

Full length article

On spectral approximation, Følner sequences and crossed products[☆]

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Abstract

In this article we study Følner sequences for operators and mention their relation to spectral approximation problems. We construct a canonical Følner sequence for the crossed product of a discrete amenable group Γ with a concrete C^* -algebra \mathcal{A} with a Følner sequence. We also state a compatibility condition for the action of Γ on \mathcal{A} . We illustrate our results with two examples: the rotation algebra (which contains interesting operators like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs) and the C^* -algebra generated by bounded Jacobi operators. These examples can be interpreted in the context of crossed products. The crossed products considered can be also seen as a more general frame that included the set of generalized band-dominated operators.

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1. Introduction

Given a sequence of linear operators $\{T_n\}_{n \in \mathbb{N}}$ in a complex separable Hilbert space \mathcal{H} that approximates an operator T in a suitable sense, a natural question is how do the spectral objects of T (the spectrum, spectral measures, numerical ranges, pseudospectra etc.) relate with those of T_n as n grows. (Excellent books that include a large number of examples and references are, e.g., [19,1].) In such generality this problem is almost impossible to address. In fact, one can easily produce examples with bad spectral approximation behavior, where the spectrum suddenly expands or contracts in the limit (see, e.g., pp. 289–291 in [43]). Also spectral pollution effects or spurious eigenvalues can appear as a consequence of the approximation process (cf. [23]). One of the standard methods to treat these problems is to compress T to a finite dimensional subspace \mathcal{H}_n (the so-called finite-section) and, then, analyze the behavior of the eigenvalues of the matrix $P_n T \upharpoonright \mathcal{H}_n$ in the limit of large n . This method requires also additional conditions in order to guarantee a good approximation behavior of spectral objects.

The following classical approximation result for Toeplitz operators due to Szegő gives an example where the finite section method can be used to approximate spectral measures. The following result can be seen as a distributive version of Szegő's classical limit theorems that involve determinants of Toeplitz matrices: denote by \mathbb{T} the unit circle with normalized Haar measure $d\theta$ and consider the real-valued functions g in $L^\infty(\mathbb{T})$ which can be thought as (selfadjoint) multiplication operators on the complex Hilbert space $\mathcal{H} := L^2(\mathbb{T})$, i.e., $M_g \varphi = g \varphi$, $\varphi \in \mathcal{H}$. Denote by P_n the finite-rank orthogonal projection onto the linear span of $\{z^l \mid z \in \mathbb{T}, l = 0, \dots, n\}$ and let $M_g^{(n)}$ be the corresponding finite section matrix. Write the corresponding eigenvalues (repeated according to multiplicity) as $\{\lambda_{0,n}, \dots, \lambda_{n,n}\}$. Then, for any continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \left(f(\lambda_{0,n}) + \dots + f(\lambda_{n,n}) \right) = \int_{\mathbb{T}} f(g(\theta)) d\theta \quad (1.1)$$

(see [50, Section 8], [25, Chapter 5] and [54] for a careful analysis of this result; a recent standard book analyzing many aspects of Toeplitz operators and containing a large number of references is [13]). The Eq. (1.1) may be also reformulated in terms of weak*-convergence of the corresponding spectral measures and it allows to approximate numerically the spectrum of M_g in terms of the eigenvalues its finite sections (see [5] as well as Chapter 7 in [44] and references cited therein).

Szegő's classical result suggests the following question: what is the reason that guarantees the convergence of spectral measures and that can be possibly useful beyond the context of Toeplitz operators? In the last two decades there has been a considerable application of methods from operator algebras (mainly C^* -algebras and von Neumann algebras¹) to this question. In Section 2.1 we will mention the Følner algebra, a unital C^* -algebra which can be naturally defined for the finite section method. Moreover, there are other natural C^* -algebras that can be associated to sequences of finite sections (see Section 2 in [45] and references cited therein).

Arveson's seminal series of articles [3–5] on these topics were directly inspired by Szegő's theorem (see, e.g., [11,32] for related developments in numerical analysis). Among other interesting results, Arveson gave conditions that guarantee that the essential spectrum of a large

¹ For the purposes of this article we will define a C^* -algebra to be a $*$ -subalgebra of $\mathcal{L}(\mathcal{H})$ (the set of bounded linear operators in \mathcal{H}) which is closed in the topology defined by the operator norm $\|\cdot\|$. If $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ we will denote by $C^*(\mathcal{T})$ the C^* -algebra generated by \mathcal{T} . A von Neumann algebra is an important subclass of the class of C^* -algebras which is closed in the weak operator topology.

class of selfadjoint operators T may be recovered from the sequence of eigenvalues of certain finite dimensional compressions T_n (see also Section 4.2). These results were then refined by Bédos who systematically applied the concept of Følner sequence to spectral approximation problems (see [6–8] as well as [26]). It is stated in Section 7.2 of [26] that SeLegue also considered Szegő-type theorems for Toeplitz operators in the context of C^* -algebras. Hansen extends some of the mentioned results to the case of unbounded operators (cf. [29, Section 7]; see also [30] for recent developments in the non-selfadjoint case). Brown shows in [15] that abstract results in C^* -algebra theory can be applied to compute spectra of important operators in mathematical physics like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs.

We recall next two related notions that are important when addressing spectral approximation problems: let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of bounded linear operators on the complex separable Hilbert space \mathcal{H} . A sequence of non-zero finite rank orthogonal projections $\{P_i\}_{i \in \mathbb{N}}$ is called a Følner sequence for \mathcal{T} , if

$$\lim_{i \rightarrow \infty} \frac{\|TP_i - P_iT\|_2}{\|P_i\|_2} = 0, \quad T \in \mathcal{T}, \quad (1.2)$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. We call $\{P_i\}_{i \in \mathbb{N}}$ a proper Følner sequence if, in addition to (1.2), it is increasing and converges strongly to $\mathbb{1}$.

The existence of a proper Følner sequence for a set of operators \mathcal{T} is a weaker notion than quasidiagonality which was introduced by Halmos in the late sixties (cf. [28]). Recall that a set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists an increasing sequence of finite-rank projections $\{P_i\}_i$ converging strongly to $\mathbb{1}$ and such that

$$\lim_{i \rightarrow \infty} \|TP_i - P_iT\| = 0, \quad T \in \mathcal{T}.$$

It is easy to show that if $\{P_i\}_{i \in \mathbb{N}}$ quasidiagonalizes the set of operators \mathcal{T} , then it is also a proper Følner sequence for \mathcal{T} (for details see the next section). Moreover, Eq. (1.2) can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the corresponding subspaces. Quasidiagonality is an important property in the analysis of the structure of C^* -algebras (see, e.g., Chapter 7 in [18] or [14,9,16,53]) and is also a very useful notion in spectral approximation problems. E.g. quasidiagonality is assumed to prove the convergence of spectra (in the selfadjoint case) and pseudospectra (cf. [17] and [29, Section 2]). Følner sequences were introduced in the context of operator algebras by Alain Connes in his seminal article [20, Section V] (see also [21,40,41]). This notion is an algebraic analogue of Følner’s characterization of amenable discrete groups (see Section 2 for precise definitions) and was used by Connes as an essential tool in the classification of injective type II_1 factors. The notion of proper Følner sequence is also interesting in the analysis of non-normal operators on Hilbert spaces and in relation of so-called finite operators (cf. [35]). In Section 2 we state the main properties of Følner sequences for operators that will be needed later and mention the relation to quasidiagonality.

The third expression in the title of this article refers to crossed products. This is a basic operator algebraic construction which is interesting in its own right (see Section 3 for details). The crossed product may be seen as a new C^* -algebra constructed from a given C^* -algebra which carries an action of a group Γ . Many important operators in mathematical physics with very interesting spectral properties can be identified as elements of certain crossed products (see, e.g., [10,33] and Section 4). The question when a crossed product of a quasidiagonal C^* -algebra by an amenable group is again quasidiagonal has been addressed several times in

the past (see, e.g., Section 11 in [14] for some partial answers). In Lemma 3.6 of [39] it is shown that if \mathcal{A} is a unital separable quasidiagonal C^* -algebra with almost periodic group action $\alpha: \mathbb{Z} \rightarrow \text{Aut } \mathcal{A}$, then the C^* -crossed-product $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}$ is again quasidiagonal. This result has been extended recently by Orfanos to C^* -crossed products $\mathcal{A} \rtimes_{\alpha} \Gamma$, where now Γ is a discrete, countable, amenable, residually finite group (cf. [37]). In this more general case there is again a certain compatibility condition on the group action of Γ on \mathcal{A} is needed. This result has been applied to generalized Bunce–Deddens algebras in [36].

The existence of Følner sequences may be established in abstract terms, but this gives in general no clues of what are the concrete matrix approximations. The aim of the present article is to give conditions and construct explicitly Følner sequences for the crossed product of a C^* -algebra \mathcal{A} that has a Følner sequence and a discrete countable amenable group Γ (see Theorem 3.4 for a precise statement). We will state a sufficient condition for the group action on the C^* -algebra \mathcal{A} relative to the choices of Følner sequences for the discrete group Γ . Our results partly extend those of Bédos for group von Neumann algebras and are related to the articles of Orfanos mentioned above. In Section 4 we will illustrate our results in two well-known examples: Theorem 3.4 can be applied to the rotation algebra (also known as non-commutative torus), since it can be seen as a crossed product of $C(\mathbb{T})$ by \mathbb{Z} . This algebra contains interesting examples from the spectral point of view, like almost Mathieu operators or discrete Schrödinger operators with magnetic potentials, that have been widely studied in the literature (see, e.g., [10] and references cited therein). The rotation algebra provides also a non-trivial example where we can verify the compatibility condition stated in Theorem 3.4. The second example refers to the C^* -algebra generated by bounded Jacobi operators on $\ell^2(\mathbb{N})$. Also in this case we can have an interpretation in terms of crossed products and we will mention some well-known results for this class of operators (cf. [44]). We conclude the article pointing out in Section 5 that Theorem 3.4 extends the class of generalized band-dominated operators as considered, e.g., in [46,42].

2. Følner sequences and quasidiagonality

The notion of Følner sequences for operators has its origins in group theory. Recall that a discrete countable group Γ is amenable if it has an invariant mean, i.e., there is a continuous linear functional ψ on $\ell^\infty(\Gamma)$ with norm one and such that

$$\psi(u_\gamma f) = \psi(f), \quad \gamma \in \Gamma, \quad f \in \ell^\infty(\Gamma),$$

where u is the left-regular representation on $\ell^2(\Gamma)$. A Følner sequence for Γ is a sequence of non-empty finite subsets $F_i \subset \Gamma$ that satisfy

$$\lim_i \frac{|(\gamma F_i) \Delta F_i|}{|F_i|} = 0 \quad \text{for all } \gamma \in \Gamma, \quad (2.1)$$

where Δ denotes the symmetric difference and $|F_i|$ is the cardinality of F_i . Then, Γ has a Følner sequence if and only if Γ is amenable (cf. Chapter 4 in [38]). Some authors require, in addition to Eq. (2.1), that the sequence is increasing and complete, i.e., $F_i \subset F_j$ if $i \leq j$ and $\Gamma = \cup_i F_i$. We will not need these additional assumptions here.

The counterpart of the previous definition in the context of operators is given as follows:

Definition 2.1. Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of bounded linear operators on the complex separable Hilbert space \mathcal{H} . A sequence of non-zero finite rank orthogonal projections $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ is

called a Følner sequence for \mathcal{T} if

$$\lim_n \frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = 0, \quad T \in \mathcal{T}, \quad (2.2)$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. If the Følner sequence $\{P_n\}_{n \in \mathbb{N}}$ satisfies, in addition, that it is increasing and converges strongly to $\mathbb{1}$, then we say it is a proper Følner sequence.

We will state next some immediate consequences of the definition that will be used later on. To simplify expressions in the rest of the article we introduce the commutator of two operators: $[A, B] := AB - BA$.

Proposition 2.2. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators and $\{P_n\}_{n \in \mathbb{N}}$ a sequence of non-zero finite rank orthogonal projections.*

- (i) *$\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{T} iff it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$ (the C^* -algebra generated by \mathcal{T} and the identity $\mathbb{1}$).*
- (ii) *Let \mathcal{T} be a selfadjoint set (i.e., $\mathcal{T}^* = \mathcal{T}$). Then $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{T} if one of the four following equivalent conditions holds for all $T \in \mathcal{T}$:*

$$\lim_n \frac{\|TP_n - P_nT\|_p}{\|P_n\|_p} = 0, \quad p \in \{1, 2\} \quad (2.3)$$

or

$$\lim_n \frac{\|(I - P_n)TP_n\|_p}{\|P_n\|_p} = 0, \quad p \in \{1, 2\}, \quad (2.4)$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the trace-class and Hilbert–Schmidt norms, respectively.

Proof. (i) We just have to show that if $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{T} and $\mathbb{1}$, then it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$. For $R, T \in \mathcal{T}$ the following elementary relations

$$\begin{aligned} \|[RT, P_n]\|_2 &\leq \|R[T, P_n]\|_2 + \|[R, P_n]T\|_2 \leq \|R\| \|[T, P_n]\|_2 + \|[R, P_n]\|_2 \|T\| \\ \|[T^*, P_n]\|_2 &= \|[T, P_n]^*\|_2 = \|[T, P_n]\|_2 \end{aligned}$$

show that $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for the $*$ -algebra $\tilde{\mathcal{T}}$ generated by \mathcal{T} . Then it is a standard $\frac{\varepsilon}{2}$ -argument to show that $\{P_n\}_{n \in \mathbb{N}}$ is still a Følner sequence for the norm closure of $\tilde{\mathcal{T}}$.

(ii) By the previous item we have that $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{T} iff it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$ and we can apply Lemma 1 in [8]. \square

The existence of a proper Følner sequence for an operator $T \in \mathcal{L}(\mathcal{H})$ has the following absorbing property for direct sums. The proof of this fact is from the first versions of [35].

Proposition 2.3. *Let \mathcal{H} and \mathcal{H}' be separable Hilbert spaces with $\dim \mathcal{H} = \infty$. If T has a proper Følner sequence, then $T \oplus X \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$ has a proper Følner sequence for any $X \in \mathcal{L}(\mathcal{H}')$.*

Proof. Let $\{P_n\}_{n \in \mathbb{N}}$ be a proper Følner sequence for T and since the sequence of projections is increasing we may assume that $\dim P_n \mathcal{H} \geq n^2$. Let $\{e_i \mid i \in \mathbb{N}\}$ be an orthonormal basis of \mathcal{H}' and denote by Q_n the orthogonal projection onto $\text{span}\{e_1, \dots, e_n\} \subset \mathcal{H}'$. Then the following

calculation shows that $\{P_n \oplus Q_n\}_n$ is a proper Følner sequence for $T \oplus X$, $X \in \mathcal{L}(\mathcal{H}')$:

$$\begin{aligned} \frac{\|[T \oplus X, P_n \oplus Q_n]\|_2^2}{\|P_n \oplus Q_n\|_2^2} &= \frac{\|[T, P_n]\|_2^2 + \|[X, Q_n]\|_2^2}{\|P_n\|_2^2 + n} \\ &\leq \frac{\|[T, P_n]\|_2^2}{\|P_n\|_2^2} + \frac{4\|Q_n\|_2^2 \|X\|^2}{n^2 + n} \\ &= \frac{\|[T, P_n]\|_2^2}{\|P_n\|_2^2} + 4\|X\| \frac{n}{n^2 + n} \rightarrow 0 \end{aligned}$$

and the proof is concluded. \square

The existence of a Følner sequence has important structural consequences. For the next result we need to recall the following notion: a hypertrace for a C^* -algebra \mathcal{A} acting on a Hilbert space \mathcal{H} is a state Ψ on $B(\mathcal{H})$ that is centralized by \mathcal{A} , i.e.

$$\Psi(XA) = \Psi(AX), \quad X \in B(\mathcal{H}), \quad A \in \mathcal{A}.$$

Hypertraces are the algebraic analogue of the invariant mean mentioned at the beginning of this section (cf. [20,21,8]). We mention some easy operator algebraic consequence of the existence of a Følner sequence for a C^* -algebra (see also [7,2]).

Proposition 2.4. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable C^* -algebra. If \mathcal{A} has a Følner sequence, then \mathcal{A} has a hypertrace.*

Example 2.5. (i) The unilateral shift is a canonical example that shows the difference between the notions of Følner sequences and quasidiagonality. On the one hand, it is a well-known fact that the unilateral shift S is not a quasidiagonal operator. (This was shown by Halmos in [27]; in fact, in this reference it is shown that S is not even quasitriangular.) If \mathcal{A} is a (unital) C^* -algebra containing a proper (i.e., non-unitary) isometry, then it *not* quasidiagonal (see, e.g. [14,18]). Finally, it can be shown that certain weighted shifts are quasidiagonal (cf. [47]).

On the other hand, it is easy to find a canonical proper Følner sequence for S . In fact, define S on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of \mathcal{H} and consider for any n the orthogonal projections P_n onto $\text{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$. Then

$$\|[P_n, S]\|_2^2 = \sum_{i=1}^{\infty} \|[P_n, S]e_i\|^2 = \|e_{n+1}\|^2 = 1$$

and

$$\frac{\|[P_n, S]\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n+1}} \xrightarrow{n \rightarrow \infty} 0.$$

A similar argument shows directly that $\{P_n\}_n$ is a proper Følner sequence for any power S^k , $k \in \mathbb{N}$. By Proposition 2.2(i) it follows that $\{P_n\}_n$ is also a proper Følner sequence for the Toeplitz algebra $C^*(S)$.

(ii) Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator. If a sequence of non-zero finite rank orthogonal operators $\{P_n\}_n$ satisfies

$$\sup_{n \in \mathbb{N}} \|(\mathbb{1} - P_n)T P_n\|_2 < \infty,$$

then Eq. (2.4) implies that $\{P_n\}_n$ is clearly a Følner sequence for T . Concrete examples satisfying the preceding condition are selfadjoint operators with a band-limited matrix representation (see, e.g., [4,5]). Band limited operators together with quasideagonal operators are the essential ingredients in the solution of Herrero's approximation problem, i.e. the characterization of the closure of block diagonal operators with bounded blocks (see Chapter 16 in [18] for a comprehensive presentation).

2.1. Følner algebra

In the study of growth properties of C^* -algebras (and motivated by previous work done by Arveson and Bédos) Vaillant defined the following natural unital C^* -algebra (see Section 3 in [52]): given an increasing sequence $\mathcal{P} := \{P_n\}_n \subset \mathcal{L}(\mathcal{H})$ of orthogonal finite rank projections strongly converging to $\mathbb{1}$, consider the set of all bounded linear operators in \mathcal{H} that have \mathcal{P} as a proper Følner sequence, i.e.,

$$\mathcal{F}_{\mathcal{P}}(\mathcal{H}) := \left\{ X \in \mathcal{L}(\mathcal{H}) \mid \lim_{n \rightarrow \infty} \frac{\|XP_n - P_nX\|_2}{\|P_n\|_2} = 0 \right\}.$$

This unital C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ (called Følner algebra by Hagen, Roch and Silbermann in Section 7.2.1 of [26]) has shown to be very useful in the analysis of the classical Szegő limit theorems for Toeplitz operators and some generalizations of them (see, e.g., Section 7.2 of [26,12]).

In general, $\mathcal{F}_{\mathcal{P}}$ will not be separable for the operator norm. This can be easily seen from the absorbing properties of proper Følner sequences for direct sums as shown in the proof of Proposition 2.3. In fact, it is possible that a Følner algebra contains a full copy of some $\mathcal{L}(\mathcal{H})$. Finally, let us mention that subalgebras of the Følner algebra are also interesting from an abstract operatoralgebraic point of view. We refer to [7,2] for further developments in this direction.

3. Følner sequences for crossed products

The crossed product may be seen as a new C^* -algebra constructed from a given C^* -algebra which carries an action of a group Γ . This procedure goes back to the pioneering work of Murray and von Neumann on rings of operators. Algebraically, this construction has some similarities with the semi-direct product of groups. Standard references which present the crossed product construction with small variations are [22, Chapter VIII], [49, Chapter 4], [51, Section V.7] or [31, Section 8.6 and Chapter 13]. Since all groups Γ considered will be amenable (and countable) we will not distinguish between crossed products and reduced crossed products.

Throughout this section \mathcal{A} denotes a concrete C^* -algebra acting on a complex separable Hilbert space \mathcal{H} . We shall assume that α is an automorphic representation of a countable discrete amenable group Γ on \mathcal{A} , i.e.

$$\alpha: \Gamma \rightarrow \text{Aut } \mathcal{A}.$$

The crossed product is a new C^* -algebra constructed with the previous ingredients and acting on the separable Hilbert space

$$\mathcal{K} := \ell^2(\Gamma) \otimes \mathcal{H} \cong \oplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma}, \quad (3.1)$$

where $\mathcal{H}_\gamma \equiv \mathcal{H}$ for all $\gamma \in \Gamma$. To make this notion precise we introduce representations of \mathcal{A} and Γ on \mathcal{K} : for $\xi = (\xi_\gamma)_\gamma \in \mathcal{K}$ we define

$$(\pi(M)\xi)_\gamma := \alpha_\gamma^{-1}(M) \xi_\gamma, \quad M \in \mathcal{A}, \quad (3.2)$$

$$(U(\gamma_0)\xi)_\gamma := \xi_{\gamma_0^{-1}\gamma}. \quad (3.3)$$

The crossed product of \mathcal{A} by Γ is the C^* -algebra on \mathcal{K} generated by these operators, i.e.,

$$\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma := C^*\left(\{\pi(M) \mid M \in \mathcal{A}\} \cup \{U(\gamma) \mid \gamma \in \Gamma\}\right) \subset \mathcal{L}(\mathcal{K}),$$

where $C^*(\cdot)$ denotes the C^* -algebra generated by its argument. A characteristic relation for the crossed product is

$$\pi(\alpha_\gamma(M)) = U(\gamma)\pi(M)U(\gamma)^{-1}, \quad (3.4)$$

that is, π is a covariant representation of the C^* -dynamical system $(\mathcal{A}, \Gamma, \alpha)$.

Remark 3.1. Later we will need the following useful operator matrix characterization of the elements in the crossed product: consider the identification $\mathcal{K} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ with $\mathcal{H}_\gamma \equiv \mathcal{H}$, $\gamma \in \Gamma$. Then, every $T \in \mathcal{L}(\mathcal{K})$ can be written as a matrix $(T_{\gamma'\gamma})_{\gamma', \gamma \in \Gamma}$ with entries $T_{\gamma'\gamma} \in \mathcal{L}(\mathcal{H})$. Any element N in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ has the form

$$N_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(A(\gamma'\gamma^{-1})\right), \quad \gamma', \gamma \in \Gamma, \quad (3.5)$$

for some mapping $A: \Gamma \rightarrow \mathcal{A} \subset \mathcal{L}(\mathcal{H})$. Roughly, this means that the “diagonals” of the operator matrices of elements in the crossed product \mathcal{N} are orbits of the group action on elements in \mathcal{A} .

For example, the matrix form of the product of generators $N := \pi(M) \cdot U(\gamma_0)$, $M \in \mathcal{A}$, $\gamma_0 \in \Gamma$ is given by

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M) \delta_{\gamma', \gamma_0\gamma} = \alpha_{\gamma'}^{-1}\left(A(\gamma'\gamma^{-1})\right), \quad \text{where } A(\tilde{\gamma}) := \begin{cases} M & \text{if } \tilde{\gamma} = \gamma_0 \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

This implies that any function $A: \Gamma \rightarrow \mathcal{A}$ with finite support determines by means of Eq. (3.5) an element in the crossed product.

3.1. Construction of a canonical Følner sequence

The aim of the present section is to give a canonical example of a Følner sequence for the crossed product C^* -algebra $\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma$ constructed above. Since $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ with $\mathcal{K} = \ell^2(\Gamma) \otimes \mathcal{H}$, our sequence is canonical in the sense that it uses explicitly a Følner sequence for Γ and a sequence of projections on \mathcal{H} (cf. Theorem 3.4). We will also assume that the unital C^* -algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is separable and has a Følner sequence $\{Q_i\}_{i \in \mathbb{N}}$.

We begin recalling some parts of Proposition 4 in [8]:

Proposition 3.2. Assume that the group Γ is countable and amenable and denote by $\{P_i\}_{i \in \mathbb{N}}$ the sequence of orthogonal finite-rank projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$ associated to a Følner sequence $\{\Gamma_i\}_{i \in \mathbb{N}}$ for the group Γ (cf. Section 2). Then $\{P_i\}_i$ is a Følner sequence for the group C^* -algebra

$$\mathcal{A}_\Gamma := C^*\{\overline{U}(\gamma) \mid \gamma \in \Gamma\} \subset \mathcal{L}(\ell^2(\Gamma)),$$

where \overline{U} is the left regular representation of Γ on $\ell^2(\Gamma)$.

Remark 3.3. (i) In Proposition 4 of [8] the author proves a stronger result. He shows that the canonical Følner net $\{P_i\}_{i \in \mathbb{N}}$ for the algebra and associated to the Følner net of the (not necessarily countable) amenable group is still a Følner net for the corresponding group von Neumann algebra, i.e., for the weak operator closure of \mathcal{A}_Γ in $\mathcal{L}(\ell^2(\Gamma))$. In general, it is not true that a Følner sequence for a concrete C^* -algebra is also a Følner sequence for its weak closure.

(ii) Recall that the preceding proposition means that the sequence $\{P_i\}_i$ satisfies the four equivalent conditions in Proposition 2.2(ii) for any element in the group von Neumann algebra.

Theorem 3.4. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable C^* -algebra which has a Følner sequence $\{Q_i\}_{i \in \mathbb{N}}$. Consider the countable and amenable group Γ and denote by $\{P_i\}_{i \in \mathbb{N}}$ the sequence of orthogonal finite-rank projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$ associated to a Følner sequence $\{\Gamma_i\}_{i \in \mathbb{N}}$ for the group Γ . Assume that there is an action of Γ on \mathcal{A} that satisfies:

$$\lim_i \left(\max_{\gamma \in \Gamma_i} \frac{\| [Q_i, \alpha_\gamma^{-1}(M)] \|_2}{\| Q_i \|_2} \right) = 0, \quad \text{for all } M \in \mathcal{A}. \quad (3.7)$$

Then the sequence $\{R_i\}_{i \in \mathbb{N}}$ with

$$R_i := P_i \otimes Q_i$$

is a Følner sequence for the crossed product $\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma$, i.e. the four equivalent conditions in Proposition 2.2(ii) are satisfied.

Proof. Step 1: we consider the identification $\mathcal{K} \cong \oplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$, $\mathcal{H}_\gamma \equiv \mathcal{H}$. In this case any element N in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ can be seen as a matrix of the form

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1} \left(A(\gamma'\gamma^{-1}) \right), \quad \gamma', \gamma \in \Gamma, \quad (3.8)$$

where $\gamma \mapsto A(\gamma)$ is a mapping from $\Gamma \rightarrow \mathcal{A}$ (cf. Remark 3.1). Moreover, defining the unitary map

$$W: \ell^2(\Gamma) \otimes \mathcal{H} \rightarrow \oplus_{\gamma \in \Gamma} \mathcal{H}_\gamma, \quad \xi \otimes \varphi \mapsto (\xi_\gamma \varphi)_{\gamma \in \Gamma}$$

it is straightforward to compute the matrix form of the projections $R_i = P_i \otimes Q_i \in \mathcal{L}(\mathcal{K})$:

$$(\widehat{R}_i)_{\gamma'\gamma} := (W R_i W^*)_{\gamma'\gamma} = \begin{cases} Q_i \delta_{\gamma'\gamma}, & \gamma', \gamma \in \Gamma_i \\ 0, & \text{otherwise.} \end{cases}$$

The commutator of \widehat{R}_i with any $N \in \mathcal{N}$ is

$$[\widehat{R}_i, N]_{\gamma'\gamma} = \begin{cases} [Q_i, N_{\gamma'\gamma}], & \gamma', \gamma \in \Gamma_i \\ Q_i N_{\gamma'\gamma}, & \gamma \notin \Gamma_i, \gamma' \in \Gamma_i \\ -N_{\gamma'\gamma} Q_i, & \gamma \in \Gamma_i, \gamma' \notin \Gamma_i \\ 0, & \gamma \notin \Gamma_i, \gamma' \notin \Gamma_i. \end{cases}$$

Step 2: we will check first the Følner condition on the product of generating elements $\pi(M) U(\gamma_0)$, $\gamma_0 \in \Gamma$, $M \in \mathcal{A}$ (cf. Eqs. (3.2) and (3.3)). The corresponding matrix elements are given according to Eq. (3.6) by

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M) \delta_{\gamma', \gamma_0 \gamma}.$$

Evaluating the commutator with \widehat{R}_i on the basis elements $\{e_l f_\gamma\}_{l,\gamma}$, with $f_\gamma(\gamma') := \delta_{\gamma\gamma'}$, we get

$$[\widehat{R}_i, (\pi(M)U(\gamma_0))] e_l f_\gamma = \begin{cases} [Q_i, \alpha_{\gamma_0\gamma}^{-1}(M)] e_l f_{\gamma_0\gamma}, & \gamma \in (\gamma_0^{-1}\Gamma_i) \cap \Gamma_i \\ Q_i \alpha_{\gamma_0\gamma}^{-1}(M) e_l f_{\gamma_0\gamma}, & \gamma \in (\gamma_0^{-1}\Gamma_i) \setminus \Gamma_i \\ -\alpha_{\gamma_0\gamma}^{-1}(M) Q_i e_l f_{\gamma_0\gamma}, & \gamma \in \Gamma_i \setminus (\gamma_0^{-1}\Gamma_i) \\ 0, & \gamma \notin \Gamma_i, \gamma \notin \gamma_0^{-1}\Gamma_i. \end{cases} \quad (3.9)$$

From this we obtain the following estimates in the Hilbert–Schmidt norm:

$$\begin{aligned} \|[\widehat{R}_i, \pi(M)U(\gamma_0)]\|_2^2 &= \sum_{l,\gamma} \|[\widehat{R}_i, (\pi(M)U(\gamma_0))] e_l f_\gamma\|^2 \\ &\leq \sum_{\gamma \in (\gamma_0^{-1}\Gamma_i) \cap \Gamma_i} \|[Q_i, \alpha_{\gamma_0\gamma}^{-1}(M)]\|_2^2 \\ &\quad + 2|(\gamma_0^{-1}\Gamma_i) \Delta \Gamma_i| \|M\|^2 \|Q_i\|_2^2 \\ &\leq |\Gamma_i| \max_{\gamma \in \Gamma_i} \|[Q_i, \alpha_\gamma^{-1}(M)]\|_2^2 \\ &\quad + 2|(\gamma_0^{-1}\Gamma_i) \Delta \Gamma_i| \|M\|^2 \|Q_i\|_2^2. \end{aligned}$$

Using now the hypothesis (3.7) as well as the amenability of Γ via Eq. (2.1) we get finally

$$\frac{\|[\widehat{R}_i, \pi(M)U(\gamma_0)]\|_2^2}{\|\widehat{R}_i\|_2^2} \leq \max_{\gamma \in \Gamma_i} \frac{\|[Q_i, \alpha_\gamma^{-1}(M)]\|_2^2}{\|Q_i\|_2^2} + 2\|M\|^2 \frac{|(\gamma_0^{-1}\Gamma_i) \Delta \Gamma_i|}{|\Gamma_i|},$$

and

$$\lim_i \frac{\|[\widehat{R}_i, \pi(M)U(\gamma_0)]\|_2^2}{\|\widehat{R}_i\|_2^2} = 0, \quad M \in \mathcal{A}, \gamma_0 \in \Gamma.$$

Thus, we have shown the Følner condition for the sequence $\{R_i\}_i$ on the product of generating elements of the crossed product. By Proposition 2.2(i) we have that $\{R_i\}_i$ is also a Følner sequence for their C^* -closure

$$\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma := C^*(\{U(\gamma_0), \pi(M) \mid \gamma_0 \in \Gamma, M \in \mathcal{A}\})$$

and the proof is concluded. \square

The preceding result extends Proposition 3.2 (proved by Bédos), since in the special case where \mathcal{H} is one-dimensional and $\mathcal{A} \cong \mathbb{C}\mathbb{1}$, the crossed product reduces to the group C^* -algebra \mathcal{A}_Γ .

Remark 3.5. The compatibility condition (3.7) in the choices of the two Følner sequences requires some comments:

- (i) Note that the compatibility condition already implies that the sequence $\{Q_i\}_i$ must be a Følner sequence for the C^* -algebra \mathcal{A} . In fact, this is one of the assumptions in Theorem 3.4 that \mathcal{A} has a Følner sequence.
- (ii) Eq. (3.7) is trivially satisfied in some cases: If Γ is finite, then the compatibility condition is a consequence of Eq. (2.2) in Definition 2.1.

Another example is given by the crossed product $\ell^\infty(\Gamma) \rtimes_\alpha \Gamma$, where Γ is a discrete amenable group, $\ell^\infty(\Gamma)$ is the von Neumann algebra acting on the Hilbert space $\ell^2(\Gamma)$ by multiplication and the action α of Γ on $\ell^\infty(\Gamma)$ is given by left translation of the argument. If $\{\Gamma_i\}_i$ is a Følner sequence for Γ and we denote by $\{P_i\}_i$ the sequence of finite rank orthogonal projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$, then it is easy to check

$$[P_i, g] = 0, \quad g \in \ell^\infty(\Gamma).$$

Therefore, we have

$$\sup_{\gamma \in \Gamma} \left\| [P_i, \alpha_\gamma^{-1}(g)] \right\|_2 = 0, \quad g \in \ell^\infty(\Gamma)$$

and we may apply Theorem 3.4 to this situation. This particular example is essentially the context in which Bédos studies crossed products in Section 3 of [8]. In fact, in this very special context one can trace back the existence of a Følner sequence for the crossed product to the amenability of the discrete group (see Proposition 14 in [8]).

The content of theorem (3.4) can be easily stated in terms of the Følner algebra introduced in Section 2.1. Let $\mathcal{Q} := \{Q_n\}_n \subset \mathcal{L}(\mathcal{H})$ be an increasing sequence of finite rank orthogonal projections converging strongly to $\mathbb{1}$ and let \mathcal{A} be a unital separable C^* -algebra contained in the Følner algebra $\mathcal{F}_{\mathcal{Q}}(\mathcal{H})$. Moreover, consider the canonical sequence of projections $\mathcal{P} := \{P_n\}_n \subset \mathcal{L}(\ell^2(\Gamma))$ associated to a Følner sequence $\{\Gamma_n\}_n$ of the amenable discrete group Γ as in theorem (3.4). (Here we also assume that the sequence Γ_n is increasing and that $\Gamma = \bigcup_n \Gamma_n$.) Then, if \mathcal{A} carries a compatible group action α (in the sense of Eq. (3.7)) we have that the corresponding crossed product is a C^* -subalgebra of the Følner algebra $\mathcal{F}_{\mathcal{R}}(\ell^2(\Gamma) \otimes \mathcal{H})$, where $\mathcal{R} := \{P_n \otimes Q_n\}_n \subset \mathcal{L}(\ell^2(\Gamma) \otimes \mathcal{H})$, i.e., we have

$$\mathcal{A} \rtimes_\alpha \Gamma \subset \mathcal{F}_{\mathcal{R}}(\ell^2(\Gamma) \otimes \mathcal{H}).$$

4. Applications and examples

We will apply next the results of the previous sections in two different situations. The two examples considered, rotation algebras and Jacobi operators, can be interpreted in the context of crossed products. Both examples are well-known and have been widely studied in the literature. The rotation algebra gives a non-trivial example, where one can verify explicitly the compatibility condition of Eq. (3.7).

4.1. The rotation algebra

The rotation algebra \mathcal{A}_θ , $\theta \in \mathbb{R}$, is the (universal) C^* -algebra generated by two unitaries U and V that satisfy the equation

$$U V = e^{2\pi i \theta} V U.$$

When θ is an integer, the algebra \mathcal{A}_θ is isomorphic to the commutative C^* -algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. For this reason, when, e.g., θ is irrational, \mathcal{A}_θ is called a non-commutative torus. Moreover, \mathcal{A}_θ has in this case a unique faithful tracial state τ which can be interpreted as a non-commutative analogue of the Haar measure on \mathbb{T}^2 . (See [10] for a thorough presentation and many results concerning spectral approximation.)

The rotation algebra is one of the fundamental examples in the theory C^* -algebras and has been extensively used in mathematical physics. Interesting examples from the spectral point of view, like the almost Mathieu operators or discrete Schrödinger operators with magnetic potentials (Harper operators), can be identified as elements of the rotation algebra (cf. [48,10]). In fact, consider for example the following representation of the generators U, V on $\mathcal{H} := \ell^2(\mathbb{Z})$:

$$(U\xi)_k := \xi_{k-1} \quad \text{and} \quad (V\xi)_k := e^{2\pi i\theta k} \xi_k,$$

where $\xi = (\xi_k)_k \in \mathcal{H}$. One defines the almost Mathieu operator with real parameters θ, λ, β as

$$H_{\theta,\lambda,\beta} := U + U^* + \frac{\lambda}{2} \left(e^{2\pi i\beta} V + e^{-2\pi i\beta} V^* \right) \in \mathcal{A}_\theta. \quad (4.1)$$

These classes of operators have a natural generalization to arbitrary graphs.

An important fact for our purposes is that the rotation algebra can also be expressed as a crossed product

$$\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z},$$

where $C(\mathbb{T})$ are the continuous function on the unit circle and the action $\alpha: \mathbb{Z} \rightarrow C(\mathbb{T})$ is given by rotation of the argument:

$$\alpha_k(f)(z) := f\left(e^{2\pi i k \theta} z\right), \quad f \in C(\mathbb{T}), \quad z \in \mathbb{T}. \quad (4.2)$$

We will apply our main result to the C^* -crossed product \mathcal{A}_θ . Let $\{\epsilon_k(z) := z^k \mid k \in \mathbb{Z}\}$ be an orthonormal basis of Hilbert space $\mathcal{H} := L^2(\mathbb{T})$ with the normalized Haar measure. We choose (as in [8, p. 216]) a Følner sequence $\{Q_n\}_{n \in \mathbb{N}_0}$, where Q_n denotes the orthogonal projection onto the subspace generated by $\{\epsilon_i \mid i = 0, \dots, n\}$. Moreover, we choose for the group $\Gamma = \mathbb{Z}$ the Følner sequence $I_n := \{-n, -(n-1), \dots, (n-1), n\}$ and denote by P_n the corresponding finite-rank orthogonal projections on $\ell^2(\mathbb{Z})$. First we verify that the compatibility condition (3.7) for our choice of Følner sequences:

Lemma 4.1. *Consider the previous Følner sequence $\{Q_n\}_n$ for the commutative C^* -algebra $\mathcal{A} := C(\mathbb{T})$ and the group action $\alpha: \mathbb{Z} \rightarrow C(\mathbb{T})$ defined in Eq. (4.2). Then for $g \in C(\mathbb{T})$ we have*

$$\left\| [Q_n, \alpha_k^{-1}(g)] \right\|_2 = \| [Q_n, g] \|_2, \quad k \in \mathbb{Z},$$

and

$$\lim_{n \rightarrow \infty} \left(\max_{k \in I_n} \frac{\left\| [Q_n, \alpha_k^{-1}(g)] \right\|_2}{\|Q_n\|_2} \right) = 0, \quad \text{for all } g \in C(\mathbb{T}).$$

Proof. The first equation is a consequence of the some elementary statements in harmonic analysis:

$$\begin{aligned}
\| [Q_n, \alpha_k^{-1}(g)] \|_2^2 &= \sum_{l=-\infty}^{\infty} \| (Q_n \alpha_{-k}(g) - \alpha_{-k}(g) Q_n) \epsilon_l \|^2 \\
&= \sum_{l=0}^n \| (\mathbb{1} - Q_n) \alpha_{-k}(g) \epsilon_l \|^2 + \sum_{l \in (\mathbb{Z} \setminus \{0, \dots, n\})} \| Q_n \alpha_{-k}(g) \epsilon_l \|^2 \\
&= \sum_{m \in (\mathbb{Z} \setminus \{0, \dots, n\})} \sum_{l=0}^n \left| e^{2\pi i k \theta(m-l)} \widehat{g}(m-l) \right|^2 \\
&\quad + \sum_{m=0}^n \sum_{l \in (\mathbb{Z} \setminus \{0, \dots, n\})} \left| e^{2\pi i k \theta(m-l)} \widehat{g}(m-l) \right|^2 \\
&= \| [Q_n, g] \|_2^2.
\end{aligned}$$

The second equation follows directly from the first equation and the fact that $\{Q_n\}_n$ is a Følner sequence for the algebra $C(\mathbb{T})$. \square

Proposition 4.2. *Let $\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, with θ irrational, be the C^* -algebra associated to the rotation algebra and acting on $\mathcal{K} = \ell^2(\mathbb{Z}) \otimes \mathcal{H}$.*

(i) *Consider the sequences $\{Q_n\}_{n \in \mathbb{N}_0}$ and $\{P_n\}_{n \in \mathbb{N}_0}$ defined before. Then*

$$\{R_n := P_n \otimes Q_n\}_{n \in \mathbb{N}_0}$$

is a Følner sequence for \mathcal{A}_θ .

(ii) *Let $T \in \mathcal{A}_\theta$ be a selfadjoint element in the rotation algebra and denote by μ_T the spectral measure associated with the unique trace of \mathcal{A}_θ . Consider the compressions $T_n := R_n T R_n$ and denote by μ_T^n the probability measures on \mathbb{R} supported on the spectrum of (T_n) . Then $\mu_T^n \rightarrow \mu_T$ in the weak*-topology, i.e.*

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \left(f(\lambda_{1,n}) + \dots + f(\lambda_{d_n,n}) \right) = \int f(\lambda) d\mu(\lambda), \quad f \in C_0(\mathbb{R}),$$

where d_n is the dimension of the R_n and $\{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}$ are the eigenvalues (repeated according to multiplicity) of T_n .

Proof. Part (i) follows from Theorem 3.4 and Lemma 4.1. To prove Part (ii) recall \mathcal{A}_θ has a unique trace [10]. The rest of the statement is a direct application of Theorem 6 (iii) in [8] to the example of the rotation algebra. \square

Since almost Mathieu operator $H_{\theta,\lambda,\beta}$ defined in Eq. (4.1) are selfadjoint elements in \mathcal{A}_θ , we can apply part (ii) of the precedent proposition. In this case the discrete measures μ_H^n are supported on the eigenvalues of the corresponding finite section matrices.

4.2. Jacobi operators

Jacobi operators have been used in many branches of mathematics. E.g., they can be interpreted as a discrete version of Schrödinger operators and appear in the approximation of differential operators by difference operators (see, e.g., [3,34]). Moreover, the relation between selfadjoint tridiagonal infinite Jacobi matrices and orthogonal polynomials is by now a standard fact. In Chapter 2 of [24] it is shown that there is a one-to-one correspondence between bounded selfadjoint Jacobi operators J and probability measures μ with compact support. The purpose of

this subsection is to illustrate how naturally the notion of a proper Følner sequence fits into the analysis of this class of operators. The results are not new and we hope that we can make these techniques accessible to other communities presenting some elementary proofs of some of the statements.

Consider on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ with canonical basis $\{e_i \mid i = 0, 1, 2, \dots\}$ the infinite Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b'_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b'_1 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.3)$$

If the diagonals $\alpha = (a_0, a_1, \dots)$, $\beta = (b_0, b_1, \dots)$, $\beta' = (b'_0, b'_1, \dots)$ are bounded (i.e., $\alpha, \beta, \beta' \in \ell^\infty(\mathbb{N}_0)$), then J is a bounded operator. Moreover, if $D_\alpha = \text{diag}(\alpha)$ denotes the diagonal operator, then J can be decomposed as

$$J = D_\alpha + D_\beta S^* + S D_{\beta'}, \quad (4.4)$$

where $Se_i := e_{i+1}$ is the shift already considered in [Example 2.5\(i\)](#).

Proposition 4.3. Denote by \mathcal{J} the set of all bounded Jacobi matrices as in Eq. (4.3). Then $\{P_n\}_n$, where P_n are the orthogonal projections onto $\text{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$ is a proper Følner sequence for $C^*(\mathcal{J})$.

Proof. By [Proposition 2.2\(i\)](#) it is enough to check Eq. (2.2) for the generating set \mathcal{J} . For any $J \in \mathcal{J}$ we have using the decomposition (4.4)

$$\begin{aligned} \| [J, P_n] \|_2 &\leq \underbrace{\| [D_\alpha, P_n] \|_2}_0 + \| [D_\beta S^*, P_n] \|_2 + \| [S D_{\beta'}, P_n] \|_2 \\ &\leq \| D_\beta \| \| [S^*, P_n] \|_2 + \| [S, P_n] \|_2 \| D_{\beta'} \|. \end{aligned}$$

Følner's condition in Eq. (2.2) follows from the computation in [Example 2.5\(i\)](#). \square

Let $J \in \mathcal{J}$ be selfadjoint and denote by μ the spectral measure associated to the cyclic vector e_0 . The support of the (discrete) spectral measures μ_n^J of the finite sections $J_n = P_n J P_n$ (with P_n as in the preceding proposition) correspond precisely with the zeros of the polynomials p_n which are orthogonal with respect to μ . For any Borel set $\Delta \subset \mathbb{R}$ define $N_n(\Delta)$ to be the number of eigenvalues of J_n counting multiplicities contained in Δ . Note that in this case we have

$$\mu_n^J(\Delta) = \frac{N_n(\Delta)}{n+1}.$$

Following Arveson [\[4,5\]](#) we say that $\lambda \in \mathbb{R}$ is essential if for every open set $\Delta \subset \mathbb{R}$ containing λ , we have

$$\lim_{n \rightarrow \infty} N_n(\Delta) = \infty.$$

The set of essential points is denoted by $\Lambda_{\text{ess}}(J)$. Recall finally that in this context the essential spectrum of the selfadjoint Jacobi operator J is defined by

$$\sigma_{\text{ess}}(J) = \sigma(J) \setminus \sigma_{\text{disc}}(J),$$

where $\lambda \in \sigma_{\text{disc}}(J)$ if it is an isolated eigenvalue in the spectrum $\sigma(J)$ whose eigenspace is finite dimensional. Then since for tridiagonal Jacobi matrices we have that

$$\sup_n \text{rank}(P_n J - J P_n) \leq 2$$

we can apply a theorem by Arveson to conclude that

$$\sigma_{\text{ess}}(J) = \Lambda_{\text{ess}}(J). \quad (4.5)$$

This result shows the way in which one can recover the essential spectrum of J out of its finite sections.

The results in this subsection have been extended far beyond the set of Jacobi operators. In fact, the equality between the essential points and the essential spectrum has been established for selfadjoint band-dominated operators (cf. Theorem 7.6 in [44]). We refer to Chapter 7 in [44] for details and a systematic analysis of many aspects of spectral approximation related to this class of operators.

5. Outlook

Jacobi operators are examples of so-called band dominated operators. This class of operators (already mentioned in the preceding section) can be identified with the crossed-product $\ell^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$, where the action α of Γ on $\ell^\infty(\mathbb{Z})$ is given by translation of the argument. This situation can be generalized to the context of discrete groups where one just replaces \mathbb{Z} by a discrete group Γ (see [46,42] and references cited therein). In Theorem 2.2 of [42] it is shown that the set of generalized band-dominated operators is precisely the reduced crossed product. In particular, if Γ is amenable (hence exact) we can apply also the result of theorem (3.4) to $\ell^\infty(\Gamma) \rtimes_\alpha \Gamma$ (with α given again by left translation) since by Remark 3.5(ii) the compatibility condition is trivially satisfied. From this point of view the result in theorem (3.4) extends the set of generalized band-dominated operators by replacing the commutative algebra $\ell^\infty(\Gamma)$ with a non-commutative C^* -algebra \mathcal{A} which carries a compatible group action of Γ on \mathcal{A} in the sense of Eq. (3.7). Moreover, our main result specifies a canonical Følner sequence where one could, in principle, address stability questions.

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