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Characterization of 1-quasi-greedy bases

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Abstract

In this note we continue the study initiated in [F. Albiac and P. Wojtaszczyk, Characterization of 1-greedy bases, J. Approx. Theory 138 (1) (2006) 65–86] of greedy-like bases in the "isometric case," i.e., in the case that the constants that arise in the context of greedy bases in their different forms are 1. Here we settle the problem to find a satisfactory characterization of 1-quasi-greedy bases in Banach spaces. We show that a semi-normalized basis in a Banach space is quasi-greedy with quasi-greedy constant 1 if and only if it is unconditional with suppression-unconditional constant 1.

Keywords: thresholding greedy algorithm, quasi-greedy basis, unconditional basis, renorming.
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1. Introduction and background

Let $(\mathbb{X}, \|\cdot\|)$ be an infinite-dimensional (real or complex) Banach space, and let $\mathcal{B} = (\mathbf{e}_n)_{n=1}^\infty$ be a semi-normalized basis for $\mathbb{X}$ with biorthogonal functionals $(\mathbf{e}_n^*)_{n=1}^\infty$. The basis $\mathcal{B}$ is *quasi-greedy* (see [1]) if for any $x \in \mathbb{X}$ the corresponding series expansion,

$$x = \sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$$

converges in norm after reordering it so that the sequence $(|\mathbf{e}_n^*(x)|)_{n=1}^\infty$ is decreasing. Wojtaszczyk showed [2] that a basis $(\mathbf{e}_n)_{n=1}^\infty$ of $\mathbb{X}$ is quasi-greedy if and only if the greedy operators $\mathcal{G}_N \colon \mathbb{X} \to \mathbb{X}$ defined by

$$x = \sum_{j=1}^\infty \mathbf{e}_j^*(x)\mathbf{e}_j \mapsto \mathcal{G}_N(x) = \sum_{j\in\Lambda_N(x)} \mathbf{e}_j^*(x)\mathbf{e}_j,$$

where $\Lambda_N(x)$ is any N-element set of indices such that

$$\min\{|\mathbf{e}_j^*(x)| \colon j \in \Lambda_N(x)\} \ge \max\{|\mathbf{e}_j^*(x)| \colon j \notin \Lambda_N(x)\},$$

are uniformly bounded, i.e.,

$$\|\mathcal{G}_N(x)\| \le C\|x\|, \quad x \in \mathbb{X},\ N \in \mathbb{N}, \tag{1}$$

for some constant C independent of x and N. Note that the operators $(\mathcal{G}_N)_{N=1}^\infty$ are neither linear nor continuous, so this is not just the Uniform Boundedness Principle!

Obviously, (1) implies that then there is a (possibly different) constant $\tilde{C}$ such that

$$\|x - \mathcal{G}_N(x)\| \le \tilde{C}\|x\|, \quad x \in \mathbb{X},\ N \in \mathbb{N}. \tag{2}$$

We will denote by $C_w = C_w[\mathcal{B}, \mathbb{X}]$ the smallest constant such that (1) holds, and by $C_\ell = C_\ell[\mathcal{B}, \mathbb{X}]$ the least constant in (2). We will refer to C_ℓ as the *suppression quasi-greedy constant of the basis*. It is rather common (cf. [3, 4]) and convenient to define the *quasi-greedy constant* of the basis as

$$C_{qg} = C_{qg}[\mathcal{B}, \mathbb{X}] = \max\{C_w[\mathcal{B}, \mathbb{X}], C_\ell[\mathcal{B}, \mathbb{X}]\}.$$

If $\mathcal{B}$ is a quasi-greedy basis and C is a constant such that $C_{qg} \le C$ we will say that $\mathcal{B}$ is C-*quasi-greedy*.

Recall also that a basis $(\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$ is *unconditional* if for any $x \in \mathbb{X}$ the series $\sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n$ converges in norm to x regardless of the order in which we arrange the terms. The property of being unconditional is easily seen to be equivalent to that of being *suppression unconditional*, which means that the natural projections onto any subsequence of the basis

$$P_A(x) = \sum_{n \in A} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad A \subset \mathbb{N},$$

are uniformly bounded, i.e., there is a constant K such that for all $x = \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n$ and all $A \subset \mathbb{N}$,

$$\left\| \sum_{n \in A} \mathbf{e}_n^*(x)\mathbf{e}_n \right\| \leq K \left\| \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n \right\|. \tag{3}$$

The smallest K in (3) is the *suppression unconditional constant* of the basis, and will be denoted by $K_{su} = K_{su}[\mathcal{B}, \mathbb{X}]$. Notice that

$$K_{su}[\mathcal{B}, \mathbb{X}] = \sup\{\|P_A\| \colon A \subset \mathbb{N} \text{ is finite}\} = \sup\{\|P_A\| \colon A \subset \mathbb{N} \text{ is cofinite}\}.$$

If a basis $\mathcal{B}$ is unconditional and K is a constant such that $K_{su} \leq K$ we will say that $\mathcal{B}$ is *K-suppression unconditional*.

Konyagin and Telmyakov [1] proved that unconditional bases with the additional property of being *democratic* are precisely the bases for which $\mathcal{G}_N(x)$ provides essentially the best N-terms approximation to x for every $N \in \mathbb{N}$ and $x \in \mathbb{X}$, i.e., there is a constant C such that

$$\|x - \mathcal{G}_N(x)\| \leq C \inf \left\{ \left\| x - \sum_{n \in A} \alpha_n \mathbf{e}_n \right\| : |A| = N, \alpha_n \in \mathbb{R}, n \in A \right\}. \tag{4}$$

These bases are known as *C-greedy bases*.

Unlike greedy bases, quasi-greedy bases are not in general unconditional. In spite of that, quasi-greedy bases preserve some vestiges of unconditionality and, for instance, they are *unconditional for constant coefficients* (see [2]). Most classical spaces contain conditional quasi-greedy bases. The first example of a conditional quasi-greedy basis was provided by Konyagin and Temlyakov in [1]. Subsequently, Wojtaszczyk gave in [2] a general construction (improved in [5]) to produce quasi-greedy bases in some Banach spaces.

His method yields the existence of conditional quasi-greedy bases in separable Hilbert spaces, in the spaces ℓ_p and $L_p[0,1]$ for $1 < p < \infty$, and in the Hardy space H_1. Dilworth and Mitra proved in [6] that ℓ_1 also has a conditional quasi-greedy basis. Subsequently, Dilworth, Kalton and Kutzarova [7] showed the existence of quasi-greedy bases in any Banach space $\mathbb{X}$ with a Schauder basis such that $\mathbb{X}$ contains a complemented subspace with a symmetric basis which is not equivalent to the unit vector basis of c_0. But, oddly enough, in all those cases the quasi-greedy constant of the basis appears to be bigger than 1.

Of course, unconditional bases are always quasi-greedy. Quantitatively, if $\mathcal{B}$ is K-suppression unconditional then $\mathcal{B}$ is K-quasi-greedy. In particular, unconditional bases with $K_{su} = 1$ are quasi-greedy with $C_w = 1$. Our aim is to show the converse of this statement, thus characterizing 1-quasi-greedy bases. The related problem of characterizing bases that are 1-greedy was solved in [8]. The question we settle is relevant since the optimality in the constants of greedy-like bases seems to improve some properties of the corresponding basis. Indeed, in the "isometric case" greedy bases gain in symmetry (they are invariant under greedy permutations instead of merely democratic). Our result reinforces this pattern by showing that "isometric" quasi-greedy basis are not only unconditional for constant coefficients, but they are unconditional. It seems surprising to us that a question that has been circulating around for quite some time now [9], ended up having a remarkably simple answer with a rather uninvolved proof.

2. The Main Theorem and its Proof

As a by-product of their research on unconditionality-type properties of quasi-greedy bases, Garrigós and Wojtaszczyk [5] have shown that bases in Hilbert spaces with $C_w = 1$ are orthogonal. A direct proof of their result can be obtained as follows.

Let $\mathcal{B} = (\mathbf{e}_n)_{n=1}^\infty$ be a basis in a (real or complex) Hilbert space with $C_w = 1$. Then, if $|\omega| = 1$, $0 < t < 1$, and $i \neq j$,

$$\|\mathbf{e}_i\|^2 \leq \|\mathbf{e}_i + \omega t \mathbf{e}_j\|^2 = \|\mathbf{e}_i\|^2 + 2t\Re(\omega\langle \mathbf{e}_i, \mathbf{e}_j\rangle) + t^2\|\mathbf{e}_j\|^2.$$

Simplifying,

$$-2\Re(\omega\langle \mathbf{e}_i, \mathbf{e}_j\rangle) \leq t\|\mathbf{e}_j\|^2.$$

Choosing ω such that $\omega\langle \mathbf{e}_i, \mathbf{e}_j\rangle = -|\langle \mathbf{e}_i, \mathbf{e}_j\rangle|$ and letting t tend to zero we obtain $|\langle \mathbf{e}_i, \mathbf{e}_j\rangle| = 0$.

A strengthening of this argument leads to the following generalization of Garrigós-Wojtaszczyk's result.

Theorem 2.1. *A quasi-greedy basis $(\mathbf{e}_n)_{n=1}^\infty$ in a Banach space $\mathbb{X}$ is quasi-greedy with $C_w = 1$ if and only if it is unconditional with suppression unconditional constant $K_{su} = 1$.*

Proof. We need only show that if x and y are vectors finitely supported in $(\mathbf{e}_n)_{n=1}^\infty$ with disjoint supports then $\|x\| \leq \|x + y\|$. This readily implies that $(\mathbf{e}_n)_{n=1}^\infty$ is unconditional with suppression unconditional constant $K_{su} = 1$.

Suppose that this is not the case and that we can pick x, $y \in \mathbb{X}$ finitely and disjointly supported in $(\mathbf{e}_n)_{n=1}^\infty$ with $\|x + y\| < \|x\|$. Consider the function $\varphi\colon \mathbb{R} \to [0, \infty)$ defined by

$$\varphi(t) = \|x + ty\|.$$

Using the definition, it is straightforward to check that φ is a convex function on the entire real line. Moreover, $\varphi(0) = \|x\|$ and, by assumption, $\varphi(1) < \|x\|$. Therefore, $\varphi(t) < \|x\|$ for all $0 < t < 1$. Choosing $t \in (0, 1)$ small enough we have $x = \mathcal{G}_N(x + ty)$, where N is the cardinality of the support of x. Consequently, for such a t,

$$\|x + ty\| = \varphi(t) < \|x\| = \|\mathcal{G}_N(x + ty)\| \leq \|x + ty\|,$$

where we used the hypothesis on the quasi-greedy constant of the basis to obtain the last inequality. This absurdity proves the result. □

We close with some consequences of Theorem 2.1, which need no further explanation.

Corollary 2.2. *Suppose $\mathcal{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a basis in a Banach space $\mathbb{X}$ with $C_w = 1$. Then $C_\ell = 1$; in particular $\mathcal{B}$ is 1-quasi-greedy.*

Corollary 2.3. *If a basis $(\mathbf{e}_n)_{n=1}^\infty$ in a Banach space $\mathbb{X}$ is 1-quasi-greedy then it is 1-suppression unconditional.*

Corollary 2.4. *Suppose $\mathcal{B} = (\mathbf{e}_n)_{n=1}^\infty$ is a basis in a Banach space $(\mathbb{X}, \|\cdot\|)$. Then $\mathbb{X}$ admits an equivalent norm $|||\cdot|||$ so that $\mathcal{B}$ is 1-quasi-greedy in the space $(\mathbb{X}, |||\cdot|||)$ if and only if $\mathcal{B}$ is unconditional.*

3. Concluding remarks, examples, and open questions

It is clear that our work opens the door to new problems within the subject of greedy approximation. In this section we describe a few avenues to continue the research reported in this paper and we also make a couple of accompanying remarks.

Theorem 2.1 imposes the restriction that $C_w[\mathcal{B},\mathbb{X}] > 1$ to any conditional basis $\mathcal{B}$ in a Banach space $\mathbb{X}$. A natural question that immediately arises is: is this the best we can say about the universal lower bound for conditional quasi-greedy bases?, i.e., is there a constant $C > 1$ such that $C \le C_w[\mathcal{B},\mathbb{X}]$ for any Banach space $\mathbb{X}$ and any conditional quasi-greedy basis $\mathcal{B}$ in $\mathbb{X}$. Furthermore, Corollary 2.2 may induce to conjecture that $C_\ell[\mathcal{B},\mathbb{X}] \le C_w[\mathcal{B},\mathbb{X}]$ for any Banach space $\mathbb{X}$ and any quasi-greedy basis $\mathcal{B}$ in $\mathbb{X}$. The following example borrowed from [10] solves in the negative both questions.

Example 3.1. Given a bounded interval $J \subset \mathbb{R}$, let $h_J = |J|^{-1}(\chi_{J^+} - \chi_{J^-})$, where J^+ and J^- are the left and right half side of J respectively, and let $h = \chi_{[0,1)}$. For $n \in \mathbb{N} \cup \{0\}$, denote by $\mathcal{D}_n$ the set of diadic subintervals of $[0,1)$ of lenght 2^{-n}, and $\mathcal{H}_n = \{h_J \colon J \in \mathcal{D}_n\}$. Given an increasing sequence $\alpha = (n_j)_{j=0}^\infty$ of nonnegative integers, we consider the L_1-normalized lacunary Haar system

$$\mathcal{H}_\alpha = \{h\} \cup \left(\cup_{j=0}^\infty \mathcal{H}_{n_j}\right).$$

Let $\mathbb{X}_\alpha$ be the closed subspace of $L_1[0,1]$ spanned by $\mathcal{H}_\alpha$. Dilworth et al. [10] proved that despite the fact that the Haar system is not a quasi-greedy basis in $L_1[0,1]$, for any $\varepsilon > 0$ there is a sequence α such that $C_w[\mathcal{H}_\alpha,\mathbb{X}_\alpha] \le 1+\varepsilon$. The inclusion of h in the lacunary Haar system is not essential but turns out to be convenient in the following argument, which is aimed to prove that $C_\ell[\mathcal{H}_\alpha,\mathbb{X}_\alpha] \ge 2$ for any α.

For each nonnegative integer j we recursively construct intervals $\mathcal{E}_j \subset \mathcal{D}_{n_j}$ such that $|\mathcal{E}_j| = 2^{n_j - j}$ and

$$K_j := \cup\{J^+ \colon J \in \mathcal{E}_{j-1}\} = \cup\{J \colon J \in \mathcal{E}_j\}, \quad j \ge 1.$$

For any $m \ge 1$ we have

$$f_m := h + \sum_{j=0}^{m-1} \sum_{J \in \mathcal{E}_j} 2^{-n_j + j} h_J = 2^m \chi_{K_m}. \tag{5}$$

Note that for $k \geq 0$, $m \geq 1$, and $N = 1 + \sum_{j=0}^{m-1} 2^{n_j - j}$, we have $\mathcal{G}_N(f_{m+k}) = f_m$. Consequently,

$$C_\ell[\mathcal{H}_\alpha, \mathbb{X}_\alpha] \geq \frac{\|f_{m+k} - f_m\|_1}{\|f_{m+k}\|_1} = 2(1 - 2^{-k}).$$

Letting k tend to ∞, we get the desired inequality.

Notice that in order to check that $\mathcal{H}_\alpha$ is a conditional basis of $\mathbb{X}_\alpha$ it suffices to compare the L_1-norm of f_m with the L_1-norm of the function obtained suppressing in (5) the summands corresponding to odd values of j.

Dilworth et al. [11] proved that a wide class of greedy bases are $(1+\varepsilon)$-greedy for any $\varepsilon > 0$ after a suitable renorming of the space. Example 3.1 and Corollary 2.4 motivate the study of the analogous questions for quasi-greedy bases.

Problem 3.2. Under which hypotheses a conditional quasi-greedy basis $\mathcal{B}$ for a Banach space $\mathbb{X}$ verifies $C_w[\mathcal{B}, \mathbb{X}] \leq 1+\varepsilon$ for any $\varepsilon > 0$ after a renorming of $\mathbb{X}$?, when does a Banach space $\mathbb{X}$ possess such a basis $\mathcal{B}$?

To the best of our knowledge it is not known whether Corollary 2.3 can be improved in the sense that there is a lower bound larger than 1 for the quasi-greedy constants of conditional bases.

Problem 3.3. Is there a constant $C > 1$ such that $C \leq C_{qg}[\mathcal{B}, \mathbb{X}]$ for any Banach space $\mathbb{X}$ and any conditional quasi-greedy basis $\mathcal{B}$ in $\mathbb{X}$?

With regard to the suppression quasi-greedy constant the main open question is:

Problem 3.4. Is there a Banach space $\mathbb{X}$ and a conditional quasi-greedy basis $\mathcal{B}$ for $\mathbb{X}$ such that $C_\ell[\mathcal{B}, \mathbb{X}] = 1$?

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