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Universal Polynomial Majorants on Convex Bodies

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Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ $(d \geqslant 2)$, and denote by $B_n(\mathbf{K})$ the set of all polynomials p_n in $\mathbb{R}^d$ of total degree $\leqslant n$ such that $|p_n| \leqslant 1$ on $\mathbf{K}$. In this paper we consider the following question: does there exist a $p_n^* \in B_n(\mathbf{K})$ which majorates *every* element of $B_n(\mathbf{K})$ *outside* of $\mathbf{K}$? In other words can we find a minimal $\gamma \geqslant 1$ and $p_n^* \in B_n(\mathbf{K})$ so that $|p_n(\mathbf{x})| \leqslant \gamma\, |p_n^*(\mathbf{x})|$ for every $p_n \in B_n(\mathbf{K})$ and $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$? We discuss the magnitude of γ and construct the universal majorants p_n^* for *even n*. It is shown that γ can be 1 only on *ellipsoids*. Moreover, $\gamma = O(1)$ on *polytopes* and has at most polynomial growth with respect to n, in general, for every convex body $\mathbf{K}$.

Key Words: convex bodies; polynomial majorants; polytopes; polytopal approximation.

Let $\mathbf{K} \subset \mathbb{R}^d$, $d \geqslant 2$, be a convex body i.e., it is a convex compact set with nonempty interior in $\mathbb{R}^d$. Consider the space P_n^d of polynomials on $\mathbb{R}^d$ of total degree $\leqslant n$, endowed with the usual supremum norm on $\mathbf{K}$. Then the unit ball in this space is given by

$$B_n(\mathbf{K}) := \{ p \in P_n^d : \| p \|_{C(\mathbf{K})} \leqslant 1 \}.$$

In this paper we address the following question: is there a "largest" polynomial in $B_n(\mathbf{K})$ which majorates *all elements* of $B_n(\mathbf{K})$ *everywhere* on $\mathbb{R}^d \backslash \mathbf{K}$? In other words does there exist a $\gamma \geqslant 1$ and $p_n^* \in B_n(\mathbf{K})$ such that

$$|p_n(\mathbf{x})| \leqslant \gamma\, |p_n^*(\mathbf{x})|, \qquad \forall p_n \in B_n(\mathbf{K}), \quad \forall \mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}? \tag{1}$$

Such a p_n^* majorates *all* $p_n \in B_n(\mathbf{K})$ at *every* point outside $\mathbf{K}$ (with the constant γ). In this sense p_n^* is a universal majorant for polynomials in $B_n(\mathbf{K})$. Naturally, we are interested in the *smallest* possible $\gamma \geqslant 1$ for which (1) holds with some $p_n^* \in B_n(\mathbf{K})$. Thus we set $\gamma_n(\mathbf{K}) := \inf\{\gamma$: there exists a $p_n^* \in B_n(\mathbf{K})$ so that (1) holds$\}$.

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The above definition is motivated by the classical inequality of Chebyshev (see [1, p. 235]) stating that when $d=1$ and $K=[-1,1]$ we have

$$|p_n(x)| \leqslant |T_n(x)|, \qquad \forall p_n \in B_n([-1,1]), \quad \forall\, |x|>1, \tag{2}$$

where $T_n(x)=\cos n \arccos x$ is the Chebyshev polynomial. This means in our terminology that $\gamma_n([-1,1])=1$ for every $n\in\mathbb{N}$, with $\pm T_n$ being the universal majorants.

In this paper we shall study the magnitude of $\gamma_n(\mathbf{K})$ when $d>1$ and $\mathbf{K}$ is a convex body in $\mathbb{R}^d$. First, it has to be noted that the above question is meaningful only for even $n\in\mathbb{N}$, because $\gamma_{2n+1}(\mathbf{K})=\infty$ whenever $d>1$ and $n\in\mathbb{N}$. Indeed, if $\gamma_{2n+1}(\mathbf{K})<\infty$, i.e., a universal majorant $p^*_{2n+1}\in B_{2n+1}(\mathbf{K})$ exists, then it follows from (1) that $\deg p^*_{2n+1}=2n+1$ (and not less), and $p^*_{2n+1}\neq 0$ on $\mathbb{R}^d\backslash\mathbf{K}$. Since $d>1$ we can easily find a line $\mathbf{L}=\{\mathbf{a}t+\mathbf{b} : t\in\mathbb{R}^1\}$ in $\mathbb{R}^d$ ($\mathbf{a}$, $\mathbf{b}\in\mathbb{R}^d$) so that $\mathbf{L}\cap\mathbf{K}=\varnothing$ and the *univariate* polynomial p^*_{2n+1} $(\mathbf{a}t+\mathbf{b})$ has degree $2n+1$. This yields that p^*_{2n+1} $(\mathbf{a}t_0+\mathbf{b})=0$ for some $t_0\in\mathbb{R}^1$ contradicting the above observation that $p^*_{2n+1}\neq 0$ on $\mathbb{R}^d\backslash\mathbf{K}$.

On the other hand for *even n* one can give a simple example of a universal majorant in $\mathbb{R}^d$, $d>1$. In what follows $|\mathbf{x}|$ denotes the Euclidean norm in $\mathbb{R}^d$ ($d\geqslant 1$), $\langle \mathbf{x}, \mathbf{y}\rangle$ stands for the inner product of $\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^d$, Bd $\mathbf{K}$ and Int $\mathbf{K}$ are the boundary and interior of $\mathbf{K}$, respectively.

EXAMPLE 1. Let $\mathbf{K}=\{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}|\leqslant 1\}$ be the Euclidean unit ball in $\mathbb{R}^d$. Then $\gamma_{2n}(\mathbf{K})=1$ with $p^*_{2n}(\mathbf{x})=T_{2n}(|\mathbf{x}|)\in B_{2n}(\mathbf{K})$ being a universal majorant. This follows immediately from (2) since $T_{2n}(t)$, $t\in\mathbb{R}^1$ is an even polynomial.

Using affine transformations of $\mathbb{R}^d$ the above example can be easily extended to arbitrary ellipsoids which means that $\gamma_{2n}(\mathbf{K})=1$ for any ellipsoid $\mathbf{K}$. Our first result gives a converse to this showing that $\gamma_{2n}(\mathbf{K})$ can attain its minimal value 1 *only* on ellipsoids.

THEOREM 1. *Let* $\mathbf{K}\subset\mathbb{R}^d$, $d\geqslant 2$, *be a convex body*; $n\in\mathbb{N}$. *Then* $\gamma_{2n}(\mathbf{K})=1$ *if and only if* $\mathbf{K}$ *is an ellipsoid, i.e.*, $\mathbf{K}=\{\mathbf{x}\in\mathbb{R}^d : |\mathbf{A}\mathbf{x}+\mathbf{b}|\leqslant 1\}$ *for some* $\mathbf{A}\in\mathbb{R}^d\times\mathbb{R}^d$ ($\det\mathbf{A}\neq 0$) *and* $\mathbf{b}\in\mathbb{R}^d$. *Moreover, in this case* $p^*_{2n}=\pm T_{2n}(|\mathbf{A}\mathbf{x}+\mathbf{b}|)$ *are the only universal majorants.*

Thus apart from ellipsoids we always have $\gamma_{2n}(\mathbf{K})>1$. It turns out that $\gamma_{2n}(\mathbf{K})=O(1)$ with a constant *independent* of n whenever $\mathbf{K}$ is a *polytope.* For a polytope $\mathbf{K}$ we shall denote by $f_j(\mathbf{K})$ the number of its j-dimensional faces, $0\leqslant j\leqslant d-1$.

THEOREM 2. *Let* $\mathbf{K}$ *be a convex polytope in* $\mathbb{R}^d$, $d \geqslant 2$. *Then for every* $n \in \mathbb{N}$

$$\gamma_{2n}(\mathbf{K}) \leqslant \sum_{j=1}^{d-2} f_j(\mathbf{K})\, f_{d-j-1}(\mathbf{K}) + 2 f_{d-1}(\mathbf{K}). \tag{3}$$

Moreover, if $\mathbf{K}$ *is central symmetric then we have* $\gamma_{2n}(\mathbf{K}) \leqslant f_{d-1}(\mathbf{K})$.

Using the above theorem and some known results on degree of approximation of convex bodies by polytopes with prescribed number of vertices or faces we can verify that $\gamma_{2n}(\mathbf{K})$ has at most polynomial growth in n for every convex body $\mathbf{K}$. Namely we have the next

THEOREM 3. *Let* $\mathbf{K}$ *be a convex body in* $\mathbb{R}^d$, $d \geqslant 2$. *Then for every* $n \in \mathbb{N}$

$$\gamma_{2n}(\mathbf{K}) \leqslant c(d, \mathbf{K})\, n^{d(d-1)}, \tag{4}$$

where $c(d, \mathbf{K}) > 0$ *depends only on* d *and* $\mathbf{K}$.

Note that in general, polynomials bounded by 1 on $\mathbf{K}$ can grow *exponentially* outside $\mathbf{K}$. Thus the polynomial growth $\gamma_{2n}(\mathbf{K}) = O(n^{d(d-1)})$ given by Theorem 3 is very small relative to the size of polynomials $p_n \in B_n(\mathbf{K})$ outside of $\mathbf{K}$. The estimate (4) can be improved further if $\mathbf{K}$ has a C^2_+-boundary, i.e., its second fundamental form exists on Bd $\mathbf{K}$ and the Gauss curvature is a positive continuous function on Bd $\mathbf{K}$.

THEOREM 4. *If* $\mathbf{K}$ *is a convex body in* $\mathbb{R}^d$ $(d \geqslant 2)$ *with a* C^2_+*-boundary then* $\gamma_{2n}(\mathbf{K}) = O(n^{2(d-1)})$.

Above estimates can be used in order to obtain results on approximation of convex surfaces by algebraic surfaces. (We call zero sets of $p_n \in P^d_n$ algebraic surfaces of order n.) Denote by $\varrho(\mathbf{A}, \mathbf{B})$ the Hausdorff distance between $\mathbf{A}, \mathbf{B} \subset \mathbb{R}^{\mathbf{d}}$.

THEOREM 5. *For any convex body* $\mathbf{K}$ *in* $\mathbb{R}^d$ $(d \geqslant 2)$ *there exists an algebraic surface* $\mathbf{\Omega}_n$ *of order* n *such that* $\varrho(Bd\, \mathbf{K}, \mathbf{\Omega}_n) \leqslant c(\frac{\log n}{n})^2$, *where* $c > 0$ *depends only on* $\mathbf{K}$ *and* d.

This paper is organized as follows. Section 1 contains some material on the geometry of convex bodies needed for our considerations. In Section 2 the proofs of Theorem 1–5 will be given. Finally, we shall conclude the paper by a discussion of some open problems.

1. GEOMETRY

First we need to introduce a certain quantity $\alpha_k(\mathbf{x})$ which measures the distance from a given $\mathbf{x} \in \mathbb{R}^d$ to the boundary $\operatorname{Bd} \mathbf{K}$ of a convex body $\mathbf{K} \subset \mathbb{R}^d$. This quantity was used in [5] and [6] for the study of multivariate Chebyshev and Bernstein Inequalities.

For given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^d$ and $\mathbf{u} \in \mathbf{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ such that $\langle \mathbf{u}, \mathbf{B} - \mathbf{A} \rangle > 0$ consider the corresponding "slab" given by

$$\mathbf{S}_{\mathbf{u}}(\mathbf{A}, \mathbf{B}) := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{A} \rangle \leqslant \langle \mathbf{u}, \mathbf{x} \rangle \leqslant \langle \mathbf{u}, \mathbf{B} \rangle\}.$$

For a fixed $\alpha > 0$ the "α-dilation" of this slab is defined by $\mathbf{S}_{\mathbf{u}}^{\alpha}(\mathbf{A}, \mathbf{B}) := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{A} \rangle - \delta_\alpha \leqslant \langle \mathbf{u}, \mathbf{x} \rangle \leqslant \langle \mathbf{u}, \mathbf{B} \rangle + \delta_\alpha\}$ where $\delta_\alpha := \frac{\alpha - 1}{2} \langle \mathbf{B} - \mathbf{A}, \mathbf{u} \rangle$. Finally, set $\mathbf{K}_\alpha := \bigcap \{\mathbf{S}_{\mathbf{u}}^{\alpha}(\mathbf{A}, \mathbf{B}) : \mathbf{S}_{\mathbf{u}}(\mathbf{A}, \mathbf{B}) \supset \mathbf{K}, \mathbf{A}, \mathbf{B} \in \mathbb{R}^d, \mathbf{u} \in \mathbf{S}^{d-1}\}$, $\alpha_{\mathbf{K}}(\mathbf{x}) := \inf\{\alpha : \mathbf{x} \in \mathbf{K}_\alpha\}$.

Clearly, $\alpha_{\mathbf{K}}(\mathbf{x}) > 1$ for $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$, $\alpha_{\mathbf{K}}(\mathbf{x}) = 1$ on $\operatorname{Bd} \mathbf{K}$, and $\alpha_{\mathbf{K}}(\mathbf{x}) < 1$ inside $\mathbf{K}$. Also, it is easy to see that when $\mathbf{K}$ is central symmetric about $\mathbf{0}$ then $\alpha_K(\mathbf{x}) = \inf\{\alpha > 0 : \frac{\mathbf{x}}{\alpha} \in \mathbf{K}\}$ is the usual Minkowski functional. It is proved in [6] that for every $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$

$$\sup\{|p_n(\mathbf{x})| : p_n \in B_n(\mathbf{K})\} = T_n(\alpha_K(\mathbf{x})). \tag{5}$$

We shall also need the following lemmas on parallel supporting hyperplanes which are proved in [5] and [6]. (A special case of Lemma 1 also appears in [7].)

LEMMA 1. *Let* $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$. *Then there exists a line* $\mathbf{L}$ *passing through* $\mathbf{x}$ *with* $\mathbf{K} \cap \mathbf{L} = [\mathbf{A}, \mathbf{B}]$, *such that* $\mathbf{K}$ *possesses parallel supporting hyperplanes at* $\mathbf{A}$ *and* $\mathbf{B}$. *Moreover, for any such line*

$$\alpha_{\mathbf{K}}(\mathbf{x}) = \left| \mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2} \right| \Big/ \frac{|\mathbf{A} - \mathbf{B}|}{2}. \tag{6}$$

For the proof of the above statement see [6], Corollary 1 and the proof of Theorem 1A on p. 422. The next lemma provides a similar statement for *inner* points of $\mathbf{K}$.

LEMMA 2. *Let* $\mathbf{x} \in \operatorname{Int} \mathbf{K}$. *Then there exists a line* $\mathbf{L}$ *passing through* $\mathbf{x}$ *with* $\mathbf{K} \cap \mathbf{L} = [\mathbf{A}, \mathbf{B}]$ *such that* (6) *holds, and* $\mathbf{K}$ *possesses parallel supporting hyperplanes at* $\mathbf{A}$ *and* $\mathbf{B}$.

Note a slight difference in the statements of Lemmas 1 and 2: when $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$ by Lemma 1 (6) holds for *every* $\mathbf{L}$ as above, while for $\mathbf{x} \in \operatorname{Int} \mathbf{K}$ by Lemma 2 (6) holds for *some* $\mathbf{L}$ as above.

The first statement of Lemma 2 asserting that (6) holds for a certain line as above is a consequence of Proposition 2 in [5]. The second statement concerning parallel supporting hyperplanes is Proposition 1 of [5].

2. PROOFS

Proof of Theorem 1. The sufficiency in Theorem 1 is straightforward, it follows by a change of variables $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ ($\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$) and Example 1.

Assume now that $\mathbf{K} \subset \mathbb{R}^d$ is such that $\gamma_{2n}(\mathbf{K}) = 1$, and $p_{2n}^* \in B_{2n}(\mathbf{K})$ is a corresponding universal majorant, so that

$$|p_{2n}(\mathbf{x})| \leqslant |p_{2n}^*(\mathbf{x})|, \qquad p_{2n} \in B_{2n}(\mathbf{K}), \quad \mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}.$$

Then it easily follows from (5) that

$$|p_{2n}^*(\mathbf{x})| \equiv T_{2n}(\alpha_{\mathbf{K}}(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}.$$

In particular, we have that for every $\mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}$ either $p_{2n}^*(\mathbf{x}) \equiv T_n(\alpha_{\mathbf{K}}(\mathbf{x}))$, or $p_{2n}^*(\mathbf{x}) \equiv -T_n(\alpha_{\mathbf{K}}(\mathbf{x}))$. Thus we may assume that

$$p_{2n}^*(\mathbf{x}) \equiv T_{2n}(\alpha_{\mathbf{K}}(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}. \tag{7}$$

First we shall verify that equality (7) holds for $\mathbf{x} \in \mathbf{K}$, as well. Choose any $\tilde{\mathbf{x}} \in \operatorname{Int} \mathbf{K}$. Then by Lemma 2 there exists a line $\mathbf{L}$ through $\tilde{\mathbf{x}}$ with $\mathbf{L} \cap \mathbf{K} = [\mathbf{A}, \mathbf{B}]$ such that

$$\alpha_{\mathbf{K}}(\tilde{\mathbf{x}}) = \frac{\left|\tilde{\mathbf{x}} - \dfrac{\mathbf{A}+\mathbf{B}}{2}\right|}{\left|\dfrac{\mathbf{A}-\mathbf{B}}{2}\right|}, \tag{8}$$

and $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}$ and $\mathbf{B}$. Let

$$\tilde{\mathbf{x}} = \frac{1-\tilde{t}}{2}\mathbf{A} + \frac{1+\tilde{t}}{2}\mathbf{B},$$

where it can be assumed that $0 \leqslant \tilde{t} \leqslant 1$. Then by (8), $\alpha_{\mathbf{K}}(\tilde{\mathbf{x}}) = \tilde{t}$. Moreover, by Lemma 1 for every $\mathbf{x} \in \mathbf{L} \backslash \mathbf{K}$ equality (6) holds, i.e. setting $\mathbf{x}_t = \frac{1-t}{2}\mathbf{A} + \frac{1+t}{2}\mathbf{B}$ we have $\alpha_{\mathbf{K}}(\mathbf{x}_t) = t$, $t > 1$. This and (7) yield that

$$p_{2n}^*(\mathbf{x}_t) \equiv T_{2n}(t), \qquad t > 1.$$

But of course the above equality of univariate polynomials has to extend from $\{t \in \mathbb{R}^1 : t > 1\}$ to the whole line, i.e.,

$$p_{2n}^*\left(\frac{1-t}{2}\mathbf{A} + \frac{1+t}{2}\mathbf{B}\right) \equiv T_{2n}(t), \qquad t \in \mathbb{R}. \tag{9}$$

In particular, setting in (9) $t = \tilde{t}$ we obtain $p_{2n}^*(\tilde{\mathbf{x}}) = T_{2n}(\tilde{t}) = T_{2n}(\alpha_{\mathbf{K}}(\tilde{\mathbf{x}}))$. Thus and by (7)

$$p_{2n}^*(\mathbf{x}) \equiv T_{2n}(\alpha_{\mathbf{K}}(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^d. \tag{10}$$

The next step is to verify that $\mathbf{K}$ is central symmetric. Set

$$\alpha_0 := \inf_{\mathbf{x} \in \mathbf{K}} \alpha_{\mathbf{K}}(\mathbf{x}), \qquad \mathbf{K}_0 := \bigcap_{\alpha > \alpha_0} \mathbf{K}_\alpha.$$

Clearly, $\alpha_0 \geqslant 0$, $\mathbf{K}_0 \neq \emptyset$ and $\operatorname{Int} \mathbf{K}_\alpha \neq \emptyset$, $\alpha > \alpha_0$. Furthermore, for every $x \in \mathbf{K}_0$ we have $\alpha_{\mathbf{K}}(\mathbf{x}) \leqslant \alpha_0$, i.e., by minimality of α_0 it follows that $\alpha_{\mathbf{K}}(\mathbf{x}) = \alpha_0$ whenever $\mathbf{x} \in \mathbf{K}_0$. This last observation implies that $\mathbf{K}_0$ must be a singleton. Indeed, if $\mathbf{a}^*, \mathbf{b}^* \in \mathbf{K}_0$ ($\mathbf{a}^* \neq \mathbf{b}^*$) then $[\mathbf{a}^*, \mathbf{b}^*] \subset \mathbf{K}_0$, and hence $\alpha_{\mathbf{K}}(\mathbf{x}) = \alpha_0$ for $\mathbf{x} \in [\mathbf{a}^*, \mathbf{b}^*]$. This and (10) yield that $p_{2n}^* \equiv T_{2n}(\alpha_0)$ on the line $\mathbf{L}^*$ through $\mathbf{a}^*$ and $\mathbf{b}^*$, in an obvious contradiction with (10). Thus $\mathbf{K}_0 = \{\mathbf{a}^*\}$. Consider now a line $\mathbf{L}^*$ through $\mathbf{a}^*$ with $\mathbf{K} \cap \mathbf{L}^* = [\mathbf{A}^*, \mathbf{B}^*]$ such that $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}^*$ and $\mathbf{B}^*$ (Lemma 2). By (9) and (10) we have with $\mathbf{x}_t^* = \frac{1-t}{2}\mathbf{A}^* + \frac{1+t}{2}\mathbf{B}^*$

$$T_{2n}(t) = p_{2n}^*(\mathbf{x}_t^*) = T_{2n}(\alpha_{\mathbf{K}}(\mathbf{x}_t^*)), \qquad t \in \mathbb{R}^1. \tag{11}$$

As t increases from -1 to 1 the continuous function $\alpha_{\mathbf{K}}(\mathbf{x}_t^*)$ decreases from 1 to α_0, and then increases from α_0 to 1. Thus in view of (11) we must have $\alpha_0 = 0$, and $\mathbf{a}^* = \mathbf{x}_0^* = (\mathbf{A}^* + \mathbf{B}^*)/2$. (In particular, $\operatorname{Int} \mathbf{K}_\alpha \neq \emptyset$ for every $\alpha > 0$.) Similarly for any $\mathbf{A} \in \operatorname{Bd} \mathbf{K}$ there exists a $\mathbf{B} \in \operatorname{Bd} \mathbf{K}$ such that $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}$ and $\mathbf{B}$. Thus using again (9) and (10)

$$T_{2n}(t) = T_{2n}\left(\alpha_{\mathbf{K}}\left(\frac{1-t}{2}\mathbf{A} + \frac{1+t}{2}\mathbf{B}\right)\right), \qquad t \in \mathbb{R}.$$

Again, as t varies in $[-1, 1]$ $\alpha_{\mathbf{K}}(\frac{1-t}{2}\mathbf{A} + \frac{1+t}{2}\mathbf{B})$ must decrease from 1 to 0 and then increase from 0 to 1. Hence $[\mathbf{A}, \mathbf{B}]$ must contain $\mathbf{a}^*$ (otherwise

$\alpha_{\mathbf{K}}(\frac{1-t}{2}\mathbf{A}+\frac{1+t}{2}\mathbf{B})$ can not attain 0), and, in addition, $\mathbf{a}^*=\frac{\mathbf{A}+\mathbf{B}}{2}$. Thus for every $\mathbf{A}\in \operatorname{Bd}\mathbf{K}$ the line through $\mathbf{A}$ and $\mathbf{a}^*$ exits $\mathbf{K}$ at $\mathbf{B}=2\mathbf{a}^*-\mathbf{A}$. This means that $\mathbf{K}$ is central symmetric about $\mathbf{a}^*$.

We may assume now that $\mathbf{a}^*=\mathbf{0}$ and $\mathbf{K}$ is symmetric about the origin. Then $\alpha_{\mathbf{K}}(t\mathbf{x})=t\alpha_{\mathbf{K}}(\mathbf{x})$ whenever $\mathbf{x}\in\mathbb{R}^d$ and $t>0$. The polynomial p_{2n}^* can be written as $p_{2n}^*(\mathbf{x})=\sum_{j=0}^{2n} h_j(\mathbf{x})$, where h_j is its jth homogeneous part, $0\leqslant j\leqslant 2n$. Furthermore $T_{2n}(t)=\sum_{j=0}^{n} c_j t^{2j}$, where $c_j\in\mathbb{R}$, $0\leqslant j\leqslant n$. Then for every $\mathbf{u}\in\mathbf{S}^{d-1}$ and $t>0$

$$p_{2n}^*(t\mathbf{u})=\sum_{j=0}^{2n} h_j(t\mathbf{u})=\sum_{j=0}^{2n} t^j h_j(\mathbf{u}),$$

$$T_{2n}(\alpha_{\mathbf{K}}(t\mathbf{u}))=T_{2n}(t\alpha_{\mathbf{K}}(\mathbf{u}))=\sum_{j=0}^{n} c_j\alpha_{\mathbf{K}}^{2j}(\mathbf{u})\, t^{2j}.$$

Hence using (10) we obtain

$$\sum_{j=0}^{2n} h_j(\mathbf{u})\, t^j=\sum_{j=0}^{n} c_j\alpha_{\mathbf{K}}^{2j}(\mathbf{u})\, t^{2j}, \qquad \mathbf{u}\in\mathbf{S}^{d-1}, \quad t>0.$$

This means that $h_{2j}(\mathbf{u})=c_j\alpha_{\mathbf{K}}^{2j}(\mathbf{u})$ for every $\mathbf{u}\in\mathbf{S}^{d-1}$. In particular

$$\alpha_{\mathbf{K}}^2(\mathbf{u})=\frac{1}{c_1}h_2(\mathbf{u}):=H_2(\mathbf{u}), \qquad u\in\mathbf{S}^{d-1}.$$

Evidently, H_2 is a positive definite quadratic form, i.e.

$$K=\{\mathbf{x}\in\mathbb{R}^d:\alpha_{\mathbf{K}}(\mathbf{x})\leqslant 1\}=\{\mathbf{x}\in\mathbb{R}^d: H_2(\mathbf{x})\leqslant 1\}$$

is an ellipsoid. In addition, by (10) the only possible majorants are $\pm T_{2n}(\alpha_{\mathbf{K}}(\mathbf{x}))$. ∎

Proof of Theorem 2. Let $\mathbf{K}\subset\mathbb{R}^d$ be a polytope. Consider $\mathbf{A}$, $\mathbf{B}\in\operatorname{Bd}\mathbf{K}$ such that $\mathbf{K}$ possesses parallel supporting hyperplanes $\mathbf{H_A}$, $\mathbf{H_B}$ at $\mathbf{A}$ and $\mathbf{B}$, and denote by $\mathscr{U}_{\mathbf{AB}}$ the set of normal vectors to such pairs of hyperplanes. Since $\mathbf{K}$ is a polytope it is easy to see that for some $\mathbf{u}\in\mathscr{U}_{\mathbf{AB}}$ the corresponding pair of hyperplanes $\mathbf{H_A}$, $\mathbf{H_B}$ has the property that the faces $\mathbf{F_A}=\mathbf{K}\cap\mathbf{H_A}$ and $\mathbf{F_B}=\mathbf{K}\cap\mathbf{H_B}$ of the polytope $\mathbf{K}$ contain a total of $d-1$ linearly independent vectors. Let $\mathscr{U}(\mathbf{K}):=\{\mathbf{u}_1, ..., \mathbf{u}_N\}\subset\mathbf{S}_+^{d-1}:=\{\mathbf{y}=(y_1, ..., y_d)\in\mathbf{S}^{d-1}: y_1\geqslant 0\}$ be the set of normal vectors to pairs of hyperplanes with the above properties. Since every $\mathbf{u}_j \in \mathscr{U}(\mathbf{K})$, $1 \leqslant j \leqslant N$, is uniquely determined by the corresponding pair of faces of $\mathbf{K}$ specified above it follows that

$$N \leqslant \tfrac{1}{2} \sum_{j=1}^{d-2} f_j(\mathbf{K})\, f_{d-j-1}(\mathbf{K}) + f_{d-1}(\mathbf{K}). \tag{12}$$

Moreover, $\mathscr{U}(\mathbf{K}) \cap \mathscr{U}_{\mathbf{AB}} \neq \varnothing$ whenever $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}$, $\mathbf{B} \in \operatorname{Bd} \mathbf{K}$. Furthermore, for every $\mathbf{u}_j \in \mathscr{U}(\mathbf{K})$ select some $\mathbf{A}_j$, $\mathbf{B}_j \in \operatorname{Bd} \mathbf{K}$ such that $\mathbf{u}_j \in \mathscr{U}_{\mathbf{A}_j \mathbf{B}_j}$, $1 \leqslant j \leqslant N$.

Finally, consider the polynomial $\tilde{T}_{2n}(t) = (T_{2n}(t)+1)/2 \in P^1_{2n}$. Obviously $\tilde{T}_{2n} \geqslant 0$ on $\mathbb{R}^1$, $\tilde{T}_{2n} \leqslant 1$ on $[-1, 1]$, and $T_{2n} \leqslant 2\tilde{T}_{2n}$ on $\mathbb{R}^1 \backslash [-1, 1]$. Now we set

$$p^*_{2n}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^{N} \tilde{T}_{2n} \left(\frac{\left\langle x - \dfrac{\mathbf{A}_j + \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle}{\left\langle \dfrac{\mathbf{A}_j - \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle} \right). \tag{13}$$

Clearly, $p^*_{2n} \in P^d_{2n}$. Moreover, we claim that $|p^*_{2n}| \leqslant 1$ on $\mathbf{K}$, i.e., $p^*_{2n} \in B_{2n}(\mathbf{K})$. Indeed, since $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}_j$ and $\mathbf{B}_j$ with normal $\mathbf{u}_j$ we have (assuming, for instance that $\langle \mathbf{A}_j, \mathbf{u}_j \rangle < \langle \mathbf{B}_j, \mathbf{u}_j \rangle$) $\langle \mathbf{A}_j, \mathbf{u}_j \rangle \leqslant \langle \mathbf{x}, \mathbf{u}_j \rangle \leqslant \langle \mathbf{B}_j, \mathbf{u}_j \rangle$, $\mathbf{x} \in \mathbf{K}$. This easily implies

$$\left| \left\langle \mathbf{x} - \frac{\mathbf{A}_j + \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle \right| \leqslant \left| \left\langle \frac{\mathbf{A}_j - \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle \right|, \qquad \mathbf{x} \in \mathbf{K}.$$

Since $|\tilde{T}_{2n}| \leqslant 1$ on $[-1, 1]$ we obtain by (13) that $p^*_{2n} \in B_{2n}(\mathbf{K})$. Now we need to show that p^*_{2n} satisfies (1) with a proper γ. Consider an arbitrary $p_{2n} \in B_{2n}(\mathbf{K})$ and $\mathbf{x}^* \in \mathbb{R}^d \backslash \mathbf{K}$. By Lemma 1 there exists a line $\mathbf{L}$ passing through $\mathbf{x}^*$ with $\mathbf{K} \cap \mathbf{L} = [\mathbf{A}^*, \mathbf{B}^*]$ such that $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{A}^*$, $\mathbf{B}^* \in \operatorname{Bd} \mathbf{K}$. As it was observed above we can choose this pair of hyperplanes $\mathbf{H}_{\mathbf{A}^*}$, $\mathbf{H}_{\mathbf{B}^*}$ (keeping $\mathbf{A}^*$, $\mathbf{B}^*$ fixed) so that some $\mathbf{u}_j \in \mathscr{U}(\mathbf{K})$, $1 \leqslant j \leqslant N$, is the normal to these hyperplanes. Then by (6) using that $\mathbf{x}^*$, $\mathbf{A}^*$, $\mathbf{B}^* \in \mathbf{L}$

$$\alpha_{\mathbf{K}}(\mathbf{x}^*) = \frac{\left| \mathbf{x}^* - \dfrac{\mathbf{A}^* + \mathbf{B}^*}{2} \right|}{\dfrac{|\mathbf{A}^* - \mathbf{B}^*|}{2}} = \left| \frac{\left\langle \mathbf{x}^* - \dfrac{\mathbf{A}^* + \mathbf{B}^*}{2}, \mathbf{u}_j \right\rangle}{\left\langle \dfrac{\mathbf{A}^* - \mathbf{B}^*}{2}, \mathbf{u}_j \right\rangle} \right|. \tag{14}$$

Recall that earlier we have already chosen $\mathbf{A}_j$, $\mathbf{B}_j$ from the pair of hyperplanes $\mathbf{H}_{\mathbf{A}^*}$, $\mathbf{H}_{\mathbf{B}^*}$ (with normal $\mathbf{u}_j$). Hence without loss of generality, $\mathbf{A}^*$, $\mathbf{A}_j \in \mathbf{H}_{\mathbf{A}^*}$, $\mathbf{B}^*$, $\mathbf{B}_j \in \mathbf{H}_{\mathbf{B}^*}$, i.e., $\mathbf{A}^* - \mathbf{A}_j$ and $\mathbf{B}^* - \mathbf{B}_j$ are normal to $\mathbf{u}_j$. Thus using (5), (14) and (13) we have for $p_{2n} \in B_{2n}(\mathbf{K})$

$$|p_{2n}(\mathbf{x}^*)| \leqslant T_{2n}(\alpha_{\mathbf{K}}(x^*)) \leqslant 2\tilde{T}_{2n}(\alpha_{\mathbf{K}}(\mathbf{x}^*))$$

$$= 2\tilde{T}_{2n}\left(\frac{\left\langle \mathbf{x}^* - \dfrac{\mathbf{A}_j + \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle}{\left\langle \dfrac{\mathbf{A}_j - \mathbf{B}_j}{2}, \mathbf{u}_j \right\rangle}\right) \leqslant 2Np^*_{2n}(\mathbf{x}^*).$$

Finally by (12) we arrive at estimate (3).

If remains to verify the sharper bound $\gamma_{2n}(\mathbf{K}) \leqslant f_{d-1}(\mathbf{K})$ in case when $\mathbf{K}$ is a central symmetric polytope. Assume that $\mathbf{0}$ is the center of symmetry of $\mathbf{K}$. Clearly, $\mathbf{K}$ has $M := f_{d-1}(\mathbf{K})/2$ pairs of parallel $(d-1)$-dimensional faces. Denote by $\boldsymbol{\omega}_j$, $1 \leqslant j \leqslant M$, the normals to these pairs of hyperplanes, and select any segments $[-\mathbf{A}_j, \mathbf{A}_j]$, $1 \leqslant j \leqslant M$ with endpoints in these pairs of faces. Finally, set

$$\tilde{p}_{2n}(\mathbf{x}) = \frac{1}{M} \sum_{j=1}^{M} \tilde{T}_{2n}\left(\frac{\langle \mathbf{x}, \boldsymbol{\omega}_j \rangle}{\langle \mathbf{A}_j, \boldsymbol{\omega}_j \rangle}\right).$$

As above, it follow that $\tilde{p}_{2n} \in B_{2n}(\mathbf{K})$. Now, for any $\mathbf{x}^* \in \mathbb{R}^d \backslash \mathbf{K}$ the line $\mathbf{L} := \{t\mathbf{x}^* : t \in \mathbb{R}^1\}$ intersects Bd $\mathbf{K}$ at some points $\pm\mathbf{B}$ which belong to a pair of parallel $(d-1)$-dimensional faces of $\mathbf{K}$ with normal $\boldsymbol{\omega}_k$ for some $1 \leqslant k \leqslant M$. Then $\mathbf{B} - \mathbf{A}_k \perp \boldsymbol{\omega}_k$ and proceeding as above we can show that for any $p_{2n} \in B_{2n}(\mathbf{K})$ $|p_{2n}(\mathbf{x}^*)| \leqslant f_{d-1}(\mathbf{K})\, \tilde{p}_{2n}(\mathbf{x}^*)$, i.e., $\gamma_{2n}(\mathbf{K}) \leqslant f_{d-1}(\mathbf{K})$. ∎

Proofs of Theorems 3 *and* 4. Now we proceed to proving Theorems 3 and 4. Their proofs are based on the "polytopal" estimate (3) for $\gamma_{2n}(\mathbf{K})$ on one side, and some known results on the rate of approximation of convex bodies by polytopes. One such result proved in [3] (see also [4]) asserts that for any convex body $\mathbf{K} \subset \mathbb{R}^d$ $(d \geqslant 2)$ and $N \in \mathbb{N}$ there exists a polytope $\mathbf{D}$ with $f_0(\mathbf{D}) = N$ vertices so that

$$\varrho(\mathbf{K}, \mathbf{D}) \leqslant \frac{c}{N^{2/(d-1)}} \tag{15}$$

with an absolute constant $c > 0$. (Here as above $\varrho(\mathbf{K}, \mathbf{D})$ stands for the Hausdorff distance between corresponding sets.) The approximating polytope $\mathbf{D}$ is constructed in [3] to be circumscribed to $\mathbf{K}$, it can be modified in an obvious way to be inscribed into $\mathbf{K}$. Moreover it is shown in [2] that if $\mathbf{K}$ is C_+^2 then for any $M \in \mathbb{N}$ there exists an inscribed polytope $\mathbf{D}$ with $\max_{0 \leqslant j \leqslant d-1} f_j(\mathbf{D}) \leqslant M$ such that

$$\varrho(\mathbf{K}, \mathbf{D}) \leqslant \frac{c_1}{M^{2/(d-1)}} \tag{16}$$

with some $c_1 > 0$ depending on $\mathbf{K}$. In principle, (16) provides a stronger bound than (15) since it is known (see e.g. [8, p. 257]) that for any polytope $\mathbf{D}$

$$f_j(\mathbf{D}) \leqslant c(d)\, f_0(\mathbf{D})^{[d/2]}, \qquad 1 \leqslant j \leqslant d-1, \tag{17}$$

with some $c(d)$ depending only on d.

We shall also need the following well known corollary of Chebyshev Inequality (2): if $p_n \in P_n^1$ is a univariate polynomial and $|p_n| \leqslant 1$ on $[-1, 1]$ then

$$|p_n(t)| \leqslant e^{c_0 n \sqrt{\delta}}, \quad |t| \leqslant 1 + \delta \qquad (0 < \delta < 1) \tag{18}$$

with some absolute constant $c_0 > 0$.

After these preliminaries we turn to the proof of Theorem 3. Consider an arbitrary convex body $\mathbf{K}$ in $\mathbb{R}^d$ ($d \geqslant 2$), and let $\mathbf{D} \subset \mathbf{K}$ be an inscribed polytope with $f_0(\mathbf{D}) = N$ vertices so that (15) holds.

By estimate (3) of Theorem 2 and (17) we have $\gamma_{2n}(\mathbf{D}) \leqslant c_1(d)\, N^d$. Thus there exists a universal majorant $p_{2n}^* \in B_{2n}(\mathbf{D})$ such that

$$|p_{2n}(\mathbf{x})| \leqslant c_1(d)\, N^d\, |p_{2n}^*(\mathbf{x})|, \qquad p_{2n} \in B_{2n}(\mathbf{D}), \quad x \in \mathbb{R}^d \backslash \mathbf{D}. \tag{19}$$

Since $|p_{2n}^*| \leqslant 1$ on $\mathbf{D} \subset \mathbf{K}$ it follows by (15) and (18) that

$$\|p_{2n}^*\|_{C(\mathbf{K})} \leqslant \exp[c_2 n N^{1/(1-d)}] \tag{20}$$

with some $c_2 > 0$ depending on d and $\mathbf{K}$. Hence setting $N := [n^{d-1}] + 1$ and $\tilde{p}_{2n} := e^{-c_2} p_{2n}^*$ we obtain by (20) that $|\tilde{p}_{2n}| \leqslant 1$ on $\mathbf{K}$, i.e., $\tilde{p}_{2n} \in B_{2n}(\mathbf{K})$. Moreover, using (19) we have for every $p_{2n} \in B_{2n}(\mathbf{K}) \subset B_{2n}(\mathbf{D})$ and $\mathbf{x} \in (\mathbb{R}^d \backslash \mathbf{K}) \subset (\mathbb{R}^d \backslash \mathbf{D})$

$$|p_{2n}(\mathbf{x})| \leqslant c_1(d)\, e^{c_2} N^d\, |\tilde{p}_{2n}(\mathbf{x})| \leqslant c_3 n^{d(d-1)}\, |\tilde{p}_{2n}(\mathbf{x})|.$$

This verifies the upper bound (4) of Theorem 3.

The proof of Theorem 4 follows similarly by using estimate (16) with $M = \max_{0 \leqslant j \leqslant d-1} f_j(\mathbf{D})$ instead of (15). This together with (3) yields the bound $\gamma_{2n}(\mathbf{D}) = O(M^2)$. Finally, setting $M := [n^{d-1}] + 1$ we arrive at $\gamma_{2n}(\mathbf{K}) = O(n^{2(d-1)})$. This completes the proof of Theorems 3 and 4. ∎

Remark. It can be shown that when $\mathbf{K}$ is central symmetric the approximating polytopes satisfying (15) and (16) can also be chosen to be central symmetric. Moreover, for central symmetric polytopes $\mathbf{D}$ by Theorem 2 the sharper estimate $\gamma_{2n}(\mathbf{D}) \leqslant f_{d-1}(\mathbf{D})$ holds. This bound leads to an improvement of the above estimates for $\gamma_{2n}(\mathbf{K})$. Indeed similarly to the proofs of Theorems 3 and 4 we can verify that in this case $\gamma_{2n}(\mathbf{K}) = O(n^{d(d-1)/2})$, and $\gamma_{2n}(\mathbf{K}) = O(n^{d-1})$ if, in addition, $\mathbf{K}$ is also C^2_+.

Proof of Theorem 5. Consider an arbitrary point $\mathbf{x}^*$ on the boundary of convex body $\mathbf{K}$. Let $p^*_{2n} \in B_{2n}(\mathbf{K})$ be a universal majorant in $B_{2n}(\mathbf{K})$. Then by Theorem 3

$$|p_{2n}(\mathbf{x})| \leqslant cn^{d(d-1)}\,|p^*_{2n}(\mathbf{x})|, \qquad p_{2n} \in B_{2n}(\mathbf{K}), \quad \mathbf{x} \in \mathbb{R}^d \backslash \mathbf{K}. \tag{21}$$

We claim that there exists a point $\tilde{\mathbf{x}} \in \mathbb{R}^d$ with $|\mathbf{x}^* - \tilde{\mathbf{x}}| = O((\frac{\log n}{n})^2)$ such that $|p^*_{2n}(\tilde{\mathbf{x}})| = 1$. In order to show this assume that $|p^*_{2n}| \leqslant 1$ in some ball $\mathbf{B}_\delta(\mathbf{x}^*)$ with center at x^* and radius $\delta > 0$. Our claim will follow if we verify that such a δ must satisfy $\delta \leqslant c(\log n/n)^2$ for some $c > 0$ independent of n. There exists $y^* \in \operatorname{Bd} \mathbf{K}$ such that $\mathbf{K}$ possesses parallel supporting hyperplanes at $\mathbf{x}^*$ and $\mathbf{y}^*$ with a normal $\mathbf{u}^* \in \mathbf{S}^{d-1}$. Let $\mathbf{L}$ be the line through $\mathbf{x}^*$ and $\mathbf{y}^*$. We may assume that $|\mathbf{x}^* - \mathbf{y}^*| = 2$. (Clearly, $|\mathbf{x}^* - \mathbf{y}^*| \geqslant \omega(\mathbf{K})$, where $\omega(\mathbf{K})$ is the minimal distance between parallel supporting hyperplanes to $\mathbf{K}$. Moreover $|\mathbf{x}^* - \mathbf{y}^*| \leqslant d(\mathbf{K}) := \max\{|\mathbf{x} - \mathbf{y}| : \mathbf{x},\ \mathbf{y} \in \mathbf{K}\}$.) Set now $\mathbf{x}_j := (1 + j\delta/2)\,\mathbf{x}^* - j\delta\,\mathbf{y}^*/2$, $j = 1, 2$. Evidently, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{L}\backslash\mathbf{K}$, $|\mathbf{x}_1 - \mathbf{x}^*| = \delta$, and $|\mathbf{x}_2 - \mathbf{x}^*| = 2\delta$.

Consider the polynomial

$$p_{2n}(\mathbf{x}) := T_{2n}\left(\frac{\left\langle \mathbf{x} - \dfrac{\mathbf{x}^* + \mathbf{y}^*}{2},\, \mathbf{u}^* \right\rangle}{\left\langle \dfrac{\mathbf{x}^* - \mathbf{y}^*}{2},\, \mathbf{u}^* \right\rangle}\right).$$

As in the proof of Theorem 2 it can be shown that $|p_{2n}| \leqslant 1$ on $\mathbf{K}$, i.e., $p_{2n} \in B_{2n}(\mathbf{K})$. Then by (21) for $\mathbf{x}_2 \in \mathbb{R}^d \backslash \mathbf{K}$

$$|p^*_{2n}(\mathbf{x}_2)| \geqslant \frac{|p_{2n}(\mathbf{x}_2)|}{cn^{d(d-1)}} = \frac{T_{2n}(1 + 2\delta)}{cn^{d(d-1)}}. \tag{22}$$

On the other hand since $|\mathbf{x}^* - \mathbf{x}_1| = \delta$ and $|p^*_{2n}| \leqslant 1$ on $\mathbf{B}_\delta(\mathbf{x}^*) \cup \mathbf{K}$, we obtain, in particular, that $|p^*_{2n}| \leqslant 1$ on $[\mathbf{y}^*, \mathbf{x}_1]$, where $|\mathbf{y}^* - \mathbf{x}_1| = (1 + \frac{\delta}{2})\,|\mathbf{x}^* - \mathbf{y}^*| = 2 + \delta$. Recall, that $\mathbf{y}^*$, $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbf{L}$ where $|(\mathbf{y}^* + \mathbf{x}_1)/2 - \mathbf{x}_2| = 1 + 3\delta/2$. Now, applying (2) to the univariate polynomial

$p^*_{2n}(t(\mathbf{x}_1-\mathbf{y}^*)+\mathbf{x}_1)\in B_n([\mathbf{y}^*,\mathbf{x}_1])$, $|\mathbf{x}_1-\mathbf{y}^*|=2+\delta$, at the point $\mathbf{x}_2$ with $|(\mathbf{x}_1+\mathbf{y}^*)/2-\mathbf{x}_2|=1+3\delta/2$ yields

$$|p^*_{2n}(\mathbf{x}_2)|\leqslant T_{2n}(1+\delta).$$

This together with (22) implies

$$cn^{d(d-1)}T_{2n}(1+\delta)\geqslant T_{2n}(1+2\delta). \tag{23}$$

Furthermore, it is well known (see [1, p. 30]) that

$$\tfrac{1}{2}(t+\sqrt{t^2-1})^{2n}\leqslant T_{2n}(t)\leqslant(t+\sqrt{t^2-1})^{2n},\qquad t>1.$$

This and (23) yield for $0<\delta\leqslant\delta_0$

$$cn^{d(d-1)}(1+\sqrt{3\delta})^{2n}\geqslant\tfrac{1}{2}(1+2\sqrt{\delta})^{2n}.$$

Hence

$$1+c_d\frac{\log n}{n}\geqslant(2cn^{d(d-1)})^{1/2n}\geqslant\frac{1+2\sqrt{\delta}}{1+\sqrt{3\delta}}\geqslant 1+c_0\sqrt{\delta},$$

i.e. we obtain that $\delta=O((\frac{\log n}{n})^2)$. Thus since $|p^*_{2n}(\mathbf{x}^*)|\leqslant 1$ there exists $\tilde{\mathbf{x}}$ such that $|\mathbf{x}^*-\tilde{\mathbf{x}}|=O((\log n/n)^2)$ and $|p^*_{2n}(\tilde{\mathbf{x}})|=1$. Consider now the polynomial $g_{4n}=(p^*_{2n})^2-1\in P^d_{4n}$. As we have shown above for every $\mathbf{x}^*\in\operatorname{Bd}\mathbf{K}$ there exists an $\tilde{\mathbf{x}}$ such that $g_{4n}(\tilde{\mathbf{x}})=0$ and $|\mathbf{x}^*-\tilde{\mathbf{x}}|\leqslant c(\log n/n)^2$. This concludes the proof. ∎

SOME OPEN PROBLEMS

The results proved above provide some insight on the magnitude of $\gamma_{2n}(\mathbf{K})$, but a number of questions remains open. Namely it would be interesting to determine for what convex bodies $\mathbf{K}$

$$\sup_{n\in\mathbb{N}}\gamma_{2n}(\mathbf{K})<\infty. \tag{24}$$

We have seen above that (24) holds for ellipsoids and polytopes. Using similar methods we can verify that (24) is true for finite intersections of central-symmetric polytopes and ellipsoids having the same center. This means that (24) holds not only for ellipsoids and polytopes. Is (24) true for every convex body $\mathbf{K} \subset \mathbb{R}^d$? Another open problem consists in characterizing those compact sets $\mathbf{K} \subset \mathbb{R}^d$ for which $\gamma_{2n}(\mathbf{K})$ has subexponential growth, i.e.,

$$\limsup_{n \to \infty} \gamma_{2n}(\mathbf{K})^{1/n} = 1. \tag{25}$$

Theorem 3 implies, in particular, that (25) holds for every convex body $\mathbf{K} \subset \mathbb{R}^d$.

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