcal{M} \cap B_{\delta_0/4}(f_1, \mathcal{C}(\mathbb{T}))$ and set $g_0 := g_1 - g_2$. Since $g_1, g_2 \in \mathcal{M} \subset \mathcal{U}_\mathrm{s}$ and because $\mathcal{U}_\mathrm{s}$ is a linear space, we have $g_0 \in \mathcal{U}_\mathrm{s}$ with

$$\|g_0\|_{U_\mathrm{s}} = \|g_1 - g_2\|_{U_\mathrm{s}} \leq \|g_1\|_{U_\mathrm{s}} + \|g_2\|_{U_\mathrm{s}} \leq 2\, C_{\boldsymbol{\Phi}} \ .$$

This shows in particular that

$$\|g_0\|_{U_\mathrm{s}} \leq 2\, C_{\boldsymbol{\Phi}} \qquad \text{for all } g_0 \in \mathcal{M}_0 \ . \tag{31}$$

Let $f \in \mathcal{M}_1$ be arbitrary and set $f_* := \frac{\delta_0}{5\,\|f\|_\infty} f$. Then $f_* \in \mathcal{M}_0$ and (31) yields

$$\|f\|_{U_\mathrm{s}} = 5\,\frac{\|f\|_\infty}{\delta_0}\,\|f_*\|_{U_\mathrm{s}} \leq 10\,\frac{\|f\|_\infty}{\delta_0}\,C_{\boldsymbol{\Phi}}$$

showing that for every $f \in \mathcal{M}_1$

$$\|\mathrm{S}_N f\|_\infty \leq \|f\|_{U_\mathrm{s}} \leq \frac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\,\|f\|_\infty \qquad \text{for all } N \in \mathbb{N} \ . \tag{32}$$

(5) Let $f \in \mathcal{C}(\mathbb{T})$ be arbitrary. Since $\mathcal{M}_1 \subset \mathcal{C}(\mathbb{T})$ is dense, there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_1$ with

$$\lim_{n\to\infty} \|f - f_n\|_\infty = 0 \ . \tag{33}$$

For every fixed $N \in \mathbb{N}$, the operator $\mathrm{S}_N : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$, as defined in (3), is bounded so that

$$\lim_{n\to\infty} \|\mathrm{S}_N(f - f_n)\|_\infty = 0 \qquad \text{for all } N \in \mathbb{N} \ . \tag{34}$$

Therefore, for every arbitrary $N \in \mathbb{N}$, one obtains

$$\begin{aligned}\|\mathrm{S}_N f\|_\infty &\leq \|\mathrm{S}_N(f - f_n)\|_\infty + \|\mathrm{S}_N f_n\|_\infty \leq \|\mathrm{S}_N(f - f_n)\|_\infty + \|f_n\|_{U_\mathrm{s}} \\ &\leq \|\mathrm{S}_N(f - f_n)\|_\infty + \tfrac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\,\|f_n\|_\infty\end{aligned}$$

using (32) to obtain the last line. Because of (33), there exists an $n_0 \in \mathbb{N}$ such that $\|f_n\|_\infty \leq \|f\|_\infty + 1$ for all $n \geq n_0$ and therefore

$$\|\mathrm{S}_N f\|_\infty \leq \|\mathrm{S}_N(f - f_n)\|_\infty + \tfrac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\big(\|f\|_\infty + 1\big) \qquad \text{for all } n \geq n_0 \ .$$

Since the left hand side of this inequality does not depend on n, one obtains from (34)

$$\|\mathrm{S}_N f\|_\infty \leq \limsup_{n\to\infty} \|\mathrm{S}_N(f - f_n)\|_\infty + \tfrac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\big(\|f\|_\infty + 1\big) = \tfrac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\big(\|f\|_\infty + 1\big) \ ,$$

and by the definition of the norm in $\mathcal{U}_\mathrm{s}$, one gets $\|f\|_{\mathcal{U}_\mathrm{s}} = \sup_{N\in\mathbb{N}} \|\mathrm{S}_N f\|_\infty \leq \frac{10\,C_{\boldsymbol{\Phi}}}{\delta_0}\big(\|f\|_\infty + 1\big)$. Since $f \in \mathcal{C}(\mathbb{T})$ was chosen arbitrary, we thus have

$$\|f\|_{\mathcal{U}_\mathrm{s}} \leq \tfrac{20\,C_{\boldsymbol{\Phi}}}{\delta_0} \qquad \text{for all } f \in \mathcal{C}(\mathbb{T}) \text{ with } \|f\|_\infty \leq 1 \ . \tag{35}$$

(6) For any arbitrary $K \in \mathbb{N}$, we define

$$f_K(t) = C_0\,\mathrm{e}^{\mathrm{i}Kt} \sum_{k=1}^{K} \frac{\sin(kt)}{k} = \mathrm{i}\,\frac{C_0}{2} \sum_{k=0, k\neq K}^{2K} \frac{\mathrm{e}^{\mathrm{i}kt}}{K - k} \ , \qquad t \in \mathbb{T} \ ,$$

with a constant $0 < C_0 < 1.2$ which can be chosen, independently of K, such that $\|f_K\|_\infty \leq 1$ for all $K \in \mathbb{N}$ [35, Chapter II.9]. For these functions, one gets

$$\|\mathrm{S}_K f_K\|_\infty \geq |(\mathrm{S}_K f_K)(0)| = \tfrac{C_0}{2} \textstyle\sum_{k=1}^{K} \frac{1}{k} \geq \frac{C_0}{2} \log(K+1) \qquad \text{for all } K \in \mathbb{N} .$$

Combining this inequality with (35), one obtains

$$\tfrac{20\,\mathcal{C}_\Phi}{\delta_0} \geq \|f_k\|_{U_s} = \sup_{N\in\mathbb{N}} \|\mathrm{S}_N f_K\|_\infty \geq \|\mathrm{S}_K f_K\|_\infty \tfrac{C_0}{2} \log(K+1) \qquad \text{for every } K \in \mathbb{N} .$$

However, this yields a contradiction for sufficiently large $K \in \mathbb{N}$, showing that assumption (28) was wrong. So the statement of the theorem is true.

The proof for $\mathcal{B} = \mathcal{U}$ follows exactly the same lines as the above proof for $\mathcal{B} = \mathcal{U}_s$ with almost no changes. Therefore, the detailed proof is omitted. ■

8.2. *Proof of Theorem 4.2*

Let $\boldsymbol{E} = \{\mathrm{E}_N\}_{N\in\mathbb{N}}$ be a sampling based Fourier approximation as assumed in the theorem. It is sufficient to verify that the operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ satisfy Properties (A) and (B) of Definition 3.2. Then the statement follows from Theorem 3.3.

First we note that the assumption in Definition 4.1 that all functionals $c_{N,n} : \mathcal{B} \to \mathbb{C}$ are continuous implies that the operators $\mathrm{E}_N : \mathcal{B} \to \mathcal{B}$ are (lower semi-) continuous. Then it is obvious that $\boldsymbol{E}$ satisfies Property (A) with the sampling sets $\mathcal{Z}_N = \bigcup_{n=-N}^{N} \mathcal{Z}_{N,n}$.

It remains to verify that $\boldsymbol{E}$ has Property (B). Let $f \in \mathcal{P}$ be an arbitrary trigonometric polynomial of degree $K \in \mathbb{N}$. Without loss of generality, we can assume that it has the form $f = \sum_{k=-K}^{K} c_k(f)\, e_k$ where some of the Fourier coefficients $c_n(f)$ may be zero. By the definition of the operators E_L, we thus have

$$\mathrm{E}_L(f) = \textstyle\sum_{n=-K}^{K} c_{L,n}(f)\, e_n \qquad \text{for all } L \geq K .$$

Then for arbitrary $N, M \in \mathbb{N}$ and using Property (b) of $\boldsymbol{E}$, one obtains

$$\lim_{L\to\infty} \mathrm{S}_{N,M}\mathrm{E}_L(f) = \lim_{L\to\infty} \sum_{n=-\min(M,K)}^{\min(N,K)} c_{L,n}(f)\, e_n = \sum_{n=-\min(M,K)}^{\min(N,K)} c_n(f)\, e_n = \mathrm{S}_{N,M} f \ ,$$

showing that $\lim_{L\to\infty} \left\|\mathrm{S}_{N,M}\left[\mathrm{E}_L(f) - f\right]\right\|_\infty = 0$ for all $N, M \in \mathbb{N}$. So therewith, one obtains

$$\lim_{L\to\infty} \sup_{N,M\in\mathbb{N}} \left\|\mathrm{S}_{N,M}\left[\mathrm{E}_L(f) - f\right]\right\|_\infty = \lim_{L\to\infty} \left\|\mathrm{E}_L(f) - f\right\|_{\mathcal{U}} = 0 \ ,$$

and because of (14), one has also $\lim_{L\to\infty} \|\mathrm{E}_L(f) - f\|_{\mathcal{U}_s} = 0$. Since $f \in \mathcal{P}$ was arbitrary, we thus have verified Property (B) of $\boldsymbol{E}$ with the set $\mathcal{M} = \mathcal{P}$. ■

8.3. *Two auxiliary lemmas*

For completeness and to make the paper self-contained, this subsection proves two simple statements which are needed in the proof of Theorem 3.3 in Section 8.1.

Lemma 8.1. *Let $\mathcal{B}$ be a Banach space, let $\mathcal{M} \subset \mathcal{B}$ be a dense subset, and let $f_1 \in \mathcal{M}$ be arbitrary. Then for any $\delta > 0$ the set*

$$\mathcal{M}_0 = \left\{g \in B_\delta(0,\mathcal{B}) \ : \ \exists g_1, g_2 \in \mathcal{M} \cap B_\delta(f_1,\mathcal{B}) \text{ so that } g = g_1 - g_2\right\}$$

is dense in $B_\delta(0,\mathcal{B})$.

Proof. Assume the statement is wrong, i.e. assume that there exists an $\epsilon > 0$ and a $q \in B_\delta(0, \mathcal{B})$ such that $B_\epsilon(q, \mathcal{B}) \subset B_\delta(0, \mathcal{B})$ and

$$\mathcal{M}_0 \cap B_\epsilon(q, \mathcal{B}) = \emptyset\ . \tag{36}$$

Then $f_2 = f_1 + q$ belongs to $B_\delta(f_1, \mathcal{B})$, and since $\mathcal{M}$ is dense in $\mathcal{B}$, one always finds $f_3 \in \mathcal{M} \cap B_\delta(f_1, \mathcal{B})$ so that $\|f_2 - f_3\|_{\mathcal{B}} < \epsilon/2$. Then $f = f_3 - f_1$ belongs obviously to $\mathcal{M}_0$ and satisfies $\|f - q\|_{\mathcal{B}} = \|f_3 - f_1 - q\|_{\mathcal{B}} = \|f_3 - f_2\|_{\mathcal{B}} < \epsilon/2$ showing that $f \in B_\epsilon(q, \mathcal{B})$, contradicting (36). ■

Lemma 8.2. *Let $\mathcal{B}$ be a Banach space, let $B_\delta(0, \mathcal{B}) = \{f \in \mathcal{B} : \|f\|_{\mathcal{B}} < \delta\}$, and let $\mathcal{M}_0$ be a dense subset of $B_\delta(0, \mathcal{B})$. Then the set $\mathcal{M}_1 = \{f = \lambda f_0 : \lambda \in \mathbb{R},\ f_0 \in \mathcal{M}_0\}$ is dense in $\mathcal{B}$.*

Proof. Assume the statement is wrong, i.e. assume there is an $\epsilon > 0$ and a $q \in \mathcal{B}$, $q \neq 0$ so that

$$\mathcal{M}_1 \cap B_\epsilon(q, \mathcal{B}) = \emptyset\ . \tag{37}$$

Let $f_1 = \frac{\delta}{2} \frac{q}{\|q\|_{\mathcal{B}}} \in \mathcal{B}_\delta(0, \mathcal{B})$, and choose μ such that $0 < \mu < \frac{\delta}{2} \min\left(1, \frac{\epsilon}{\|q\|_{\mathcal{B}}}\right)$. Then $B_\mu(f_1, \mathcal{B}) \subset B_\delta(0, \mathcal{B})$ and since $\mathcal{M}_0$ is dense in $B_\delta(0, \mathcal{B})$ one always finds an $f_2 \in \mathcal{M}_0$ such that $\|f_1 - f_2\|_{\mathcal{B}} < \mu$. Inserting the definition of f_1, one obtains

$$\left\| q - \tfrac{2}{\delta} \|q\|_{\mathcal{B}}\, f_2 \right\|_{\mathcal{B}} < \frac{2\mu}{\delta}\ \|q\|_{\mathcal{B}} < \epsilon\ . \tag{38}$$

So if $f_0 := \lambda f_2$ with $\lambda = \frac{2}{\delta} \|q\|_{\mathcal{B}}$ then $f_0 \in \mathcal{M}_1$ and (38) shows that $f_0 \in B_\epsilon(q, \mathcal{B})$ contradicting assumption (37). ■

Appendix. Properties of the Banach Spaces U, U_{s} and $\mathcal{U}$, $\mathcal{U}_{\mathrm{s}}$

This appendix proves the properties of the spaces U, U_{s}, $\mathcal{U}$, and $\mathcal{U}_{\mathrm{s}}$ as stated in Section 2.2. The first subsection verifies the statements concerning U and U_{s} whereas the second subsection presents the proof concerning $\mathcal{U}$ and $\mathcal{U}_{\mathrm{s}}$.

A.1. The Banach spaces of U and U_{s} – proof Theorem 2.2

(i) By the definitions (11) and (12) of $\|\cdot\|_U$ and $\|\cdot\|_{U_{\mathrm{s}}}$, one has $\|f\|_{U_{\mathrm{s}}} \leq \|f\|_U$ for every $f \in U$, showing in particular that $U \subseteq U_{\mathrm{s}}$. Let $f \in U \subset \mathcal{C}(\mathbb{T})$ be arbitrary, and for any $M \in \mathbb{N}$ let

$$(\mathrm{V}_M f)\,(t) = \tfrac{1}{M} \textstyle\sum_{N=0}^{M-1} (\mathrm{S}_N f)(t)\ , \qquad t \in \mathbb{T}\ ,$$

be the first arithmetic means of the partial Fourier series $\mathrm{S}_N f$. Then the triangle inequality yields

$$\|\mathrm{V}_M f\|_\infty \leq \tfrac{1}{M} \textstyle\sum_{N=0}^{M-1} \|\mathrm{S}_N f\|_\infty \leq \tfrac{1}{M} \textstyle\sum_{N=0}^{M-1} \|f\|_{U_{\mathrm{s}}} = \|f\|_{U_{\mathrm{s}}} \leq \|f\|_U\ ,$$

and one gets $\|f\|_\infty \leq \|\mathrm{V}_M f\|_\infty + \|f - \mathrm{V}_M f\|_\infty \leq \|f\|_{U_{\mathrm{s}}} + \|f - \mathrm{V}_M f\|_\infty$. By the Theorem of Fejér (see, e.g., [35, Chapter III.3]), the last term on the right hand side converges to zeros as $M \to \infty$, proving that $\|f\|_\infty \leq \|f\|_{U_{\mathrm{s}}} \leq \|f\|_U$ for any $f \in U$.

(ii) First, we verify the second statement for U_{s}. It is easy to see that U_{s} is a linear space. So it remains to show that U_{s} is complete. Let $\{f_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in U_{s}.

Then (14) implies that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}(\mathbb{T})$. Consequently, there exists an $f \in \mathcal{C}(\mathbb{T})$ such that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, and it remains to show that $f \in U_s$.

Since $\{f_n\}_{n\in\mathbb{N}} \subset U_s$ is a Cauchy sequence, there exists an $n_0 \in \mathbb{N}$ so that $\|f_n - f_m\|_{U_s} \leq 1$ for all $n, m \geq n_0$. Therefore, one gets

$$\left\| f_n \right\|_{U_s} \leq \left\| f_n - f_{n_0} \right\|_{U_s} + \left\| f_{n_0} \right\|_{U_s} \leq 1 + \left\| f_{n_0} \right\|_{U_s} \qquad \text{for all } n \geq n_0 \ ,$$

and this implies certainly

$$\left\| f_n \right\|_{U_s} \leq \max\left(1 + \left\| f_{n_0} \right\|_{U_s} \ , \ \max_{1\leq m<n_0} \|f_m\|_{U_s}\right) =: C_2 \qquad \text{for all } n \in \mathbb{N} \ .$$

Let $N \in \mathbb{N}$ be arbitrary. Then, because $\mathrm{S}_N : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ is bounded, we have $\lim_{n\to\infty} \|\mathrm{S}_N f - \mathrm{S}_N f_n\|_\infty = 0$ and so

$$\begin{aligned}\|\mathrm{S}_N f\|_\infty &= \lim_{n\to\infty} \|\mathrm{S}_N f_n\|_\infty \leq \sup_{n\in\mathbb{N}} \|\mathrm{S}_N f_n\|_\infty \leq \sup_{n\in\mathbb{N}}\left(\sup_{N\in\mathbb{N}} \|\mathrm{S}_N f_n\|_\infty\right)\\ &= \sup_{n\in\mathbb{N}} \|f_n\|_{U_s} \leq C_2 \ ,\end{aligned}$$

showing that $f \in U_s$, i.e. showing that U_s is complete. The proof for U follows exactly the same lines and is therefore omitted. ■

A.2. Properties of $\mathcal{U}$ and $\mathcal{U}_s$ – proof of Theorem 2.4, Lemma 2.3, and Corollary 2.5

This section verifies the properties of the spaces $\mathcal{U}$ and $\mathcal{U}_s$ given in Theorem 2.4. The proof of this theorem follows easily from Lemma 2.3, which we will prove first. However, some necessary technicalities in the proof of Lemma 2.3 are handled by the following auxiliary lemma.

Lemma A.1. *Let $\check{N} \in \mathbb{N}$ be arbitrary. To every $r \in \mathbb{R}_+$ there exists a trigonometric polynomial $f_{\check{N},r}$ and an $\hat{N} \in \mathbb{N}$ with $\hat{N} > \check{N}$ such that*

(a) $\left\| f_{\check{N},r} \right\|_\infty \leq 1$
(b) $\left(\mathrm{S}_{N,M} f_{\check{N},r}\right)(t) = 0$ *for all $t \in \mathbb{T}$ and for every $N < \check{N}$ and all $M \in \mathbb{N}_0$*
(c) $\max_{N,M\in\mathbb{N}_0} \left\| \mathrm{S}_{N,M} f_{\check{N},r} \right\|_\infty > r$
(d) $\mathrm{S}_{N,M} f_{\check{N},r} = f_{\check{N},r}$ *for all $N > \hat{N}$ and all $M \in \mathbb{N}_0$.*

Proof. Throughout this proof, we consider polynomials $p_{K,L} \in \mathcal{P}$ of the form

$$p_{K,L}(t) = C_0\, \mathrm{e}^{\mathrm{i}Lt} \left(\sum_{k=1}^{K} \frac{\sin(kt)}{k}\right) = \mathrm{i}\frac{C_0}{2}\left[\sum_{n=L-K}^{L-1} \frac{\mathrm{e}^{\mathrm{i}nt}}{L-n} + \sum_{n=L+1}^{L+K} \frac{\mathrm{e}^{\mathrm{i}nt}}{L-n}\right] , \qquad t \in \mathbb{T} \ , \tag{A.1}$$

with $K, L \in \mathbb{N}_0$. Therein, $C_0 >$ is chosen such that $\left\| p_{K,L} \right\|_\infty \leq 1$ for all $K, L \in \mathbb{N}_0$. Such a constant exists (cf. [35, Chapter II.9] and the discussion in the proof of Theorem 3.3).

Let $\check{N} \in \mathbb{N}$ be arbitrary and choose $K \in \mathbb{N}$ so that $\frac{C_0}{2}\log(K+1) > r$. Then we set $L = \check{N} + K$ and $\hat{N} = L + K$. With these values for K and L, we consider the polynomial $p_{K,L}$ given in (A.1). Note that by its construction all Fourier coefficients $c_n(p_{K,L})$ with $n \leq 0$ are necessarily zero. Next, $\mathrm{S}_{N,M}\, p_{K,L}$ is investigated on three different sets for the index N and for an arbitrary $M \in \mathbb{N}_0$.

(1) For $N < \check{N} = L - K$, Definition (A.1) of $p_{K,L}$ shows immediately that $\mathrm{S}_{N,M}\, p_{K,L} = 0$.
(2) Similarly, for $N > \hat{N}$, one obtains immediately $\mathrm{S}_N\, p_{K,L} = p_{K,L}$.
(3) From the interval $\check{N} \leq N \leq \hat{N}$, we pick for the moment the value $N = L$. Then (A.1) yields

$$\left(\mathrm{S}_{N,M}\, p_{K,L}\right)(0) = \left(\mathrm{S}_{L,0}\, p_{K,L}\right)(0) = \tfrac{C_0}{2} \textstyle\sum_{k=1}^{K} \tfrac{1}{k} \geq \tfrac{C_0}{2} \log(K+1)$$

showing that

$$\max_{\check{N} \leq N \leq \hat{N}, M \in \mathbb{N}0} \left\| \mathrm{S}_{N,M}\, p_{K,L} \right\|_\infty \geq \tfrac{C_0}{2} \log(K+1) > r\ .$$

Finally, we set $f_{\check{N},r} = p_{K,L}$. Then the previously derived properties of $p_{K,L}$ show that $f_{\check{N},r}$ satisfies the statement of the lemma. ■

Remark A.1. The trigonometric polynomials $f_{\check{N},r}$, constructed in the previous proof, are the building blocks in the following proof of Lemma 2.3. These polynomials have the form

$$f_{\check{N},r}(t) = \textstyle\sum_{n=\check{N}}^{\hat{N}} c_n(f_{\check{N},r})\, \mathrm{e}^{int}\ , \qquad t \in \mathbb{T}\ ,$$

with $\check{N} > 0$. So all of its coefficients $c_n(f_{\check{N},r})$ with $n < \check{N}$ and with $n > \hat{N}$ are equal to zero, i.e. the support of the sequence of Fourier coefficients of $f_{\check{N},r}$ satisfies

$$\operatorname{supp}\left(\left\{c_n(f_{\check{N},r})\right\}_{n\in\mathbb{Z}}\right) \subset [\check{N}, \hat{N}] \subset \mathbb{N}$$

and the degree of $f_{\check{N},r}$ is $\hat{N} = \deg(f_{\check{N},r})$.

With Lemma A.1 at our disposal, we can now prove Lemma 2.3 showing that there exist functions $f \in U_{\mathrm{s}}$ for which the Fourier series $\mathrm{S}_N f$ does not converge uniformly to f.

Proof (*Lemma* 2.3)**.** We construct a particular $f \in U$ with the property claimed by the lemma. This construction is based on the trigonometric polynomials $f_{\check{N},r}$ used in the proof of Lemma A.1.

(1) At the beginning, we construct recursively a sequence $\{p_k\}_{k=1}^{\infty}$ of trigonometric polynomials and associate sequences $\{\check{N}_k\}_{k=1}^{\infty} \subset \mathbb{N}$ and $\{\hat{N}_k\}_{k=1}^{\infty} \subset \mathbb{N}$ and $\{R_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$. Starting with $k = 1$, we define $\check{N}_1 = 0$, $r_1 = (3+1)^2$, and $p_1 = f_{\check{N}_1,r_1}$ where $f_{\check{N}_1,r_1}$ is given by Lemma A.1 and satisfies therefore $\|p_1\|_\infty \leq 1$. Then we set $R_1 = \max_{N\in\mathbb{N}} \|\mathrm{S}_N\, p_1\|_\infty$ and $\hat{N}_1 = \deg(p_1)$, and we define the trigonometric polynomial

$$\varphi_1 = \tfrac{1}{R_1}\, p_1\ .$$

For $k = 2$, we define $\check{N}_2 = \hat{N}_1 + 1$, $r_2 = (3+2)^2$, and $p_2 = f_{\check{N}_2,r_2}$ with $\|p_2\|_\infty \leq 1$. Then we set $R_2 = \max_{N\in\mathbb{N}} \|\mathrm{S}_N\, p_2\|_\infty$ and $\hat{N}_2 = \deg(p_2)$ and we define

$$\varphi_2 = \varphi_1 + \tfrac{1}{R_2}\, p_2\ .$$

Assume we already defined p_{k-1}, $\check{N}_{k-1}$, $\hat{N}_{k-1}$, and R_{k-1}. Then we set $\check{N}_k = \hat{N}_{k-1} + 1$, $r_k = (3+k)^2$ and define the polynomial $p_k = f_{\check{N}_k,r_k}$ according to Lemma A.1 and which satisfies $\|p_k\|_\infty \leq 1$. Then we set $R_k = \max_{N\in\mathbb{N}} \|\mathrm{S}_N\, p_k\|_\infty$ and $\hat{N}_k = \deg(p_k)$ and define

$$\varphi_k = \varphi_{k-1} + \tfrac{1}{R_k}\, p_k = \textstyle\sum_{n=1}^{k} \tfrac{1}{R_n}\, p_n\ .$$

Notice that the sequences $\{\check{N}_n\}_{n\in\mathbb{N}}$ and $\{\hat{N}_n\}_{n\in\mathbb{N}}$ are related as follows

$$\cdots < \hat{N}_{n-1} < \check{N}_n < \hat{N}_n < \check{N}_{n+1} < \cdots , \qquad n \in \mathbb{N} .$$

So the intervals $[\check{N}_n, \hat{N}_n]$ are mutually disjoint, and since $[\check{N}_n, \hat{N}_n]$ is the support set of the Fourier coefficients of the trigonometric polynomials $\{p_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$, it is clear that these support sets are mutually disjoint, i.e.

$$\operatorname{supp}\left(\{c_n(p_k)\}_{n\in\mathbb{Z}}\right) \cap \operatorname{supp}\left(\{c_n(p_\ell)\}_{n\in\mathbb{Z}}\right) = \emptyset \qquad \text{for } k \neq \ell .$$

Moreover, since the polynomials p_n were constructed according to Lemma A.1, Properties *b*) and *d*) from this lemma yield in particular

$$\mathrm{S}_{N,M}\, p_m = \begin{cases} 0 & \text{for all } N < \check{N}_m \\ p_m & \text{for all } N > \hat{N}_m \end{cases} \qquad \text{for all } M \in \mathbb{N}_0 , \tag{A.2}$$

and since $\mathrm{S}_N = \mathrm{S}_{N,N}$ the above relation holds in particular if we replace $\mathrm{S}_{N,M}$ by S_N. Property (A.2) of the polynomials p_n will be used extensively during the remainder of this proof.

(2) Based on the previously constructed sequences, we define the function

$$f(t) = \textstyle\sum_{n=1}^{\infty} \frac{1}{R_n} p_n(t) , \qquad t \in \mathbb{T} . \tag{A.3}$$

Using that $R_n > r_n$ for every $n \in \mathbb{N}$ (cf. Point (c) of Lemma A.1), one obtains

$$\|f\|_\infty \leq \sum_{n=1}^{\infty} \frac{\|p_n\|_\infty}{r_n} \leq \sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{(3+n)^2} = \sum_{k=4}^{\infty} \frac{1}{k^2} \leq \int_3^\infty \frac{\mathrm{d}t}{t^2} = \frac{1}{3} , \tag{A.4}$$

showing that the sum, defining f, converges absolutely on $\mathbb{T}$. Next, we verify that $f \in U_{\mathrm{s}}$. To this end, let $N \in \mathbb{N}$ be arbitrary. By the construction of the sequences $\{\check{N}_n\}_{n\in\mathbb{N}}$ and $\{\hat{N}_n\}_{n\in\mathbb{N}}$ in Step 1, there exists exactly one $m \in \mathbb{N}$ such that $N \in [\check{N}_m, \hat{N}_m]$, and so (A.2) yields

$$\mathrm{S}_N f = \textstyle\sum_{n=1}^{\infty} \frac{1}{R_n} \mathrm{S}_N p_n = \sum_{n=1}^{m-1} \frac{1}{R_n} \mathrm{S}_N p_n + \frac{1}{R_m} \mathrm{S}_N p_m = \sum_{n=1}^{m-1} \frac{1}{R_n} p_n + \frac{1}{R_m} \mathrm{S}_N p_m .$$

Then the definition of the numbers R_n and the fact $\|p_n\|_\infty \leq 1$ give the bound

$$\begin{aligned} \|\mathrm{S}_N f\|_\infty &= \sum_{n=1}^{m-1} \frac{1}{R_n} \|p_n\|_\infty + \frac{1}{R_m} \|\mathrm{S}_N p_m\|_\infty \leq \sum_{n=1}^{\infty} \frac{1}{r_n} + \frac{1}{R_m} \max_{N\in\mathbb{N}} \|\mathrm{S}_N p_m\|_\infty \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{r_n} \leq 4/3 , \end{aligned}$$

using (A.4) to get the last equation. So $\|f\|_{U_{\mathrm{s}}} = \sup_{N\in\mathbb{N}} \|\mathrm{S}_N f\|_\infty \leq 4/3 < \infty$ showing that $f \in U_{\mathrm{s}}$.

(3) For an arbitrary $m \in \mathbb{N}$, (A.2) yields

$$\mathrm{S}_{\check{N}_m} f = \mathrm{S}_{\check{N}_m} \left(\sum_{n=1}^{\infty} \frac{1}{R_n} p_n \right) = \sum_{n=1}^{\infty} \frac{1}{R_n} \mathrm{S}_{\check{N}_m} p_n = \sum_{n=1}^{m-1} \frac{1}{R_n} \mathrm{S}_{\check{N}_m} p_n = \sum_{n=1}^{m-1} \frac{1}{R_n} p_n .$$

In particular, since $\|p_n\|_\infty \leq 1$ for every $n \in \mathbb{N}$, we get

$$\left\|\mathrm{S}_{\check{N}_m} f\right\|_\infty \leq \textstyle\sum_{n=1}^{m-1} \frac{1}{R_n} \leq \sum_{n=1}^{\infty} \frac{1}{r_n} \leq \frac{1}{3} < \infty \qquad \text{for all } m \in \mathbb{N} ,$$

showing that

$$\liminf_{N\to\infty} \|\mathrm{S}_N f\|_\infty \leq \textstyle\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty . \tag{A.5}$$

(4) As in Step 2, let $N \in \mathbb{N}$ be arbitrary. Then there exists exactly one $m \in \mathbb{N}$ such that $N \in [\check{N}_m, \hat{N}_m]$, and so (A.2) yields again

$$\mathrm{S}_N f = \sum_{n=1}^{\infty} \frac{1}{R_n} \mathrm{S}_N p_n = \sum_{n=1}^{m-1} \frac{1}{R_n} \mathrm{S}_N p_n + \frac{1}{R_m} \mathrm{S}_N p_m$$

and consequently

$$\|\mathrm{S}_N f\|_\infty \geq \frac{1}{r_m} \|\mathrm{S}_N p_m\|_\infty - \sum_{n=1}^{m-1} \frac{1}{r_n} \|\mathrm{S}_N p_n\|_\infty \geq \frac{1}{r_m} \|\mathrm{S}_N p_m\|_\infty - \sum_{n=1}^{m-1} \frac{1}{r_n} .$$

So by Part (c) of Lemma A.1, we have $\max_{N \in [\check{N}_n, \hat{N}_n]} \|\mathrm{S}_N f\|_\infty \geq 1 - \sum_{n=1}^{\infty} \frac{1}{r_n}$ showing that

$$\limsup_{N \to \infty} \|\mathrm{S}_N f\|_\infty \geq 1 - \sum_{n=1}^{\infty} \frac{1}{r_n} . \tag{A.6}$$

(5) Using (A.5) and (A.6), one obtains

$$\limsup_{N \to \infty} \|\mathrm{S}_N f\|_\infty - \liminf_{N \to \infty} \|\mathrm{S}_N f\|_\infty \geq 1 - 2 \sum_{n=1}^{\infty} \frac{1}{r_n} \geq 1 - \frac{2}{3} = \frac{1}{3} > 0 ,$$

applying (A.4) to get the numerical value on the right. This proves the statement for U_s.

The second statement of Lemma 2.3 holds for the same function f as constructed in (A.3). Since all Fourier coefficients $c_n(f)$ with $n < 0$ of f are equal to zero, one has $\mathrm{S}_{N,M} f = \mathrm{S}_N f$ for all $N, M \in \mathbb{N}_0$. In particular, $\|f\|_U = \|f\|_{U_s}$ showing that $f \in U$, and in all other equations of the above proof the operator S_N can be replaced by $\mathrm{S}_{N,M}$ without changing any of the conclusions. ■

Proof (*Theorem* 2.4)**.** The completeness of $\mathcal{U}$ and $\mathcal{U}_s$ follows already from Theorem 2.2. To verify the norm convergence of $\mathrm{S}_N f$ and $\mathrm{S}_{N,M} f$, we have to verify that the norms of the operators $\mathrm{S}_N : \mathcal{U}_s \to \mathcal{U}_s$ and $\mathrm{S}_{N,M} : \mathcal{U} \to \mathcal{U}$ are uniformly bounded. For the first operator, we have

$$\begin{aligned} \|\mathrm{S}_N\|_{\mathcal{U}_s \to \mathcal{U}_s} &= \sup_{\|f\|_{\mathcal{U}_s} \leq 1} \|\mathrm{S}_N f\|_{\mathcal{U}_s} = \sup_{\|f\|_{\mathcal{U}} \leq 1} \sup_{M \in \mathbb{N}} \|\mathrm{S}_M \mathrm{S}_N f\|_\infty = \sup_{\|f\|_{\mathcal{U}} \leq 1} \sup_{K \in \mathbb{N}} \|\mathrm{S}_K f\|_\infty \\ &= \sup_{\|f\|_{\mathcal{U}} \leq 1} \|f\|_{\mathcal{U}_s} = 1 , \end{aligned} \tag{A.7}$$

and for the second operator one obtains exactly the same result. Then the norm convergence of the Fourier series follows from the fact that the trigonometric polynomials $\mathcal{P}$ are dense in $\mathcal{U}_s$ and $\mathcal{U}$ and that the Fourier series converges for all $p \in \mathcal{P}$.

Next, we verify Statement (3). To this end, let $f \in \mathcal{U}_s$ be arbitrary. Then (14) and (15) imply $\lim_{N \to \infty} \|f - \mathrm{S}_N f\|_\infty = 0$ proving that $\mathcal{U}_s \subseteq \left\{ f \in U_s \ : \ \lim_{N \to \infty} \|f - \mathrm{S}_N f\|_\infty = 0 \right\}$. Conversely, let $f \in U_s$ be an arbitrary function which satisfies

$$\lim_{N \to \infty} \|f - \mathrm{S}_N f\|_\infty = 0 \tag{A.8}$$

and set $p_N = \mathrm{S}_N f \in \mathcal{P}$. Then for any $N \in \mathbb{N}$, one has

$$\begin{aligned} \|f - p_N\|_{U_s} &= \sup_{M \in \mathbb{N}} \|\mathrm{S}_M f - \mathrm{S}_M p_N\|_\infty = \sup_{M \in \mathbb{N}} \|\mathrm{S}_M f - \mathrm{S}_M \mathrm{S}_N f\|_\infty \\ &= \sup_{M \geq N} \|\mathrm{S}_M f - \mathrm{S}_N f\|_\infty \end{aligned}$$

because $\mathrm{S}_M p_N = \mathrm{S}_N p_N = \mathrm{S}_N f$ for every $M \geq N$ and because $\mathrm{S}_M f = \mathrm{S}_M \mathrm{S}_N f$ for all $M < N$. It follows from (A.8) that to every $\epsilon > 0$ there exists an $N_0 = N_0(\epsilon)$ such that $\|\mathrm{S}_M f - \mathrm{S}_N f\|_\infty < \epsilon$ for all $M, N \geq N_0$, and so one gets for $p_{N_0} = \mathrm{S}_{N_0} f$

$$\left\| f - p_{N_0} \right\|_{U_s} = \sup_{M \geq N} \left\| \mathrm{S}_M f - \mathrm{S}_{N_0} f \right\|_\infty < \epsilon .$$

So to the arbitrarily chosen $f \in U_{\mathrm{s}}$, which satisfies (A.8), and an arbitrary $\epsilon > 0$, we found a polynomial $p_{N_0} \in \mathcal{P}$ such that $\left\| f - p_{N_0} \right\|_{U_{\mathrm{s}}} < \epsilon$, proving that

$$\left\{ f \in U_{\mathrm{s}} \;:\; \lim_{N\to\infty} \| f - \mathrm{S}_N f \|_\infty \right\} \subseteq \overline{\operatorname{span}\{f \in \mathcal{P}\}}^{\|\cdot\|_{U_{\mathrm{s}}}} = \mathcal{U}_{\mathrm{s}} \;.$$

Thus $\mathcal{U}_{\mathrm{s}}$ is precisely the set of all $f \in \mathcal{C}(\mathbb{T})$ with uniformly convergent Fourier series.

It remains to verify that $\mathcal{U}_{\mathrm{s}}$ is a proper subset of U_{s}, i.e. that there exists an $f \in U_{\mathrm{s}}$ which belongs not to $\mathcal{U}_{\mathrm{s}}$. We prove the statement by contradiction. Let $f \in U_{\mathrm{s}}$ be arbitrary and assume that f belongs also to $\mathcal{U}_{\mathrm{s}}$. The polynomials $\mathcal{P}$ are dense in $\mathcal{U}_{\mathrm{s}}$. So to every $\epsilon > 0$ there exists a $p \in \mathcal{P}$ such that $\| f - p \|_{U_{\mathrm{s}}} < \epsilon/2$, and because of (14), we have $\| f - p \|_\infty < \epsilon/2$. Therewith, one obtains for every $N \in \mathbb{N}$

$$\begin{aligned}
&\| f - \mathrm{S}_N f \|_\infty \\
&\leq \| f - p \|_\infty + \| p - \mathrm{S}_N p \|_\infty + \| \mathrm{S}_N p - \mathrm{S}_N f \|_\infty \\
&\leq \epsilon/2 + \| p - \mathrm{S}_N p \|_\infty + \| \mathrm{S}_N (p - f) \|_\infty \leq \epsilon/2 + \| p - \mathrm{S}_N p \|_\infty + \| p - f \|_{U_{\mathrm{s}}} \\
&\leq 2 \cdot \epsilon/2 + \| p - \mathrm{S}_N p \|_\infty \;.
\end{aligned}$$

Because $p \in \mathcal{P}$ is a trigonometric polynomial, one has $\lim_{N\to\infty} \| p - \mathrm{S}_N p \|_\infty = 0$ showing that $\limsup_{N\to\infty} \| f - \mathrm{S}_N f \|_\infty \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\lim_{N\to\infty} \| f - \mathrm{S}_N f \|_\infty = 0$. In particular, because $f \in U_{\mathrm{s}}$ was arbitrary, we have $\lim_{N\to\infty} \| \mathrm{S}_N f \|_\infty = \| f \|_\infty$ for all $f \in U_{\mathrm{s}}$. But this contradicts Lemma 2.3 and so there exist functions $f \in U_{\mathrm{s}}$ which do not belong to $\mathcal{U}_{\mathrm{s}}$.

Therewith, Statement (3) of the theorem is proved. The proof of Statement (4) follows the same lines and is therefore omitted. ■

Proof (*Corollary* 2.5)**.** First, we rearrange the sequence $\{e_n\}_{n\in\mathbb{Z}}$ according to the corollary, i.e.

$$\widetilde{e}_n := \begin{cases} e_{-(n-1)/2} & : \quad n = 1, 3, 5, 7, \ldots \\ e_{n/2} & : \quad n = 2, 4, 6, 8, \ldots \;. \end{cases} \tag{A.9}$$

Therewith $\widetilde{\boldsymbol{e}} = \{\widetilde{e}_n\}_{n\in\mathbb{N}_0}$ and we have to show that this sequence is a basis for $\mathcal{U}_{\mathrm{s}}$. To this end, we consider the partial sum operators $\widetilde{\mathrm{S}}_N : \mathcal{U}_{\mathrm{s}} \to \mathcal{U}_{\mathrm{s}}$ associated with $\widetilde{\boldsymbol{e}}$ and given by

$$\widetilde{\mathrm{S}}_N f = \textstyle\sum_{n=0}^{N} \widetilde{c}_n(f)\, e_n \;, \qquad N \in \mathbb{N}_0 \;,$$

wherein the coefficients $\{\widetilde{c}_n(f)\}_{n\in\mathbb{N}_0}$ are defined in terms of the Fourier coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$ of f in the same way as $\{\widetilde{e}_n\}_{n\in\mathbb{N}_0}$ was defined in (A.9) in terms of $\{e_n\}_{n\in\mathbb{Z}}$. One easily verifies that these operators are related to the symmetric partial Fourier series (3) by

$$\begin{aligned} \widetilde{\mathrm{S}}_{2K+1} f &= \mathrm{S}_K f && \text{for} \quad K = 0, 1, 2, 3, \ldots \\ \widetilde{\mathrm{S}}_{2K} f &= \mathrm{S}_{K-1} f + c_K(f)\, e_K && \text{for} \quad K = 1, 2, 3, 4, \ldots \;. \end{aligned} \tag{A.10}$$

It is well known that $\widetilde{\boldsymbol{e}}$ is a Schauder basis for $\mathcal{U}_{\mathrm{s}}$ if and only if the partial sum operators are uniformly bounded. Using (A.7), it follows from (A.10) that $\left\| \widetilde{\mathrm{S}}_N \right\|_{\mathcal{U}_{\mathrm{s}}\to\mathcal{U}_{\mathrm{s}}} = 1$ for all odd N. For even indices, i.e. for $N = 2K$ with $K \in \mathbb{N}$, the second line of (A.10) yields

$$\left\| \widetilde{\mathrm{S}}_{2K} \right\|_{\mathcal{U}_{\mathrm{s}}\to\mathcal{U}_{\mathrm{s}}} = \sup_{\|f\|_{\mathcal{U}_{\mathrm{s}}}\leq 1} \left\| \widetilde{S}_{2K} f \right\|_{\mathcal{U}_{\mathrm{s}}} \leq \sup_{\|f\|_{\mathcal{U}_{\mathrm{s}}}\leq 1} \left\| \widetilde{S}_{K-1} f \right\|_{\mathcal{U}_{\mathrm{s}}} + \sup_{\|f\|_{\mathcal{U}_{\mathrm{s}}}\leq 1} |c_K(f)| \; \|e_K\|_{\mathcal{U}_{\mathrm{s}}} \leq 2$$

using again (A.7) and that $\|e_K\|_{\mathcal{U}_{\mathrm{s}}} = 1$ for all $K \in \mathbb{N}_0$ and noting that (14) implies $|c_K(f)| \leq \|f\|_\infty \leq \|f\|_{\mathcal{U}_{\mathrm{s}}}$ for all $K \in \mathbb{N}_0$. So the norms of the partial sum operators $\widetilde{S}_N$ are uniformly bounded for all $N \in \mathbb{N}_0$, proving that $\widetilde{\boldsymbol{e}}$ is a Schauder basis for $\mathcal{U}_{\mathrm{s}}$. ■

Proof (*Lemma* 2.6)**.** Let $t_0 \in \mathbb{T}$ be arbitrary. For any $N \in \mathbb{N}$, we define the functional $\Psi_{t_0,N} : \mathcal{S} \to \mathbb{C}$ by $\Psi_{t_0,N}(f) = (\mathrm{S}_N f)(t_0)$. By the assumption of the corollary, it follows that $\sup_{N\in\mathbb{N}} \left|\Psi_{t_0,N}(f)\right| = \sup_{N\in\mathbb{N}} |(\mathrm{S}_N f)(t_0)| < \infty$. Then the uniform boundedness principle (Theorem of Banach–Steinhaus) implies that $\sup_{N\in\mathbb{N}} \left\|\Psi_{t_0,N}\right\| = C(\mathcal{S}, t_0) < \infty$. Moreover, since $\mathcal{S}$ is shift-invariant, the constant on the right hand side is in fact independent on t_0, i.e. there is a $C(\mathcal{S}) < \infty$ such that $\sup_{N\in\mathbb{N}} \left\|\Psi_{t,N}\right\| \leq C(\mathcal{S})$ for all $t \in \mathbb{T}$. So for a fixed $N \in \mathbb{N}$, the operator norm of S_N is

$$\|\mathrm{S}_N\|_{\mathcal{S}\to\mathcal{C}(\mathbb{T})} = \sup_{f\in\mathcal{S}} \sup_{t\in\mathbb{T}} \frac{|(\mathrm{S}_N f)(t)|}{\|f\|_{\mathcal{S}}} = \sup_{t\in\mathbb{T}} \sup_{f\in\mathcal{S}} \frac{\left|\Psi_{t,N}(f)\right|}{\|f\|_{\mathcal{S}}} = \sup_{t\in\mathbb{T}} \left\|\Psi_{t,N}\right\| = C(\mathcal{S}) ,$$

and since the right hand side is independent on N, one has $\|f\|_{\mathcal{U}_s} = \sup_{N\in\mathbb{N}} \|\mathrm{S}_N f\|_\infty = C(\mathcal{S})$, i.e. $\|f\|_{\mathcal{U}_s} \leq C(\mathcal{S}) \|f\|_{\mathcal{S}}$ for all $f \in \mathcal{S}$ which proves the lemma. ■

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