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Polytopal Approximation Bounding the Number of k-Faces

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Assume that M is a convex body with C^2 boundary in $\mathbb{R}^d$. The paper considers polytopal approximation of M with respect to the most commonly used metrics, like the symmetric difference metric δ_S, the L_p metric, $1 \leqslant p \leqslant \infty$, or the Banach–Mazur metric. In case of δ_S, the main result states that if P_n is a polytope whose number of k faces is at most n then

$$\delta_S(M, P_n) > \frac{1}{67e^2\pi} \cdot \frac{1}{d} \cdot \left(\int_{\partial M} \kappa(x)^{1/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.$$

The analogous estimates are proved for all the other metrics. Finally, the optimality of these estimates is verified up to a constant depending on the metric and the dimension. © 2000 Academic Press

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1. INTRODUCTION

The history of approximation of convex bodies by polytopes with given number of vertices or facets dates back to the middle of the century, and even asymptotic formulae are known for the best approximation if the boundary has positive curvature (see the comprehensive surveys [8] and [9] of P. Gruber for detailed history of the problem).

In this paper, we consider the problem if the number of flags or the number of k–faces is given. We determine the value of the best approximation up to a multiplicative constant depending on the problem (and independent of the convex body). This problem was raised by I. Bárány and D. Larman in [1].

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There are various metrics used in the theory of polytopal approximation. It is convenient to phrase some of the definitions in terms of the support function $h_K(u) = \max_{x \in K} \langle x, u \rangle$ of a convex body K. Let M and P be convex bodies in $\mathbb{R}^d$.

Symmetric difference metric. $\delta_S(M, P)$ is the volume of the symmetric difference of M and P.

Banach–Mazur metric. Assume that M and P contains the origin o. Then $\delta_{BM}(M, P)$ is the minimum of $\ln \lambda$ such that there exists a linear transformation T satisfying $TP \subset M \subset \lambda \cdot TP$.

Hausdorff metric. $\delta_H(M, P) = \max_{u \in S^{d-1}} |h_M(u) - h_P(u)|$

Schneider's distance. If M contains P then $\delta_{SCH}(M, P)$ is the maximum volume of a cap cut off from M by a supporting hyperplane of P

L_p *metric for* $p \geqslant 1$. $\delta_p(M, P) = (\int_{S^{d-1}} |h_M(u) - h_P(u)|^p \, du)^{1/p}$

In case of the Banach–Mazur metric, one usually considers o-symmetric bodies. What is essential, that the origin is contained in the interior of both bodies.

Observe that $\delta_H(M, P)$ (which is actually the L_∞ metric) is the maximum of the distances of the points of M from P and the distances of the points of P from M.

Assume that M is smooth and $P \subset M$ is a polytope and ∂P is close to ∂M. It can be proved that the supporting hyperplane of P which cuts off the cap with maximum volume from M is the affine hull of a facet of P.

If $P \subset M$ then $\delta_1(M, P)$ is actually proportional with the deviation of the mean width. The L_p metric, $p > 1$, has no obvious geometric meaning, but it is a useful tool. For example, various applications for the stability results related to the isoperimetric inequality are presented in [7].

We note that there exists a general version of the symmetric difference metric: If $w(x)$ is a positive continuous function a neighborhood of ∂M in $\mathbb{R}^d$, then $\delta_w(M, P)$ is the integral of $w(x)$ on the symmetric difference of M and P.

Assume that the boundary ∂M of the convex body M is C^2; namely, the second fundamental form Q_x exists at each $x \in \partial M$. In particular, the Gauß curvature $\kappa(x) = \det Q_x$ is a continuous function (see the book [14] of R. Schneider, or Section 2 for some basic facts).

Next we define the integral expressions appearing in the formulae below. The integration is with respect to the induced measure on ∂M, and the exterior unit normal at a point $x \in \partial M$ is denoted by $v(x)$.

$$\mathscr{A}_S(\partial M) = \mathscr{A}_{SCH}(\partial M) = \left(\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)}$$

$$\mathscr{A}_{BM}(\partial M) = \left(\int_{\partial M} \frac{\kappa(x)^{1/2}}{h(\nu(x))^{(d-1)/2}}\, dx \right)^{2/(d-1)}$$

$$\mathscr{A}_H(\partial M) = \left(\int_{\partial M} \kappa(x)^{1/2}\, dx \right)^{2/(d-1)}$$

$$\mathscr{A}_p(\partial M) = \left(\int_{\partial M} \kappa(x)^{(d-1+p)/(d-1+2p)}\, dx \right)^{(d-1+2p)/(p(d-1))}$$

Note that the integral expression in $\mathscr{A}_S(\partial M)$ is is the so called affine surface area, and it is invariant under volume preserving affine transformations (see W. Blaschke [3], E. Lutwak [11] or K. Leichtweiß [10]). The integral in $\mathscr{A}_{BM}$, the so called centro affine surface area, is invariant under linear transformations (see K. Leichtweiß [10]).

In case of δ_w, the corresponding set function is

$$\mathscr{A}_w(\partial M) = \left(\int_{\partial M} w(x)^{(d+1)/(d-1)} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)}.$$

THEOREM A. *Let δ be one of the metrics above in $\mathbb{R}^d$, $d \geqslant 4$, together with the corresponding set function $\mathscr{A}$, and set $0 \leqslant k \leqslant d-1$. Assume that M is a convex body with C^2 boundary. For large n, let P_n be the polytope whose number of k faces is at most n and $\delta(M, P_n)$ is minimal under this condition.*

(i) *If $\delta = \delta_S$ or $\delta = \delta_p$ then*

$$\delta(M, P_n) > \frac{1}{67e^2\pi} \cdot \frac{1}{\mathrm{d}} \cdot \mathscr{A}(\partial M) \cdot \frac{1}{n^{2/(d-1)}}.$$

(ii) *If $\delta = \delta_H$, or that $\delta = \delta_{BM}$ and $o \in \operatorname{int} M$ then*

$$\delta(M, P_n) > \frac{1}{34e\pi} \cdot d \cdot \mathscr{A}(\partial M) \cdot \frac{1}{n^{2/(d-1)}}.$$

(iii) *If $P_n \subset M$ and $\delta = \delta_{SCH}$ then there exists c_0 depending on d such that*

$$\delta_{SCH}(M, P_n) > c_0 \cdot \mathscr{A}_{SCH}(\partial M) \cdot \frac{1}{n^{(d+1)/(d-1)}}.$$

We compare the value of the constant above to the constants appearing in the known formulae: If $k \in \{0, d-1\}$ and the metric is δ_S, δ_1, δ_H or δ_{BM} then

$$\delta(M, P_n) < \gamma \cdot d \cdot \mathscr{A}(\partial M) \cdot \frac{1}{n^{2/(d-1)}}$$

where γ is an absolute constant (see [9] if the Gauß curvature is positive, and [4] for general smooth convex bodies). The lower bound for δ_S and for δ_p are new even if $k \in \{0, d-1\}$.

In case of $d = 3$, Theorem A follows from the known asymptotic results and by Euler's formula. Theorem A was independently proved by I. Bárány (unpublished) if $P \subset M$ and $\delta = \delta_S$ using the theory of cap covering. His proof gives no estimates on the constants.

Note that the assumption in Theorem A that ∂M is C^2 can not be relaxed if one wants to have a lower bound of order $1/n^{2/(d-1)}$ for $\delta(M, P_n)$ (see [9]).

A flag of a polytope P is a sequence $F^0 \subset F^1 \subset \cdots \subset F^{d-1}$ where F^k is a k-face of P. Observe that the number of flags is larger than the number of k-faces for any k.

Next we present a partial converse to Theorem A. Call ∂M to be C^2_+ if $\kappa(x) > 0$ for $x \in \partial M$.

THEOREM B. *Let M be a convex body with C^2_+ boundary in $\mathbb{R}^d$, $d \geqslant 4$, and let δ be one of the metrics above but δ_{SCH} together with the corresponding set function $\mathscr{A}$. If $\delta = \delta_{BM}$ then assume that $o \in \operatorname{int} M$. Then there exists a c depending on the problem and a polytope Q_n with at most n flags such that*

$$\delta(M, Q_n) < c \cdot \mathscr{A}(\partial M) \cdot \frac{1}{n^{2/(d-1)}}.$$

The same formula holds assuming that Q_n is inscribed or circumscribed.

If $\delta = \delta_{SCH}$ then assume that $Q_n \subset M$, and replace $n^{-2/(d-1)}$ by $n^{-(d+1)/(d-1)}$.

Why not to have an asymptotic formula? The reason is that it is very hard to control the number of k–faces, $1 \leqslant k \leqslant d-2$. On the other hand, it will be done in a subsequent paper if $d = 3$ and $k = 1$.

Assume that M is a general convex body (with no assumption on the boundary). P. M. Gruber conjectures (personal communication) that for any metric δ there exists a constant $\gamma(M)$ depending on M such that there exists a polytope Q_n with at most n flags satisfying

$$\delta(Q_n, M) < \frac{\gamma(M)}{n^{2/(d-1)}}$$

(and replace $n^{-2/(d-1)}$ by $n^{-(d+1)/(d-1)}$ if $\delta = \delta_{SCH}$).

In case of the symmetric difference metric, I conjecture a more precise statement: Let $0 < t < 1/\mathrm{e} \cdot V(M)$. The so called floating body $M_t \subset M$ is defined so that the closure of $M \backslash M_t$ is the union of the caps cut off from M with volume t. If ∂M is C^2 then

$$V(M \backslash M_t) \sim \frac{1}{2} \left(\frac{d+1}{\kappa_{d-1}} \right)^{2/(d+1)} \int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \cdot t^{2/(d+1)}$$

where κ_{d-1} is the content of the unit $(d-1)$-ball (see [3] or [10]).

Conjecture. If M is a convex body in $\mathbb{R}^d$, $d \geqslant 4$, then there exists a polytope Q for small t such that $M_t \subset Q \subset M$ and the number of flags of Q is at most

$$c_1 \cdot V(M \backslash M_t) \cdot \frac{1}{t}$$

where c_1 depends only on d.

C. Schütt [15], and independently I. Bárány (personal communication) proved that a suitable Q exists if not the number of flags but the number of vertices or facets is bounded. These results verify the conjecture if $d = 2, 3$.

In general, we present the proof for the symmetric difference metric in full detail, and sketch the necessary changes for the other metrics afterwards. We start with the proof of Theorem B in Sections 2 and 3 because it is much simpler. After that we verify Theorem A if the curvature is positive everywhere in Sections 4–8, and then we complete the proof of Theorem A in Section 9 by separating the "flat part".

2. THEOREM B FOR THE SYMMETRIC DIFFERENCE METRIC

We write $f \ll g$ or $f = O(g)$ if there exists a constant $c > 0$ depending on the dimension d such that $|f| < c \cdot g$. If $f \ll g$ and $f \gg g$ then write $f \approx g$.

Let the convex body M have a C^2_+ boundary, and let $x \in \partial M$. Identify the tangent hyperplane at x with $\mathbb{R}^{d-1}$. Then an open neighborhood U of

x in ∂M is the graph of a convex C^2 function f defined in the projection V of U into $\mathbb{R}^{d-1}$.

Denote by l_z the derivative of f at z and by q_z the quadratic form representing the second derivative of f. Fix some $y \in V$. We deduce using the Taylor expansion of f that

$$f(z) = f(y) + l_y(z-y) + \tfrac{1}{2} q_w(z-y)$$

where $w = y + t(z-y)$ for some $0 < t < 1$. Now q_z is positive definite, and a continuous function of z. Note that the second fundamental form at x is $Q_x = q_x$.

2.1. *Inscribed Polytope*

If $x \in \partial M$ and $s > 0$ then define $C(x, s)$ as $H^+(x, s) \cap \partial M$ where $H^+(x, s)$ is the open half space containing x whose bounding hyperplane is parallel to the tangent at x and the distance of the bounding hyperplane and x is s.

Let $V > 0$ be small (we specify the meaning of "small" in (3) below). For any $x \in \partial M$, set

$$s(x) = \kappa(x)^{1/(d+1)} \cdot V^{2/(d+1)}. \tag{1}$$

Let $m = m(V, M)$ be maximal with the property that there exists a family of pairwise disjoint sets $C(x_1, s(x_1))$, ..., $C(x_m, s(x_m))$ on ∂M. Denoting $s(x_i)$ by s_i and the complement of $H_i^+(x_i, 5s_i)$ by H_i^-, the approximating polytope is defined as $P = \bigcap_{i=1}^m H_i^-$.

Now consider a $y \in \partial M$. Identify the tangent hyperplane H at y with $\mathbb{R}^{d-1}$, and let $\mathscr{E}_y(t)$ be the $(d-1)$-ellipsoid in H defined by $\frac{1}{2} Q_y(u) \leqslant t$. Since the second fundamental form Q_x is a continuous function of x, and ∂M is compact, there exist a positive t_0 depending only on M, and a neighborhood U_y of y on ∂M with the following property: If $t < t_0$ and $x \in U_y$ then

$$v + 0.99 \cdot \mathscr{E}_y(t) \subset \pi_H(C(x, t)) \subset v + 1.01 \cdot \mathscr{E}_y(t) \tag{2}$$

for some v where $\pi_H(\cdot)$ is the projection into H. In particular, the condition on V is that

$$t_0 > 50 \cdot \max_{x \in \partial M} \kappa(x)^{1/(d+1)} \cdot V^{2/(d+1)}. \tag{3}$$

Now the maximality of the system $\{C(x_i, s_i)\}$ yields that for any $x \in \partial M$ there exists an x_i such that $C(x_i, s_i)$ intersects $C(x, s(x))$. We deduce by (3) that

$$x \in C(x_i, 5s_i) \text{ and } x_i \in C(x, 5s(x)), \tag{4}$$

and hence $P \subset M$.

Set $F_i = H_i^- \cap \partial P$, and observe that the facets of P form a subset of $\{F_i\}$. We deduce by (2) and (4) that if F_i and F_j intersect then $x_j \in C(x_i, 21s_i)$. Therefore there exists a constant c_0 depending only on the dimension such that the number of flags containing F_i is at most c_0, which in turn yields that the total number of flags of P is at most $c_0 \cdot m$.

Let n be large. Choose V to be minimal such that $n > c_0 \cdot m$ for the corresponding m. Since

$$|C(x_i, s_i)| \approx |\mathscr{E}_{x_i}(s_i)| \approx \frac{s_i^{(d-1)/2}}{\kappa(x_i)^{1/2}}, \tag{5}$$

we deduce the estimate

$$\begin{aligned}\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx &> \sum_{i=1}^{m} \int_{C(x_i, s_i)} \kappa(x)^{1/(d+1)}\, dx \\ &\gg \sum_{i=1}^{m} \kappa(x_i)^{1/(d+1)} \cdot \frac{s_i^{(d-1)/2}}{\kappa(x_i)^{1/2}} = m \cdot V^{(d-1)/(d+1)}. \end{aligned}\tag{6}$$

Since (5) yields that

$$V(\mathrm{conv}\; C(x_i, 5s_i)) \ll \frac{s_i^{(d+1)/2}}{\kappa(x_i)^{1/2}} = V,$$

we conclude by $n \approx m$ that

$$\begin{aligned}\delta_S(M, P) &\ll m \cdot V \ll m \cdot \left(\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \cdot \frac{1}{m} \right)^{(d+1)/(d-1)} \\ &\ll \left(\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.\end{aligned}$$

2.2. *Circumscribed Polytope*

The proof is very similar to the inscribed case, only the necessary changes are sketched.

For small V, the sets $\{C(x_i, s_i)\}$, $i = 1, \ldots, m$, are constructed exactly the same way. Now the approximating polytope Q is the intersection of the supporting half spaces at $x_1, \ldots, x_m$.

The basic change is to replace (2) with the corresponding statement about cones: There exist a positive t_0 depending only on M, and a neighborhood U_y of y on ∂M with the following property: If $t < t_0$, the distance of z from M is t, and the closest point of M to z lies in U_y then the tangent cone $\sigma(z)$ to M with apex z satisfies

$$v + 0.99 \cdot \mathscr{E}_y(t) \subset \pi_H(\sigma(z) \cap \partial M) \subset v + 1.01 \cdot \mathscr{E}_y(t)$$

for some v.

Denote by F_i the facet of Q touching at x_i. Then the projection of F_i into ∂M along the normals to ∂M is contained in $C(x_i, 5s_i)$. Therefore the volume between F_i and its projection is at most $5s_i\,|C(x_i, 5s_i)| \ll V$. On the other hand, if $F_i \cap F_j \neq \emptyset$ then $x_j \in C(x_i, 21s_i)$, and hence Q has at most $c_0 \cdot m$ flags. Now the proof can be finished as above.

2.3. *The Inscribed Case Revisited*

In case of the symmetric difference metric, there exists a rather natural way to construct well approximating inscribed polytopes; namely, using lattice polytopes.

Based on arguments of I. Bárány and D. Larman in [1], the paper [2] proves the existence of c_1 depending on d with the following property: For large n, define r by

$$n = c_1 \cdot \int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \cdot r^{d(d-1)/(d+1)}.$$

Then the number of flags of the polytope

$$P_r = \operatorname{conv}\left(\frac{1}{r}\,\mathbb{Z}^d \cap M\right)$$

is at most n. On the other hand, [2] also proves that

$$\begin{aligned}\delta_S(M, P_r) &\ll \int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \cdot \frac{1}{r^{2d/(d+1)}}\\ &\ll \left(\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx\right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.\end{aligned}$$

Unfortunately, this method does not work for the other metrics because the Hausdorff distance of M and P_r can be as large as $1/r$.

I would like to point out that the proof of the results quoted above are independent of this paper. On the other hand, [2] uses the lower bound of Corollary 1 in Section 7 in order to verify that P_r has many k-faces.

3. THEOREM B FOR THE OTHER METRICS

The arguments are quite analogous to the case of the symmetric difference metric. Therefore here we consider only the case of inscribed polytopes.

In all cases, the number of flags is at most $c_0 \cdot m$ with the constant c_0 above. We provide only the corresponding definition of $s(x)$, and sometimes the analogue of (6) or the final estimate.

In the case of Schneider's notion of distance, the same argument applies word by word. Since $\delta_{SCH}(M, P) \ll V$, (6) gives the right estimate directly.

For δ_w, set

$$s(x) = \frac{\kappa(x)^{1/(d+1)}}{w(x)^{2/(d+1)}} \cdot V^{2/(d+1)}.$$

Then the weighted volume of a cap is

$$\int_{\operatorname{conv} C(x_i, 5s_i)} w(x)\, dx \ll \frac{s_i^{(d+1)/2}}{\kappa(x_i)^{1/2}} \cdot w(x_i) = V,$$

and the analogue of (6) is

$$\int_{\partial M} w(x)^{(d-1)/(d+1)} \kappa(x)^{1/(d+1)} dx$$

$$\gg \sum_{i=1}^{m} w(x)^{(d-1)/(d+1)} \cdot \kappa(x_i)^{1/(d+1)} \cdot \frac{s_i^{(d-1)/2}}{\kappa(x_i)^{1/2}} = m \cdot V^{(d-1)/(d+1)}.$$

Next consider the Hausdorff distance. This is actually the simplest case because $s(x)$ can be chosen a constant s for small s, and

$$\int_{\partial M} \kappa(x)^{1/2}\, dx \gg \sum_{i=1}^{m} \kappa(x_i)^{1/2} \cdot \frac{s^{(d-1)/2}}{\kappa(x_i)^{1/2}} = m \cdot s^{(d-1)/2}.$$

Since $\delta_H(M, P) \ll s$, this completes the proof.

Very similar is the case of the Banach–Mazur distance. Now let $b > 0$ be small, and set

$$s(x) = h_M(v(x)) \cdot b.$$

Then we have

$$\int_{\partial M} \frac{\kappa(x)^{1/2}\, dx}{h_M(v(x))^{(d-1)/2}} \gg \sum_{i=1}^{m} \frac{\kappa(x_i)^{1/2}\, dx}{h_M(v(x_i))^{(d-1)/2}} \cdot \frac{s^{(d-1)/2}}{\kappa(x_i)^{1/2}} = m \cdot b^{(d-1)/2}.$$

On the other hand,

$$\delta_{BM}(M, P) \ll \max_i \ln \frac{h_M(\nu(x_i))}{h_P(\nu(x_i))} \ll b.$$

Finally, consider the L_p metric, $p \geqslant 1$. Note that

$$\delta_p(M, P) = \left(\int_{\partial M} |h_M(\nu(x)) - h_P(\nu(x))| \, \kappa(x) \, dx \right)^{1/p}.$$

So let $\lambda > 0$ be small, and set

$$s(x) = \frac{\lambda^{2/(d-1)}}{\kappa(x)^{1/(d-1+2p)}}.$$

The value of λ is chosen in a way that the analogous argument as for (6) gives

$$\int_{\partial M} \kappa(x)^{(d-1+p)/(d-1+2p)} dx \gg m \cdot \lambda.$$

Therefore we conclude that

$$\begin{aligned} \delta_p(M, P) &\ll \left(\sum_{i=1}^{m} \int_{\mathrm{conv}\, C(x_i, 5s_i)} s_i^p \kappa(x) \, dx \right)^{1/p} \\ &\ll \left(\sum_{i=1}^{m} s_i^p \kappa(x_i) \cdot \frac{s_i^{(d-1)/2}}{\kappa(x_i)^{1/2}} \right)^{1/p} = (m \cdot \lambda^{d-1+p/d-1})^{1/p} \\ &\ll \left(\int_{\partial M} \kappa(x)^{(d-1+p)/(d-1+2p)} \, dx \right)^{(d-1+p)/p(d-1)} \cdot \frac{1}{m^{2/(d-1)}}. \end{aligned}$$

With this last inequality, Theorem B has been established.

4. SOME AUXILIARY LEMMAS

This section collects some technical statements which are needed in the proof of Theorem A. I advise the reader to skip this section just now, and to return when a statement is needed.

A set S is called star shaped with respect to x if conv $\{x, y\}$ is a subset of S for any $y \in S$. Star shaped sets will occur as the union subsets of the tiles of a power diagram containing a fixed face F, which set $St(F)$ will be star shaped with respect to the centroid v of F. Observe that for any rays starting from v, the intersection of the ray and $St(F)$ is contained in one of the tiles.

PROPOSITION 4.1. *Let S be a Jordan measurable star shaped set in $\mathbb{R}^{d-1}$ with respect to the origin o with non-empty interior. Consider the measurable functions $a(z) \in \mathbb{R}^{d-1}$ and $r(z) \in \mathbb{R}$ which are constant along the open rays starting from o. Then for any positive definite quadratic form q, we have*

$$\max_{z \in S} |q(z - a(z)) - r(z)| \geqslant \frac{1}{16e\pi} \cdot d \cdot (\det q)^{1/(d-1)} \cdot |S|^{2/(d-1)} \tag{7}$$

$$\int_S |q(z - a(z)) - r(z)|\, dz > \frac{1}{32e^2\pi} \cdot \frac{1}{d} \cdot (\det q)^{1/(d-1)} \cdot |S|^{(d+1)/(d-1)}. \tag{8}$$

Proof. We may assume that $q(z) = z^2$. Note that

$$\min_{a, r \in \mathbb{R}} \max_{t \in [0, 1]} |(t-a)^2 - r| = \tfrac{1}{8}$$

where the optimal values are $a = 1/2$ and $r = 1/16$.

The radial function of S is $R(u) = \sup_{tu \in S} t$ for $u \in S^{d-2}$. Then

$$\max_{z \in S} |(z - a(z))^2 - r(z)| \geqslant \tfrac{1}{8} \max_{u \in S^{d-2}} R(u)^2,$$

while using polar coordinates yields that

$$|S| = \frac{1}{d-1} \cdot \int_{S^{d-2}} R(u)^{d-1}\, du.$$

In particular, we may assume that S is the unit $(d-1)$-ball B^{d-1}. Now Stirling's formula

$$|B^{d-1}| = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2} + 1\right)} < \left(\frac{2e\pi}{d-1}\right)^{(d-1)/2} \cdot \frac{1}{\sqrt{\pi(d-1)}}$$

yields (7) by simple computations.

Turning to (8), we start with

$$\min_{a, r \in \mathbb{R}} \int_0^1 |(t-a)^2 - r|\, dt = \tfrac{1}{16}$$

where the optimal values of a and r are again $a = \frac{1}{2}$ and $r = \frac{1}{16}$. We deduce that

$$\min_{a, r \in \mathbb{R}} \int_0^1 |(t-a)^2 - r|\, t^{d-2}\, dt > \frac{1}{\mathrm{e}} \cdot \min_{a, r \in \mathbb{R}} \int_{1-1d}^1 |(t-a)^2 - r|\, dt = \frac{1}{16e} \cdot \frac{1}{d^3} \cdot$$

Therefore using polar coordinates yield that

$$\int_S |(z-a(z))^2 - r(z)|\, dz \geqslant \frac{1}{16e} \cdot \frac{1}{d^3} \cdot \int_{S^{d-2}} R(u)^{d+1}\, du.$$

Now we may assume by Hölder's inequality that S is the unit $(d-1)$-ball, and (8) is the consequence of Stirling's formula. ∎

The proof of Theorem A uses functions which are very close to be a quadratic form.

For any $a \in \mathbb{R}^{d-1}$, let g_a be a continuous non-negative function. Consider $a_1, ..., a_m \in \mathbb{R}^{d-1}$, $r_1, ..., r_m \in \mathbb{R}$ and Jordan measurable sets $\Omega_1, ..., \Omega_m$ such that $\Omega_1, ..., \Omega_m$ cover the sets $g_{a_i}(z-a_i) \leqslant r_i$ and $g_{a_i}(z-a_i) - r_i = \min_j g_{a_j}(z-a_j) - r_j$ for $z \in \Omega_i$.

Proposition 4.2. *Assume that q is a positive definite quadratic form, the sets $\Omega_1, ..., \Omega_m$ are as above and $q(z-a_i) \leqslant g_{a_i}(z-a_i) \leqslant 2q(z-a_i)$ for every a_i. Then*

(i) $\max_i \max_{z \in \Omega_i} g_{a_i}(z-a_i) \ll \max_i \max_{z \in \Omega_i} |g_{a_i}(z-a_i) - r_i|$.

(ii) $\sum_i \int_{\Omega_i} g_{a_i}(z-a_i)\, dz \ll \sum_i \int_{\Omega_i} |g_{a_i}(z-a_i) - r_i|\, dz$.

Proof. We prove (ii) since (i) readily holds.

Denote by σ_0 the part of $\sigma = \bigcup \Omega_i$ which is contained in the union of the sets $g_{a_i}(z-a_i) < 2r_i$, and set $\sigma_1 = \sigma \backslash \sigma_0$. Readily,

$$\int_{\sigma_1} g_{a_i}(z-a_i)\, dz \leqslant 2 \cdot \int_{\sigma_1} |g_{a_i}(z-a_i) - r_i|\, dz.$$

Now number $r_1, ..., r_m$, so that r_1 is maximal, and $g_{a_i}(z-a_i) \leqslant r_i$, $i = 1, ..., l$, is a maximal disjoint family with the property that if the set $g_{a_j}(z-a_j) \leqslant r_j$ intersects the set $g_{a_i}(z-a_i) \leqslant r_i$ for $j > i$ then $r_j \leqslant r_i$. We deduce that

$$\begin{aligned}\int_{\sigma_0} g_{a_i}(z-a_i)\, dz &\ll \sum_{i=0}^{l} \int_{q(z-a_i) < r_i} q(z-a_i)\, dz \\ &\ll \sum_{i=0}^{l} \int_{g_{ai}(z-a_i) < r_i} |g_{a_i}(z-a_i) - r_i|\, dz\end{aligned}$$

where the last expression is readily at most $\int_{\sigma_0} |g_{a_i}(z-a_i) - r_i|\, dz$. ∎

We close the section with the following simple inequalities:

PROPOSITION 4.3. *Assume that μ_i, n_i, $i=1\dots,k$, are positive numbers. Then*

(i) $\max_i\ (\mu_i/n_i) \geqslant \sum_i \mu_i / \sum_i n_i$

(ii) *for any* $p \geqslant 1$,

$$\left(\sum_i \mu_i^{(d-1+2p)/(d-1)} \cdot \frac{1}{n_i^{2p/(d-1)}}\right)^{1/p} \geqslant \left(\sum_i \mu_i\right)^{(d-1+2p)/p(d-1)} \cdot \frac{1}{(\sum_i n_i)^{2/(d-1)}}$$

Proof. The first inequality readily holds. Turning to the second one, Hölder's inequality yields that

$$\left(\sum_i n_i\right)^{2p/(d-1+2p)} \cdot \left(\sum_i \mu_i^{(d-1+2p)/(d-1)} \cdot \frac{1}{n_i^{2p/(d-1)}}\right)^{(d-1)/(d-1+2p)} \geqslant \sum_i \mu_i,$$

and hence (ii) also holds. ∎

5. THEOREM A IF $\delta=\delta_S$ AND $\kappa(x)>0$

Let the convex body M have a C^2_+ boundary.

The following lemma, which allows us to consider parts of ∂M which are essentially paraboloid, can be proved *via* the standard methods.

LEMMA 4. *Assume that ψ (ϱ) are positive continuous functions in a neighborhood of ∂M in $\mathbb{R}^d$ (on ∂M). Let $\varepsilon>0$.*

Then there exist finitely many pairs of open Jordan measurable subsets $\Sigma_\beta \subset \tilde{\Sigma}_\beta$ of ∂M and hyperplanes H_β, quadratic forms q_β and constants ψ_β with the following properties:

(i) *The closures of $\tilde{\Sigma}_\beta$ are disjoint subsets of ∂M and*

$$\sum_\beta \int_{\Sigma_\beta} \varrho(x)\,dx > (1-\varepsilon) \cdot \int_X \varrho(x)\,dx;$$

(ii) *$\tilde{\Sigma}_\beta(\Sigma_\beta)$ is the graph of a C^2 function f_β on some $\tilde{\Phi}_\beta \subset H_\beta$ ($\Phi_\beta \subset H_\beta$), and* $\operatorname{cl} \Phi_\beta \subset \tilde{\Phi}_\beta$;

(iii) *if l_z is the derivative of f_β at z then $\|l_z\| < \varepsilon$;*

(iv) *For the quadratic form q_z representing the second derivative of f_β at z, we have $q_\beta \leqslant \frac{1}{2} q_z \leqslant (1+\varepsilon)\cdot q_\beta$;*

(v) *There exists a neighborhood of $\tilde{\Sigma}_\beta$ in $\mathbb{R}^d$ such that each point z in this neighborhood satisfies $\psi_\beta < \psi(z) < (1+\varepsilon)\cdot\psi_\beta$.*

For some small $\varepsilon > 0$, consider the parts $\{\Sigma_\beta, \tilde{\Sigma}_\beta\}$ of ∂M provided by Lemma 13 if $\varrho(x) = \kappa(x)^{1/(d+1)}$.

Since the number of flags is at least say the number of vertices, we consider only the case if the number of k-faces of P_n is at most n, $0 \leqslant k \leqslant d-1$.

Let P_n be the polytope with at most n k–faces minimizing $\delta_S(M, P_n)$. Since ∂M is C^2_+, the maximal diameter of a facet of P_n tends to zero as n tends to infinity. Assume that the projection of a facet G into H_β intersects Φ_β and G is on the same side of ∂P_n where H_β lies. Then the projection lies completely in $\tilde{\Phi}_\beta$. In addition, if the facet G' has the same property with respect to $H_{\beta'}$ for some $\beta' \neq \beta$ then $G \cap G' = \emptyset$.

Fix some β. We want to work in H_β so denote by n_β the number of k-faces of P_n which are on the side of H_β and whose orthogonal projection into H_β meets $\tilde{\Phi}_\beta$. Observe that $\sum_\beta n_\beta < n$.

Identify H_β with $\mathbb{R}^{d-1}$. To any facet G_i of P_n whose orthogonal projection Π_i onto H_β intersects $\tilde{\Phi}_\beta$, define $a_i \in \mathbb{R}^{d-1}$ by the fact that $\operatorname{aff} G_i$ is parallel to the tangent hyperplane at the point $(a_i, f_\beta(a_i)) \in \partial M$. In particular, there exists some $r_i \in \mathbb{R}$ such that $\operatorname{aff} G_i$ is the graph of $f_\beta(a_i) + l_{a_i}(z - a_i) + r_i$. Now the part of ∂P_n above $\tilde{\Phi}_\beta$ is the graph of φ_β where

$$\varphi_\beta(z) = f_\beta(a_i) + l_{a_i}(z - a_i) + r_i$$

if $z \in \Pi_i$.

Define the functions g_{a_i} by

$$f_\beta(z) = f_\beta(a_i) + l_{a_i}(z - a_i) + g_{a_i}(z - a_i).$$

We deduce by Taylor's formula that $g_{a_i}(z - a_i) = \frac{1}{2} q_w(z - a_i)$ for some w between a and z. Assume that $z \in \Pi_i$. Then

$$f_\beta(z) - \varphi_\beta(z) = g_{a_i}(z - a_i) - r_i \qquad \text{and} \qquad q_\beta \leqslant g_{a_i} \leqslant (1 + \varepsilon)\, q_\beta,$$

and in addition,

$$g_{a_i}(z - a_i) - r_i = \min_j g_{a_j}(z - a_j) - r_j.$$

Finally, set

$$\omega(z) = |g_{a_i}(z - a_i) - r_i|.$$

Using these notions, the volume between ∂M and ∂P_n and above $\tilde{\Phi}_\beta$ is $\int_{\tilde{\Phi}_\beta} \omega(z)\, dz$.

Number the facets so that $\Pi_1, ..., \Pi_t$ are the ones which intersect Φ_β. Denote by T^+ the union of the sets Π_i and the sets $g_{a_i}(z - a_i) \leqslant r_i$,

$i=1, ..., t$, and let Ω_i be the set of $z \in T^+$ satisfying $g_{a_i}(z-a_i)-r_i \leqslant g_{a_j}(z-a_j)-r_j$, $j=1, ..., t$. We may assume that $\Omega_i \subset \tilde{\Phi}_\beta$.

For any face F of some Π_i, denote by $s(F)$ its center of mass. To each sequence $F^k \subset F^{k+1} \subset \cdots \subset F^{d-1}$ where F^j is a j-face (and hence F^{d-1} is some Π_i), assign the polytope $\sigma = \operatorname{conv}\{F, s(F^{k+1}), ..., s(F^{d-1})\}$. If the k-face F intersect Φ_β then denote by $St(F)$ the union of all σ which contain F, where $St(F)$ is star shaped with respect to $s(F)$. The family of these star shaped sets cover Φ_β, and the interiors of any two of them are disjoint.

If a k-face F is contained in a Π_i intersecting Φ_β, the estimate

$$\sum_i \int_{\Pi_i \cap F} |q_\beta(z-a_i)-r_i| \, dz \geqslant \frac{1}{32e^2\pi d} \cdot (\det q_\beta)^{1/(d-1)} \cdot |St(F)|^{(d+1)/(d-1)} \tag{9}$$

holds by (8). For large n, we have at most n_β k-face of this kind, and their stars naturally covers $|\Phi_\beta|$. We deduce by Jensen's inequality that

$$\sum_{i=1}^{t} \int_{\Omega_i} |q_\beta(z-a_i)-r_i| \, dz$$
$$\geqslant \frac{1}{32e^2\pi d} \cdot (\det q_\beta)^{1/(d-1)} \cdot |\Phi_\beta|^{(d+1)/(d-1)} \cdot \frac{1}{n_\beta{}^{2/(d-1)}}.$$

Observe that for $z \in \Omega_i$, we have

$$|q_\beta(z-a_i)-r_i| \leqslant |g_{a_i}(z-a_i)-r_i| + \varepsilon \cdot g_{a_i}(z-a_i).$$

Therefore Proposition 4.2(ii) yields that

$$\sum_{i=1}^{t} \int_{\Omega_i} |g_{a_i}(z-a_i)-r_i| \, dz$$
$$\geqslant (1-O(\varepsilon)) \cdot \frac{1}{32e^2\pi d} \cdot (\det q_\beta)^{1/(d-1)} \cdot |\Phi_\beta|^{(d+1)/(d-1)} \cdot \frac{1}{n_\beta{}^{2/(d-1)}}.$$

Since $\omega(z) \geqslant |g_{a_i}(z-a_i)-r_i|$ holds for $z \in \Omega_i$ by the definition of Ω_i, we conclude the estimate

$$\int_{\tilde{\Phi}_\beta} \omega(z) \, dz \geqslant (1-O(\varepsilon)) \cdot \frac{1}{32e^2\pi d} \cdot (\det q_\beta)^{1/(d-1)} \cdot |\Phi_\beta|^{(d+1)/(d-1)} \cdot \frac{1}{n_\beta{}^{2/(d-1)}}.$$

It is time to transfer the estimates onto the boundary of ∂M. The Gauß curvature at $x=(z, f_\beta(z))$ is

$$\kappa(x) = \frac{\det q_z}{(1+\|l_z\|^2)^{(d+1)/2}} = (1+O(\varepsilon)) \cdot 2^{d-1} \det q_\beta.$$

Therefore summing up the estimates for various β yields that

$$\delta_S(M, P_n) \geqslant (1 - O(\varepsilon)) \cdot \frac{1}{64e^2\pi d} \sum_{\beta} \left(\int_{\Sigma_\beta} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n_\beta^{2/(d-1)}}.$$

Now apply Proposition 4.3(ii) (with $p = 1$) and Lemma 5.1 in order to obtain the global estimate; namely, the inequality

$$\delta_S(M, P_n) \geqslant (1 - O(\varepsilon)) \cdot \frac{1}{64e^2\pi d} \cdot \left(\int_{\partial M} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.$$

Therefore choosing ε small enough initially, we conclude Theorem A in this case if $n > n(\varepsilon)$.

6. THEOREM A FOR MOST METRICS IF $\kappa(x) > 0$

First observe that the case of δ_w can be dealt with exactly the same way as δ_S, only one assumes that $w_\beta < w(x) < (1 + \varepsilon)\, w_\beta$ if x is in a small neighborhood of $\tilde{\Sigma}_\beta$ in $\mathbb{R}^d$.

6.1. *The Hausdorff Related Metrics*

We start with the Hausdorff metric. Let $\varepsilon > 0$, and apply Lemma 5.1 with $\varrho(x) = \kappa(x)^{1/2}$. Denote by Δ_β the maximum of the distances of a point of $\tilde{\Sigma}_\beta$ from P_n and a point of ∂P_n near $\tilde{\Sigma}_\beta$ from M. Since the derivative $\|l_z\| < \varepsilon$ at each $z \in \tilde{\Phi}_\beta$, we deduce that

$$\Delta_\beta = (1 + O(\varepsilon)) \cdot \max_{z \in \tilde{\Sigma}_\beta} \omega(z).$$

On the other hand, $\delta_H(M, P_n)$ is at least the maximum of these values. Now apply the argument above, using maximum instead of integration or summation. In particular, use (7) instead of (8), and (i) of Proposition 4.2 and Proposition 4.3 instead of (ii), and deduce Theorem A in this case.

Let us consider the Banach–Mazur distance. First we rewrite it in a form which is more suitable in our context.

Let P be some polytope containing the origin in its interior. Assume that $\{F_j\}$ is the family of facets of P, and $x_j \in \partial M$ is the point where the exterior normals coincide with the exterior normals to F_j. Then $M \subset P$ is equivalent saying that for each j, we have $h_M(v(x_j)) \leqslant h_M(v(x_j))$. Now assume that

$P \subset M$ and s_j is the distance of x_j from aff F_j (the depth of the corresponding cap). Therefore the minimal λ satisfying $M \subset P_n$ is

$$\lambda = 1 + \max_j \left(1 - \frac{s_j}{h_M(v(x_j))}\right)^{-1} \cdot \frac{s_j}{h_M(v(x_j))}.$$

To prove the lower bound, we may assume that $P_n \subset M$ and $\delta_{BM}(M, P_n)$ is the minimum of $\ln \lambda$ such that $M \subset \lambda P_n$.

Apply Lemma 5.1 with $\varrho(x) = \kappa(x)^{1/2}$ and $\psi(x) = h(v(x))$. Therefore the optimal λ corresponding to Σ_β is

$$\lambda_\beta = 1 + (1 + O(\varepsilon)) \cdot \frac{\max_{z \in \tilde{\Sigma}_\beta} \omega(z)}{h_\beta},$$

and similar argument works as above.

Finally, in case of Schneider's notion of distance, we search for the maximum volume of a cap cut off from M by the affine hull of some facet of P_n. In this case, apply Lemma 5.1 with $\varrho(x) = \kappa(x)^{1/(d+1)}$.

For a positive definite quadratic form $q(u)$, $u \in \mathbb{R}^{d-1}$, and for $s > 0$, let C be the cap of the graph of q of depth s at the origin. In other words, $(u, t) \in C$ if and only if $q(u) \leqslant t \leqslant s$. Then the volume of the cap is

$$V(C) = \frac{2\kappa_{d-1}}{d+1} \cdot \frac{1}{\sqrt{\det q}} \cdot s^{(d+1)/2}.$$

Therefore around Σ_β, the maximum volume of a cap cut off by the affine hull of a facet of P_n is

$$(1 + O(\varepsilon)) \cdot \frac{2\kappa_{d-1}}{d+1} \cdot \frac{1}{\sqrt{\det q_\beta}} \cdot \left(\max_{z \in \tilde{\Sigma}_\beta} \omega(z)\right)^{(d+1)/2}.$$

Using this estimate, Theorem A follows also in this case.

6.2. L_1 *Metric*

In this case, the main tool is polarity. Assume that K is a convex body containing the origin o in its interior. Then the polar K^* of K is

$$K^* = \{ y \mid \forall x \in K \langle x, y \rangle \leqslant 1 \}.$$

The polar is also a convex body, and $K^{**} = K$. If ∂K is C^2_+ then ∂K^* is also C^2_+. On the other hand, if K is a polytope then K^* is also a polytope, and the number of k-faces of K is the same as the number of $(d-1-k)$-faces of K^*. All these properties can be found say in [14].

Set $w(x) = \|x\|^{-(d+1)}$ for $x \neq o$. The fundamental observation is that

$$\delta_1(M, P) = \delta_w(M^*, P^*)$$

(see [6]). In particular, the polytope P minimizing $\delta_1(M, P)$ with at most k n-faces is the polar of the best approximating polytope of M^* with respect to δ_w having at most n $(d-1-k)$-faces. Now Theorem A is a consequence of (see [6])

$$\int_{\partial M^*} \frac{1}{\|x\|^{d-1}} \cdot \kappa(x)^{1/(d+1)}\, dx = \int_{\partial M} \kappa(x)^{d/(d+1)}\, dx.$$

So far Theorem A has been proved for all the metrics but L_p, $p > 1$ in the C^2_+ case. In order to handle L_p, $p > 1$, we need to understand how to approximate subsets of ∂M.

7. SMOOTH SUBSETS OF A CONVEX HYPERSURFACE

We say that X is a convex C^2_+ hypersurface if it is an open, Jordan measurable subset of a convex body M with C^2 boundary, the origin lies in the interior of M, and $\kappa(x) > 0$ if $x \in \operatorname{cl} X$.

Similarly, Y is called a convex polytopal hypersurface if it is a Jordan measurable subset of a polytope P and the origin lies in the interior of P. If Y approximates X then we make the following assumptions: If the approximation is with respect to the symmetric difference metric (or δ_w) then define $Y \subset \partial P$ as the radial projection of X. Otherwise, for any $x \in X$ consider the points $y \in \partial P$ where the exterior normals at x to M are also exterior normals at y to P, and Y is the union of these sets. We say that Y is inscribed if $Y \subset M$, and Y is circumscribed if $Y \cap \operatorname{int} M = \emptyset$. The faces of Y are the intersections of the faces of P with the interior of Y.

Now we extend the notions of distances to X and Y. Observe that for $x \in X$, we have

$$h_P(\nu(x)) - h_M(\nu(x)) = \max_{y \in Y} \langle \nu(x), y - x \rangle.$$

Symmetric difference metric and δ_w. $\delta_S(X, Y)$ is the volume of the part of the cone over X which lies between X and Y, and $\delta_w(X, Y)$ is the integral of w on this part.

Banach–Mazur metric. Assume that Y is inscribed. Then $\delta_{BM}(X, Y)$ is the minimum of $\ln \lambda$ such that λY is circumscribed.

L$_1$ *metric.* $\delta_1(X, Y) = \int_X |\max_{y \in Y} \langle \nu(x), y - x \rangle| \, \kappa(x) \, dx$.

Hausdorff metric. $\delta_H(X, Y) = \max_{x \in X} | \max_{y \in Y} \langle \nu(x), y - x \rangle |$

Schneider's distance. If Y is inscribed into X then $\delta_{SCH}(X, Y)$ is the maximum volume of a cap cut off from M by the affine hull of a facet of Y.

Let us justify the definitions above. Observe that for almost all the metrics, if $X = \partial M$ (and hence $Y = \partial P$) then $\delta(X, Y) = \delta(M, P)$. In case of Schneider's notion of distance we have to assume that ∂P is close to ∂M.

The only real exception is the Banach–Mazur distance, in this case the definition has been substantially altered. On the other hand, the lack of linear transformation is irrelevant in our context because linear transformations keep the family of faces.

Fix X, and an open, Jordan measurable subset X' of X whose closure lies in X. For some polytope P, let Y be the subset of ∂P corresponding to X. Now the positive curvature condition yields $\varepsilon > 0$ for any metric δ with the following property: Assume that Y is a convex polytopal surface satisfying $\delta(X, Y) < \varepsilon$. If the projection of X' intersects the facet F then the closure of F lies in the interior of Y. Here projection is the same as the one to be used to define Y.

Therefore the arguments for closed hypersurfaces generalize in a direct way to our case for all metrics but δ_1. Observe that the boundary of X on ∂M causes no problem because it has zero $(d-1)$-measure.

Turning to the L_1 metric, we have a closer look at the properties of polarity. If $u \neq o$ then define u^* to be the hyperplane $H = \{z \colon \langle z, u \rangle = 1\}$, and set $H^* = u$. Observe that if $v \in u^*$ then $u \in v^*$.

Let X be a convex C^2_+ hypersurface, which then lies on the boundary of a convex body M where M contains the origin in its interior. Define X^* to be the set of polar images of the tangent hyperplanes at the points of X. Then X^* is also convex C^2_+ hypersurface lying on the boundary of M^* (see [14]). Observe that $X^{**} = X$.

Let $Y \subset \partial P$ be a convex polytopal surface approximating X with respect to δ_1. Consider the tangent hyperplanes at the points of Y which are parallel to the tangent hyperplane at some point of X, and denote by Y^* the set of polar images of them. Then $Y^* \subset \partial P^*$ is a convex polytopal hypersurface approximating X^* in the sense of δ_w.

Now there exists a one to one correspondence between the k-faces of Y whose closure does not intersect the boundary of Y and the $(d-1-k)$-faces of Y^*.

Set $w(x) = \|x\|^{-(d+1)}$ for $x \neq o$. Then the same argument as above yields that

$$\delta_1(X, Y) = \delta_w(X^*, Y^*).$$

This way the problem of best approximation of X with respect to δ_1 bounding the number of k-faces is translated into best approximation of X^* with respect to δ_w bounding the number of $(d-1-k)$-faces.

Define the expressions $\mathscr{A}(X)$ by replacing the integration over ∂M by integration over X in the definitions for all the metrics we have considered. Then the arguments above lead to

COROLLARY 1. *Let δ be any of the metrics of this section but δ_{SCH} together with the corresponding set function $\mathscr{A}$. Assume that Y_n is the polytopal surface whose number of k faces or the number of flags is at most n and $\delta(X, Y_n)$ is minimal under this condition. One may impose the additional restriction that Y_n is inscribed or circumscribed. Then*

$$\frac{1}{65e^2\pi}\cdot\frac{1}{d}\cdot\mathscr{A}(X)\cdot\frac{1}{n^{2/(d-1)}}<\delta(X, Y_n)<c\cdot\mathscr{A}(\partial X)\cdot\frac{1}{n^{2/(d-1)}}$$

where c depends only on the problem.

If $\delta=\delta_{SCH}$ then the analogous statement holds with different constants, only $n^{2/(d-1)}$ should be replaced by $n^{(d+1)/(d-1)}$.

8. L_p METRIC, $p>1$

Let X be a convex C^2_+ hypersurface, and consider a convex polytope P containing the origin in its interior. Then the part Y of ∂P associated to X is defined as for the L_1 metric: for any $x\in X$ consider the points $y\in\partial P$ where the exterior normals at x to M are also exterior normals at y to P, and Y is the union of these sets. In particular, the L_p distance of X and Y is

$$\delta_p(X, Y)=\left(\int_X |\max_{y\in Y}\langle v(x), y-x\rangle|^p\,\kappa(x)\,dx\right)^{1/p}.$$

Now let M be a convex body with C^2_+ boundary and fix $0\leqslant k\leqslant d-1$ and $p>1$.

For large n, consider the polytope P_n which has at most n k-faces and $\delta_p(M, P_n)$ is minimal under this condition. We prove that

$$\delta_p(M, P_n)>\frac{1}{66e^2\pi\, d}\left(\int_{\partial M}\kappa(x)^{(d-1+p)/(d-1+2p)}\,dx\right)^{(d-1+2p)/p(d-1)}\cdot\frac{1}{n^{2/(d-1)}}. \tag{10}$$

So let $\varepsilon > 0$, and consider the open, Jordan measurable subsets Σ_β of ∂M given by Lemma 5.1. We need the following properties in the sequel: the closures of Σ_β's are pairwise disjoint,

$$\sum_\beta \int_{\Sigma_\beta} \kappa(x)^{(d-1+p)/(d-1+2p)}\, dx > (1-\varepsilon) \cdot \int_{\partial M} \kappa(x)^{(d-1+p)/(d-1+2p)}\, dx \tag{11}$$

and there exists a positive κ_β for each Σ_β such that

$$\kappa(x) = (1 + O(\varepsilon)) \cdot \kappa_\beta \quad \text{for} \quad x \in \Sigma_\beta. \tag{12}$$

Denote by Y_β the part of ∂P_n associated to Σ_β. Observe that for large n, if the facets G and G' of ∂P_n intersect Y_β and $Y_{\beta'}$, respectively, and $\beta \neq \beta'$ then $G \cap G' = \emptyset$. Therefore the number n_β of k-faces of Y_β satisfies $\sum n_\beta < n$.

Now start the estimate on $\delta_p(M, P_n)$ by estimating each $\delta_p(\Sigma_\beta, Y_\beta)$. We deduce applying first Hölder's inequality, then using Corollary 1 that

$$\delta_p(\Sigma_\beta, Y_\beta) \geqslant \frac{\int_{\Sigma_\beta} |\max_{y \in Y} \langle v(x), y - x \rangle|\, \kappa(x)\, dx}{(\int_{\Sigma_\beta} \kappa(x)\, dx)^{(p-1)/p}}$$

$$> \frac{\frac{1}{65 e^2 \pi\, d} \cdot \left(\int_{\Sigma_\beta} \kappa(x)^{d/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{{n_\beta}^{2/(d-1)}}}{(\int_{\Sigma_\beta} \kappa(x)\, dx)^{p-1/p}}.$$

Here the condition (12) yields that

$$\frac{(\int_{\Sigma_\beta} \kappa(x)^{d/(d+1)}\, dx)^{(d+1)/(d-1)}}{(\int_{\Sigma_\beta} \kappa(x)\, dx)^{(p-1)/p}}$$
$$= (1 + O(\varepsilon)) \cdot \left(\int_{\Sigma_\beta} \kappa(x)^{(d-1+p)/(d-1+2p)}\, dx \right)^{(d-1+2p)/p(d-1)}$$

where $O(\cdot)$ depends on d and p.

Therefore we conclude by Proposition 4.3 and (11) the estimate

$$\delta_p(M, P_n) > \left(\sum_\beta \delta_p(\Sigma_\beta, Y_\beta)^p \right)^{1/p} > (1 + O(\varepsilon)) \cdot \frac{1}{65 e^2 \pi\, d}$$
$$\times \left(\int_{\partial M} \kappa(x)^{(d-1+p)/(d-1+2p)}\, dx \right)^{(d-1+2p)/p(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.$$

Choosing ε small enough initially, this proves (10) for large n. In particular, the proof of Theorem A in the C^2_+ case is now complete.

Similar arguments can be used for general convex C^2_+ hypersurfaces to verify

Corollary 2. *Let X be a Jordan measurable C^2_+ convex hypersurface and let $p > 1$. Assume that Y_n is the polytopal surface whose number of k faces or the number of flags is at most n and $\delta_p(X, Y_n)$ is minimal under this condition. One may impose the additional restriction that Y_n is inscribed or circumscribed. Then for large n,*

$$\frac{1}{66e^2\pi d} \cdot \mathscr{A}_p(X) \cdot \frac{1}{n^{2/(d-1)}} < \delta_p(X, Y_n) < c \cdot \mathscr{A}_p(X) \cdot \frac{1}{n^{2/(d-1)}}$$

where c depends only on p and d.

9. IF $\kappa(x)$ MIGHT BE ZERO

Let δ be one of the metrics defined in the Introduction, and denote by $\mathscr{A}$ the corresponding set function.

For a convex body M with C^2 boundary in $\mathbb{R}^d$, assume that the origin is contained in the interior of M. Choose a Jordan measurable, open $X \subset \partial M - t$ such that $\kappa(x) > 0$ if x lies in the closure of X and $\mathscr{A}(X) > 0.99 \cdot \mathscr{A}(\partial M)$.

Assume that P_n is the polytope minimizing $\delta(M, P_n)$ under the condition that the number of k-faces of P_n is at most n for large n where $P_n \subset M$ if $\delta = \delta_{SCH}$ or $\delta = \delta_{BM}$. Denote by Y_n the part of P_n corresponding to X (see Section 7). Then the number of k-faces of Y_n is at most n. Since $\delta(M, P_n) \geqslant \delta(X, Y_n)$, we conclude Theorem A by Corollaries 1 and 2.

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