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Some Expansions in Basic Fourier Series and Related Topics

Sergei K. Suslov[1]

Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804, U.S.A.
E-mail: sks@asu.edu

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We consider explicit expansions of some elementary and q-functions in basic Fourier series introduced recently by Bustoz and Suslov. Natural q-extensions of the Bernoulli and Euler polynomials, numbers, and the Riemann zeta function are discussed as a by-product.

Key Words: Trigonometric functions; basic trigonometric functions; orthogonality relations; Fourier series; q-Fourier series; the Bernoulli and Euler polynomials and numbers and their q-extensions; the Riemann zeta function and its q-extension.

1. INTRODUCTION

A periodic function $f(x)$ with period $2l$ can be represented, under certain conditions, as the Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n}{l} x + b_n \sin \frac{\pi n}{l} x \right). \tag{1.1}$$

For the extensive theory of these series see, for example, [2, 10, 57, 64, 66].

Basic Fourier series have been introduced recently [14] as certain extensions of the classical Fourier series (1.1). The following functions $C_q(x; \omega)$ and $S_q(x; \omega)$ given by

$$C_q(x; \omega) = \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \, {}_2\varphi_1 \left(\begin{matrix} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{matrix} ; q^2, -\omega^2 \right) \tag{1.2}$$

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and

$$S_q(x;\omega)=\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\,\frac{2q^{1/4}\omega}{1-q}\cos\theta\,{}_2\varphi_1\left(\begin{matrix}-q^2e^{2i\theta},\,-q^2e^{-2i\theta}\\ q^3\end{matrix};q^2,-\omega^2\right), \tag{1.3}$$

were discussed in [8, 36, 48] as q-analogs of $\cos\omega x$ and $\sin\omega x$.

We use the standard notations [23] for the basic hypergeometric series

$${}_r\varphi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,t\right):=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r}t^n, \tag{1.4}$$

$$(a;q)_0:=1,\qquad (a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k), \tag{1.5}$$

$$(a_1,a_2,\ldots,a_m;q)_n:=\prod_{l=1}^{m}(a_l;q)_n, \tag{1.6}$$

where $n=1,2,\ldots,$ or ∞, when $|q|<1$. Also,

$$(a;q)_\alpha:=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}. \tag{1.7}$$

Functions $C_q(x;\omega)$ and $S_q(x;\omega)$ are defined by (1.2)–(1.3) for $|\omega|<1$ only. For the analytic continuation of these functions in a larger domain see, for example, [14, 36, 48].

The above q-trigonometric functions appear in recent literature from different contexts. Atakishiyev and Suslov [8, 48] found solutions of a q-analog of the equation for harmonic motion. Ismail and Zhang [36] expanded the corresponding basic exponential function in terms of "q-spherical harmonics" and later, together with Rahman [31], they extended this analog of the expansion formula of the plane wave from q-ultraspherical to continuous q-Jacobi polynomials. "Addition" theorems for the basic trigonometric functions were found in [33, 48, 50]; see also [53] for a review. Relations of the quadratic q-exponentials with the connection coefficient problem have been investigated in [30]. Bustoz and Suslov [14] have established the following orthogonality property,

$$\int_0^\pi C_q(\cos\theta;\omega)\, C_q(\cos\theta;\omega')(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta = 0, \tag{1.8}$$

$$\int_0^\pi S_q(\cos\theta;\omega)\, S_q(\cos\theta;\omega')(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta = 0, \tag{1.9}$$

$$\int_0^\pi C_q(\cos\theta;\omega)\, S_q(\cos\theta;\omega')(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta = 0, \tag{1.10}$$

and

$$\begin{aligned}\int_0^\pi & C_q^2(\cos\theta;\omega)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \\ &= \int_0^\pi S_q^2(\cos\theta;\omega)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \\ &= \pi \frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2} \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty} \sum_{k=0}^\infty \frac{q^{k/2}}{1+\omega^2 q^k}.\end{aligned} \tag{1.11}$$

Here ω and ω' are different solutions of the transcendental equation

$$S_q\left(\frac{1}{2}(q^{1/4}+q^{-1/4});\omega\right) = \frac{(-i\omega;q^{1/2})_\infty - (i\omega;q^{1/2})_\infty}{2i(-q\omega^2;q^2)_\infty} = 0. \tag{1.12}$$

The method of [14] shows that these orthogonality relations hold also when ω and ω' are different solutions of

$$C_q\left(\frac{1}{2}(q^{1/4}+q^{-1/4});\omega\right) = \frac{(-i\omega;q^{1/2})_\infty + (i\omega;q^{1/2})_\infty}{2(-q\omega^2;q^2)_\infty} = 0. \tag{1.13}$$

See also [53] for an elementary proof of the orthogonality property of the basic trigonometric system on the basis of an integral evaluated by Ismail and Stanton [33].

The basic trigonometric functions (1.2)–(1.3) are solutions of a very special case of a general difference equation of hypergeometric type on nonuniform lattices; see [8, 43, 47, 48]. The Askey–Wilson polynomials and their special and limiting cases [7, 39, 43] are well-known as the simplest and the most important orthogonal solutions of this difference equation of hypergeometric type. Recently, Ismail, Masson and Suslov [29], and Suslov [49, 51] have found another type of orthogonal solutions of this difference equation at the higher ${}_2\varphi_1$ and ${}_8\varphi_7$-levels, respectively.

Bustoz and Suslov [14] introduced the corresponding basic Fourier series as

$$f(\cos\theta) = a_0 + \sum_{n=1}^{\infty} (a_n C_q(\cos\theta;\omega_n) + b_n S_q(\cos\theta;\omega_n)), \tag{1.14}$$

where $\omega_0 = 0, \omega_1, \omega_2, \omega_3, \ldots$ are nonnegative zeros of (1.12) arranged in ascending order of magnitude, and

$$a_0 = \frac{1}{2k(0)} \int_0^{\pi} f(\cos\theta)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta, \tag{1.15}$$

$$a_n = \frac{1}{k(\omega_n)} \int_0^{\pi} f(\cos\theta)\, C_q(\cos\theta;\omega_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta, \tag{1.16}$$

$$b_n = \frac{1}{k(\omega_n)} \int_0^{\pi} f(\cos\theta)\, S_q(\cos\theta;\omega_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \tag{1.17}$$

are the corresponding q-Fourier coefficients. The complex form of the basic Fourier series (1.14) is

$$f(\cos\theta) = \sum_{n=-\infty}^{\infty} c_n \mathscr{E}_q(\cos\theta; i\omega_n) \tag{1.18}$$

with

$$c_n = \frac{1}{2k(\omega_n)} \int_0^{\pi} f(\cos\theta)\, \mathscr{E}_q(\cos\theta; -i\omega_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta, \tag{1.19}$$

where $\omega_n = 0, \pm\omega_1, \pm\omega_2, \pm\omega_3, \ldots$ and $\omega_0 = 0 < \omega_1 < \omega_2 < \omega_3 \ldots$ are nonnegative zeros of (1.12); the normalization constants $k(\omega_n)$ are defined by the expression in the right side of (1.11),

$$\begin{aligned} k(\omega) &= \frac{1}{2} \int_0^{\pi} (C_q^2(\cos\theta;\omega) + S_q^2(\cos\theta;\omega))(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \\ &= \pi \frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2} \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k/2}}{1+\omega^2 q^k}, \end{aligned} \tag{1.20}$$

and the basic analog of the exponential function,

$$\mathscr{E}_q(x; i\omega) = C_q(x;\omega) + i S_q(x;\omega), \tag{1.21}$$

is introduced with the help of the q-analog of Euler's formula. It is convenient for the further consideration to rewrite (1.20) in a compact form

$$k(\omega) = \pi \frac{(q^{1/2}; q)_\infty^2}{(q; q)_\infty^2} \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \kappa(\omega), \tag{1.22}$$

where by the definition

$$\kappa(\omega) = \sum_{k=0}^{\infty} \frac{q^{k/2}}{1+\omega^2 q^k}. \tag{1.23}$$

See [30, 31, 33, 36, 48, 50], and our review paper [53] for the detailed investigation of the properties of the q-exponential function $\mathscr{E}_q(x; \alpha)$. The case of the basic Fourier series on a q-linear grid is considered in [13].

Although classical Fourier series have a long and distinguished history, not much is known about the basic Fourier series. In the original paper [14] Bustoz and Suslov have proved that the basic trigonometric system $\{\mathscr{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$ is complete; see also [54], [56] for an extension and an independent proof of this result. They have also established some elementary facts about convergence of these series. In particular, if $f(x)$ is continuous and the series (1.14) or (1.18) converge uniformly on $[-1, 1]$, then its sum is $f(x)$ by Theorems 12 and 13 of [14]. Numerical investigation of the basic Fourier series was recently started by Gosper and Suslov [24] with the help of the Macsyma computer algebra system.

Explicit expansions of functions in the basic Fourier series lead, in a natural way, to a new class of formulas never investigated before from the analytical and/or numerical point of view. Bustoz and Suslov [14] have discussed just a few simple examples. Our main objective in the present paper is to give explicit expansions of many elementary and q-functions in basic Fourier series (1.14)–(1.19). Some integrals, evaluated recently by Ismail and Stanton [33], play an important role in this analysis.

We shall also consider expansions of certain functions with respect to the basic trigonometric system $\{\mathscr{E}_q(x; i\varpi_n)\}_{n=-\infty}^{\infty}$, where $\varpi_n = \pm\varpi_1, \pm\varpi_2, \pm\varpi_3, \ldots$ and $0 < \varpi_1 < \varpi_2 < \varpi_3 < \ldots$ are nonnegative zeros of (1.13). The corresponding "modified" basic Fourier series are

$$f(\cos\theta) = \sum_{n=1}^{\infty} (a_n C_q(\cos\theta; \varpi_n) + b_n S_q(\cos\theta; \varpi_n)), \tag{1.24}$$

where

$$a_n = \frac{1}{k(\varpi_n)} \int_0^\pi f(\cos\theta)\, C_q(\cos\theta; \varpi_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta, \tag{1.25}$$

$$b_n = \frac{1}{k(\varpi_n)} \int_0^\pi f(\cos\theta)\, S_q(\cos\theta; \varpi_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta, \tag{1.26}$$

and the complex form is

$$f(\cos\theta)=\sum_{n=-\infty}^{\infty} c_n \mathcal{E}_q(\cos\theta; i\varpi_n) \tag{1.27}$$

with

$$c_n=\frac{1}{2k(\varpi_n)}\int_0^{\pi} f(\cos\theta)\, \mathcal{E}_q(\cos\theta; -i\varpi_n)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta. \tag{1.28}$$

The normalization constants $k(\varpi_n)$ are defined by (1.20) and (1.22)–(1.23). The basic trigonometric system $\{\mathcal{E}_q(x; i\varpi_n)\}_{n=-\infty}^{\infty}$ under consideration is complete due to Theorem 9 of [54]. Therefore analogs of Theorems 12 and 13 of [14] hold for the series (1.24)–(1.28); if $f(x)$ is continuous and the series converges uniformly on $[-1, 1]$, then its sum is $f(x)$.

The large ω-asymptotics of the zeros of (1.12) and (1.13), which are important for study of the convergence of q-Fourier series (1.14)–(1.19) and (1.24)–(1.28), are

$$\omega_n=q^{1/4-n}-c_1(q)+o(1) \tag{1.29}$$

and

$$\varpi_n=q^{3/4-n}-c_1(q)+o(1) \tag{1.30}$$

as $n\to\infty$, where

$$c_1(q)=\frac{q^{1/4}}{2(1-q^{1/2})}\,\frac{(q; q^2)_\infty^2}{(q^2; q^2)_\infty^2}. \tag{1.31}$$

These formulas have been found numerically in [24], see also [14] for the leading terms; the rigorous proof is given in [52].

Certain q-Fourier expansions found in this paper give us a possibility to introduce analogs of the Bernoulli and Euler polynomials, numbers and Riemann zeta function and investigate some of their properties.

The paper is organized as follows. In Section 2, we obtain the q-Fourier expansions for some polynomials. Expansions of the basic trigonometric and exponential functions are found in Sections 3 to 5, consequences of Parseval's identity are discussed in Section 6. In Sections 7 to 9, we introduce, in a natural way, q-analogs of the classical Bernoulli and Euler polynomials, numbers, and the zeta function, respectively. Basic Fourier series expansions of certain functions and some miscellaneous results are established in Sections 10 and 11. We close the paper in Appendix, providing details of the rigorous proofs of the uniform convergence of the q-Fourier expansions discussed in Sections 2 to 5.

2. EXPANSIONS OF SOME POLYNOMIALS

The continuous q-ultraspherical polynomials $C_m(\cos\theta;\beta\mid q)$ are defined as

$$C_m(\cos\theta;\beta\mid q)=\sum_{k=0}^{m}\frac{(\beta;q)_k\,(\beta;q)_{m-k}}{(q;q)_k\,(q;q)_{m-k}}\,e^{i(m-2k)\,\theta}, \tag{2.1}$$

see, for example, [23]. Bustoz and Suslov [14] have established the following q-Fourier expansion,

$$\begin{aligned}C_m(x;q^{1/2}\mid q)&\\&=\pi\,\frac{(q^{1/2};q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-i)^m\,q^{m^2/4}\,\frac{\omega_n^{-1/2}}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}\\&\quad\times J_{m+1/2}^{(2)}(2\omega_n;q)\,\mathscr{E}_q(x;i\omega_n),\qquad m>0,\end{aligned} \tag{2.2}$$

for the special case of these polynomials, namely, $C_m(x;q^{1/2}\mid q)$, which are the basic analogs of the Legendre polynomials. Here $J_\nu^{(2)}(x;q)$ is Jackson's q-Bessel function defined as [26–28]

$$J_\nu^{(2)}(x;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{(\nu+n)\,n}\,\frac{(-1)^n\,(x/2)^{\nu+2n}}{(q;q)_n(q^{\nu+1};q)_n}. \tag{2.3}$$

Expansion (2.2) is an "inversion" of the following special case,

$$\begin{aligned}\mathscr{E}_q(x;i\omega_n)&=\frac{(q;q)_\infty}{(q^{1/2};q)_\infty}\,\frac{\omega_n^{-1/2}}{(-q\omega_n^2;q^2)_\infty}\\&\quad\times\sum_{m=0}^{\infty}i^m\,(1-q^{m+1/2})\,q^{m^2/4}J_{m+1/2}^{(2)}(2\omega_n;q)\,C_m(x;q^{1/2}\mid q),\end{aligned} \tag{2.4}$$

of Ismail and Zhang's basic analog of the expansion formula of the plane wave in terms of the spherical harmonics [36].

As two special cases of the formula (2.2), the basic Fourier expansions for the linear and quadratic functions,

$$x=(q^{1/4}+q^{-1/4})\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,S_q(x;\omega_n) \tag{2.5}$$

and

$$\begin{aligned}x^2&=\frac{(1+q^{1/2})^2}{4(1+q^{1/2}+q)}-\frac{(1+q^{1/2})(1-q^2)}{2q}\\&\quad\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n^2}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,C_q(x;\omega_n),\end{aligned} \tag{2.6}$$

have been discussed in [14], [24]; see [24] for the numerical investigation of the convergence of these series using the Mascyma computer algebra system.

We shall continue to study formula (2.2) here, emphasizing now the relation with the q-Lommel polynomials $h_{n,\nu}(x; q)$ introduced by Ismail in [27]. These polynomials arise from the expansion

$$q^{n\nu+n(n-1)/2} J^{(2)}_{\nu+n}(x; q) = h_{n,\nu}\left(\frac{1}{x}; q\right) J^{(2)}_{\nu}(x; q) - h_{n-1,\nu+1}\left(\frac{1}{x}; q\right) J^{(2)}_{\nu-1}(x; q). \tag{2.7}$$

They are generated by the three-term recurrence relation

$$h_{n+1,\nu}(x; q) = (1-q^{\nu+n})(2x) h_{n,\nu}(x; q) - q^{\nu+n-1} h_{n-1,\nu}(x; q), \tag{2.8}$$

$n > 0$ with

$$h_{0,\nu}(x; q) = 1, \qquad h_{1,\nu}(x; q) = (1-q^{\nu})(2x). \tag{2.9}$$

The q-Lommel polynomials $h_{n,\nu}(x; q)$ are orthogonal with respect to a purely discrete measure whose masses are located at the reciprocals of the zeros of $J^{(2)}_{\nu-1}(x; q)$ [27]. Ismail gave also an explicit formula for them,

$$h_{n,\nu}(x; q) = \sum_{k=0}^{[n/2]} (-1)^k q^{(\nu+k-1)k} \frac{(q; q)_{n-k} (q^{\nu}; q)_{n-k}}{(q; q)_k (q; q)_{n-2k} (q^{\nu}; q)_k} (2x)^{n-2k}, \tag{2.10}$$

which implies the symmetry relation,

$$h_{n,\nu}(-x; q) = (-1)^n h_{n,\nu}(x; q). \tag{2.11}$$

The q-spherical Bessel function $J^{(2)}_{m+1/2}(2\omega_n; q)$ appearing in (2.2) can be reduced to the q-Lommel polynomials and $J^{(2)}_{\pm 1/2}(2\omega_n; q)$ by (2.7),

$$\begin{aligned} q^{m^2/2} J^{(2)}_{m+1/2}(2\omega_n; q) = {} & h_{m,1/2}\left(\frac{1}{2\omega_n}; q\right) J^{(2)}_{1/2}(2\omega_n; q) \\ & - h_{m-1,3/2}\left(\frac{1}{2\omega_n}; q\right) J^{(2)}_{-1/2}(2\omega_n; q). \end{aligned} \tag{2.12}$$

But due to (5.34)–(5.35) of [14], namely,

$$S_q(\eta; \omega) = \frac{(q; q)_\infty}{(q^{1/2}; q)_\infty} \frac{\omega^{1/2}}{(-q\omega^2; q^2)_\infty} J^{(2)}_{1/2}(2\omega; q), \tag{2.13}$$

$$C_q(\eta; \omega) = \frac{(q; q)_\infty}{(q^{1/2}; q)_\infty} \frac{\omega^{1/2}}{(-q\omega^2; q^2)_\infty} J^{(2)}_{-1/2}(2\omega; q) \tag{2.14}$$

the basic sine $S_q(\eta;\omega)$ and basic cosine $C_q(\eta;\omega)$ functions at $\eta=(q^{1/4}+q^{-1/4})/2$ are just multiples of $J^{(2)}_{1/2}(2\omega;q)$ and $J^{(2)}_{-1/2}(2\omega;q)$. (This is a direct analog of the well-known fact that the classical Bessel functions of the form $J_{m+1/2}(z)$ can be reduced in the same manner to the elementary functions [63].) Also, by (1.12) and Eq. (13.10) of [14],

$$J^{(2)}_{1/2}(2\omega_n;q)=0, \tag{2.15}$$

$$J^{(2)}_{-1/2}(2\omega_n;q)=(-1)^n\,\frac{(q^{1/2};q)_\infty}{(q;q)_\infty}\,(-q\omega_n^2;q^2)_\infty\,\omega_n^{-1/2}\,\sqrt{\frac{(-\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty}}\,. \tag{2.16}$$

Therefore

$$\begin{aligned}J^{(2)}_{m+1/2}(2\omega_n;q)=&(-1)^{n-1}\,q^{-m^2/2}\,\frac{(q^{1/2};q)_\infty}{(q;q)_\infty}\,(-q\omega_n^2;q^2)_\infty\\&\times\omega_n^{-1/2}h_{m-1,\,3/2}\left(\frac{1}{2\omega_n};q\right)\sqrt{\frac{(-\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty}}\,,\end{aligned} \tag{2.17}$$

$m>0$, which reduces the infinite series representation (2.3) to a finite sum in the case of the q-spherical Bessel function at the points $x=2\omega_n$. It is worth mentioning also that the orthogonality properties (12.16)–(12.17) of [14] for these functions imply the explicit orthogonality,

$$\begin{aligned}&(1+(-1)^{m+p})\\&\times\sum_{n=1}^{\infty}h_m\left(\frac{1}{2\omega_n};q\right)h_p\left(\frac{1}{2\omega_n};q\right)\frac{1}{\omega_n^2\,\kappa(\omega_n)}=\frac{q^{(m+1)^2/2}}{1-q^{m+3/2}}\,\delta_{mp},\end{aligned} \tag{2.18}$$

and the completeness,

$$\sum_{m=0}^{\infty}h_m\left(\frac{1}{2\omega_n};q\right)h_m\left(\frac{1}{2\omega_l};q\right)\frac{1-q^{m+3/2}}{q^{(m+1)^2/2}}=\omega_n^2\,\kappa(\omega_n)\,\delta_{nl}, \tag{2.19}$$

relations for our special case of the q-Lommel polynomials, namely,

$$h_m(x;q):=h_{m,\,3/2}(x;q). \tag{2.20}$$

This shorter notation will be used throughout the paper. The measure can also be found explicitly for the special case $\nu=1/2$ of the q-Lommel polynomials; see Eqs. (2.31) and (2.35)–(2.36) below.

In view of (2.17), the Ismail and Zhang formula (2.4) takes the form

$$\mathscr{E}_q(x; i\omega_n) = \frac{(-1)^{n-1}}{\omega_n} \sum_{m=1}^{\infty} i^m (1-q^{m+1/2})\, q^{-m^2/4} \times h_{m-1}\left(\frac{1}{2\omega_n}; q\right) \sqrt{\frac{(-\omega_n^2; q^2)_\infty}{(-q\omega_n^2; q^2)_\infty}}\, C_m(x; q^{1/2} \mid q), \tag{2.21}$$

and, finally, expansion (2.2) can be rewritten in terms of the q-Lommel polynomials as

$$C_m(x; q^{1/2} \mid q) = \sum_{n=-\infty}^{\infty}{}' (-i)^m \frac{q^{-m^2/4}}{\kappa(\omega_n)\,\omega_n}\, h_{m-1}\left(\frac{1}{2\omega_n}; q\right) \times (-1)^{n-1} \sqrt{\frac{(-q\omega_n^2; q^2)_\infty}{(-\omega_n^2; q^2)_\infty}}\, \mathscr{E}_q(x; i\omega_n), \tag{2.22}$$

$m > 0$. (The prime in the sum here means that the term with $n = 0$ should be omitted; we could also formally assume that $h_{-1} = 0$.)

"Odd" and "even" parts of this expansion are

$$C_{2m+1}(x; q^{1/2} \mid q) = 2\, q^{-m(m+1)-1/4} \sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{\kappa(\omega_n)\,\omega_n}\, h_{2m}\left(\frac{1}{2\omega_n}; q\right) \times \sqrt{\frac{(-q\omega_n^2; q^2)_\infty}{(-\omega_n^2; q^2)_\infty}}\, S_q(x; \omega_n) \tag{2.23}$$

and

$$C_{2m}(x; q^{1/2} \mid q) = 2q^{-m^2} \sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{\kappa(\omega_n)\,\omega_n}\, h_{2m-1}\left(\frac{1}{2\omega_n}; q\right) \times \sqrt{\frac{(-q\omega_n^2; q^2)_\infty}{(-\omega_n^2; q^2)_\infty}}\, C_q(x; \omega_n). \tag{2.24}$$

The series in (2.23) and (2.24) converge absolutely and uniformly in x on every closed subinterval of $(-\eta, \eta)$ and $(-\eta_1, \eta_1)$, respectively, where $\eta = x(1/4)$, $\eta_1 = x(1/2)$ and $x(\varepsilon) = (q^{\varepsilon} + q^{-\varepsilon})/2$. This can be shown with the help of Lemma 1 from Appendix and asymptotic formula (1.29); by analytic continuation these expansions hold in the open disks $|x| < \eta$ and $|x| < \eta_1$ of the complex x-plane, respectively.

These formulas are convenient for expansion of the elementary powers $x, x^2, x^3, x^4, \ldots$ in basic Fourier series. Equations (2.5) and (2.6) are, obviously, the special cases $m=0$ and $m=1$ of these formulas, respectively. The case $m=1$ of (2.23) gives rise to

$$x^3 = \frac{(1+q^{1/2})^3}{4q^{3/4}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n} \sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\, S_q(x;\omega_n)$$
$$- \frac{(1+q^{1/2})(1-q^2)(1-q^3)}{4q^{9/4}}$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n^3} \sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\, S_q(x;\omega_n). \tag{2.25}$$

The case $m=2$ of (2.24) results in

$$x^4 = \frac{(1+q^{1/2})^2\,(1-q)^2\,(2-q^{1/2}+2q)}{16(1-q^{3/2})(1-q^{5/2})}$$
$$- \frac{(1+q^{1/2})^3\,(1-q^2)(1-2q^{1/2}+4q-2q^{3/2}+q^2)}{8q^{5/2}}$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n^2} \sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\, C_q(x;\omega_n)$$
$$+ \frac{(1+q^{1/2})(1-q^2)(1-q^3)(1-q^4)}{8q^4}$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n^4} \sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\, C_q(x;\omega_n). \tag{2.26}$$

In a similar fashion, one can obtain expansions of the higher order powers in the basic Fourier series using a computer algebra system. Convenient relation between (2.23)–(2.24) and q-analogs of the Bernoulli polynomials will be discussed in Section 7; see (7.44)–(7.45). Expansions of some "generalized power functions" in the q-Fourier series are found in Section 4; see (4.18) and (4.21).

Equations (2.22)–(2.24) give us the explicit expansion of the analog of the Legendre polynomials, namely, $C_m(x; q^{1/2} \mid q)$, in the q-Fourier series. In order to find expansion for the continuous q-ultraspherical polynomials $C_m(x; \gamma \mid q)$ of the general form, one can just use Rogers' connection coefficient formula,

$$C_m(x;\gamma \mid q) = \sum_{k=0}^{[m/2]} \beta^k\, \frac{(\gamma/\beta;q)_k\,(\gamma;q)_{m-k}\,(1-\beta q^{m-2k})}{(q;q)_k\,(q\beta;q)_{m-k}\,(1-\beta)}\, C_{m-2k}(x;\beta \mid q) \tag{2.27}$$

(see, for example, (7.6.14) of [23]), for $\beta=q^{1/2}$ and our expansions (2.22)–(2.24) together with the explicit representation for the q-Lommel polynomials (2.10).

In a similar manner, the connection coefficient formulas for the Askey–Wilson polynomials found in [7], [30], see also (7.6.2)–(7.6.3) of [23], give rise to the q-Fourier expansion of these general classical orthogonal polynomials. We would like to leave the details to the reader.

Let us discuss also similar expansions in the "modified" Fourier series (1.24)–(1.28). One can use the same arguments as above replacing ω_n by ϖ_n and eliminating the constant term. Analog of (2.2) has the form

$$C_m(x;q^{1/2}\mid q)=\pi\,\frac{(q^{1/2};q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-i)^m\,q^{m^2/4}\,\frac{\varpi_n^{-1/2}}{k(\varpi_n)(-q\varpi_n^2;q^2)_\infty}\times J^{(2)}_{m+1/2}(2\varpi_n;q)\,\mathcal{E}_q(x;i\varpi_n),\qquad m\geqslant 0, \tag{2.28}$$

which is an "inversion" of

$$\mathcal{E}_q(x;i\varpi_n)=\frac{(q;q)_\infty}{(q^{1/2};q)_\infty}\,\frac{\varpi_n^{-1/2}}{(-q\varpi_n^2;q^2)_\infty}\times\sum_{m=0}^{\infty}i^m\,(1-q^{m+1/2})\,q^{m^2/4}J^{(2)}_{m+1/2}(2\varpi_n;q)\,C_m(x;q^{1/2}\mid q). \tag{2.29}$$

Relation with the q-Lommel polynomials is

$$J^{(2)}_{m+1/2}(2\varpi_n;q)=(-1)^{n-1}\,q^{-m^2/2}\,\frac{(q^{1/2};q)_\infty}{(q;q)_\infty}\,(-q\varpi_n^2;q^2)_\infty\times\varpi_n^{-1/2}h_{m,\,1/2}\left(\frac{1}{2\varpi_n};q\right)\sqrt{\frac{(-\varpi_n^2;q^2)_\infty}{(-q\varpi_n^2;q^2)_\infty}}, \tag{2.30}$$

$m\geqslant 0$; cf. (2.17). We shall use the shorter notation

$$\bar{h}_m(x;q):=h_{m,\,1/2}(x;q), \tag{2.31}$$

for this special case of the q-Lommel polynomials throughout the paper.

As a result, expansion (2.28) can be rewritten as

$$C_m(x;q^{1/2}\mid q)=\sum_{n=-\infty}^{\infty}(-i)^m\,\frac{q^{-m^2/4}}{\kappa(\varpi_n)\,\varpi_n}\,\bar{h}_m\left(\frac{1}{2\varpi_n};q\right)\times(-1)^{n-1}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\,\mathcal{E}_q(x;i\varpi_n), \tag{2.32}$$

$m \geqslant 0$. "Odd" and "even" parts of this expansion are

$$C_{2m+1}(x;\, q^{1/2}\,|\, q) = 2\, q^{-m(m+1)-1/4} \sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{\kappa(\varpi_n)\,\varpi_n}\, \bar{h}_{2m+1}\left(\frac{1}{2\varpi_n};\, q\right) \times \sqrt{\frac{(-q\varpi_n^2;\, q^2)_\infty}{(-\varpi_n^2;\, q^2)_\infty}}\, S_q(x;\, \varpi_n) \tag{2.33}$$

and

$$C_{2m}(x;\, q^{1/2}\,|\, q) = 2q^{-m^2} \sum_{n=1}^{\infty} \frac{(-1)^{m+n-1}}{\kappa(\varpi_n)\,\varpi_n}\, \bar{h}_{2m}\left(\frac{1}{2\varpi_n};\, q\right) \times \sqrt{\frac{(-q\varpi_n^2;\, q^2)_\infty}{(-\varpi_n^2;\, q^2)_\infty}}\, C_q(x;\, \varpi_n). \tag{2.34}$$

The series in (2.33) and (2.34) converge absolutely and uniformly in x on every closed subinterval of $(-\eta_1, \eta_1)$ and $(-\eta, \eta)$, respectively, where $\eta = x(1/4)$, $\eta_1 = x(1/2)$ and $x(\varepsilon) = (q^{\varepsilon} + q^{-\varepsilon})/2$. This can be shown with the help of Lemma 1 from Appendix and (1.30); by the analytic continuation these expansions hold in the open disks $|x| < \eta_1$ and $|x| < \eta$ of the complex x-plane, respectively.

It is worth noting the explicit orthogonality,

$$(1+(-1)^{m+p}) \sum_{n=1}^{\infty} \bar{h}_m\left(\frac{1}{2\varpi_n};\, q\right) \bar{h}_p\left(\frac{1}{2\varpi_n};\, q\right) \frac{1}{\varpi_n^2\, \kappa(\varpi_n)} = \frac{q^{m^2/2}}{1-q^{m+1/2}}\, \delta_{mp}, \tag{2.35}$$

and the completeness,

$$\sum_{m=0}^{\infty} \bar{h}_m\left(\frac{1}{2\varpi_n};\, q\right) \bar{h}_m\left(\frac{1}{2\varpi_l};\, q\right) \frac{1-q^m}{q^{m^2/2}} = \varpi_n^2\, \kappa(\varpi_n)\, \delta_{nl}, \tag{2.36}$$

relations for the special case of the q-Lommel polynomials (2.31).

Formulas (2.32)–(2.34) give us the possibility to find the expansions of the elementary powers, similar to (2.5), (2.6), (2.25), and (2.26). For example,

$$1 = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n} \sqrt{\frac{(-q\varpi_n^2;\, q^2)_\infty}{(-\varpi_n^2;\, q^2)_\infty}}\, C_q(x;\, \varpi_n), \tag{2.37}$$

$$x = \frac{1-q}{q^{1/4}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^2} \sqrt{\frac{(-q\varpi_n^2;\, q^2)_\infty}{(-\varpi_n^2;\, q^2)_\infty}}\, S_q(x;\, \varpi_n), \tag{2.38}$$

$$x^2=\frac{(1+q^{1/2})^2}{2q^{1/2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\,C_q(x;\varpi_n)$$
$$-\frac{(1-q)(1-q^2)}{2q}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^3}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\,C_q(x;\varpi_n), \tag{2.39}$$

$$x^3=\frac{(1-q^{1/2})(1+q^{1/2})^3\,(1-q^{1/2}+3q-q^{3/2}+q^2)}{4q^{7/2}}$$
$$\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^2}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\,S_q(x;\varpi_n)$$
$$-\frac{(1-q)(1-q^2)(1-q^3)}{4q^{9/4}}$$
$$\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^4}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\,S_q(x;\varpi_n). \tag{2.40}$$

Remark 1. It is worth mentioning that the connection with the q-Lommel polynomials leads to a proof of the completeness of the basic trigonometric system $\{\mathscr{E}_q(x;i\omega_n)\}_{n=-\infty}^{\infty}$ on the zeros of the Jackson q-Bessel functions (2.3) due to the completeness of the continuous q-ultraspherical and the q-Lommel polynomials. See also [14] for the original proof of the completeness property using the methods of the theory of entire functions [11, 41, 42]. These results on the completeness of the basic trigonometric system in the higher spaces of functions will be discussed in details in our forthcoming papers [53, 54, 56].

3. BASIC COSINE AND SINE FUNCTIONS

The Fourier series of the cosine function $\cos\omega x$ on the interval $(-\pi,\pi)$ has the form

$$\cos\omega x=\frac{2}{\pi}\sin\omega\pi\left[\frac{1}{2\omega}+\sum_{n=1}^{\infty}(-1)^n\frac{\omega\cos nx}{\omega^2-n^2}\right]. \tag{3.1}$$

Here we shall establish a q-analog of this formula, namely, basic Fourier expansion for $C_q(x;\omega)$, as

$$C_q(x;\omega)=2S_q(\eta;\omega)$$
$$\times\left[\frac{1}{2\kappa(0)\,\omega}+\sum_{n=1}^{\infty}\frac{(-1)^n\omega}{\kappa(\omega_n)(\omega^2-\omega_n^2)}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,C_q(x;\omega_n)\right]. \tag{3.2}$$

Let us first derive this q-Fourier series formally and then discuss its convergence. Bustoz and Suslov evaluated the following integral,

$$\int_0^\pi C_q(\cos\theta;\omega)\,C_q(\cos\theta;\omega')(e^{2i\theta},e^{-2i\theta};q)_{1/2}\,d\theta$$
$$=\frac{2\pi}{\omega^2-\omega'^2}\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}$$
$$\times[\omega C_q(\eta;\omega')\,S_q(\eta;\omega)-\omega' C_q(\eta;\omega)\,S_q(\eta;\omega')], \qquad (3.3)$$

see (3.29) of [14]. This gives the values of the q-Fourier coefficients in (3.2) when $\omega'=\omega_n$ are nonnegative zeros of (1.12). Using the uniform bounds for the basis functions (12.1)–(12.2) from Lemma 1 in Appendix and the leading term, $\lim_{n\to\infty} q^n\omega_n=q^{1/4}$, in the asymptotic expression (1.29) for the zeros ω_n, one can see that when $\omega\neq\omega_n$ the series in the right side of (3.2) converges absolutely and uniformly on every closed subinterval of $(-\eta_1,\eta_1)$ with $\eta_1=(q^{1/2}+q^{-1/2})/2$ by the Weierstrass M-Test and the Limit Comparison Test. So, it represents $C_q(x;\omega)$ on $[-1,1]$ due to Theorem 13 of [14]. By analytic continuation expansion (3.2) holds then in the open disk $|x|<\eta_1$ of the complex x-plane. We leave the details to the reader.

The Fourier series for the sine function $\sin\omega x$ on the interval $(-\pi,\pi)$ has the form

$$\sin\omega x=\frac{2}{\pi}\sin\omega\pi\sum_{n=1}^\infty(-1)^n\frac{n\sin nx}{\omega^2-n^2}. \qquad (3.4)$$

An analog of this formula for the basic sine function $S_q(x;\omega)$ is

$$S_q(x;\omega)=2S_q(\eta;\omega)$$
$$\times\sum_{n=1}^\infty\frac{(-1)^n\omega_n}{\kappa(\omega_n)(\omega^2-\omega_n^2)}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,S_q(x;\omega_n). \qquad (3.5)$$

Integral (3.30) of [14],

$$\int_0^\pi S_q(\cos\theta;\omega)\,S_q(\cos\theta;\omega')(e^{2i\theta},e^{-2i\theta};q)_{1/2}\,d\theta$$
$$=\frac{2\pi}{\omega^2-\omega'^2}\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}$$
$$\times[\omega' S_q(\eta;\omega)\,C_q(\eta;\omega')-\omega S_q(\eta;\omega')\,C_q(\eta;\omega)], \qquad (3.6)$$

results in the q-Fourier coefficients in (3.5) when $\omega' = \omega_n$ are nonnegative zeros of (1.12). When $\omega \neq \omega_n$ the series in the right side of (3.5) converges absolutely and uniformly on every closed subinterval of $(-\eta, \eta)$ with $\eta = (q^{1/4} + q^{-1/4})/2$, so it represents $S_q(x; \omega)$ on $[-1, 1]$ and, by analytic continuation, in the open disk $|x| < \eta$.

In a similar manner one can obtain expansions of the q-trigonometric functions in the series (1.24)–(1.28). Expansion for $C_q(x; \omega)$ has the form

$$C_q(x; \omega) = 2C_q(\eta; \omega) \times \left[\sum_{n=1}^{\infty} \frac{(-1)^n \varpi_n}{\kappa(\varpi_n)(\omega^2 - \varpi_n^2)} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}}\, C_q(x; \varpi_n) \right]. \tag{3.7}$$

The q-Fourier coefficients can be evaluated with the help of (3.3) when $\omega' = \varpi_n$ are nonnegative zeros of (1.13). If $\omega \neq \varpi_n$ the series in the right side of (3.7) converges absolutely and uniformly on every closed subinterval of $(-\eta, \eta)$ by the Weierstrass M-Test and the Limit Comparison Test. Thus, it represents $C_q(x; \omega)$ on $[-1, 1]$ due to Theorem 13 of [14]. By the analytic continuation expansion (3.7) holds then in the open disk $|x| < \eta$ of the complex x-plane.

Expansion of the basic sine function $S_q(x; \omega)$ is

$$S_q(x; \omega) = 2C_q(\eta; \omega) \times \sum_{n=1}^{\infty} \frac{(-1)^n \omega}{\kappa(\varpi_n)(\omega^2 - \varpi_n^2)} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}}\, S_q(x; \varpi_n) \tag{3.8}$$

in view of the integral (3.6). When $\omega \neq \varpi_n$ the series in the right side of (3.8) converges absolutely and uniformly on every closed subinterval of $(-\eta_1, \eta_1)$ with $\eta_1 = (q^{1/2} + q^{-1/2})/2$, so it represents $S_q(x; \omega)$ on $[-1, 1]$ and, by the analytic continuation, in the open disk $|x| < \eta_1$.

4. BASIC EXPONENTIAL FUNCTION

Expansion of the exponential function $\exp(\alpha x)$ in Fourier series on $(-\pi, \pi)$ is

$$e^{\alpha x} = \frac{e^{\alpha\pi} - e^{-\alpha\pi}}{\pi} \left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2 + n^2} (\alpha \cos nx - n \sin nx) \right]. \tag{4.1}$$

Expansion for $\mathscr{E}_q(x; \alpha)$ in the basic Fourier series has a similar form

$$\mathscr{E}_q(x; \alpha) = \frac{(-\alpha; q^{1/2})_\infty - (\alpha; q^{1/2})_\infty}{(q\alpha^2; q^2)_\infty} \times \left[\frac{1}{2\kappa(0)\,\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\kappa(\omega_n)(\alpha^2+\omega_n^2)} \times \sqrt{\frac{(-q\omega_n^2; q^2)_\infty}{(-\omega_n^2; q^2)_\infty}}\, (\alpha C_q(x; \omega_n) - \omega_n S_q(x; \omega_n)) \right]. \tag{4.2}$$

It follows directly from the basic Fourier series (3.2) and (3.5) for the $C_q(x; \omega)$ and $S_q(x; \omega)$ and the analog of Euler's formula (1.21).

Formula (4.2) admits further generalization, namely,

$$\begin{aligned} &\mathscr{E}_q(\cos\theta; \alpha)\, \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty}\, C_m(\cos\theta; \gamma \mid q) \\ &= \frac{\pi(\gamma, \gamma q^{m+1}; q)_\infty}{(q; q)_m\, (q, \gamma^2 q^m; q)_\infty\, (q\alpha^2; q^2)_\infty} \\ &\times \sum_{n=-\infty}^{\infty} \frac{(i\alpha\omega_n q^{(m+1)/2}; q)_\infty}{k(\omega_n)\,(-q\omega_n^2; q^2)_\infty}\, \alpha^m q^{m^2/4} \\ &\times (iq^{(1-m)/2}\omega_n/\alpha; q)_m\, \mathscr{E}_q(\cos\theta; i\omega_n) \\ &\times {}_2\varphi_2\left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha, -iq^{(m+1)/2}\alpha/\omega_n \\ \gamma q^{m+1}, \quad iq^{(m+1)/2}\alpha\omega_n \end{matrix} ; q, iq^{(m+1)/2}\alpha\gamma\omega_n \right), \end{aligned} \tag{4.3}$$

where $C_m(\cos\theta; \gamma \mid q)$ are the continuous q-ultraspherical polynomials. We derive this q-Fourier expansion only formally here. In order to complete the proof one can see Appendix for the details on the uniform convergence of this series.

Ismail and Stanton [33] evaluated the following important integral

$$\begin{aligned} &\int_0^\pi \mathscr{E}_q(\cos\theta; \alpha)\, \mathscr{E}_q(\cos\theta; \beta)\, C_m(\cos\theta; \gamma \mid q) \\ &\quad \times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty}\, d\theta \\ &= \frac{2\pi(\gamma, \gamma q^{m+1}, -\alpha\beta q^{(m+1)/2}; q)_\infty\, (-q^{(1-m)/2}\beta/\alpha; q)_m}{(q; q)_m\, (q, \gamma^2 q^m; q)_\infty\, (q\alpha^2, q\beta^2; q^2)_\infty}\, \alpha^m q^{m^2/4} \\ &\times {}_2\varphi_2\left(\begin{matrix} -q^{(m+1)/2}\alpha/\beta, -q^{(m+1)/2}\beta/\alpha \\ \gamma q^{m+1}, \quad -q^{(m+1)/2}\alpha\beta \end{matrix} ; q, -q^{(m+1)/2}\alpha\beta\gamma \right). \end{aligned} \tag{4.4}$$

The proof of (4.4) is also given in [53]. The special case $\beta=0$ takes the form

$$\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\,\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\,C_m(\cos\theta;\gamma\mid q)\,d\theta$$

$$=\frac{2\pi(\gamma,\gamma q^{m+1};q)_\infty}{(q;q)_m\,(q,\gamma^2q^m;q)_\infty\,(q\alpha^2;q^2)_\infty}\,\alpha^m q^{m^2/4}$$

$$\times\,{}_0\varphi_1\left(\begin{matrix}-\\ \gamma q^{m+1}\end{matrix};q,\alpha^2\gamma q^{m+1}\right). \tag{4.5}$$

Integral (4.4) is one of our "key tools" in the present paper, we shall call it the Ismail and Stanton integral. One can easily show that the special case $m=0$ and $\gamma=q^{1/2}$ of (4.4) leads to (1.8)–(1.11). This gives an independent proof of Bustoz and Suslov's orthogonality property for the basic trigonometric system $\{\mathcal{E}_q(x;i\omega_n)\}_{n=-\infty}^{\infty}$; see [53].

Expansion (4.3) follows formally from the definition of the complex form of q-Fourier series (1.18)–(1.19) and the integral (4.4) by Ismail and Stanton. When $0<|\alpha|<1$ and $0\leqslant\gamma<1$ the series in the right side of (4.3) converges absolutely and uniformly on $[-1,1]$ which completes the proof; see Appendix for more details.

It is worth mentioning a few special cases of (4.3). When $m=0$ and $\gamma=q^{1/2}$ we obtain q-Fourier expansion (4.2) again. Case $\alpha=0$ and $\gamma=q^{1/2}$ corresponds to the expansion of the q-Legendre polynomials $C_m(x;q^{1/2}\mid q)$ in basic Fourier series (2.22); see also (12.8) of [14]. When $\gamma=0$ expansion (4.3) takes the form

$$\mathcal{E}_q(\cos\theta;\alpha)(q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty\,H_m(\cos\theta\mid q)$$

$$=\frac{\pi}{(q;q)_\infty\,(q\alpha^2;q^2)_\infty}\,\alpha^m q^{m^2/4}$$

$$\times\sum_{n=-\infty}^{\infty}\frac{(i\alpha\omega_n q^{(m+1)/2};q)_\infty}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}\,(iq^{(1-m)/2}\omega_n/\alpha;q)_m$$

$$\times\,\mathcal{E}_q(\cos\theta;i\omega_n), \tag{4.6}$$

where

$$H_m(\cos\theta\mid q)=(q;q)_m\,C_m(\cos\theta;0|q) \tag{4.7}$$

are the continuous q-Hermite polynomials. Expansion (4.6) follows directly from the integral (5.4) of [33]. Two special cases of (4.6), namely, $m=0$,

$$\begin{aligned} &\mathscr{E}_q(\cos\theta;\alpha)(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty \\ &\quad = \frac{\pi}{(q;q)_\infty\,(q\alpha^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(i\alpha\omega_n q^{1/2};q)_\infty}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}\,\mathscr{E}_q(\cos\theta; i\omega_n), \end{aligned} \tag{4.8}$$

and $\alpha=0$,

$$\begin{aligned} &(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty\, H_m(\cos\theta \mid q) \\ &\quad = \frac{\pi}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{m^2/4}\,(-i\omega_n)^m}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}\,\mathscr{E}_q(\cos\theta; i\omega_n), \end{aligned} \tag{4.9}$$

are of interest to note. Use of the generating function (9.5) of [48],

$$\sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m}\,\alpha^m H_m(\cos\theta \mid q) = (q\alpha^2;q^2)_\infty \mathscr{E}_q(\cos\theta;\alpha) \tag{4.10}$$

see also [36], in (4.8) leads to (4.9) again. Roger's generating relation for the continuous q-Hermite polynomials,

$$\sum_{m=0}^{\infty} H_m(\cos\theta \mid q)\,\frac{\alpha^m}{(q;q)_m} = \frac{1}{(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty}, \qquad |\alpha|<1 \tag{4.11}$$

and (4.9) results in the following expansion,

$$\begin{aligned} &\frac{(q,\, q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{\pi(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty} \\ &\quad = \sum_{n=-\infty}^{\infty} \frac{\varepsilon_q(-i\alpha\omega_n q^{1/4})\,\mathscr{E}_q(\cos\theta; i\omega_n)}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}, \end{aligned} \tag{4.12}$$

where $|\alpha|<1$ and

$$\varepsilon_q(x) = \sum_{m=0}^{\infty} \frac{q^{m(m-1)/4}}{(q;q)_m}\, x^m \tag{4.13}$$

is an analog of the exponential function on a q-linear grid (see, for example, [48] and references therein).

Multiplying both sides of (4.9) by $r^m\ H_m(\cos\varphi\mid q)/(q;q)_m$ and then summing over m from zero to infinity with the help of the Poisson kernel for the continuous q-Hermite polynomials

$$\sum_{m=0}^{\infty}\frac{r^m}{(q;q)_m}H_m(\cos\theta\mid q)\,H_m(\cos\varphi\mid q)$$
$$=\frac{(r^2;q)_\infty}{(re^{i\theta+i\varphi},re^{i\theta-i\varphi},re^{i\varphi-i\theta},re^{-i\theta-i\varphi};q)_\infty},\qquad |r|<1, \tag{4.14}$$

(see, for example, [23]) and the generating function (4.10), we obtain the following bilinear generating relation for the basic exponential functions,

$$\sum_{n=-\infty}^{\infty}\frac{(-qr^2\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty}k^{-1}(\omega_n)\,\mathscr{E}_q(\cos\theta;i\omega_n)\,\mathscr{E}_q(\cos\varphi;-ir\omega_n)$$
$$=\frac{(q,r^2,q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty}{\pi(re^{i\theta+i\varphi},re^{i\theta-i\varphi},re^{i\varphi-i\theta},re^{-i\theta-i\varphi};q)_\infty},\qquad |r|<1, \tag{4.15}$$

originally found in [14]. Bustoz and Suslov have also introduced a method of summation of basic Fourier series on the basis of this explicit "Poisson-type" kernel. Some integrals related to the kernels (4.12) and (4.15) are discussed in Section 11.

Equation (4.3) gives us also the possibility to find q-Fourier expansions of "generalized power functions." The case $m=0$ and $\alpha=0$ results in

$$\frac{(q,\gamma^2,q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty}{\pi(\gamma,\gamma q,\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}$$
$$=\sum_{n=-\infty}^{\infty}\frac{1}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}\,{}_0\varphi_1\left(\begin{matrix}-\\ \gamma q\end{matrix};q,-\gamma q\omega_n^2\right)$$
$$\times\mathscr{E}_q(\cos\theta;i\omega_n)$$
$$=\frac{1}{k(0)}+2\sum_{n=1}^{\infty}\frac{1}{k(\omega_n)(-q\omega_n^2;q^2)_\infty}$$
$$\times{}_0\varphi_1\left(\begin{matrix}-\\ \gamma q\end{matrix};q,-\gamma q\omega_{nn}^2\right)C_q(\cos\theta;\omega), \tag{4.16}$$

where the left side is, essentially, the quotient of the weight functions for the two systems of the continuous q-ultraspherical polynomials, $C_m(\cos\theta;\gamma\mid q)$ and $C_m(\cos\theta;q^{1/2}\mid q)$. When $\gamma=0$ we arrive at the generating function (10.7) of [14] (factor π is missing in the left side of this formula). Substituting $\gamma = q^{m+1/2}$ in (4.16), with the help of (1.4)–(1.7), (2.3), and (2.17), one gets the q-Fourier cosine expansion for the even "generalized power function" of the form,

$$(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_m = \prod_{k=0}^{m-1} ((1+q^{k+1/2})^2 - 4q^{k+1/2}x^2), \tag{4.17}$$

$x = \cos\theta$, namely,

$$\begin{aligned}
&\prod_{k=0}^{m-1} ((1+q^{k+1/2})^2 - 4q^{k+1/2}x^2) \\
&\quad = \frac{(q; q)_{2m}}{(q^{1/2}, q^{3/2}; q)_m} + 2\frac{(q; q)_{2m}}{(q^{1/2}; q)_m}\, q^{-m^2/2} \\
&\quad \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\, \omega_n^{m+1}}\, h_{m-1}\left(\frac{1}{2\omega_n}; q\right) \sqrt{\frac{(-q\omega_n^2; q^2)_\infty}{(-\omega_n^2; q^2)_\infty}}\, C_q(x; \omega_n), \quad m > 1.
\end{aligned} \tag{4.18}$$

The case $m = 1$ and $\alpha = 0$ of (4.3) gives rise to the following q-Fourier expansion,

$$\begin{aligned}
&\frac{(q, \gamma^2 q, q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{\pi(\gamma q, \gamma q^2, \gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} \cos\theta \\
&\quad = \sum_{n=-\infty}^{\infty} \frac{q^{1/4}(-i\omega_n)}{2k(\omega_n)(-q\omega_n^2; q^2)_\infty}\, {}_0\varphi_1\left(\begin{matrix} - \\ \gamma q^2 \end{matrix}; q, -\gamma q^2 \omega_n^2\right) \\
&\quad \times \mathscr{E}_q(\cos\theta; i\omega_n) \\
&\quad = \sum_{n=1}^{\infty} \frac{q^{1/4}\omega_n}{k(\omega_n)(-q\omega_n^2; q^2)_\infty}\, {}_0\varphi_1\left(\begin{matrix} - \\ \gamma q^2 \end{matrix}; q, -\gamma q^2 \omega_n^2\right) \\
&\quad \times S_q(\cos\theta; \omega_n).
\end{aligned} \tag{4.19}$$

Substituting $\gamma = q^{m-1/2}$ we obtain, in a similar manner, a "counterpart" of (4.18), the q-Fourier sine expansion for the odd "generalized power function" of the form,

$$(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_{m-1} \cos\theta = x \prod_{k=0}^{m-2} ((1+q^{k+1/2})^2 - 4q^{k+1/2}x^2), \tag{4.20}$$

$x = \cos\theta$, namely,

$$x \prod_{k=0}^{m-2} ((1+q^{k+1/2})^2 - 4q^{k+1/2}x^2)$$
$$= \frac{(q;q)_{2m-1}}{(q^{1/2};q)_m} q^{-m^2/2+1/4}$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n^m} h_{m-1}\left(\frac{1}{2\omega_n}; q\right) \sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\, S_q(x;\omega_n), \qquad (4.21)$$

$m>0$. One can obtain expansions (2.5)–(2.6) and (2.25)–(2.26) as the special cases $m=1$ and $m=2$ of (4.18) and (4.21). These formulas are very convenient for expansion of the elementary powers of the higher degrees in basic Fourier series.

On the other hand, the connection relation [30],

$$(\alpha e^{i\chi}, \alpha e^{-i\chi}; q)_m = \sum_{k=0}^{m} q^k \frac{(q^{-m};q)_k\,(\alpha a, \alpha/a;q)_m}{(q, \alpha a, q^{1-m}a/\alpha;q)_k} (ae^{i\chi}, ae^{-i\chi};q)_k, \qquad (4.22)$$

see also (II.12) of [23], with $a=q^{1/2}$ and $\chi = 2\theta$ gives rise to one parameter extensions of (4.18) and (4.21).

Expansion for $\mathscr{E}_q(x;\alpha)$ in the "modified" basic Fourier series (1.24), which will be needed in Section 8, has the form

$$\mathscr{E}_q(x;\alpha) = \frac{(-\alpha;q^{1/2})_\infty + (\alpha;q^{1/2})_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\varpi_n)(\alpha^2+\varpi_n^2)}$$
$$\times \sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\, (\varpi_n C_q(x;\varpi_n) + \alpha S_q(x;\varpi_n)). \qquad (4.23)$$

It follows directly from the series (3.7), (3.8) and the q-analog of Euler's formula (1.21). An extension of (4.23) is

$$\mathscr{E}_q(\cos\theta;\alpha) \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta};q)_\infty} C_m(\cos\theta;\gamma\,|\,q)$$
$$= \frac{\pi(\gamma, \gamma q^{m+1};q)_\infty}{(q;q)_m\,(q,\gamma^2 q^m;q)_\infty\,(q\alpha^2;q^2)_\infty}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(i\alpha\varpi_n q^{(m+1)/2};q)_\infty}{k(\omega_n)\,(-q\varpi_n^2;q^2)_\infty} \alpha^m q^{m^2/4}$$
$$\times (iq^{(1-m)/2}\varpi_n/\alpha;q)_m\, \mathscr{E}_q(\cos\theta; i\varpi_n)$$
$$\times {}_2\varphi_2\left(\begin{matrix} iq^{(m+1)/2}\varpi_n/\alpha, -iq^{(m+1)/2}\alpha/\varpi_n \\ \gamma q^{m+1}, \quad iq^{(m+1)/2}\alpha\varpi_n \end{matrix}; q, iq^{(m+1)/2}\alpha\gamma\varpi_n\right). \qquad (4.24)$$

This follows formally from the definition of the "modified" q-Fourier series (1.27) and the integral (4.4). When $0<|\alpha|<1$ and $0\leqslant\gamma<1$ the series in the right side of (4.24) converges absolutely and uniformly on $[-1,1]$ which completes the proof; see Appendix.

The q-Fourier expansion (4.24) has many interesting special and limiting cases. We leave the details to the reader and discuss the q-Fourier expansions of "generalized power functions" only. The special case $m=0$ and $\alpha=0$ of (4.24) is

$$\begin{aligned}
&\frac{(q, \gamma^2, q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{\pi(\gamma, \gamma q, \gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty}\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{1}{k(\varpi_n)(-q\varpi_n^2; q^2)_\infty}\,{}_0\varphi_1\left(\begin{matrix}-\\ \gamma q\end{matrix}; q, -\gamma q\varpi_n^2\right)\\
&\qquad\times\mathscr{E}_q(\cos\theta; i\varpi_n)\\
&\quad=2\sum_{n=1}^{\infty}\frac{1}{k(\varpi_n)(-q\varpi_n^2; q^2)_\infty}\\
&\qquad\times{}_0\varphi_1\left(\begin{matrix}-\\ \gamma q\end{matrix}; q, -\gamma q\varpi_n^2\right)C_q(\cos\theta; \varpi_n).
\end{aligned}\tag{4.25}$$

Substituting $\gamma=q^{m+1/2}$, one gets the q-Fourier cosine expansion for the even "generalized powers"

$$\begin{aligned}
&\prod_{k=0}^{m-1}((1+q^{k+1/2})^2-4q^{k+1/2}x^2)=2\,\frac{(q; q)_{2m}}{(q^{1/2}; q)_m}\,q^{-m^2/2}\\
&\quad\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^{m+1}}\,\bar{h}_m\left(\frac{1}{2\varpi_n}; q\right)\sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}}\,C_q(x; \varpi_n).
\end{aligned}\tag{4.26}$$

In a similar fashion, letting $m=1$, $\alpha=0$ in (4.24) and then substituting $\gamma=q^{m-1/2}$, we obtain the q-Fourier sine expansion for the odd "generalized powers"

$$\begin{aligned}
&x\prod_{k=0}^{m-2}((1+q^{k+1/2})^2-4q^{k+1/2}x^2)=\frac{(q; q)_{2m-1}}{(q^{1/2}; q)_m}\,q^{-m^2/2+1/4}\\
&\quad\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n^{m}}\,\bar{h}_m\left(\frac{1}{2\varpi_n}; q\right)\sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}}\,S_q(x; \varpi_n),
\end{aligned}\tag{4.27}$$

$m>0$. These formulas are very convenient for deriving the expansions (2.37)–(2.40).

5. BASIC COSECANT AND COTANGENT FUNCTIONS

The partial fraction decompositions for the cosecant and cotangent functions are

$$\frac{1}{\sin \omega}=\frac{1}{\omega}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\omega-\pi n}+\frac{1}{\omega+\pi n}\right) \tag{5.1}$$

and

$$\cot \omega=\frac{1}{\omega}+\sum_{n=1}^{\infty}\left(\frac{1}{\omega-\pi n}+\frac{1}{\omega+\pi n}\right). \tag{5.2}$$

The special cases $x=0$ and $x=\eta=(q^{1/4}+q^{-1/4})/2$ of the expansion of $C_q(x;\omega)$ in the basic Fourier series (3.2), result in natural q-extensions of these classical formulas

$$\begin{aligned}\frac{1}{S_q(\eta;\omega)}&=\frac{1}{2\kappa(0)\,\omega}\\ &\quad+\sum_{n=1}^{\infty}\frac{(-1)^n}{\kappa(\omega_n)}\left(\frac{1}{\omega-\omega_n}+\frac{1}{\omega+\omega_n}\right)\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\mathrm{Cot}_q(\omega):&=\frac{C_q(\eta;\omega)}{S_q(\eta;\omega)}\\ &=\frac{1}{\kappa(0)\,\omega}+\sum_{n=1}^{\infty}\frac{1}{\kappa(\omega_n)}\left(\frac{1}{\omega-\omega_n}+\frac{1}{\omega+\omega_n}\right),\end{aligned} \tag{5.4}$$

respectively.

These partial fraction decompositions can also be established with the help of Mittag-Leffler's theorem; see, for example, [25] and [37]. Indeed, functions $f(\omega)=1/S_q(\eta;\omega)$ and $g(\omega)=C_q(\eta;\omega)/S_q(\eta;\omega)$ are meromorphic functions in ω with simple poles at zeros of $S_q(\eta;\omega)$, $\omega=0,\ \pm\omega_1,\ \pm\omega_2,\ \pm\omega_3,\ \ldots$. The residues at these poles $\omega=\omega_n$ are

$$\begin{aligned}\operatorname{Res} f(\omega)\big|_{\omega=\omega_n}&=\lim_{\omega\to\omega_n}\frac{\omega-\omega_n}{S_q(\eta;\omega)}=\lim_{\omega\to\omega_n}\frac{1}{\dfrac{d}{d\omega}S_q(\eta;\omega)}\\ &=\pi\,\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}\,\frac{(-1)^n}{k(\omega_n)}\,\sqrt{\frac{(-\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty}}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res} g(\omega)|_{\omega=\omega_n} &= \lim_{\omega\to\omega_n} (\omega-\omega_n)\,\frac{C_q(\eta;\omega)}{S_q(\eta;\omega)}\\ &= \lim_{\omega\to\omega_n} \frac{C_q(\eta;\omega)}{\dfrac{d}{d\omega}\,S_q(\eta;\omega)}\\ &= \pi\,\frac{(q^{1/2};q)^2_\infty}{(q;q)^2_\infty}\,\frac{1}{k(\omega_n)}\,\frac{(-\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty},\end{aligned}\tag{5.5}$$

respectively. We have used here (13.10) and the following consequence of (3.31)–(3.32) and (6.6), all of [14],

$$k(\omega_n)=\pi\,\frac{(q^{1/2};q)^2_\infty}{(q;q)^2_\infty}\,C_q(\eta;\omega_n)\,\frac{d}{d\omega}\,S_q(\eta;\omega_n).\tag{5.6}$$

To complete the proof of (5.3) and (5.4) on the basis of the Mittag–Leffler theorem, one has to show that the still undetermined additive entire functions in the partial fraction decompositions for $f(\omega)$ and $g(\omega)$ are identically zero.

Special cases $x=0$ of (3.7) and $x=\eta$ of (3.8) result in the following partial fraction decompositions

$$\frac{1}{C_q(\eta;\omega)}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\kappa(\varpi_n)}\left(\frac{1}{\omega-\varpi_n}-\frac{1}{\omega+\varpi_n}\right)\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}}\tag{5.7}$$

and

$$\operatorname{Tan}_q(\omega):=\frac{S_q(\eta;\omega)}{C_q(\eta;\omega)}=-\sum_{n=1}^{\infty}\frac{1}{\kappa(\varpi_n)}\left(\frac{1}{\omega-\varpi_n}+\frac{1}{\omega+\varpi_n}\right),\tag{5.8}$$

respectively.

6. SOME CONSEQUENCES OF PARSEVAL'S IDENTITY

Parseval's formula,

$$\begin{aligned}&2k(0)\,a_0^2+\sum_{n=1}^{\infty}k(\omega_n)\,(a_n^2+b_n^2)\\ &\qquad=\sum_{n=-\infty}^{\infty}2k(\omega_n)|c_n|^2=\int_{-1}^{1}|f(x)|^2\,\rho(x)\,dx,\end{aligned}\tag{6.1}$$

holds for basic Fourier series due to the completeness of the q-trigonometric system $\{\mathscr{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$; see [2, 14]. Here c_n are the q-Fourier coefficients of $f(x)$ defined by (1.19), $2k(\omega_n)$ and $\rho(x)$ are the $\mathscr{L}^2$-norm and the weight function in the orthogonality relations (1.8)–(1.11), respectively; we have corrected a misprint in (9.24) of [14]. Equation (6.1) gives rise to some new series. We consider only a few examples here.

Expansion of the basic cosine function $C_q(x; \omega)$ in basic Fourier series (3.2) and (6.1) result in the following partial fraction decomposition

$$\frac{1}{2k(0)\,\omega^2}+\sum_{n=1}^{\infty}\frac{\omega^2}{k(\omega_n)(\omega^2-\omega_n^2)^2}\,\frac{(-\omega_n^2; q^2)_\infty}{(-q\omega_n^2; q^2)_\infty}$$
$$=\frac{(q; q)_\infty^4}{4\pi^2\,(q^{1/2}; q)_\infty^4}\left[\frac{k(\omega)}{S_q^2(\eta; \omega)}+\frac{\pi(q^{1/2}; q)_\infty^2}{\omega(q; q)_\infty^2}\,\frac{C_q(\eta; \omega)}{S_q(\eta; \omega)}\right] \tag{6.2}$$

in view of the integral (6.10) of [14], namely,

$$\int_0^\pi C_q^2(\cos\theta; \omega)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\,d\theta$$
$$=k(\omega)+\frac{\pi(q^{1/2}; q)_\infty^2}{\omega(q; q)_\infty^2}\,C_q(\eta; \omega)\,S_q(\eta; \omega). \tag{6.3}$$

Expansion of the basic sine function $S_q(x; \omega)$ in basic Fourier series (3.5) together with (6.1) gives another partial fraction decomposition

$$\sum_{n=1}^{\infty}\frac{\omega_n^2}{k(\omega_n)(\omega^2-\omega_n^2)^2}\,\frac{(-\omega_n^2; q^2)_\infty}{(-q\omega_n^2; q^2)_\infty}$$
$$=\frac{(q; q)_\infty^4}{4\pi^2\,(q^{1/2}; q)_\infty^4}\left[\frac{k(\omega)}{S_q^2(\eta; \omega)}-\frac{\pi(q^{1/2}; q)_\infty^2}{\omega(q; q)_\infty^2}\,\frac{C_q(\eta; \omega)}{S_q(\eta; \omega)}\right] \tag{6.4}$$

due to the integral (6.11) of [14],

$$\int_0^\pi S_q^2(\cos\theta; \omega)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\,d\theta$$
$$=k(\omega)-\frac{\pi(q^{1/2}; q)_\infty^2}{\omega(q; q)_\infty^2}\,C_q(\eta; \omega)\,S_q(\eta; \omega). \tag{6.5}$$

Subtracting (6.4) from (6.2) we obtain expansion (5.4) once again. This gives an independent proof of this formula. The addition of two equations (6.2) and (6.4) leads to a new result,

$$\frac{1}{2k(0)\,\omega^2}+\sum_{n=1}^{\infty}\frac{(\omega^2+\omega_n^2)}{k(\omega_n)(\omega^2-\omega_n^2)^2}\frac{(-\omega_n^2;q^2)_\infty}{(-q\omega_n^2;q^2)_\infty}$$
$$=\frac{(q;q)_\infty^4}{2\pi^2\,(q^{1/2};q)_\infty^4}\frac{k(\omega)}{S_q^2(\eta;\omega)}. \tag{6.6}$$

Writing

$$\frac{(\omega^2+\omega_n^2)}{(\omega^2-\omega_n^2)^2}=\frac{1}{2}\left[\frac{1}{(\omega-\omega_n)^2}+\frac{1}{(\omega+\omega_n)^2}\right],$$

one gets, finally,

$$\sum_{n=-\infty}^{\infty}\frac{1}{\kappa(\omega_n)(\omega-\omega_n)^2}=\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\frac{\kappa(\omega)}{S_q^2(\eta;\omega)}, \tag{6.7}$$

as a q-analog of

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\omega-\pi n)^2}=\frac{1}{\sin^2\omega}. \tag{6.8}$$

It is worth noting that—as in the classical case—expansion (6.7) can be obtain by a formal differentiation of the both sides of (5.4) using (2.18)–(2.19) and (2.22)–(2.23) of [24] or (2.4)–(2.5) and (2.8) of [52].

Similarly, expansion of the basic exponential function $\mathscr{E}_q(x;\alpha)$ in basic Fourier series (4.2) results in the following partial fraction decomposition

$$\frac{1}{2\kappa(0)\,\alpha}+\sum_{n=1}^{\infty}\frac{\alpha}{\kappa(\omega_n)(\alpha^2+\omega_n^2)}=\frac{1}{2}\frac{(-\alpha;q^{1/2})_\infty+(\alpha;q^{1/2})_\infty}{(-\alpha;q^{1/2})_\infty-(\alpha;q^{1/2})_\infty}. \tag{6.9}$$

Here we have used integral (6.8) from [53]

$$\int_0^{\pi}\mathscr{E}_q(\cos\theta;\alpha)\,\mathscr{E}_q(\cos\theta;\beta)(e^{2i\theta},e^{-2i\theta};q)_{1/2}\,d\theta$$
$$=\pi\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(-\alpha,-\beta;q^{1/2})_\infty-(\alpha,\beta;q^{1/2})_\infty}{(\alpha+\beta)(q\alpha^2,q\beta^2;q^2)_\infty}. \tag{6.10}$$

Expansion (5.4) for the basic cotangent function appears once again if one changes α by $i\omega$. This gives another proof of (5.4).

Parseval's identity (6.1) for expansions (2.5) and (2.6) results in

$$\sum_{n=1}^{\infty} \frac{1}{\kappa(\omega_n)\,\omega_n^2} = \frac{q^{1/2}}{2(1-q^{3/2})} \tag{6.11}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\kappa(\omega_n)\,\omega_n^4} = \frac{q^2}{2(1-q^{3/2})^2\,(1-q^{5/2})}, \tag{6.12}$$

respectively. We have used (2.6) and (2.26) in order to evaluate the integrals; see [24] for numerical investigation of the convergence of these series. Relations (6.11) and (6.12) are q-analogs of Euler's sums

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \tag{6.13}$$

respectively. See further discussion of these and related series in Section 9.

We can also derive (6.11) if we rewrite (6.7) in the form

$$\sum_{n=1}^{\infty} \frac{2}{\kappa(\omega_n)(\omega-\omega_n)^2} = \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty}\,\frac{\kappa(\omega)}{S_q^2(\eta; \omega)} - \frac{1}{\kappa(0)\,\omega^2} \tag{6.14}$$

and then take the limit of the both sides as $\omega \to 0$. The termwise limit of the left side gives the series in (6.11) up to a constant multiple. In order to take the limit in the right side of (6.14), we use the Laurent expansion

$$\frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty}\,\frac{\kappa(\omega)}{S_q^2(\eta; \omega)} = \frac{1-q^{1/2}}{\omega^2} + \frac{q^{1/2}}{1-q^{3/2}} + O(\omega^2) \tag{6.15}$$

as $\omega \to 0$, which can be obtain from (6.5) and (6.15) of [14]. Equation (6.11) follows also from (6.9) in a similar fashion, using the constant term in the Laurent expansion of the right side at $\alpha = 0$.

It is worth mentioning, finally, that the Parseval identity (6.1) for expansions (2.23)–(2.24) gives the $\mathscr{L}^2$-norm of the q-Lommel polynomials (2.20) in the orthogonality relation (2.18). Similar considerations hold in the case of the basic Fourier series (1.27); we leave the details to the reader.

7. BERNOULLI POLYNOMIALS, NUMBERS, AND THEIR q-EXTENSIONS

The classical Bernoulli polynomials can be introduced by means of the generating function

$$\frac{\alpha e^{\alpha x}}{e^{\alpha}-1}=1+\sum_{m=1}^{\infty} B_m(x)\,\frac{\alpha^m}{m!}, \qquad |\alpha|<2\pi. \tag{7.1}$$

They have the following Fourier expansions

$$B_{2m-1}(x)=(-1)^m\,\frac{2(2m-1)!}{(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n^{2m-1}} \tag{7.2}$$

and

$$B_{2m}(x)=(-1)^{m-1}\,\frac{2(2m)!}{(2\pi)^{2m}}\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^{2m}} \tag{7.3}$$

with $m=1, 2, 3, \ldots$ (see, for example, [1, 2, 5, 22, 64]). The Bernoulli numbers are defined as

$$B_m=B_m(0). \tag{7.4}$$

One can see that $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$; $B_{2m+1}=0$ when $m=1, 2, \ldots$. These numbers appear as coefficients in Taylor's expansion

$$\begin{aligned}\frac{\omega}{2}\cot\frac{\omega}{2}&=1+\sum_{m=1}^{\infty}(-1)^m\,\frac{B_{2m}}{(2m)!}\,\omega^{2m}\\&=1-\frac{1}{12}\,\omega^2-\frac{1}{720}\,\omega^4-\frac{1}{30240}\,\omega^6-\cdots.\end{aligned} \tag{7.5}$$

Also, the special case $x=0$ of (7.1) is

$$\frac{\alpha}{e^{\alpha}-1}=1+\sum_{m=1}^{\infty} B_m\,\frac{\alpha^m}{m!} \tag{7.6}$$

and, in view of (7.1) and (7.6), we arrive at the explicit expression for the Bernoulli polynomials

$$B_m(x)=\sum_{k=0}^{m} B_{m-k}\,C_m^k\,x^k, \qquad C_m^k=\frac{m!}{k!\,(m-k)!} \tag{7.7}$$

in terms of Bernoulli's numbers and binomial coefficients. See [1, 5, 22, 64] for an account of further properties of the Bernoulli polynomials and numbers.

The basic Fourier series (4.2) give us the possibility to introduce q-analogs of the Bernoulli polynomials and numbers in the following manner. Let us start from the classical case $q=1$, rewrite expansion (4.1) as

$$\frac{\alpha\pi e^{\alpha x}}{e^{\alpha\pi}-e^{-\alpha\pi}}=\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{\alpha^2}{\alpha^2+n^2}\cos nx+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\alpha n}{\alpha^2+n^2}\sin nx \tag{7.8}$$

and substitute

$$\frac{\alpha^2}{\alpha^2+n^2}=\sum_{m=1}^{\infty}(-1)^{m-1}\frac{\alpha^{2m}}{n^{2m}}, \qquad \frac{\alpha n}{\alpha^2+n^2}=\sum_{m=1}^{\infty}(-1)^{m-1}\frac{\alpha^{2m-1}}{n^{2m-1}}.$$

Thus,

$$\frac{\alpha\pi e^{\alpha x}}{e^{\alpha\pi}-e^{-\alpha\pi}}=\frac{1}{2}+\sum_{m=1}^{\infty}\alpha^{2m-1}b_{2m-1}(x)+\sum_{m=1}^{\infty}\alpha^{2m}b_{2m}(x), \tag{7.9}$$

where

$$b_{2m-1}(x)=(-1)^{m-1}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n^{2m-1}}, \tag{7.10}$$

$$b_{2m}(x)=(-1)^{m}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^{2m}} \tag{7.11}$$

are, essentially, Bernoulli's polynomials

$$b_m(x)=\frac{(2\pi)^m}{2(m!)}B_m\left(\frac{x+\pi}{2\pi}\right), \tag{7.12}$$

see [2], or just compare (7.2)–(7.3) with (7.10)–(7.11). The multiples of the Bernoulli numbers appear here when one substitutes $x=-\pi$ due to (7.4).

We can now repeat this consideration in the q-case, starting from the expansion (4.2). It gives the following generating relation

$$\frac{\alpha(q\alpha^2;q^2)_\infty\,\mathscr{E}_q(x;\alpha)}{(-\alpha;q^{1/2})_\infty-(\alpha;q^{1/2})_\infty}$$
$$=\frac{1}{2}(1-q^{1/2})+\sum_{m=1}^{\infty}\alpha^{2m-1}B_{2m-1}(x;q)+\sum_{m=1}^{\infty}\alpha^{2m}B_{2m}(x;q), \tag{7.13}$$

where

$$B_{2m-1}(x;q):=(-1)^{m-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n{}^{2m-1}}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,S_q(x;\omega_n), \quad (7.14)$$

$$B_{2m}(x;q):=(-1)^{m}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n{}^{2m}}\sqrt{\frac{(-q\omega_n^2;q^2)_\infty}{(-\omega_n^2;q^2)_\infty}}\,C_q(x;\omega_n) \quad (7.15)$$

are q-analogs of polynomials (7.10)–(7.11).

In order to prove that functions $B_m(x;q)$ defined in (7.14)–(7.15) are, indeed, polynomials of exact degrees m in x, let us use the explicit representation (2.10) for Ismail's q-Lommel polynomials in expansions (4.18) and (4.21),

$$\begin{aligned} x\prod_{k=0}^{m-2}&((1+q^{k+1/2})^2-4q^{k+1/2}x^2)\\ &=(-1)^{m-1}\frac{(q;q)_{2m-1}}{(q^{1/2};q)_m}\,q^{-m^2/2+1/4}\\ &\quad\times\sum_{k=0}^{[(m-1)/2]}q^{k(k+1/2)}\frac{(q;q)_{m-k-1}(q^{3/2};q)_{m-k-1}}{(q;q)_k\,(q;q)_{m-2k-1}(q^{3/2};q)_k}\,B_{2m-2k-1}(x;q) \end{aligned}\quad (7.16)$$

and

$$\begin{aligned} \prod_{k=0}^{m-1}&((1+q^{k+1/2})^2-4q^{k+1/2}x^2)-\frac{(q;q)_{2m}}{(q^{1/2},q^{3/2};q)_m}\\ &=2(-1)^{m}\frac{(q;q)_{2m}}{(q^{1/2};q)_m}\,q^{-m^2/2}\\ &\quad\times\sum_{k=0}^{[(m-1)/2]}q^{k(k+1/2)}\frac{(q;q)_{m-k-1}(q^{3/2};q)_{m-k-1}}{(q;q)_k\,(q;q)_{m-2k-1}(q^{3/2};q)_k}\,B_{2m-2k}(x;q) \end{aligned}\quad (7.17)$$

for the odd and even degrees, respectively. Equations (7.16) and (7.17) can be thought as systems of linear equations for $B_m(x;q)$ with triangular matrices which can be solved by the method of elimination. The left sides of (7.16) and (7.17) are polynomials of exact degrees $2m-3$ and $2m-2$, respectively. This completes the proof.

We shall call $B_m(x;q)$ as the q-Bernoulli polynomials. One can also introduce the q-Bernoulli numbers as

$$B_m(q):=B_m(-\eta;q), \qquad \eta=(q^{1/4}+q^{-1/4})/2 \quad (7.18)$$

in accordance with (7.4) and (7.12). By using the generating function (7.13) and the limiting relation

$$\lim_{q\to 1^-} \mathcal{E}_q(x; (1-q)\,\alpha/2) = e^{\alpha x}$$

it can be shown that the classical Bernoulli polynomials and numbers are the $q\to 1^-$ limits of (7.14)–(7.15) and (7.18), respectively. We leave the details to the reader.

Let us discuss some properties of the q-Bernoulli polynomials and numbers. The explicit form of the first five q-Bernoulli's polynomials follows directly from the generating relation (7.13) and expansions (2.5)–(2.6) and (2.25)–(2.26):

$$B_0(x;q) = \frac{1}{2}(1-q^{1/2}), \qquad B_1(x;q) = q^{1/4}\,\frac{1-q^{1/2}}{(q;q)_1}\,x, \tag{7.19}$$

$$B_2(x;q) = 2q\,\frac{1-q^{1/2}}{(q;q)_2}\left[x^2-\frac{(1+q^{1/2})^2}{4(1+q^{1/2}+q)}\right], \tag{7.20}$$

$$B_3(x;q) = 4q^{9/4}\,\frac{1-q^{1/2}}{(q;q)_3}\,x\left[x^2-\frac{(1+q^{1/2})^2}{4q^{1/2}}\right], \tag{7.21}$$

$$\begin{aligned} B_4(x;q) &= 8q^4\,\frac{1-q^{1/2}}{(q;q)_4} \\ &\quad\times\left[x^4-\frac{(1+q^{1/2})^2\,(1-2q^{1/2}+4q-2q^{3/2}+q^2)}{4q^{3/2}}\,x^2\right. \\ &\quad\left.+\frac{(1+q^{1/2})(1+q^{3/2})(1-q)^2\,(1+2q+q^{3/2}+2q^2+q^3)}{16q^{3/2}\,(1-q^{3/2})(1-q^{5/2})}\right]. \end{aligned} \tag{7.22}$$

The corresponding q-Bernoulli numbers defined by (7.18) are

$$B_0(q) = \frac{1}{2}(1-q^{1/2}), \qquad B_1(q) = -\frac{1}{2}, \tag{7.23}$$

$$B_2(q) = \frac{q^{1/2}}{2(1-q^{3/2})}, \qquad B_3(q) = 0, \tag{7.24}$$

$$B_4(q) = -\frac{q^2}{2(1-q^{3/2})^2\,(1-q^{5/2})}. \tag{7.25}$$

It can be shown that as in the classical case,

$$B_{2m+1}(q) = 0, \qquad m = 1, 2, 3\ldots. \tag{7.26}$$

Indeed, substituting $x=-\eta$ in the generating relation (7.13) and taking into account that

$$\mathscr{E}_q(-\eta; \alpha)=\frac{(\alpha; q^{1/2})_\infty}{(q\alpha^2; q^2)_\infty} \tag{7.27}$$

by (1.21) from this paper and (5.37)–(5.38) of [14], one gets

$$\begin{aligned}
&\alpha\,\frac{(-\alpha; q^{1/2})_\infty+(\alpha; q^{1/2})_\infty}{(-\alpha; q^{1/2})_\infty-(\alpha; q^{1/2})_\infty}\\
&\quad=1-q^{1/2}+2\sum_{m=2}^{\infty}\alpha^{2m-1}B_{2m-1}(q)+2\sum_{m=1}^{\infty}\alpha^{2m}B_{2m}(q).
\end{aligned} \tag{7.28}$$

The left side here is an even function of α, so (7.26) takes place.

Equating the coefficients of α^{2n+1} in the both sides of

$$\begin{aligned}
&\alpha\,\frac{(-\alpha; q^{1/2})_\infty+(\alpha; q^{1/2})_\infty}{(-\alpha; q^{1/2})_\infty-(\alpha; q^{1/2})_\infty}\,((-\alpha; q^{1/2})_\infty-(\alpha; q^{1/2})_\infty)\\
&\quad=\alpha\,((-\alpha; q^{1/2})_\infty+(\alpha; q^{1/2})_\infty)
\end{aligned}$$

by (7.28) and (II.2) of [23], we obtain the recurrence relation for the q-Bernoulli numbers,

$$2\sum_{k=0}^{n}B_{2k}(q)\,\frac{q^{(n-k)(n-k+1/2)}}{(q^{1/2}; q^{1/2})_{2n-2k+1}}=\frac{q^{n(n-1/2)}}{(q^{1/2}; q^{1/2})_{2n}},$$

which is convenient for their evaluation.

Substituting $\alpha=i\omega$ in (7.28), we obtain a q-analog of Taylor's expansion (7.5) as

$$\omega\,\mathrm{Cot}_q(\omega)=1-q^{1/2}+2\sum_{m=1}^{\infty}(-1)^m B_{2m}(q)\,\omega^{2m}. \tag{7.29}$$

This follows also from (6.9) in a similar manner.

Our generating function (7.13) can be rewritten in the form

$$\frac{\alpha(q\alpha^2; q^2)_\infty\,\mathscr{E}_q(-\eta; \alpha)}{(-\alpha; q^{1/2})_\infty-(\alpha; q^{1/2})_\infty}\,(q\alpha^2; q^2)_\infty\,\mathscr{E}_q(x; \alpha)\,\frac{1}{(\alpha; q^{1/2})_\infty}$$

in view of (7.27). Expanding this function in the powers of α with the help of (7.13) with $x=-\eta$, (7.18), (4.10), all of this paper, and (II.1) of [23]; we get an explicit expression of the q-Bernoulli polynomials

$$B_m(x; q)=\sum_{n=0}^{m}B_{m-n}(q)\sum_{k=0}^{n}\frac{q^{k^2/4}}{(q^{1/2}; q^{1/2})_{n-k}\,(q; q)_k}\,H_k(x\,|\,q) \tag{7.30}$$

in terms of the q-Bernoulli numbers and the continuous q-Hermite polynomials.

The generating relation (7.13) implies the following contour integral representation

$$B_m(x;q) = \frac{1}{2\pi i} \int_{|\alpha| = r < \omega_1} \frac{(q\alpha^2;q^2)_\infty \, \mathcal{E}_q(x;\alpha)}{(-\alpha;q^{1/2})_\infty - (\alpha;q^{1/2})_\infty} \, \alpha^{-m} \, d\alpha \tag{7.31}$$

for the q-Bernoulli polynomials. The special case $x = -\eta$ is

$$B_m(q) = \frac{1}{2\pi i} \int_{|\alpha| = r < \omega_1} \frac{(\alpha;q^{1/2})_\infty}{(-\alpha;q^{1/2})_\infty - (\alpha;q^{1/2})_\infty} \, \alpha^{-m} \, d\alpha \tag{7.32}$$

by (7.18) and (7.27). On the other hand, the generating relation (7.28) and (7.26) imply

$$B_{2m}(q) = \frac{1}{4\pi i} \int_{|\alpha| = r < \omega_1} \frac{(-\alpha;q^{1/2})_\infty + (\alpha;q^{1/2})_\infty}{(-\alpha;q^{1/2})_\infty - (\alpha;q^{1/2})_\infty} \, \alpha^{-2m} \, d\alpha \tag{7.33}$$

in a similar fashion.

Since the q-trigonometric functions satisfy the difference-differentiation formulas

$$\frac{\delta}{\delta x} C_q(x;\omega) = -\frac{2q^{1/4}}{1-q} \, \omega S_q(x;\omega) \tag{7.34}$$

and

$$\frac{\delta}{\delta x} S_q(x;\omega) = \frac{2q^{1/4}}{1-q} \, \omega C_q(x;\omega) \tag{7.35}$$

(see [36, 48]); where the operator $\delta/\delta x$ is the standard Askey–Wilson divided difference operator defined by

$$\frac{\delta u(z)}{\delta x(z)} = \frac{u(z+1/2) - u(z-1/2)}{x(z+1/2) - x(z-1/2)} \tag{7.36}$$

with $x(z) = (q^z + q^{-z})/2 = \cos\theta$, $q^z = e^{i\theta}$; applying the divided-difference operator $\delta/\delta x$ to the both sides of (7.14) and (7.15) we obtain

$$\frac{\delta}{\delta x} B_m(x;q) = \frac{2q^{1/4}}{1-q} \, B_{m-1}(x;q), \tag{7.37}$$

which is an analog of

$$B'_m(x) = m\,B_{m-1}(x). \tag{7.38}$$

Formula (7.37) follows also from (7.30) or (7.31) if one uses the difference-differentiation formulas (e.g., [48]),

$$\frac{\delta}{\delta x}\, H_k(x \mid q) = 2q^{(1-k)/2}\, \frac{1-q^k}{1-q}\, H_{k-1}(x \mid q) \tag{7.39}$$

and

$$\frac{\delta}{\delta x}\, \mathscr{E}_q(x;\alpha) = \frac{2q^{1/4}\alpha}{1-q}\, \mathscr{E}_q(x;\alpha), \tag{7.40}$$

for the continuous q-Hermite polynomials and q-exponential function, respectively.

Expansions (7.14)–(7.15) and the orthogonality relations for the basic trigonometric system (1.8)–(1.11) together with (7.18) of this paper and (13.10) of [14] result in

$$\begin{aligned}&\int_0^\pi B_m(\cos\theta;q)\, B_n(\cos\theta;q)(e^{2i\theta}, e^{-2i\theta};q)_{1/2}\, d\theta\\ &\qquad = (-1)^{m-1}\pi\, \frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}\, B_{m+n}(q).\end{aligned} \tag{7.41}$$

This implies that the q-Bernoulli polynomials are orthogonal for $m+n$ odd due to (7.26), which is also true due to the symmetry property

$$B_m(-x;q) = (-1)^m\, B_m(x;q). \tag{7.42}$$

Equation (7.41) is an analog of the property

$$\int_0^1 B_m(x)\, B_n(x)\, dx = (-1)^{m-1}\, \frac{m!\, n!}{(m+n)!}\, B_{m+n}, \tag{7.43}$$

$m, n = 1, 2, 3, \ldots,$ holding for the classical Bernoulli polynomials [1].

Expansions (2.23)–(2.24) and (2.10) lead to the following relations between the q-Legendre polynomials $C_m(x;q^{1/2} \mid q)$ and the q-Bernoulli polynomials

$$\begin{aligned}&C_{2m+1}(x;q^{1/2} \mid q)\\ &\quad = 2q^{-m(m+1)-1/4} \sum_{k=0}^{m} q^{k(k+1/2)}\, \frac{(q;q)_{2m-k}\, (q^{3/2};q)_{2m-k}}{(q;q)_k\, (q;q)_{2m-2k}\, (q^{3/2};q)_k}\, B_{2m-2k+1}(x;q)\end{aligned} \tag{7.44}$$

and

$$C_{2m}(x; q^{1/2} \mid q) = 2q^{-m^2} \sum_{k=0}^{m-1} q^{k(k+1/2)} \frac{(q; q)_{2m-k-1}(q^{3/2}; q)_{2m-k-1}}{(q; q)_k\, (q; q)_{2m-2k-1}\, (q^{3/2}; q)_k} B_{2m-2k}(x; q) \tag{7.45}$$

for the odd and even degrees, respectively. On the other hand, expansions of the q-Bernoulli polynomials in terms of the q-Legendre polynomials are

$$\begin{aligned} B_{2m+1}(x; q) = \sum_{n=0}^{m} C_{2n+1}(x; q^{1/2} \mid q)\, q^{-n(n+1)-1/4}\, (1-q^{2n+3/2}) \\ \times \sum_{k=0}^{n} q^{k(k+1/2)} \frac{(q; q)_{2n-k}(q^{3/2}; q)_{2n-k}}{(q; q)_k\, (q; q)_{2n-2k}\, (q^{3/2}; q)_k} B_{2m+2n-2k+2}(q) \end{aligned} \tag{7.46}$$

and

$$\begin{aligned} B_{2m}(x; q) = -\sum_{n=0}^{m-1} C_{2n}(x; q^{1/2} \mid q)\, q^{-n^2}\, (1-q^{2n+1/2}) \\ \times \sum_{k=0}^{n-1} q^{k(k+1/2)} \frac{(q; q)_{2n-k-1}(q^{3/2}; q)_{2n-k-1}}{(q; q)_k\, (q; q)_{2n-2k-1}(q^{3/2}; q)_k} B_{2m+2n-2k}(q) \end{aligned} \tag{7.47}$$

due to (7.44)–(7.45), (7.41), and the orthogonality relation

$$\begin{aligned} \int_0^{\pi} C_m(\cos\theta; q^{1/2} \mid q)\, C_n(\cos\theta; q^{1/2} \mid q)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \\ = 2\pi \frac{(q^{1/2}; q)_\infty^2}{(q; q)_\infty^2} (1-q^{m+1/2})^{-1}\, \delta_{mn} \end{aligned} \tag{7.48}$$

(see, for example, [23]).

The property

$$B_m(x+1) - B_m(x) = mx^{m-1} \tag{7.49}$$

follows directly from the generating function (7.1) for the classical Bernoulli polynomials. In order to establish a q-analog of this formula we need to introduce an extension of the q-Bernoulli polynomials by means of the generating relation

$$\frac{\alpha(q\alpha^2;q^2)_\infty\,\mathscr{E}_q(x,y;\alpha)}{(-\alpha;q^{1/2})_\infty-(\alpha;q^{1/2})_\infty}$$
$$=\frac{1}{2}(1-q^{1/2})+\sum_{m=1}^{\infty}\alpha^{2m-1}B_{2m-1}(x,y;q)+\sum_{m=1}^{\infty}\alpha^{2m}B_{2m}(x,y;q), \qquad (7.50)$$

where $\mathscr{E}_q(x,y;\alpha)$ is the q-quadratic exponential in two independent variables x and y; see, for example, [36, 48] or a review paper [53]. This function appears in the generating relation

$$\sum_{m=0}^{\infty}\frac{q^{m^2/4}}{(q;q)_m}\alpha^m H_m(x,y\mid q)=(q\alpha^2;q^2)_\infty\,\mathscr{E}_q(x,y;\alpha), \qquad (7.51)$$

where $H_m(x,y\mid q)$ are the q-analogs of the classical Hermite polynomials $H_m(x+y)$ introduced in [48]; see, for example, (9.1)–(9.2) of [48] for the explicit representation of these polynomials in terms of terminating ${}_6\varphi_5$-series. Using the same arguments as in (7.30) with the help (7.51) one obtains

$$B_m(x,y;q)=\sum_{n=0}^{m}B_{m-n}(q)\sum_{k=0}^{n}\frac{q^{k^2/4}}{(q^{1/2};q^{1/2})_{n-k}\,(q;q)_k}H_k(x,y\mid q). \qquad (7.52)$$

It is natural to view the generalized q-Bernoulli polynomials $B_m(x,y;q)$ here as extensions of $B_m(x+y)$. The following properties hold

$$B_m(x,y;q)=B_m(y,x;q), \qquad B_m(x,0;q)=B_m(x;q). \qquad (7.53)$$

Substituting $y=\pm\eta$ in (7.50) and then subtracting the results we obtain

$$\alpha\mathscr{E}_q(x;\alpha)=\sum_{n=1}^{\infty}\alpha^n\,(B_n(x,\eta;q)-B_n(x,-\eta;q)) \qquad (7.54)$$

with the help of the addition theorem [48, 53],

$$\mathscr{E}_q(x, y; \alpha) = \mathscr{E}_q(x; \alpha)\, \mathscr{E}_q(y; \alpha), \tag{7.55}$$

the symmetry relation $\mathscr{E}_q(-x; \alpha) = \mathscr{E}_q(x; -\alpha)$, which follows directly from (4.10); and (7.27). By (3.7) of [53]

$$\begin{aligned} \mathscr{E}_q(x; \alpha) &= {}_2\varphi_1\left(\begin{matrix} -e^{2i\theta}, -e^{-2i\theta} \\ q \end{matrix}; q^2, q\alpha^2\right) \\ &\quad + \frac{2q^{1/4}}{1-q}\, \alpha \cos\theta\, {}_2\varphi_1\left(\begin{matrix} -qe^{2i\theta}, -qe^{-2i\theta} \\ q^3 \end{matrix}; q^2, q\alpha^2\right) \\ &= \sum_{n=0}^{\infty} \frac{q^{n/2}\alpha^n}{(q; q)_n}\, \phi_n(x; q), \end{aligned} \tag{7.56}$$

where by the definition

$$\phi_{2k}(x; q) = \prod_{p=0}^{k-1} (4x^2 q^{2p} + (1-q^{2p})^2), \tag{7.57}$$

$$\phi_{2k+1}(x; q) = 2q^{-1/4} x \prod_{p=0}^{k-1} (4x^2 q^{2p+1} + (1-q^{2p+1})^2). \tag{7.58}$$

(We assume that the empty products when $k = 0$ here are equal to 1.) Substituting (7.56) to the left hand side of (7.54) and equating the coefficients of α^n we obtain an interesting analog of (7.49):

$$B_n(x, \eta; q) - B_n(x, -\eta; q) = \frac{q^{(n-1)/2}}{(q; q)_{n-1}}\, \phi_{n-1}(x; q). \tag{7.59}$$

In a similar manner one can derive

$$B_n(x, y; q) = \sum_{k=0}^{n} \frac{q^{(n-k)/2}}{(q; q)_{n-k}}\, B_k(x; q)\, \phi_{n-k}(y; q) \tag{7.60}$$

as the q-analog of

$$B_n(x+y) = \sum_{k=0}^{n} C_n^k B_k(x)\, y^{n-k}. \tag{7.61}$$

This shows that $B_n(x, y; q)$ are polynomials in x and y. The first q-Bernoulli polynomials $B_n(x, y; q)$ are

$$B_0(x, y; q) = \frac{1}{2}(1-q^{1/2}), \tag{7.62}$$

$$B_1(x, y; q) = q^{1/4}\frac{1-q^{1/2}}{(q;q)_1}(x+y), \tag{7.63}$$

$$B_2(x, y; q) = 2q\frac{1-q^{1/2}}{(q;q)_2} \times\left[x^2+y^2+\frac{1+q}{q^{1/2}}xy-\frac{(1+q^{1/2})^2}{4(1+q^{1/2}+q)}\right], \tag{7.64}$$

$$B_3(x, y; q) = 4q^{9/4}\frac{1-q^{1/2}}{(q;q)_3} \times\left[x^3+y^3+\frac{1+q+q^2}{q}xy(x+y)-\frac{(1+q^{1/2})^2}{4q^{1/2}}(x+y)\right], \quad (7$$

$$\begin{aligned}B_4(x, y; q) = 8q^4\frac{1-q^{1/2}}{(q;q)_4} &\times\left[x^4+y^4+\frac{1+q+q^2+q^3}{q^{3/2}}xy(x+y)\right.\\ &+\frac{(1+q^2)(1+q+q^2)}{q^2}x^2y^2\\ &-\frac{(1+q^{1/2})^2(1-2q^{1/2}+4q-2q^{3/2}+q^2)}{4q^{3/2}}(x^2+y^2)\\ &+\frac{(1+q+q^2)(1+q^{1/2})^2(1-3q^{1/2}+q)}{4q^{5/2}}xy\\ &\left.+\frac{(1+q^{1/2})(1+q^{3/2})(1-q)^2(1+2q+q^{3/2}+2q^2+q^3)}{16q^{3/2}(1-q^{3/2})(1-q^{5/2})}\right].\end{aligned} \tag{7.66}$$

This follows directly from (7.60); we leave the details to the reader.

Remark 2. In the classical case, expression (7.7) leads directly to a trivial identity, $B_m = B_m$, when $x = 0$. Substituting $x = -\eta$ in (7.30) in a similar manner, one gets

$$B_m(q) = \sum_{n=0}^{m} B_{m-n}(q) \sum_{k=0}^{n} \frac{q^{k^2/4}}{(q^{1/2}; q^{1/2})_{n-k}\,(q; q)_k} H_k(-\eta \mid q). \tag{7.67}$$

But

$$H_k(-\eta \mid q) = (-1)^k \frac{(q; q)_k}{(q^{1/2}; q^{1/2})_k}\, q^{-k/4} \tag{7.68}$$

in view of (4.10), (7.27) of this paper, and expansion (II.2) of [23]. Therefore

$$B_m(q) = \sum_{n=0}^{m} B_{m-n}(q) \sum_{k=0}^{n} \frac{(-1)^k\, q^{k(k-1)/4}}{(q^{1/2}; q^{1/2})_k\,(q^{1/2}; q^{1/2})_{n-k}}, \tag{7.69}$$

which leads to the trivial identity $B_m(q) = B_m(q)$ again due to

$$\sum_{k=0}^{n} \frac{(-1)^k\, q^{k(k-1)/4}}{(q^{1/2}; q^{1/2})_k\,(q^{1/2}; q^{1/2})_{n-k}} = \delta_{n,0}. \tag{7.70}$$

The last relation appears if one expands the both sides of the identity $1 = (\alpha; q^{1/2})/(\alpha; q^{1/2})$ in the powers of α with the help of (II.1)–(II.2) of [23].

Remark 3. Relations (4.10) and (7.56) give rise to the following summation formula

$$\phi_n(x; q) = q^{n(n-2)/4} \sum_{k=0}^{[n/2]} q^{k(k-n+1)} \frac{(q; q)_n}{(q^2; q^2)_k\,(q; q)_{n-2k}} H_{n-2k}(x \mid q), \tag{7.71}$$

which allows to simplify (9.12) of [48]

$$H_n(x, y \mid q) = \sum_{k=0}^{n} q^{(k-n)(k+n-2)/4} \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} H_k(x \mid q)\,\phi_{n-k}(y;q). \quad (7.72)$$

Remark 4. In the recent preprints [34, 35] Ismail and Stanton have found the following representations for the q-exponential function

$$\begin{aligned} \mathscr{E}_q(x;\alpha) &= \frac{(-\alpha;q^{1/2})_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{(q;q)_n} \varphi_n(x;q) \\ &= \frac{(\alpha;q^{1/2})_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n}{(q;q)_n} \varphi_n(-x;q), \end{aligned} \quad (7.73)$$

where

$$\varphi_n(\cos\theta;q) = (q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n. \quad (7.74)$$

An independent proof of these formulas can be given using the same method as in the proof of the generating relation (4.10) in the Appendix of [53]. Equation

$$B_n(x;q) = \sum_{k=0}^{n} \frac{B_{n-k}(q)}{(q;q)_k} \varphi_k(-x;q) \quad (7.75)$$

follows directly from (7.13), (7.18) and (7.73)–(7.74) as a q-analog of (7.7). One can also see in view of (7.75) that summation formula

$$\varphi_k(-x;q) = \sum_{k=0}^{n} q^{k^2/4} \frac{(q;q)_n}{(q;q)_k\,(q^{1/2};q^{1/2})_{n-k}} H_k(x \mid q) \quad (7.76)$$

allows to simplify our expression (7.30) for the q-Bernoulli polynomials.

We have discussed here some properties of the q-Bernoulli polynomials and numbers. They deserve a more detailed consideration. Other q-analogs of the Bernoulli polynomials and numbers were studied in [3, 15, 16–17, 38, 46, 59, 62].

8. BASIC EULER POLYNOMIALS AND NUMBERS

The classical Euler polynomials can be introduced by means of the generating function

$$\frac{2e^{\alpha x}}{e^{\alpha}+1}=1+\sum_{m=1}^{\infty} E_m(x)\,\frac{\alpha^m}{m!}, \qquad |\alpha|<\pi. \tag{8.1}$$

They have the following Fourier expansions

$$E_{2m-1}(x)=(-1)^m\,\frac{4(2m-1)!}{\pi^{2m}}\sum_{n=1}^{\infty}\frac{\cos \pi(2n+1)\,x}{(2n+1)^{2m}} \tag{8.2}$$

and

$$E_{2m}(x)=(-1)^m\,\frac{4(2m)!}{\pi^{2m+1}}\sum_{n=1}^{\infty}\frac{\sin \pi(2n+1)\,x}{(2n+1)^{2m+1}} \tag{8.3}$$

with $m=1, 2, 3, \ldots$ (see, for example, [1, 22]). The Euler numbers are defined by

$$E_m=2^m E_m(1/2). \tag{8.4}$$

One can see that $E_0=1$, $E_2=-1$, $E_4=5$; $E_{2m+1}=0$ when $m=0, 1, 2, \ldots$. These numbers appear as coefficients in Taylor's expansion

$$\begin{aligned}\frac{1}{\cos\omega}&=1+\sum_{m=1}^{\infty}(-1)^m\,\frac{E_{2m}}{(2m)!}\,\omega^{2m}\\&=1+\frac{1}{2}\,\omega^2+\frac{5}{24}\,\omega^4+\frac{61}{720}\,\omega^6+\ldots.\end{aligned} \tag{8.5}$$

Also, the special case $x=1/2$ of (8.1) is

$$\frac{2e^{\alpha/2}}{e^{\alpha}+1}=1+\sum_{m=1}^{\infty} E_m\,\frac{(\alpha/2)^m}{m!} \tag{8.6}$$

and, in view of (8.1) and (8.6), we arrive at the explicit expression for the Euler polynomials

$$E_m(x)=\sum_{k=0}^{m} 2^{-k}E_k\,C_m^k\,(x-1/2)^{m-k}, \qquad C_m^k=\frac{m!}{k!\,(m-k)!} \tag{8.7}$$

in terms of Euler's numbers and the binomial coefficients. See [1] and [22] for an account of further properties of the Euler polynomials and numbers.

The basic Fourier series (4.23) gives us the possibility to introduce q-analogs of the Euler polynomials and numbers. Let us rewrite (4.23) as

$$\begin{aligned}
&\frac{(q\alpha^2; q^2)_\infty \, \mathscr{E}_q(x; \alpha)}{(-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty} \\
&\qquad = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \varpi_n}{\kappa(\varpi_n)(\alpha^2+\varpi_n^2)} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}} \, C_q(x; \varpi_n) \\
&\qquad\quad + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \alpha}{\kappa(\varpi_n)(\alpha^2+\varpi_n^2)} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}} \, S_q(x; \varpi_n)
\end{aligned} \tag{8.8}$$

and substitute

$$\frac{\varpi_n}{\alpha^2+\varpi_n^2} = \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{\varpi_n^{2m+1}}, \qquad \frac{\alpha}{\alpha^2+\varpi_n^2} = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\alpha^{2m-1}}{\varpi_n^{\;2m}},$$

when $|\alpha| < \varpi_1$. As a result, we arrive at the following generating relation

$$\begin{aligned}
&\frac{(q\alpha^2; q^2)_\infty \, \mathscr{E}_q(x; \alpha)}{(-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty} \\
&\qquad = \frac{1}{2} + \sum_{m=1}^{\infty} \alpha^{2m-1} E_{2m-1}(x; q) + \sum_{m=1}^{\infty} \alpha^{2m} E_{2m}(x; q),
\end{aligned} \tag{8.9}$$

where

$$E_{2m-1}(x; q) := (-1)^{m-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\varpi_n)\, \varpi_n^{2m}} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}} \, S_q(x; \varpi_n), \tag{8.10}$$

$$E_{2m}(x; q) := (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\varpi_n)\, \varpi_n^{2m+1}} \sqrt{\frac{(-q\varpi_n^2; q^2)_\infty}{(-\varpi_n^2; q^2)_\infty}} \, C_q(x; \varpi_n) \tag{8.11}$$

are q-analogs of the Euler polynomials (8.2)–(8.3).

If we introduce q-Euler numbers as

$$E_m(q) = E_m(0; q), \tag{8.12}$$

the generating relation (8.9) takes the form

$$\frac{(q\alpha^2; q^2)_\infty}{(-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty} = \frac{1}{2} + \sum_{m=1}^{\infty} \alpha^m E_m(q), \tag{8.13}$$

due to $\mathcal{E}_q(0;\alpha)=1$, and $E_{2m+1}(q)=0$, $m=0, 1, 2, \ldots$. By using the generating relations (8.9) and (8.13) one can show that the classical Euler polynomials (8.2)–(8.3) and numbers (8.4) are the $q\to 1^-$ limits of (8.10)–(8.11) and (8.12), respectively.

Equating the coefficients of α^n in the both sides of (8.9) with the help of (8.13) and (7.56) we obtain the following analog of (8.7):

$$E_n(x;q)=\sum_{k=0}^{n} E_k(q)\,\frac{q^{(n-k)/2}}{(q;q)_{n-k}}\,\phi_{n-k}(x;q). \tag{8.14}$$

The last equation shows that the $E_n(x;q)$ defined by (8.10)–(8.11) are, indeed, polynomials of degree n in x. We shall call them the q-Euler polynomials.

Let us discuss some properties of the q-Euler polynomials and numbers that are similar to the classical ones. In view of (8.9)–(8.11) or (8.14), the following symmetry relation holds

$$E_n(-x;q)=(-1)^n E_n(x;q). \tag{8.15}$$

The first five q-Euler's numbers can be found directly from the generating relation (8.13):

$$E_0(q)=\frac{1}{2},\qquad E_1(q)=0, \tag{8.16}$$

$$E_2(q)=-\frac{q^{1/2}}{2(1-q^{1/2})^2\,(1+q)},\qquad E_3(q)=0, \tag{8.17}$$

$$E_4(q)=\frac{q(1+2q-q^{3/2}+2q^2+q^3)}{2(1-q^{1/2})^3\,(1-q^{3/2})(1+q)^2\,(1+q^2)}. \tag{8.18}$$

The corresponding q-Euler polynomials are

$$E_0(x;q)=\frac{1}{2},\qquad E_1(x;q)=\frac{q^{1/4}}{(q;q)_1}\,x, \tag{8.19}$$

$$E_2(x;q)=\frac{2q}{(q;q)_2}\left[x^2-\frac{(1+q^{1/2})^2}{4q^{1/2}}\right], \tag{8.20}$$

$$E_3(x;q)=\frac{q^{5/4}}{(q;q)_3}\,x\left[4qx^2-\frac{(1+q^{1/2})^2}{q^{1/2}}\,(1-q^{1/2}+3q-q^{3/2}+q^2)\right], \tag{8.21}$$

$$\begin{aligned}E_4(x;q)=\frac{2q^2}{(q;q)_4}\Bigg[&4q^2x^4-\frac{(1+q^{1/2})^2}{q^{1/2}}\\&\times(1-q^{1/2}+3q-3q^{3/2}+6q^2-3q^{5/2}+3q^3-q^{7/2}+q^4)\,x^2\\&+\frac{(1+q^{1/2})^3\,(1+q^{3/2})(1+2q-q^{3/2}+2q^2+q^3)}{4q}\Bigg].\end{aligned} \tag{8.22}$$

This follows directly from (8.14) and (8.16)–(8.18).

Equating the coefficients of α^{2n} in the both sides of

$$\frac{(q\alpha^2; q^2)_\infty}{(-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty} ((-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty) = (q\alpha^2; q^2)_\infty$$

by (8.13) and (II.2) of [23], we obtain the recurrence relation for the q-Euler numbers,

$$2 \sum_{k=0}^{n} E_{2k}(q) \frac{q^{(n-k)(n-k-1/2)}}{(q^{1/2}; q^{1/2})_{2n-2k}} = (-1)^n \frac{q^{n^2}}{(q^2; q^2)_n}, \tag{8.23}$$

which is a q-analog of

$$\sum_{k=0}^{n} E_{2k} C_{2n}^{2k} = 0. \tag{8.24}$$

Substituting $\alpha = i\omega$ in (8.13) we arrive at the following analog of (8.5):

$$\frac{1}{C_q(\eta; \omega)} = 1 + 2 \sum_{m=1}^{\infty} (-1)^m E_{2m}(q)\, \omega^{2m}. \tag{8.25}$$

The q-exponential function satisfies the difference-differentiation formula (7.40) and applying the operator (7.36) to the both sides of (7.56) one gets

$$\frac{\delta}{\delta x} \phi_m(x; q) = 2q^{-1/4} \frac{1-q^m}{1-q} \phi_{m-1}(x; q). \tag{8.26}$$

With the help of the last equation we obtain from (8.14) that

$$\frac{\delta}{\delta x} E_m(x; q) = \frac{2q^{1/4}}{1-q} E_{m-1}(x; q), \tag{8.27}$$

which is an analog of

$$E'_m(x) = m E_{m-1}(x). \tag{8.28}$$

The property

$$E_m(x+1) + E_m(x) = 2x^m \tag{8.29}$$

follows directly from the generating function (8.1) for the classical Euler polynomials. In order to establish a q-analog of this formula we need to introduce an extension of the q-Euler polynomials by means of the generating relation

$$\frac{(q\alpha^2; q^2)_\infty \mathcal{E}_q(x, y; \alpha)}{(-\alpha; q^{1/2})_\infty + (\alpha; q^{1/2})_\infty} = \frac{1}{2} + \sum_{m=1}^{\infty} \alpha^{2m-1} E_{2m-1}(x, y; q) + \sum_{m=1}^{\infty} \alpha^{2m} E_{2m}(x, y; q), \tag{8.30}$$

where $\mathcal{E}_q(x, y; \alpha)$ is the q-quadratic exponential in two independent variables x and y; see, for example, [36, 48, 53]; which is an analog of $\exp[\alpha(x+y)]$. Thus, it is natural to view the generalized q-Euler polynomials $E_m(x, y; q)$ here as an extension of $E_m(x+y)$. The following properties hold

$$E_m(x, y; q) = E_m(y, x; q), \qquad E_m(x, 0; q) = E_m(x; q). \tag{8.31}$$

Substituting $y = \pm\eta$ in (8.30) and then adding the results we obtain

$$\mathcal{E}_q(x; \alpha) = 1 + \sum_{n=1}^{\infty} \alpha^n \left(E_n(x, \eta; q) + E_n(x, -\eta; q)\right) \tag{8.32}$$

with the help of the addition theorem (7.55), the symmetry relation $\mathcal{E}_q(-x; \alpha) = \mathcal{E}_q(x; -\alpha)$, and (7.27). Using the expansion (7.56) in the left hand side of (8.32) and equating the coefficients of α^n we obtain an analog of (8.29):

$$E_n(x, \eta; q) + E_n(x, -\eta; q) = \frac{q^{n/2}}{(q; q)_n} \phi_n(x; q). \tag{8.33}$$

In a similar manner, using (7.55) in (8.30) one can derive the following extension of (8.14),

$$E_n(x, y; q) = \sum_{k=0}^{n} \frac{q^{(n-k)/2}}{(q; q)_{n-k}} E_k(x; q)\, \phi_{n-k}(y; q), \tag{8.34}$$

which is the q-analog of

$$E_n(x+y) = \sum_{k=0}^{n} C_n^k E_k(x)\, y^{n-k}. \tag{8.35}$$

This consideration shows that $E_n(x, y; q)$ are polynomials in x and y. They satisfy the difference-differentiation formula (8.27). The first q-Euler polynomials $E_n(x, y; q)$ are

$$E_0(x, y; q) = \frac{1}{2}, \qquad E_1(x, y; q) = \frac{q^{1/4}}{(q; q)_1}(x+y), \tag{8.36}$$

$$E_2(x, y; q) = \frac{2q}{(q; q)_2}\left[x^2+y^2+\frac{1+q}{q^{1/2}}xy-\frac{(1+q^{1/2})^2}{4q^{1/2}}\right], \tag{8.37}$$

$$\begin{aligned} E_3(x, y; q) = \frac{4q^{9/4}}{(q; q)_3}\bigg[& x^3+y^3+\frac{1+q+q^2}{q}xy(x+y) \\ & -\frac{(1+q^{1/2})^2}{4q^{3/2}}(1-q^{1/2}+3q-q^{3/2}+q^2)(x+y)\bigg], \end{aligned} \tag{8.38}$$

$$\begin{aligned} E_4(x, y; q) = \frac{8q^4}{(q; q)_4}\bigg[& x^4+y^4+\frac{1+q+q^2+q^3}{q^{3/2}}xy(x+y) \\ & +\frac{(1+q^2)(1+q+q^2)}{q^2}x^2y^2-\frac{(1+q^{1/2})^2}{4q^{5/2}}(x^2+y^2) \\ & \times(1-q^{1/2}+3q-3q^{3/2}+6q^2-3q^{5/2}+3q^3-q^{7/2}+q^4) \\ & -\frac{1+q+q^2}{4q^{5/2}}(1+q^{1/2})^2(1-2q^{1/2}+5q-2q^{3/2}+q^2)\,xy \\ & +\frac{(1+q^{1/2})^3(1+q^{3/2})(1+2q-q^{3/2}+2q^2+q^3)}{16q^3}\bigg]. \end{aligned} \tag{8.39}$$

Expansions (8.10)–(8.11) and the orthogonality relations for the basic trigonometric system (1.8)–(1.11) together with (13.11) of [14] result in

$$\begin{aligned} & \int_0^\pi E_m(\cos\theta; q)\, E_n(\cos\theta; q)(e^{2i\theta}, e^{-2i\theta}; q)_{1/2}\, d\theta \\ & \quad = (-1)^{[(m+n)/2]}\,\pi\,\frac{(q^{1/2}; q)_\infty^2}{(q; q)_\infty^2}\,E_{m+n+1}(\eta; q). \end{aligned} \tag{8.40}$$

This implies that the q-Euler polynomials are orthogonal when $m+n$ is odd due to (8.11); that is also true due to the symmetry property (8.15). Equation (8.40) is an analog of the property

$$\int_0^1 E_m(x)\, E_n(x)\, dx = (-1)^m\, 2\,\frac{m!\, n!}{(m+n+1)!}\,E_{m+n+1}(1), \tag{8.41}$$

$m, n = 0, 1, 2, \ldots,$ holding for the classical Euler polynomials [1].

Expansions (4.26)–(4.27) together with the explicit representation (2.10) for the q-Lommel polynomials give

$$\prod_{k=0}^{m-1} ((1+q^{k+1/2})^2-4q^{k+1/2}x^2) = 2(-1)^m \frac{(q;q)_{2m}}{(q^{1/2};q)_m} q^{-m^2/2} \times \sum_{k=0}^{[m/2]} q^{k(k-1/2)} \frac{(q;q)_{m-k}\,(q^{1/2};q)_{m-k}}{(q;q)_k\,(q;q)_{m-2k}(q^{1/2};q)_k} E_{2m-2k}(x;q), \tag{8.42}$$

and

$$x \prod_{k=0}^{m-2} ((1+q^{k+1/2})^2-4q^{k+1/2}x^2) = \frac{(q;q)_{2m-1}}{(q^{1/2};q)_m} q^{-m^2/2+1/4} \times (-1)^{m-1} \sum_{k=0}^{[m/2]} q^{k(k-1/2)} \frac{(q;q)_{m-k}\,(q^{1/2};q)_{m-k}}{(q;q)_k\,(q;q)_{m-2k}(q^{1/2};q)_k} E_{2m-2k-1}(x;q) \tag{8.43}$$

for even and odd degrees, respectively.

In a similar manner, expansions (2.33)–(2.34) and (2.10) lead to relations between the q-Legendre polynomials $C_m(x;q^{1/2}\,|\,q)$ and the q-Euler polynomials

$$C_{2m}(x;q^{1/2}\,|\,q) = 2q^{-m^2} \sum_{k=0}^{m} q^{k(k-1/2)} \frac{(q;q)_{2m-k}\,(q^{1/2};q)_{2m-k}}{(q;q)_k\,(q;q)_{2m-2k}\,(q^{1/2};q)_k} E_{2m-2k}(x;q) \tag{8.44}$$

and

$$C_{2m+1}(x;q^{1/2}\,|\,q) = 2q^{-m(m+1)-1/4} \times \sum_{k=0}^{m} q^{k(k-1/2)} \frac{(q;q)_{2m-k+1}\,(q^{1/2};q)_{2m-k+1}}{(q;q)_k\,(q;q)_{2m-2k+1}\,(q^{1/2};q)_k} E_{2m-2k+1}(x;q). \tag{8.45}$$

On the other hand, using these formulas and the orthogonality property one can find expansions of the q-Euler polynomials in terms of the q-Legendre polynomials; cf. (7.46)–(7.47).

We have discussed here some properties of the q-Euler polynomials and numbers. They deserve more detailed consideration. Other analogs of Euler's numbers were discussed in [16, 45] and [46].

9. EXTENSIONS OF RIEMANN ZETA AND RELATED FUNCTIONS

The Riemann zeta function is, usually, introduced as the Dirichlet series of the form

$$\zeta(z)=\sum_{n=1}^{\infty} n^{-z}. \tag{9.1}$$

This series converges uniformly and absolutely for $\operatorname{Re} z>1$ and, therefore, defines a holomorphic function in the half-plane $\operatorname{Re} z>1$. For an analytic continuation of this function in the entire complex plane, other properties and applications, see, for example, [1, 5, 22, 37, 40, 64]. In view of the relation,

$$\zeta(2m)=(-1)^{m-1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m}, \qquad m>0 \tag{9.2}$$

(cf. (9.1) and (7.3) for $x=0$), the zeta function can be thought as an extension of Bernoulli's numbers to an arbitrary complex index.

In a similar manner, our equation (7.15), considered at $x=-\eta$, and (13.10) of [14] give rise to a natural extension of Riemann's zeta function as

$$\zeta_q(z)=\sum_{n=1}^{\infty}\frac{1}{\kappa(\omega_n)\,\omega_n{}^{z}}. \tag{9.3}$$

The right side here is a uniformly and absolutely convergent series of analytic functions in any domain $\operatorname{Re} z>1$ and consequently the series is an analytic function in such a domain. Indeed, for $z=x+iy$,

$$\left|\frac{\omega_n{}^{-z}}{\kappa(\omega_n)}\right|=\frac{\omega_n{}^{-x}}{\kappa(\omega_n)}=M_n, \tag{9.4}$$

and

$$\frac{M_{n+1}}{M_n}=\left(\frac{\omega_n}{\omega_{n+1}}\right)^{x}\frac{\kappa(\omega_n)}{\kappa(\omega_{n+1})}. \tag{9.5}$$

Also,

$$\lim_{n\to\infty}\frac{\kappa(\omega_n)}{\kappa(\omega_{n+1})}=\lim_{n\to\infty}\frac{(-q^{1/2}\omega_n^2,-\omega_{n+1}^2;q)_\infty}{(-\omega_n^2,-q^{1/2}\omega_{n+1}^2;q)_\infty}$$
$$=\lim_{n\to\infty}\frac{(-q^{1/2-2n}(q^n\omega_n)^2;q)_{2n}}{(-q^{-2n}(q^n\omega_n)^2;q)_{2n}}$$
$$\times\lim_{n\to\infty}\frac{(-q^{-2n-2}(q^{n+1}\omega_{n+1})^2;q)_{2n+2}}{(-q^{-3/2-2n}(q^{n+1}\omega_{n+1})^2;q)_{2n+2}}$$
$$\times\lim_{n\to\infty}\frac{(-q^{1/2+2n}\omega_n^2,-q^{2n+2}\omega_{n+1}^2;q)_\infty}{(-q^{2n}\omega_n^2,-q^{5/2+2n}\omega_{n+1}^2;q)_\infty}$$
$$=\lim_{n\to\infty}\frac{(-q^{1/2}/(q^n\omega_n)^2;q)_{2n}}{(-q/(q^n\omega_n)^2;q)_{2n}}$$
$$\times\lim_{n\to\infty}\frac{(-q/(q^{n+1}\omega_{n+1})^2;q)_{2n+2}}{(-q^{1/2}/(q^{n+1}\omega_{n+1})^2;q)_{2n+2}}$$
$$\times q^{-1}\frac{(-q,-q^{1/2};q)_\infty}{(-q^{1/2},-q;q)_\infty}=q^{-1} \tag{9.6}$$

by (I.5) and (I.9) of [23], (12.7) from Appendix, and the asymptotic formula

$$\lim_{m\to\infty} q^m\omega_m=q^{1/4} \tag{9.7}$$

for the zeros of (1.12) established in [14, 24] and [52] (cf. also [18]). Therefore

$$\lim_{n\to\infty}\frac{M_{n+1}}{M_n}=q^{\operatorname{Re}z-1}<1, \tag{9.8}$$

which completes the proof.

The series (9.3) can also be rewritten as a contour integral of the form

$$\frac{1}{2\pi i}\int_C \operatorname{Cot}_q(\omega)\,\omega^{-z}\,d\omega=\zeta_q(z) \tag{9.9}$$

by Cauchy's residue theorem and (5.6). Here C is a contour which starts at the infinity in the upper half-plane, encircles the line $\omega>\omega_1$ in the positive direction excluding the origin, and returns to the infinity in the lower half-plane.

Also,

$$\zeta_q(2m)=(-1)^{m-1}B_{2m}(q), \tag{9.10}$$

which is an analog of the classical relation (9.2) between the zeta function and the Bernoulli numbers. This relation can be thought as a consequence of Parseval's identity (6.1) for expansions (7.14) and (7.15), the corresponding integral can be evaluated by (7.41). On the other hand, due to the orthogonality relation (2.18) and (9.3), formula (9.10) gives explicit representation for the moments of the q-Lommel polynomials (2.20) in terms of the q-Bernoulli numbers. One can write Gram's determinant expression for these polynomials in terms of the q-zeta function under consideration.

As a special case, due to (6.11) and (6.12), we have

$$\zeta_q(2) = \frac{q^{1/2}}{2(1-q^{3/2})} \tag{9.11}$$

and

$$\zeta_q(4) = \frac{q^2}{2(1-q^{3/2})^2\,(1-q^{5/2})}, \tag{9.12}$$

which are analogs of the classical Euler's results (6.13), namely,

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}. \tag{9.13}$$

Euler considered also the following series

$$\phi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \tag{9.14}$$

see [9] for a nice review on Euler's discovery of the main properties of the zeta and related functions. It is connected to the zeta function as

$$\phi(z) = (1-2^{1-z})\,\zeta(z). \tag{9.15}$$

Special values are

$$\phi(2) = \frac{\pi^2}{12}, \qquad \phi(4) = \frac{7\pi^4}{720} \tag{9.16}$$

in view of (9.13) and (9.15).

Equation (7.15) for $x=0$ gives us the possibility to introduce a q-extension of $\phi(z)$ as

$$\phi_q(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\kappa(\omega_n)\,\omega_n{}^z} \sqrt{\frac{(-q\omega_n^2;\, q^2)_\infty}{(-\omega_n^2;\, q^2)_\infty}} \tag{9.17}$$

and evaluate the following sums

$$\phi_q(2m) = (-1)^m B_{2m}(0; q) \tag{9.18}$$

in terms of the values of the q-Bernoulli polynomials $B_{2m}(x; q)$ at $x=0$. These values can be found from (7.30) in terms of the q-Bernoulli numbers as

$$\begin{aligned} B_{2m}(0; q) &= \sum_{n=0}^{2m} B_{2m-n}(q) \sum_{k=0}^{[n/2]} \frac{(-1)^k q^{k^2}}{(q^{1/2}; q^{1/2})_{n-2k}\,(q^2; q^2)_k} \\ &= \sum_{n=0}^{m} B_{2m-2n}(q) \sum_{k=0}^{n} \frac{(-1)^k q^{k^2}}{(q^{1/2}; q^{1/2})_{2n-2k}\,(q^2; q^2)_k} \\ &\quad + B_1(q) \sum_{k=0}^{m-1} \frac{(-1)^k q^{k^2}}{(q^{1/2}; q^{1/2})_{2m-2k-1}\,(q^2; q^2)_k} \end{aligned} \tag{9.19}$$

because

$$H_{2k+1}(0|q) = 0, \qquad H_{2k}(0|q) = (-1)^k\,(q; q^2)_k \tag{9.20}$$

due to the generating relation (4.10) and (II.2) of [23]. On the other hand, formulas (7.75)–(7.76) result in the following closed form

$$B_n(0; q) = \sum_{k=0}^{n} \frac{(-q^{1/2}; q)_k}{(q; q)_k}\, B_{n-k}(q).$$

Extensions of (9.16) are

$$\phi_q(2) = \frac{q}{2(1+q)(1-q^{3/2})} \tag{9.21}$$

and

$$\phi_q(4) = \frac{q^{5/2}\,(1+2q+q^{3/2}+2q^2+q^3)}{2(1+q)^2\,(1+q^2)(1-q^{3/2})^2\,(1-q^{5/2})} \tag{9.22}$$

in view of (7.20), (7.22), and (9.18). From (9.11)–(9.12) and (9.21)–(9.22) one gets

$$\phi_q(2)/\zeta_q(2) = \frac{q^{1/2}}{1+q} \to \frac{1}{2}, \tag{9.23}$$

$$\phi_q(4)/\zeta_q(4) = q^{1/2}\,\frac{1+2q+q^{3/2}+2q^2+q^3}{(1+q)^2\,(1+q^2)} \to \frac{7}{8} \tag{9.24}$$

as $q \to 1^-$ (cf. (9.15) for $z=2$ and $z=4$, respectively).

Let us also introduce

$$\xi_q(z)=\sum_{n=1}^{\infty}\frac{1}{\kappa(\varpi_n)\,\varpi_n{}^{z}}, \tag{9.25}$$

$$\chi_q(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\kappa(\varpi_n)\,\varpi_n{}^{z}}\sqrt{\frac{(-q\varpi_n^2;q^2)_\infty}{(-\varpi_n^2;q^2)_\infty}} \tag{9.26}$$

as q-analogs of

$$\xi(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^z},\qquad \chi(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}, \tag{9.27}$$

respectively. Letting $x=\eta$ and $x=0$ in (8.10) and (8.11) we arrive at the following relations with the q-Euler polynomials and numbers

$$\xi_q(2m)=(-1)^{m-1}\,E_{2m-1}(\eta;q), \tag{9.28}$$

$$\chi_q(2m+1)=(-1)^m\,E_{2m}(q). \tag{9.29}$$

In particular,

$$\xi_q(2)=\frac{1}{2(1-q^{1/2})}, \tag{9.30}$$

$$\xi_q(4)=\frac{q^{1/2}}{2(1-q^{1/2})^2\,(1-q^{3/2})}, \tag{9.31}$$

$$\chi_q(1)=\frac{1}{2},\qquad \chi_q(3)=\frac{q^{1/2}}{2(1-q^{1/2})^2\,(1+q)}, \tag{9.32}$$

which are q-analogs of the classical sums

$$\xi(2)=\frac{\pi^2}{8},\qquad \xi(4)=\frac{\pi^4}{96}, \tag{9.33}$$

$$\chi(1)=\frac{\pi}{4},\qquad \chi(3)=\frac{\pi^3}{32}. \tag{9.34}$$

Functions $\zeta_q(z)$, $\phi_q(z)$, $\xi_q(z)$ and $\chi_q(z)$ deserve more detailed consideration.

It is hoped that this observation will lead, eventually, to a q-extension of the theory of the zeta and related functions. Other q-analogs of the zeta function were discussed by Al-Salam [3], Satoh [45], Tsumura [58–61] and in the recent papers by Cherednik [19–21].

10. MORE EXPANSIONS

10.1. *Al-Salam and Chihara Polynomials.* The Al-Salam and Chihara polynomials are

$$p_m(x; a, b \mid q) = a^{-m} \frac{(ab; q)_m}{(q; q)_m} \, {}_3\varphi_2 \left(\begin{matrix} q^{-m}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix} ; q, q \right), \tag{10.1}$$

see [4, 39]. They have the following generating function

$$\sum_{m=0}^{\infty} t^m p_m(\cos\theta; a, b \mid q) = \frac{(ta,\ tb; q)_\infty}{(te^{i\theta},\ te^{-i\theta}; q)_\infty}. \tag{10.2}$$

Ismail and Stanton evaluated another important integral [33],

$$\begin{aligned} &\int_0^\pi \mathscr{E}_q(\cos\theta; \alpha)\, \mathscr{E}_q(\cos\theta; \beta)\, p_m(\cos\theta; a, -a|q) \\ &\qquad \times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(a^2e^{2i\theta}, a^2e^{-2i\theta}; q^2)_\infty}\, d\theta \\ &\quad = \frac{2\pi(-\alpha\beta q^{(m+1)/2}; q)_\infty\, (-q^{(1-m)/2}\beta/\alpha; q)_m}{(q; q)_m\, (q, -a^2q^m; q)_\infty\, (q\alpha^2, q\beta^2; q^2)_\infty}\, \alpha^m q^{m^2/4} \\ &\qquad \times {}_2\varphi_2 \left(\begin{matrix} -q^{(m+1)/2}\alpha/\beta, -q^{(m+1)/2}\beta/\alpha \\ -q, \quad -q^{(m+1)/2}\alpha\beta \end{matrix} ; q, -q^{(m+1)/2}\alpha\beta a^2 \right), \end{aligned} \tag{10.3}$$

which implies the following q-Fourier expansion,

$$\begin{aligned} &\mathscr{E}_q(\cos\theta; \alpha)\, \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{(a^2e^{2i\theta}, a^2e^{-2i\theta}; q^2)_\infty}\, p_m(\cos\theta; a, -a|q) \\ &\quad = \frac{\pi}{(q; q)_m\, (q, -a^2q^m; q)_\infty\, (q\alpha^2; q^2)_\infty} \\ &\qquad \times \sum_{n=-\infty}^{\infty} \frac{(i\alpha\omega_n q^{(m+1)/2}; q)_\infty}{k(\omega_n)(-q\omega_n^2; q^2)_\infty}\, \alpha^m q^{m^2/4} \\ &\qquad \times (iq^{(1-m)/2}\omega_n/\alpha; q)_m\, \mathscr{E}_q(\cos\theta; i\omega_n) \\ &\qquad \times {}_2\varphi_2 \left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha, -iq^{(m+1)/2}\alpha/\omega_n \\ -q, \quad iq^{(m+1)/2}\alpha\omega_n \end{matrix} ; q, iq^{(m+1)/2}\alpha a^2\omega_n \right), \end{aligned} \tag{10.4}$$

somewhat similar to (4.3). Using (10.2) in (10.4), we arrive at

$$\mathscr{E}_q(\cos\theta;\alpha)\,\frac{(q\alpha^2,\ a^2t^2;\ q^2)_\infty\ (q,\ q^{1/2}e^{2i\theta},\ q^{1/2}e^{-2i\theta};\ q)_\infty}{\pi(a^2e^{2i\theta},\ a^2e^{-2i\theta};\ q^2)_\infty\ (te^{i\theta},\ te^{-i\theta};\ q)_\infty}$$
$$=\sum_{n=-\infty}^{\infty}\frac{1}{k(\omega_n)\,(-q\omega_n^2;\ q^2)_\infty}\ \mathscr{E}_q(\cos\theta;\ i\omega_n)$$
$$\times\sum_{m=0}^{\infty}\frac{(iq^{(1-m)/2}\omega_n/\alpha;\ q)_m\ (i\alpha\omega_n q^{(m+1)/2};\ q)_\infty}{(q;\ q)_m\ (-a^2q^m;\ q)_\infty}\ (\alpha t)^m\, q^{m^2/4}$$
$$\times\,{}_2\varphi_2\left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha,\ -iq^{(m+1)/2}\alpha/\omega_n \\ -q,\qquad iq^{(m+1)/2}\alpha\omega_n\end{matrix};\ q,\ iq^{(m+1)/2}\alpha a^2\omega_n\right). \tag{10.5}$$

10.2. *Continuous q-Ultraspherical Polynomials.* Using Rogers' generating function for the continuous q-ultraspherical polynomials,

$$\sum_{m=0}^{\infty} C_m(\cos\theta;\gamma\mid q)\,t^m=\frac{(\gamma te^{i\theta},\ \gamma te^{-i\theta};\ q)_\infty}{(te^{i\theta},\ te^{-i\theta};\ q)_\infty}, \tag{10.6}$$

$|t|<1$, and (4.3), one gets the expansion

$$\mathscr{E}_q(\cos\theta;\alpha)\,\frac{(q,\ q^{1/2}e^{2i\theta},\ q^{1/2}e^{-2i\theta},\ \gamma te^{i\theta},\ \gamma te^{-i\theta};\ q)_\infty}{\pi(\gamma,\ \gamma e^{2i\theta},\ \gamma e^{-2i\theta},\ te^{i\theta},\ te^{-i\theta};\ q)_\infty}$$
$$=\sum_{n=-\infty}^{\infty}\frac{(q\alpha^2;\ q^2)_\infty^{-1}}{k(\omega_n)\,(-q\omega_n^2;\ q^2)_\infty}\ \mathscr{E}_q(\cos\theta;\ i\omega_n)$$
$$\times\sum_{m=0}^{\infty}\frac{(iq^{(1-m)/2}\omega_n/\alpha;\ q)_m\ (\gamma q^{m+1},\ i\alpha\omega_n q^{(m+1)/2};\ q)_\infty}{(q;\ q)_m\ (\gamma^2q^m;\ q)_\infty}\ q^{m^2/4}\,(\alpha t)^m$$
$$\times\,{}_2\varphi_2\left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha,\ -iq^{(m+1)/2}\alpha/\omega_n \\ \gamma q^{m+1},\qquad iq^{(m+1)/2}\alpha\omega_n\end{matrix};\ q,\ iq^{(m+1)/2}\alpha\gamma\omega_n\right). \tag{10.7}$$

The continuous q-ultraspherical polynomials $C_m(\cos\theta;\gamma\mid q)$ have also the following bilinear generating relation,

$$\sum_{m=0}^{\infty}\frac{(q;\ q)_m}{(\gamma;\ q)_m}\,C_m(\cos\theta;\gamma\mid q)\,C_m(\cos\varphi;\gamma\mid q)\,t^m\,{}_2\varphi_1\left(\begin{matrix}\gamma,\ \gamma^2q^m\\ \gamma q^{m+1}\end{matrix};\ q,\ t^2\right)$$
$$=\frac{(\gamma te^{i\theta+i\varphi},\ \gamma te^{i\theta-i\varphi},\ \gamma te^{i\varphi-i\theta},\ \gamma te^{-i\theta-i\varphi};\ q)_\infty}{(te^{i\theta+i\varphi},\ te^{i\theta-i\varphi},\ te^{i\varphi-i\theta},\ te^{-i\theta-i\varphi};\ q)_\infty}, \tag{10.8}$$

$|t|<1$; see [12, 32, 44]. Expansions (4.3) and (10.8) give

$$\frac{(q, \gamma te^{i\theta+i\varphi}, \gamma te^{i\theta-i\varphi}, \gamma te^{i\varphi-i\theta}, \gamma te^{-i\theta-i\varphi}; q)_\infty}{\pi(\gamma, te^{i\theta+i\varphi}, te^{i\theta-i\varphi}, te^{i\varphi-i\theta}, te^{-i\theta-i\varphi}; q)_\infty} \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} \mathscr{E}_q(\cos\theta; \alpha)$$

$$= \sum_{n=-\infty}^{\infty} \frac{(q\alpha^2; q^2)_\infty^{-1}}{k(\omega_n)(-q\omega_n^2; q^2)_\infty} \mathscr{E}_q(\cos\theta; i\omega_n)$$

$$\times \sum_{m=0}^{\infty} \frac{(iq^{(1-m)/2}\omega_n/\alpha; q)_m (\gamma q^{m+1}, i\alpha\omega_n q^{(m+1)/2}; q)_\infty}{(\gamma^2 q^m; q)_\infty (\gamma; q)_m} q^{m^2/4} (\alpha t)^m$$

$$\times C_m(\cos\varphi; \gamma \mid q) {}_2\varphi_1 \left(\begin{matrix} \gamma, \gamma^2 q^m \\ \gamma q^{m+1} \end{matrix} ; q, t^2 \right)$$

$$\times {}_2\varphi_2 \left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha, -iq^{(m+1)/2}\alpha/\omega_n \\ \gamma q^{m+1}, \quad iq^{(m+1)/2}\alpha\omega_n \end{matrix} ; q, iq^{(m+1)/2}\alpha\gamma\omega_n \right). \tag{10.9}$$

11. MISCELLANEOUS RESULTS

Expansion (4.12) and the orthogonality property of the basic trigonometric system (7.5) of [14] result in the following connecting relation,

$$\varepsilon_q(i\alpha\omega_n q^{1/4}) = \frac{(q; q)_\infty (-q\omega_n^2; q^2)_\infty}{2\pi} \times \int_0^\pi \mathscr{E}_q(\cos\theta; i\omega_n) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty} d\theta, \tag{11.1}$$

between the analogs of the exponential function on q-linear and q-quadratic grids. Using (4.12) twice, with the help of the orthogonality property of the basic trigonometric system, one gets

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q^{1/2})_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta} \beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty} d\theta = \frac{2\pi^2}{(q; q)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{\varepsilon_q(i\alpha\omega_n q^{1/4}) \varepsilon_q(-i\beta\omega_n q^{1/4})}{k(\omega_n)(-q\omega_n^2; q^2)_\infty^2}. \tag{11.2}$$

The same consideration for (4.15) gives us a possibility to evaluate another similar integral,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q^{1/2})_\infty}{(\alpha e^{i\theta+i\varphi}, \alpha e^{i\theta-i\varphi}, \alpha e^{i\varphi-i\theta}, \alpha e^{-i\theta-i\varphi}; q)_\infty} \frac{d\theta}{(\beta e^{i\theta+i\phi}, \beta e^{i\theta-i\phi}, \beta e^{i\phi-i\theta}, \beta e^{-i\theta-i\phi}; q)_\infty}$$

$$= \frac{2\pi^2}{(q, q, \alpha^2, \beta^2; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q\alpha^2\omega_n^2, -q\beta^2\omega_n^2; q^2)_\infty}{k(\omega_n)(-q\omega_n^2; q^2)_\infty^2} \times \mathscr{E}_q(\cos\varphi; i\alpha\omega_n) \mathscr{E}_q(\cos\phi; -i\beta\omega_n). \tag{11.3}$$

It is of interest to compare these expansions with the classical Askey–Wilson integral [7].

The limiting case $\beta \to 1^-$ of the last formula leads to the bilinear generating relation (4.15) again. Indeed,

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{\pi} \frac{f(\cos\varphi)(q, r^2, e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(re^{i\theta+i\varphi}, re^{i\theta-i\varphi}, re^{i\varphi-i\theta}, re^{-i\theta-i\varphi}; q)_\infty}\, d\varphi = f(\cos\theta), \tag{11.4}$$

for every bounded function $f(\cos\theta)$ that is continuous on $0<\theta<\pi$. This has been shown in [6] with the help of a result from [65].

Substituting $\phi=\varphi$ and integrating the both sides of (11.3) over φ with the help of (6.10), we arrive at the following double integral

$$\begin{aligned}
&\int_0^{\pi}\int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\alpha e^{i\theta+i\varphi}, \alpha e^{i\theta-i\varphi}, \alpha e^{i\varphi-i\theta}, \alpha e^{-i\theta-i\varphi}; q)_\infty}\\
&\quad\times \frac{(e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(\alpha e^{i\theta+i\varphi}, \alpha e^{i\theta-i\varphi}, \alpha e^{i\varphi-i\theta}, \alpha e^{-i\theta-i\varphi}; q)_\infty}\\
&\quad\times \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{(q^{1/2}e^{2i\varphi}, q^{1/2}e^{-2i\varphi}; q)_\infty}\, d\theta d\varphi\\
&\quad= \frac{2\pi^3}{(q, q, \alpha^2, \beta^2; q)_\infty(\alpha-\beta)}\, \frac{(q^{1/2}; q)_\infty^2}{(q; q)_\infty^2}\\
&\quad\times \sum_{n=-\infty}^{\infty} \frac{(-i\alpha\omega_n, i\beta\omega_n; q^{1/2})_\infty - (i\alpha\omega_n, -i\beta\omega_n; q^{1/2})_\infty}{i\,k(\omega_n)\,\omega_n\,(-q\omega_n^2; q^2)_\infty^2}.
\end{aligned} \tag{11.5}$$

12. APPENDIX: UNIFORM BOUNDS AND UNIFORM CONVERGENCE

This section contains the details of the proof of the uniform convergence of the q-Fourier series (4.3). Similar arguments can be used for (4.24) and for many other series in this paper.

12.1. *Uniform Bounds.* In order to verify the uniform convergence of the q-Fourier series we first provide convenient uniform upper bounds for the q-exponential function.

LEMMA 1. *Let* $-x(\varepsilon) < -1 \leqslant x = \cos\theta \leqslant 1 < x(\varepsilon)$, *where* $x(\varepsilon) = (q^{\varepsilon} + q^{-\varepsilon})/2$, $0 < q < 1$ *and* $0 < \varepsilon < 1/2$. *The following inequalities hold*

$$|\mathscr{E}_q(\cos\theta; i\omega)| < \frac{(-q^{1/4-\varepsilon}\,|\omega|; q^{1/2})_\infty}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty (-q\omega^2; q^2)_\infty} \tag{12.1}$$

and

$$|\mathscr{E}_q(\cos\theta; i\omega)| < \frac{(q^{1/2}, -q^{1/4-\varepsilon}\,|\omega|, -q^{1/4+\varepsilon}/|\omega|; q^{1/2})_\infty}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty\, (-q\omega^2; q^2)_\infty} \tag{12.2}$$

for all real values of the ω.

Proof. By Lemma 2 of [14] for the continuous q-Hermite polynomials $H_n(x\,|\,q)$ one can write

$$H_n(\cos\theta\,|\,q) \leqslant H_n(x(\varepsilon)|q), \tag{12.3}$$

where $x(\varepsilon) = (q^{\varepsilon} + q^{-\varepsilon})/2$. Using (7.4.5) of [23] with $e^{i\theta} = q^{\varepsilon}$

$$\begin{aligned} H_n(x(\varepsilon)|q) &= \frac{q^{-n\varepsilon}}{(q^{2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(1+n)k}}{(q, q^{1-2\varepsilon}; q)_k} \\ &\quad - \frac{q^{(n+2)\varepsilon}}{(1-q^{2\varepsilon})(q^{1-2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(1+n)k}}{(q, q^{1+2\varepsilon}; q)_k} \\ &< \frac{q^{-n\varepsilon}}{(q^{2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(1+n)k}}{(q, q^{1-2\varepsilon}; q)_k}. \end{aligned} \tag{12.4}$$

But $(a; q)_k > (a; q)_\infty$ for $0 < a < 1$, and

$$\sum_{k=0}^{\infty} \frac{q^{(1+n)k}}{(q, q^{1-2\varepsilon}; q)_k} < \frac{1}{(q^{1-2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(1+n)k}}{(q; q)_k} = \frac{(q; q)_n}{(q, q^{1-2\varepsilon}; q)_\infty}.$$

Therefore

$$|H_n(\cos\theta\,|\,q)| < q^{-n\varepsilon} \frac{(q; q)_n}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty}. \tag{12.5}$$

Similar inequality holds for the continuous q-ultraspherical polynomials.

The uniform bounds (12.1)–(12.2) for the $\mathscr{E}_q(\cos\theta; i\omega)$ can be obtain now by means of the generating function (4.10). Indeed, for all real ω the following inequalities hold

$$
\begin{aligned}
(-q\omega^2; q^2)_\infty\, |\mathscr{E}_q(x; i\omega)| &\leqslant \sum_{k=0}^{\infty} \frac{q^{k^2/4}\, |\omega|^k}{(q; q)_k}\, |H_k(x \mid q)| \\
&< \frac{1}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} q^{k^2/4-k\varepsilon}\, |\omega|^k \\
&< \frac{1}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(k^2-k)/4}\, (q^{1/4-\varepsilon}\, |\omega|)^k}{(q^{1/2}; q^{1/2})_k} \\
&= \frac{(-q^{1/4-\varepsilon}\, |\omega|; q^{1/2})_\infty}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty},
\end{aligned} \tag{12.6}
$$

which proves (12.1). In order to derive (12.2), one can rewrite the third line here as

$$
\begin{aligned}
\sum_{k=0}^{\infty} q^{k^2/4}\, (q^{-\varepsilon}\, |\omega|)^k &< \sum_{k=-\infty}^{\infty} q^{k^2/4}\, (q^{-\varepsilon}\, |\omega|)^k \\
&= (q^{1/2}, -q^{1/4-\varepsilon}\, |\omega|, -q^{1/4+\varepsilon}/|\omega|; q^{1/2})_\infty
\end{aligned}
$$

by the Jacobi triple product identity. ∎

12.2. *Complete Asymptotic Expansion of* $\kappa(\omega)$. We also provide here more details on the asymptotics of the $\mathscr{L}^2$-norm for the basic trigonometric system $\{\mathscr{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$ as $|\omega_n| \to \infty$. The leading term is given in [14] without the proof.

It is more convenient to use (1.23) instead of (1.22). One can easily see that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{q^{k/2}}{1+\omega^2 q^k} &= \kappa(\omega) + \frac{q^{1/2}}{\omega^2}\, \kappa\left(\frac{q^{1/2}}{\omega}\right) \\
&= \frac{1}{1+\omega^2}\, {}_1\psi_1\left(\begin{matrix} -\omega^2 \\ -q\omega^2 \end{matrix}; q, q^{1/2}\right) \\
&= \frac{(q, q, -q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{(q^{1/2}, q^{1/2}, -\omega^2, -q/\omega^2; q)_\infty}
\end{aligned}
$$

by the Ramanujan summation theorem. Therefore

$$
\kappa(\omega) = \frac{(q, q, -q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{(q^{1/2}, q^{1/2}, -\omega^2, -q/\omega^2; q)_\infty} - \frac{q^{1/2}}{\omega^2}\, \kappa\left(\frac{q^{1/2}}{\omega}\right) \tag{12.7}
$$

which gives the complete asymptotic expansion of the constant $k(\omega)$ in (1.19) and (1.21) in terms of q/ω^2:

$$
\begin{aligned}
k(\omega) = \pi \, & \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \frac{(-q^{1/2}\omega^2; q)_\infty}{(-\omega^2; q)_\infty} \\
& \times \sum_{k=0}^{\infty} \frac{(q^{-1/2}; q)_k}{(q; q)_k} \left(-\frac{q}{\omega^2} \right)^k \\
& - \pi \frac{(q^{1/2}; q)_\infty^2}{(q; q)_\infty^2} \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \\
& \times \frac{q^{1/2}}{\omega^2} \sum_{k=0}^{\infty} \frac{1}{1-q^{k+1/2}} \left(-\frac{q}{\omega^2} \right)^k.
\end{aligned}
\tag{12.8}
$$

This expansion confirms that we can use the leading term (6.8) of [14].

12.3. *Uniform Convergence in* (4.3). Let $0<\gamma<1$ and $0<|\alpha|<1$. By (III.4) of [23] one can rewrite (4.3) as

$$
\begin{aligned}
\mathscr{E}_q(\cos\theta; \alpha) & \frac{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} C_m(\cos\theta; \gamma \mid q) \\
= & \frac{\pi(\gamma, \gamma q^{m+1}, \alpha^2; q)_\infty \alpha^m q^{m^2/4}}{(q; q)_m \, (q, \gamma^2 q^m; q)_\infty \, (q\alpha^2; q^2)_\infty} \\
& \times \sum_{n=-\infty}^{\infty} \frac{(iq^{(1-m)/2}\omega_n/\alpha; q)_m}{k(\omega_n)(-q\omega_n^2; q^2)_\infty} \mathscr{E}_q(\cos\theta; i\omega_n) \\
& \times {}_2\varphi_1 \left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha, \, iq^{(m+1)/2}\gamma\omega_n/\alpha \\ \gamma q^{m+1} \end{matrix} ; q, \alpha^2 \right).
\end{aligned}
\tag{12.9}
$$

We have already discussed the uniform bounds for $\mathscr{E}_q(\cos\theta; i\omega_n)$ and the large ω-asymptotic of $k(\omega_n)$ in (12.1)–(12.2) and (12.8), respectively.

Introduce

$$
\begin{aligned}
C_n = & (iq^{(1-m)/2}\omega_n/\alpha; q)_m \\
& \times {}_2\varphi_1 \left(\begin{matrix} iq^{(m+1)/2}\omega_n/\alpha, \, iq^{(m+1)/2}\gamma\omega_n/\alpha \\ \gamma q^{m+1} \end{matrix} ; q, \alpha^2 \right).
\end{aligned}
\tag{12.10}
$$

Then

$$
\begin{aligned}
|C_n| &\leqslant (-q^{(1-m)/2}\,|\omega_n/\alpha|;\, q)_m \\
&\quad \times \sum_{k=0}^{\infty} \frac{(-q^{(m+1)/2}\,|\omega_n/\alpha|,\, -q^{(m+1)/2}\,|\gamma\omega_n/\alpha|;\, q)_k}{(q,\, \gamma q^{m+1};\, q)_k}\, |\alpha|^{2k} \\
&< (-q^{(1-m)/2}\,|\omega_n/\alpha|;\, q)_m\, \frac{(-q^{(m+1)/2}\,|\gamma\omega_n/\alpha|;\, q)_\infty}{(\gamma q^{m+1};\, q)_\infty} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(-q^{(m+1)/2}\,|\omega_n/\alpha|;\, q)_k}{(q;\, q)_k}\, |\alpha|^{2k}
\end{aligned}
$$

and by the q-binomial theorem

$$
\begin{aligned}
|C_n| &< (-q^{(1-m)/2}\,|\omega_n/\alpha|;\, q)_m \\
&\quad \times \frac{(-q^{(m+1)/2}\,|\alpha\omega_n|,\, -q^{(m+1)/2}\,|\gamma\omega_n/\alpha|;\, q)_\infty}{(|\alpha|^2,\, \gamma q^{m+1};\, q)_\infty}.
\end{aligned}
\tag{12.11}
$$

Therefore the n-th term in the series (12.9) can be estimated as

$$
\begin{aligned}
&\left| \frac{C_n \mathscr{E}_q(\cos\theta;\, i\omega_n)}{k(\omega_n)\,(-q\omega_n^2;\, q^2)_\infty} \right| \\
&\quad < A\, \frac{(-q^{1/4-\varepsilon}\,|\omega_n|;\, q^{1/2})_\infty}{\kappa(\omega_n)\,(-\omega_n^2;\, q)_\infty}\, (-q^{(1-m)/2}\,|\omega_n/\alpha|;\, q)_m \\
&\quad\quad \times (-q^{(m+1)/2}\,|\alpha\omega_n|,\, -q^{(m+1)/2}\,|\gamma\omega_n/\alpha|;\, q)_\infty = A a_n
\end{aligned}
\tag{12.12}
$$

for $-(q^{\varepsilon}+q^{-\varepsilon})/2 < x < (q^{\varepsilon}+q^{-\varepsilon})/2$ and $0<\varepsilon<1/2$. The value of the positive constant A here can be easily found explicitly but it is not essential for our consideration. Suppose $n \geqslant 0$, the case of negative n is similar. Consider

$$
b_n = B(\gamma q^{-2\varepsilon})^n, \qquad B = \frac{(q^{1/2};\, q)_\infty^2}{(q;\, q)_\infty^2}
$$

with $\gamma q^{-2\varepsilon} < 1$ and $\sum_{n=0}^{\infty} b_n < \infty$. In view of (12.7) and (9.7) from this paper and (I.7) of [23],

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left[\frac{(-q^{1/4-\varepsilon}\omega_n; q^{1/2})_\infty \, (-q^{(1-m)/2}\,|\omega_n/\alpha|; q)_\infty}{(-q^{1/2}\omega_n^2; q)_\infty \, (-q^{(1+m)/2}\,|\omega_n/\alpha|; q)_\infty} \right. \\
&\qquad \left. \times \frac{(-q^{(m+1)/2}\,|\alpha\omega_n|, -q^{(m+1)/2}\,|\gamma\omega_n/\alpha|; q)_\infty}{(\gamma q^{-2\varepsilon})^n} \right] \\
&= (-q^{1/2-\varepsilon}; q^{1/2})_\infty \, (-q^{(1-m)/2+1/4}\,|\alpha|; q)_m \\
&\qquad \times \frac{(-q^{(m+1)/2+1/4}\,|\alpha|, -q^{(m+1)/2+1/4}\,|\gamma/\alpha|; q)_\infty}{(-q; q)_\infty} \\
&\qquad \times \lim_{n\to\infty} \left[\frac{(-q^{1/4-\varepsilon}\omega_n; q^{1/2})_{2n}(-q^{(1-m)/2}\,|\omega_n/\alpha|; q)_n}{(-q^{1/2}\omega_n^2; q)_{2n}(-q^{(1+m)/2}\,|\omega_n/\alpha|; q)_n} \right. \\
&\qquad \left. \times \frac{(-q^{(m+1)/2}\,|\alpha\omega_n|, -q^{(m+1)/2}\,|\gamma\omega_n/\alpha|; q)_n}{(\gamma q^{-2\varepsilon})^n} \right] \\
&= (-q^{1/2-\varepsilon}; q^{1/2})_\infty \, (-q^{(1-m)/2+1/4}/|\alpha|; q)_m \\
&\qquad \times \frac{(-q^{(m+1)/2+1/4}\,|\alpha|, -q^{(m+1)/2+1/4}\,|\gamma/\alpha|; q)_\infty}{(-q; q)_\infty} \\
&\qquad \times \lim_{n\to\infty} \frac{(-q^{1/4+\varepsilon-n}/\omega_n; q^{1/2})_{2n}(-q^{(1+m)/2-n}|\alpha/\omega_n|; q)_n}{(-q^{1/2-2n}/\omega_n^2; q)_{2n}(-q^{(1-m)/2-n}|\alpha/\omega_n|; q)_n} \\
&\qquad \times \lim_{n\to\infty} (-q^{(1-m)/2-n}/|\alpha\omega_n|, -q^{(1-m)/2-n}|\alpha/\gamma\omega_n|; q)_n \\
&= \frac{(-q^{\varepsilon}, -q^{1/2-\varepsilon}, -q^{1/4+m/2}\,|\alpha|, -q^{1/4-m/2}/|\alpha|; q^{1/2})_\infty}{(-q, -1; q)_\infty} \\
&\qquad \times \frac{(-q^{3/4+m/2}\,|\gamma/\alpha|, -q^{1/4-m/2}\,|\alpha/\gamma|; q)_\infty}{(-q^{1/4-m/2}\,|\alpha|, -q^{3/4+m/2}/|\alpha|; q)_\infty}.
\end{aligned}$$

As a result

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

and the series (12.9) converges absolutely and uniformly on $[-1, 1]$ when $0 < \gamma < 1$ due to the Weierstrass M-Test and the Limit Comparison Test. This completes the proof of the pointwise convergence of the q-Fourier series (4.3) by Theorem 13 of [14].

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