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A Szegő type theorem for truncated Toeplitz operators

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Abstract

Truncated Toeplitz operators are compressions of multiplication operators on L^2 to model spaces (that is, subspaces of H^2 which are invariant with respect to the backward shift). For this class of operators we prove certain Szegő type theorems concerning the asymptotics of their compressions to an increasing chain of finite dimensional model spaces.

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The Toeplitz operators are compressions of multiplication operators on the space $L^2(\mathbb{T})$ to the Hardy space H^2 ; the multiplier is called the symbol of the operator. With respect to the standard exponential basis, their matrices are constant along diagonals; if we truncate such a matrix considering only its upper left finite corner, we obtain classical Toeplitz matrices.

It does not come as a surprise that there are connections between the asymptotics of these Toeplitz matrices and the whole Toeplitz operator, or its symbol. A central result is Szegő's strong limit theorem and its variants (see, for instance, [2] and the references within), which deal

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with the asymptotics of the eigenvalues of the Toeplitz matrix. The version that is a starting point for our investigation states that, if T_n are truncated Toeplitz matrices corresponding to the symbol a , then

$$\frac{1}{n} \operatorname{Tr} T_n^p \rightarrow \int_{\mathbb{T}} a^p dm.$$

On the other hand, certain generalizations of Toeplitz matrices have attracted a great deal of attention in the last decade, namely compressions of multiplication operators to subspaces of the Hardy space which are invariant under the backward shift. These “model spaces” are of the form $H^2 \ominus uH^2$ with u an inner function, and the compressions are called truncated Toeplitz operators. They have been formally introduced in [10]; see [6] for a more recent survey. Although classical Toeplitz matrices have often been a starting point for investigating truncated Toeplitz operators, the latter may exhibit surprising properties.

It thus seems natural to see whether an analogue of Szegő’s strong limit theorem can be obtained in this more general context. Viewed as truncated Toeplitz operators, the Toeplitz matrices act on model spaces corresponding to the inner functions $u(z) = z^n$, and Szegő’s theorem is about the asymptotical situation when $n \rightarrow \infty$. The natural generalization is then to consider a sequence of zeros (λ_j) in $\mathbb{D}$, and to let the truncations act on the model space corresponding to the finite Blaschke product associated to λ_j , $1 \leq j \leq n$.

Such a result has been obtained in [1]; it deals with the asymptotics of the determinant of a truncated Toeplitz operator. Let us note that in the case of classical Toeplitz operators and matrices one has different variants of Szegő’s Theorem, either in terms of the determinant of the truncation, or in terms of the trace, and one can pass from one to the other. However, this is no longer true in our generalized context, where the two different classes of results do not have a visible connection.

The purpose of this paper is to find an analogue of the trace type Szegő theorem. We manage to obtain a complete result in the case when (λ_j) is not a Blaschke sequence. The Blaschke case seems to be less prone to an elegant solution, and we have only partial conclusions.

The technique we use is inspired by one of the approaches to the Szegő Theorem that is based on approximation by circulants (see, for instance, [8]). In our case we use the analogue of circulants for truncated Toeplitz operators, namely elements in the so-called Sedlock algebras [11].

The plan of the paper is the following. After a rather extensive preliminary section that introduces the basic notions, we discuss the special case of finite dimensional model spaces. We introduce then the Sedlock algebras in Section 3, and prove a result important in its own right, Theorem 3.2, which gives an alternate identification of these algebras. Sedlock algebras on finite dimensional model spaces are briefly discussed in Section 4, after which Section 5 develops the approximation technique based on them. The main result, Theorem 6.1, is proved in Section 6. The last section discusses through some examples the problems that appear when considering Blaschke sequences.

1. Preliminaries

1.1. Model spaces

Let H^2 be the Hardy space of square integrable functions on the circle with negative Fourier coefficients equal to 0. We recall that a *model space* is a subspace of H^2 which is invariant for the backwards shift, and that every such space is of the form $K_B = H^2 \ominus BH^2$ where B is an inner

function, i.e., an element of H^2 of modulus 1 almost everywhere. We write P_B for orthogonal projection from L^2 onto K_B .

We will often make the supplementary assumption that $B(0) = 0$. In this case $\widehat{B}$ defined by $\widehat{B} = B/z$, is also an inner function.

The reproducing kernels for K_B at the points $\lambda \in \mathbb{D}$ are the functions

$$k_\lambda^B(z) = \frac{1 - \overline{B(\lambda)}B(z)}{1 - \overline{\lambda}z}, \quad z \in \mathbb{D}.$$

When B has an angular derivative in the sense of Caratheodory at a point in $\zeta \in \mathbb{T}$, that is, when the nontangential limits of B and B' exist in ζ , with the limit of B in ζ of modulus 1, then all functions in K_B have a radial limit in ζ , and the corresponding reproducing kernel k_ζ^B belongs to K_B . We have

$$k_\zeta^B(z) = \frac{1 - \overline{B(\zeta)}B(z)}{1 - \overline{\zeta}z} = \zeta \overline{B(\zeta)} \frac{B(\zeta) - B(z)}{\zeta - z}, \quad z \in \mathbb{D}.$$

We also use the fact that

$$\|k_\zeta^B\|^2 = k_\zeta^B(\zeta) = \lim_{r \rightarrow 1} \frac{1 - \overline{B(\zeta)}B(r\zeta)}{1 - r} = \frac{\zeta B'(\zeta)}{B(\zeta)} = |B'(\zeta)|. \quad (1.1)$$

(The last equality follows from the fact that all of the above terms are positive real numbers.)

1.2. Truncated Toeplitz operators

For a function $\phi \in L^\infty$, M_ϕ denotes the operator of multiplication by ϕ on the space L^2 ; it is bounded and $\|M_\phi\| = \|\phi\|_\infty$.

Let us now assume only $\phi \in L^2$. The map $f \mapsto P_B \phi f$ is then linear from the dense subspace $H^\infty \cap K_B$ to K_B . When this map is bounded, it can be extended to a bounded linear operator on the whole K_B , that will be denoted by $T_B[\phi] = P_B M_\phi|_{K_B}$ and called a *truncated Toeplitz operator*. For $\phi(z) = z$ one obtains the *compressed shift*, which we will denote by S_B . The closed linear space of all bounded truncated Toeplitz operators on K_B will be denoted by $\mathcal{T}_B$.

The function ϕ is called the *symbol* of the operator. It is known [10] that any operator in $\mathcal{T}_B$ has a symbol ϕ in $K_B + \overline{K_B}$, which is unique in case $B(0) = 0$. Such a symbol is called *standard*. Also, if $B(0) = 0$ we see that $1 \in K_B$, so that the orthogonal complement of the constants in K_B equals $zK_{\widehat{B}}$. Thus:

$$K_B + \overline{K_B} = K_B \oplus \overline{zK_{\widehat{B}}}$$

and standard symbols can be uniquely written in the form $\phi_+ + \overline{\phi_-}$, with $\phi_+ \in K_B$ and $\phi_- \in zK_{\widehat{B}}$. Moreover, using the fact that for any inner function U the map $h \mapsto \bar{z}\bar{h}U$ is an involution on K_U one sees that

$$\{\overline{\phi_-} : \phi_- \in zK_{\widehat{B}}\} = \{\bar{B}\psi_- : \psi_- \in zK_{\widehat{B}}\}$$

and so we will write our standard symbol (uniquely) as $\psi_+ + \bar{B}\psi_-$ with $\psi_+ \in K_B$ and $\psi_- \in zK_{\widehat{B}}$.

For $\alpha \in \mathbb{T}$ one defines $S_B^\alpha = S_B + \alpha(1 \otimes \widehat{B})$. One can check easily that the operators S_B^α (called *modified compressed shifts*) are unitary. It is proved in [10] that the modified compressed shifts belong to $\mathcal{T}_B$.

Suppose now that $a \in \mathbb{D}$, and define $b_a(z) = \frac{z-a}{1-\bar{a}z}$; $b_a(z)$ is an automorphism of the unit disc. We set $B^a = B \circ b_a$. The following result, that we will have the opportunity to use below, is Proposition 4.1 of [3].

Lemma 1.1. *The formula*

$$U_a(f) = \sqrt{b'_a} \cdot f \circ b_a = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} f \circ b_a \quad (1.2)$$

defines a unitary operator U_a from K_B to K_{B^a} , and for any symbol ϕ we have

$$U_a T_B[\phi] U_a^* = T_{B^a}[\phi \circ b_a].$$

1.3. Clark measures

The following measures have been introduced by Clark [5]. For $\alpha \in \mathbb{T}$, the function $\frac{\alpha+B(z)}{\alpha-B(z)}$ has positive real part, and therefore, by Herglotz theorem, there exists a finite positive measure μ_α^B on $\mathbb{T}$ such that

$$\Re \left(\frac{\alpha + B(z)}{\alpha - B(z)} \right) = \int_{\mathbb{T}} P_{r,t}(\zeta) d\mu_\alpha^B(\zeta), \quad (1.3)$$

where $P_{r,t}(\zeta) = \frac{1-r^2}{|\zeta - re^{it}|^2}$ is the Poisson kernel. The measure is singular, and

$$\|\mu_\alpha^B\| = \frac{1 - |B(0)|^2}{|\alpha - B(0)|^2}; \quad (1.4)$$

in particular, if $B(0) = 0$ then $\|\mu_\alpha^B\| = 1$. A result of Aleksandrov states that for every continuous function f

$$\int_{\mathbb{T}} f dm = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f d\mu_\alpha \right) dm(\alpha), \quad (1.5)$$

where m is the normalized Lebesgue measure. We will often write μ_α instead of μ_α^B when there is a single inner function B involved.

We note that the Clark measures μ_α^b are defined in the same way for any function b in the unit ball of H^∞ . In particular for the zero function $b = 0$, for every α , μ_α^b is the normalized Lebesgue measure.

The next theorem combines results from [5,9].

Theorem 1.2. *Suppose that B is an inner function, $\alpha \in \mathbb{T}$, and μ_α is defined by (1.3). Then any function $f \in K_B$ has a radial limit f^* almost everywhere with respect to μ_α , the operator $V : K_u \rightarrow L^2(\mu_\alpha)$ defined by $Vf = f^*$ is unitary, and $V S_B^\alpha = M_z V$.*

Suppose now that $\lambda_j \in \mathbb{D}$, $j \geq 1$. Define, for $n \geq 1$,

$$B_n = \prod_{j=1}^n \left(-\frac{|\lambda_j|}{\lambda_j} b_{\lambda_j} \right). \quad (1.6)$$

(By convention, in case $\lambda_j = 0$, the corresponding factor will be $-b_0 = -z$.) The following lemma is well known; we provide a proof for completeness.

Lemma 1.3. Fix $\alpha \in \mathbb{T}$, and consider for each n the measure $\mu_\alpha^{B_n}$.

- (i) If $\sum(1 - |\lambda_j|) = \infty$, then $\mu_\alpha^{B_n}$ converges in the weak star topology to normalized Lebesgue measure.
- (ii) If $\sum(1 - |\lambda_j|) < \infty$, then $\mu_\alpha^{B_n}$ converges in the weak star topology to μ_α^B , where $B = \prod_{j=1}^\infty (-\frac{|\lambda_j|}{\lambda_j}) b_{\lambda_j}$ is the infinite Blaschke product with zeros λ_j .

Proof. The sequence $(B_n)_n$ converges uniformly on every compact set of $\mathbb{D}$ to the function B , where B is the zero function in case (i) and the infinite Blaschke product with zeros λ_j in case (ii). For $0 \leq r < 1$ and $t \in \mathbb{R}$, the Poisson kernel is defined by $P_{re^{it}}(\zeta) = \frac{1-r^2}{|\zeta - re^{it}|^2} = \sum_{n \in \mathbb{Z}} r^{|n|} e^{-int} \zeta^n$. We have

$$\int_{\mathbb{T}} P_{re^{it}}(\zeta) d\mu_\alpha^{B_n}(\zeta) = \Re \left(\frac{\alpha + B_n(re^{it})}{\alpha - B_n(re^{it})} \right) \rightarrow \Re \left(\frac{\alpha + B(re^{it})}{\alpha - B(re^{it})} \right) = \int_{\mathbb{T}} P_{re^{it}}(\zeta) d\mu_\alpha^B(\zeta).$$

Since $\text{span}\{P_{re^{it}}; 0 \leq r < 1, t \in \mathbb{R}\}$ is dense in the space of continuous functions and the norms of the measures $\mu_\alpha^{B_n}$ are uniformly bounded by (1.4), we obtain that $(\mu_\alpha^{B_n})_n$ converges in the weak star topology to μ_α^B . Now the proof is finished since μ_α^B is the normalized Lebesgue measure when $B \equiv 0$. $\square$

The behavior of the Clark measures with respect to change of variable by an automorphism of the disc is given by the next lemma, whose proof is a simple change of variable in (1.3).

Lemma 1.4. With the above notations, we have $\mu_\alpha^{B^a} = (b_{-a})_*(|b'_{-a}| \mu_\alpha^B)$ (that is, the pushforward of the measure $|b'_{-a}| \mu_\alpha^B$ by the map b_{-a}).

1.4. The Cauchy transform

For the next facts about Cauchy transforms we refer to [4]. If μ is a Borel measure on $\mathbb{T}$, its Cauchy transform $\mathcal{K}_\mu$ is the analytic function on $\mathbb{D}$ defined by

$$\mathcal{K}_\mu(z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\zeta}z} d\mu(\zeta).$$

The following lemma summarizes the properties of the Cauchy transform that we need.

Lemma 1.5. Suppose μ is a Borel measure on $\mathbb{T}$. Then:

- (1) $\mathcal{K}_\mu \in H^p$ for all $0 < p < 1$.
- (2) $\mathcal{K}_\mu \equiv 0$ if and only if $d\mu = \bar{\phi} dm$ for some $\phi \in H_0^1$, where $H_0^1 = \{f \in H^1, f(0) = 0\}$.

Suppose $\Lambda = (\lambda_j)_{j \geq 1}$ is a sequence of points in $\mathbb{D}$, and each λ_j is repeated m_j times, with m_j a finite integer or infinite. We will denote by $\mathfrak{L}_\Lambda$ the linear span of $z^m k_{\lambda_j}^{m+1}$ and $\bar{z}^m \bar{k}_{\lambda_j}^{m+1}$, with $j \geq 1$ and $0 \leq m < m_j$. The next lemma is probably known, but we have not found an appropriate reference.

Lemma 1.6. Suppose $\sum_{j=1}^\infty (1 - |\lambda_j|) = \infty$. Then $\mathfrak{L}_\Lambda$ is dense in $C(\mathbb{T})$.

Proof. Suppose μ is a Borel measure on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} z^m k_{\lambda_j}^{m+1}(z) d\mu = \int_{\mathbb{T}} \bar{z}^m \bar{k}_{\lambda_j}^{m+1}(z) d\mu = 0$$

for all $j \geq 1$ and $0 \leq m < m_j$. By replacing μ , if necessary, with $\frac{1}{2}(\mu + \bar{\mu})$ and $\frac{1}{2i}(\mu - \bar{\mu})$, we may assume that μ is real. We have then $\mathcal{K}_{\mu}^{(m)}(\lambda_j) = 0$ for all $j \geq 1$ and $0 \leq m < m_j$. Since $\mathcal{K}_{\mu} \in H^p$ for $0 < p < 1$, if it is not identically zero its zeros would have to satisfy the Blaschke condition. So $\mathcal{K}_{\mu} \equiv 0$, and then by Lemma 1.5 $\mu = \bar{\phi} dm$ for some $\phi \in H_0^1$. Since μ is real, ϕ is real almost everywhere. Therefore $\phi \equiv 0$ and $\mu \equiv 0$, which finishes the proof. $\square$

If B_n is defined by (1.6), then each of the functions $z^m k_{\lambda_j}^{m+1}$ belongs to K_{B_n} for sufficiently large n . This remark yields the next corollary.

Corollary 1.7. Suppose $\sum_{j=1}^{\infty} (1 - |\lambda_j|) = \infty$, and B_n is defined by (1.6). Then $\bigvee_{n=1}^{\infty} (K_{B_n} + \overline{K_{B_n}})$ is dense in $C(\mathbb{T})$.

1.5. Schatten–von Neumann classes

We recall in this subsection several well known facts about Schatten–von Neumann classes (see, for instance, [12]). If A is a positive operator on a separable Hilbert space, then its trace is given by

$$\text{Tr } A = \sum_k \langle A e_k, e_k \rangle,$$

where the sum on the right hand side does not depend on the orthonormal basis $(e_k)_k$.

If T is a compact operator, we denote by $(s_k(T))$ the sequence of singular values of T ; that is, the sequence of positive eigenvalues of $(T^*T)^{\frac{1}{2}}$. For $q > 0$, the Schatten–von Neumann class $\mathcal{C}_q$ is formed by the compact operators T with $\sum_k s_k(T)^q < \infty$. If $T \in \mathcal{C}_1$, then the series $\sum_k \langle T e_k, e_k \rangle$ is absolutely convergent, once again it does not depend on the orthonormal basis $(e_k)_k$, and defines the trace $\text{Tr } T$. If R is a bounded operator, then $RT \in \mathcal{C}_1$ and $\text{Tr}(RT) \leq \|R\| \cdot \|T\|_1$. Finally, if p is a positive integer and $T \in \mathcal{C}_p$, then $T^p \in \mathcal{C}_1$.

2. Finite dimensional model spaces

Let us suppose now that $u = B$ is a finite Blaschke product of order N . In this case K_B is of dimension N and contains only rational functions with poles outside the closed unit disc (more precisely, in the reciprocals of the zeros of B); we have $\dim K_B = N$ and $\dim \mathcal{T}_B = 2N + 1$. The measure μ_{α}^B is concentrated on the roots ζ_k^{α} ($k = 1, \dots, N$) of the equation $B(\zeta) = \alpha$, and is given by the formula

$$\mu_{\alpha}^B = \sum_{k=1}^N \frac{1}{|B'(\zeta_k^{\alpha})|} \delta_{\zeta_k^{\alpha}}. \quad (2.1)$$

The functions in K_B as well as the standard symbols of TTOs have well defined values with respect to μ_{α}^B .

We will have the opportunity to use the next lemma, which completes (1.1).

Lemma 2.1. Suppose $B = \prod_{j=1}^n b_{\lambda_j}$. Then we have:

(i)

$$|B'(e^{it})| = e^{it} \frac{B'(e^{it})}{B(e^{it})} = \sum_{j=1}^n \frac{1 - |\lambda_j|^2}{|e^{it} - \lambda_j|^2} = \frac{d}{dt} \text{Arg } B(e^{it}).$$

(ii) For all $\zeta \in \mathbb{T}$ we have $|B'(\zeta)| \geq \frac{1}{2} \sum_{j=1}^n (1 - |\lambda_j|)$.

Proof. (i) The second equality is just a computation, using the formula for B . Taking absolute values, we deduce the first equality.

For the third equality, note that

$$\text{Arg } B(e^{it}) = \Im \log(B(e^{it})) = \Im \left(\sum_{j=1}^n \log b_{\lambda_j}(e^{it}) \right).$$

Differentiating with respect to t , we obtain

$$\frac{d}{dt} \text{Arg } B(e^{it}) = \Im \left(\sum_{j=1}^n i \frac{1 - |\lambda_j|^2}{|e^{it} - \lambda_j|^2} \right) = \sum_{j=1}^n \frac{1 - |\lambda_j|^2}{|e^{it} - \lambda_j|^2}$$

as required.

Part (ii) is a consequence of (i), once we note that for all $\zeta \in \mathbb{T}$ and $\lambda \in \mathbb{D}$ we have $1 - |\lambda| \leq |\zeta - \lambda| \leq 1 + |\lambda|$, and therefore

$$\frac{1 - |\lambda|}{1 + |\lambda|} \leq \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} \leq \frac{1 + |\lambda|}{1 - |\lambda|}. \quad \square$$

For each $\alpha \in \mathbb{T}$ the reproducing kernels $k_{\zeta_k^\alpha}^B$, $k = 1, \dots, N$, form an orthogonal base of K_B . The operator S_B^α has as eigenvalues ζ_k^α ($k = 1, \dots, N$), with corresponding eigenvectors $k_{\zeta_k^\alpha}^B$.

Consider a measure $\nu = \sum_{k=1}^N \nu_k \delta_{\zeta_k^\alpha}$ concentrated on the points ζ_k^α and define the operator D_ν by

$$D_\nu = \sum_{k=1}^N \nu_k \left(\frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \otimes \frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \right). \quad (2.2)$$

Since, by [10, Section 5], the rank one operators $k_{\zeta_k^\alpha}^B \otimes k_{\zeta_k^\alpha}^B$ are in $\mathcal{T}_B$, we also have $D_\nu \in \mathcal{T}_B$. In particular, we will denote $\Delta_B^\alpha = D_{\mu_\alpha^B}$; thus

$$\Delta_B^\alpha := \sum_{k=1}^N \frac{1}{|B'(\zeta_k^\alpha)|} \left(\frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \otimes \frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \right). \quad (2.3)$$

The next lemma gives more information about the unitary operators that have appeared in Lemma 1.1.

Lemma 2.2. Suppose $a \in \mathbb{D}$, $\alpha \in \mathbb{T}$, and the operator $U_a : K_B \rightarrow K_{B^a}$ is defined by (1.2).

(i) For each $k = 1, \dots, N$,

$$U_a(k_{\zeta_k^\alpha}^B) = \sqrt{b'_{-a}(\zeta_k^\alpha)} k_{\eta_k^\alpha}^{B^a}$$

where $\eta_k^\alpha = b_{-a}(\zeta_k^\alpha)$.

(ii) If $v = \sum_{k=1}^N v_k \delta_{\zeta_k^\alpha}$, then

$$U_a D_v U_a^* = D_{v^a},$$

where $v^a = \sum_{k=1}^N v_k \delta_{\eta_k^\alpha}$.

(iii) We have

$$U_a \Delta_B^\alpha U_a^* = D_{\tilde{v}_\alpha^a},$$

where

$$\tilde{v}_\alpha^a = \frac{1}{|b'_{-a} \circ b_a|} \mu_\alpha^{B^a} = |b'_a| \mu_\alpha^{B^a}.$$

Proof. For $g \in K_{B^a}$ we have

$$\langle g, U_a(k_{\zeta_k^\alpha}^B) \rangle = \langle U_a^* g, k_{\zeta_k^\alpha}^B \rangle = (U_a^* g)(\zeta_k^\alpha) = \sqrt{b'_{-a}(\zeta_k^\alpha)} g(\eta_k^\alpha),$$

which proves (i).

To prove (ii), we use (i) to check the action of the left hand side operator on the reproducing kernels. Thus

$$U_a D_v U_a^*(k_{\eta_k^\alpha}^{B^a}) = \sqrt{b'_a(\eta_k^\alpha)} U_a D_v(k_{\zeta_k^\alpha}^B) = v_k \sqrt{b'_a(\eta_k^\alpha)} \sqrt{b'_{-a}(\zeta_k^\alpha)} k_{\eta_k^\alpha}^{B^a} = v_k k_{\eta_k^\alpha}^{B^a}.$$

(iii) is a consequence of (ii): if $v = \mu_\alpha^B$, then

$$\begin{aligned} v^a &= \sum_{k=1}^N \frac{1}{|B'(\zeta_k^\alpha)|} \delta_{\eta_k^\alpha} = \sum_{k=1}^N \frac{1}{|(B^a)'(\eta_k^\alpha)| \cdot |b'_{-a}(\zeta_k^\alpha)|} \delta_{\eta_k^\alpha} \\ &= \sum_{k=1}^N \frac{1}{|(B^a)'(\eta_k^\alpha)| \cdot |b'_{-a} \circ b_a(\eta_k^\alpha)|} \delta_{\eta_k^\alpha} = \frac{1}{|b'_{-a} \circ b_a|} \mu_\alpha^{B^a} = |b'_a| \mu_\alpha^{B^a}. \quad \square \end{aligned}$$

3. Sedlock algebras

Sedlock algebras have been introduced in [11], where it was shown that they are the only maximal subalgebras of $\mathcal{T}_B$; see also [7] for related results. The family of Sedlock algebras is indexed by a parameter $\alpha \in \mathbb{C} \cup \{\infty\}$. We will be interested only in $\alpha \in \mathbb{T}$, in which case the Sedlock algebra $\mathcal{B}_\alpha^B$ is defined to be the commutant of the unitary operator S_B^α .

In [11, Proposition 3.2] Sedlock characterizes the algebra $\mathcal{B}_\alpha^B$ as the set of truncated Toeplitz operators with symbols of the form

$$\phi + \alpha \overline{S_B(Bz\phi)} = \phi + \alpha \overline{P_B(B\phi)},$$

where $\phi \in K_B$. If $B(0) = 0$ we have $P_B(B\phi) = B(\phi - \phi(0))$, and thus Sedlock's result yields the following lemma.

Lemma 3.1. Suppose $B(0) = 0$. A bounded operator A is in $\mathcal{B}_\alpha^B$ if and only if $A = T_B[\phi + \alpha \bar{B}(\phi - \phi(0))]$ for some $\phi \in K_B$.

Using Theorem 1.2, one can connect the two descriptions of the Sedlock algebras, as shown by the next theorem.

Theorem 3.2. Let B be an arbitrary inner function satisfying $B(0) = 0$. Suppose $T = T_B[\phi + \alpha \bar{B}(\phi - \phi(0))] \in \mathcal{B}_\alpha^B$. Then the function ϕ has radial limits almost everywhere with respect to μ_α . If we denote the limit function by ϕ^* , then $\phi^* \in L^\infty(\mu_\alpha)$, and $T = \phi^*(S_B^\alpha)$.

Proof. The first part of the statement follows from Theorem 1.2, since $\phi \in K_B$. The operator $V : K_B \rightarrow L^2(\mu_\alpha)$ defined by $Vf = f^*$ is unitary, and $VS_B^\alpha = M_z V$. Since $\mathcal{B}_\alpha^B = \{S_B^\alpha\}'$, we have $VT V^* \in \{M_z\}'$, and thus $VT V^* = \Phi(M_z) = M_\Phi$ for some $\Phi \in L^\infty(\mu_\alpha)$.

On the other hand, it follows from Proposition 3.2 of [11] that, when we write an operator $T \in \mathcal{B}_\alpha$ as $T = T_B[\phi + \alpha \bar{B}(\phi - \phi(0))]$, we can identify ϕ as $\phi = T\mathbf{1}$. So

$$\phi^* = V\phi = VT\mathbf{1} = M_\Phi V\mathbf{1} = M_\Phi \mathbf{1} = \Phi.$$

Thus

$$T = V^* \phi^*(M_z) V = \phi^*(S_B^\alpha),$$

which ends the proof of the theorem. $\square$

Some consequences for operators in the Sedlock class $\mathcal{B}_\alpha^B$ can be immediately deduced.

Corollary 3.3. With the above notations, if we decompose the singular measure μ_α as $\mu_\alpha = \mu_\alpha^c + \mu_\alpha^a$ in its continuous and atomic parts, and the support of μ_α^a is the sequence $(\eta_n)_n$, then:

- (i) $T_B[\phi + \alpha \bar{B}(\phi - \phi(0))]$ is compact if and only if $\phi^* = 0$ μ_α^c -almost everywhere, while $\phi^*(\eta_j) \rightarrow 0$.
- (ii) If q is a positive real number, $T_B[\phi + \alpha \bar{B}(\phi - \phi(0))]$ is in the class $\mathcal{C}_q$ if and only if $\phi^* = 0$ μ_α^c -almost everywhere and $\sum_n |\phi^*(\eta_n)|^q < \infty$.
- (iii) If $p \geq 1$ is an integer and $T = T_B[\phi + \bar{B}(\phi - \phi(0))] \in \mathcal{C}_p$, then

$$\text{Tr } T^p = \sum_n \phi^*(\eta_n)^p.$$

4. Sedlock algebras and finite Blaschke products

Let us suppose now that B is a finite Blaschke product such that $B(0) = 0$. Then, for $\alpha \in \mathbb{T}$, the measure μ_α^B is given by (2.1). As noted above, for each k , $k_{\zeta_k^\alpha}^B$ is an eigenvector of S_B^α associated to the eigenvalue ζ_k^α . Therefore, if $\phi \in K_B$, Theorem 3.2 implies that

$$T_B[\phi + \alpha \bar{B}(\phi - \phi(0))] = \sum_{k=1}^N \phi(\zeta_k^\alpha) \left(\frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \otimes \frac{k_{\zeta_k^\alpha}^B}{\|k_{\zeta_k^\alpha}^B\|} \right). \quad (4.1)$$

If $T = T_B[\phi + \bar{B}(\phi - \phi(0))]$ and $\nu = \sum_{k=1}^N \nu_k \delta_{\zeta_k^\alpha}$, it follows from (4.1) and (2.2) that

$$\text{Tr}(D_\nu T^p) = \sum_{k=1}^N \nu_k \phi^p(\zeta_k^\alpha) = \int_{\mathbb{T}} \phi^p d\nu. \quad (4.2)$$

In particular,

$$\text{Tr}(\Delta_B^\alpha T^p) = \int_{\mathbb{T}} \phi^p d\mu_\alpha^B \quad (4.3)$$

and therefore, using also (1.3),

$$\|\Delta_B^\alpha\|_1 = \text{Tr}(\Delta_B^\alpha) = \int_{\mathbb{T}} d\mu_\alpha^B = 1. \quad (4.4)$$

We may integrate with respect to α to obtain some interesting consequences.

Lemma 4.1. *With the above notations, we have*

$$\int_{\mathbb{T}} \Delta_B^\alpha dm(\alpha) = T_B \left[\frac{1}{|B'|} \right].$$

Proof. For $f \in K_B$, we have $f = \sum_{k=1}^N \left\langle f, \frac{k_{\zeta_k}^B}{\|k_{\zeta_k}^B\|} \right\rangle \frac{k_{\zeta_k}^B}{\|k_{\zeta_k}^B\|}$. By (1.5), we get

$$\begin{aligned} \int_{\mathbb{T}} \Delta_B^\alpha(f) dm(\alpha) &= \int_{\mathbb{T}} \left(\sum_{k=1}^N \frac{1}{|B'(\zeta_k^\alpha)|} \left\langle f, \frac{k_{\zeta_k}^B}{\|k_{\zeta_k}^B\|} \right\rangle \frac{k_{\zeta_k}^B}{\|k_{\zeta_k}^B\|} \right) dm(\alpha) \\ &= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \left\langle f, \frac{k_{\zeta}^B}{\|k_{\zeta}^B\|} \right\rangle \frac{k_{\zeta}^B}{\|k_{\zeta}^B\|} d\mu_\alpha^B(\zeta) \right) dm(\alpha) \\ &= \int_{\mathbb{T}} \frac{1}{|B'(\zeta)|} f(\zeta) k_{\zeta}^B dm(\zeta) \\ &= T_B \left[\frac{1}{|B'|} \right] (f). \quad \square \end{aligned}$$

From (4.3) and Lemma 4.1 follows the next corollary.

Corollary 4.2. *With the above notations (including $T = T_B[\phi + \alpha \bar{B}(\phi - \phi(0))]$), we have*

$$\text{Tr} \left(T_B \left[\frac{1}{|B'|} \right] T^p \right) = \int \phi^p dm.$$

5. Approximating by circulants

In this section we will develop a procedure, in the case $B(0) = 0$, for deriving properties of a truncated Toeplitz operator from a closely associated element of a Sedlock algebra. The idea comes from the case of classical Toeplitz matrices (see, for instance, [8, Chapter 4]) and it is based on an enlargement of the model space.

Fix $\alpha \in \mathbb{T}$, and let B be a finite Blaschke product with $B(0) = 0$ and ψ be a (general) standard symbol for a truncated Toeplitz on K_B . Thus $\psi \in K_B + \overline{K_B}$ is of the form $\psi = \psi_+ + \bar{B}\psi_-$ with $\psi_+ \in K_B$ and $\psi_- \in zK_{\widehat{B}}$. Let u be an inner function that is a multiple of B ; that is, $u = Bv$ for some inner function v . Obviously, ψ is also a standard symbol for a truncated Toeplitz operator on K_u (since $K_B + \overline{K_B} \subset K_u + \overline{K_u}$). We intend to show that, if we set

$$\phi = \psi_+ + \bar{\alpha}v\psi_-$$

then we can have good estimates for the difference between $T_u[\psi]$ and the operator $T_u[\phi + \alpha \bar{u}(\phi - \phi(0))]$ in $\mathcal{B}_\alpha^u$. Notice that $\phi \in K_B \subset K_u$ and that $\phi(0) = \psi_+(0)$ (since $\psi_-(0) = 0$).

We see that:

$$\begin{aligned}\phi + \alpha \bar{u}(\phi - \phi(0)) &= \psi_+ + \bar{\alpha} v \psi_- + \alpha \bar{u}(\psi_+ + \bar{\alpha} v \psi_- - \psi_+(0)) \\ &= \psi_+ + \bar{B} \psi_- + \bar{\alpha} v \psi_- + \alpha \bar{u}(\psi_+ - \psi_+(0)) \\ &= \psi + \bar{\alpha} v \psi_- + \alpha \bar{u}(\psi_+ - \psi_+(0))\end{aligned}$$

and thus

$$T_u[\phi + \alpha \bar{u}(\phi - \phi(0))] - T_u[\psi] = \bar{\alpha} T_u[v \psi_-] + \alpha T_u[\bar{u}(\psi_+ - \psi_+(0))]. \quad (5.1)$$

We can now show that the norms of the two operators on the right hand side of (5.1) do not depend on which inner function u we choose. We will interpret these operators as operators on all of L_2 , rather than just on the subspace K_u . We use throughout the computations below the fact that ψ , ψ_+ , ψ_- are all bounded functions.

We use repeatedly that $K_u = K_B \oplus B K_v = K_v \oplus v K_B$ and that $P_{v K_B} = M_v P_B M_{\bar{v}}$. This gives us that:

$$T_u[v \psi_-] = P_u M_{v \psi_-} P_u = P_u M_{v \psi_-} (P_B + P_{B K_v}).$$

But, it is clear that the range of $M_{v \psi_-} P_{B K_v}$ is contained in $u H^2$ and so $P_u M_{v \psi_-} P_{B K_v} = 0$. Thus:

$$T_u[v \psi_-] = P_u M_{v \psi_-} P_B = (P_v + P_{v K_B}) M_{v \psi_-} P_B.$$

Using the fact that the range of $M_{v \psi_-} P_B$ is contained in $v H^2$, so that $P_v M_{v \psi_-} P_B = 0$, it follows that:

$$P_u M_{v \psi_-} P_u = P_{v K_B} M_{v \psi_-} P_B = M_v P_B M_{\psi_-} P_B. \quad (5.2)$$

So $T_u[v \psi_-]$ is obtained by multiplying the operator $T_B[\psi_-]$ by a unitary operator. In particular, the norm (uniform or Schatten–von Neumann) of $T_u[v \psi_-]$ depends only on B and ψ and is independent of the choice of v .

In our analysis of the second operator, we simplify the notation by denoting $\psi_0 = \psi_+ - \psi_+(0)$ so that $\psi_0 \in z K_{\bar{B}}$. We have

$$P_u M_{\bar{u} \psi_0} P_u = P_u M_{\bar{u} \psi_0} (P_v + P_{v K_B}).$$

Now, using the fact that $K_v \subset v H_-^2$ and that $\psi_0 \in K_B \subset B H_-^2$ we see that the range of $M_{\bar{u} \psi_0} P_v$ is contained in $\bar{u} B v H_-^2 = H_-^2$. So it is clear that $P_u M_{\bar{u} \psi_0} P_v = 0$. Thus

$$P_u M_{\bar{u} \psi_0} P_u = P_u M_{\bar{u} \psi_0} P_{v K_B}.$$

Next we notice that:

$$P_u M_{\bar{u} \psi_0} P_{v K_B} = P_u M_{\bar{u} \psi_0} M_v P_B M_{\bar{v}} = P_u M_{\bar{B} \psi_0} P_B M_{\bar{v}}.$$

Since $\psi_0 \in K_B$ we know that $\bar{B} \psi_0 \subset H_-^2$. But, an element of K_B is a function $h \in H^2$ which is orthogonal to $B H^2$ and when we multiply such a function by a (bounded) element of H_-^2 we get an element h_1 of L^2 which is orthogonal to $B H^2$ and is therefore orthogonal to $B K_v$. Thus we can conclude that

$$P_u M_{\bar{u} \psi_0} P_u = (P_{B K_v} + P_B) M_{\bar{B} \psi_0} P_B M_{\bar{v}} = P_B M_{\bar{B} \psi_0} P_B M_{\bar{v}}. \quad (5.3)$$

So, just as in the first case, we have obtained that any norm (uniform or Schatten–von Neumann) of $T_u[\bar{u}(\psi_+ - \psi_+(0))]$ is independent of the choice of v . We summarize the conclusion of our argument in the following lemma.

Lemma 5.1. Suppose $\alpha \in \mathbb{T}$, $\psi = \psi_+ + \bar{B}\psi_- \in K_B + \overline{K_B}$. If $u = Bv$ for some inner function v , and $\phi = \psi_+ + \bar{\alpha}v\psi_-$, then $T_u[\phi + \alpha\bar{u}(\phi - \phi(0))] \in \mathcal{B}_\alpha^u$ and, for any $0 < p \leq \infty$, we have

$$\|T_u[\phi + \alpha\bar{u}(\phi - \phi(0))] - T_u[\psi]\|_p \leq C_p,$$

where C_p is a finite constant independent of v .

6. A Szegő theorem for non-Blaschke sequences

Suppose now that $\lambda_j \in \mathbb{D}$, $j \geq 1$. As in Section 1.3, define, for $n \geq 1$, $B_n = \prod_{j=1}^n (-\frac{|\lambda_j|}{\lambda_j} b_{\lambda_j})$.

The next theorem may be considered the central result of this paper; it gives a Szegő theorem for truncated Toeplitz operators.

Theorem 6.1. Suppose $\sum(1 - |\lambda_j|) = \infty$, B_n is defined as above, $\psi \in C(\mathbb{T})$, and $p \in \mathbb{N}$. Then

$$\mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi])^p \right) \rightarrow \int_{\mathbb{T}} \psi^p dm. \quad (6.1)$$

If ψ is real-valued, then for every continuous function g on $[\inf \psi, \sup \psi]$ we have

$$\mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] g(T_{B_n}[\psi]) \right) \rightarrow \int_{\mathbb{T}} g \circ \psi dm. \quad (6.2)$$

The proof of the theorem will use a series of intermediate results which have interest in themselves. The basic technique is introduced in the next lemma.

Lemma 6.2. Fix $N \in \mathbb{N}$, $\alpha \in \mathbb{T}$, assume $B_N(0) = 0$, and denote by $\zeta_{n,k}^\alpha$, $k = 1, \dots, n$, the roots of the equation $B_n(z) = \alpha$. Suppose that for each $n \geq N$ we are given an atomic measure $\nu_n = \sum_{k=1}^n \nu_k^\alpha \delta_{\zeta_{n,k}^\alpha}$; denote by D_{ν_n} the corresponding diagonal operator as defined by (2.2). Assume that

- (i) ν_n is weakly convergent to some measure ν .
- (ii) $\|D_{\nu_n}\| \rightarrow 0$.

Then for any $\psi \in K_{B_N} + \overline{K_{B_N}}$ and $p \in \mathbb{N}$ we have

$$\mathrm{Tr}(D_{\nu_n}(T_{B_n}[\psi])^p) \rightarrow \int \psi^p d\nu.$$

Proof. If $\psi = \psi_+ + \bar{B}_N\psi_- \in K_{B_N} + \overline{K_{B_N}}$, we define $\phi_n = \psi_+ + \bar{\alpha}\frac{B_n}{B_N}\psi_-$ and $\tilde{\phi}_n = \phi_n + \alpha\bar{B}_n(\phi_n - \phi_n(0))$ for $n \geq N$. Applying Lemma 5.1 for $B = B_N$ and $u = B_n$ we obtain that the trace norm $\|T_{B_n}[\psi] - T_{B_n}[\tilde{\phi}_n]\|_1$ is bounded by a constant independent of n . Therefore, using condition (ii),

$$\begin{aligned} & |\mathrm{Tr}(D_{\nu_n}(T_{B_n}[\psi])^p) - \mathrm{Tr}(D_{\nu_n}(T_{B_n}[\tilde{\phi}_n])^p)| \\ & \leq \sum_{k=0}^{p-1} \left| \mathrm{Tr} \left(D_{\nu_n} \left(T_{B_n}[\psi]^k \left(T_{B_n}[\psi] - T_{B_n}[\tilde{\phi}_n] \right) T_{B_n}[\tilde{\phi}_n]^{p-k-1} \right) \right) \right| \\ & \leq M \|D_{\nu_n}\| \cdot \|T_{B_n}[\psi] - T_{B_n}[\tilde{\phi}_n]\|_1 \rightarrow 0. \end{aligned}$$

Now, according to (4.2), we have

$$\mathrm{Tr}(D_{v_n} T_{B_n} [\tilde{\phi}_n]^p) = \int \phi_n^p dv_n.$$

Since $B_n(\zeta_{n,k}^\alpha) = \alpha$, we have $\psi(\zeta_j^{(n)}) = \phi_n(\zeta_j^{(n)})$ for all $j = 1, \dots, n$, and thus the last integral is $\int \psi dv_n$, which by (i) converges to $\int \psi dv$. $\square$

Next, with a suitable assumption on the symbol, we eliminate the restriction $B(0) = 0$.

Lemma 6.3. *Let $\sum(1 - |\lambda_j|) = \infty$, B_n be as above, $\alpha \in \mathbb{T}$ fixed, and $\Delta_{B_n}^\alpha$ as defined by (2.3). Suppose $\psi \in (K_{B_N^{-\lambda_1}} + \overline{K_{B_N^{-\lambda_1}}}) \circ b_{\lambda_1}$ and $p \in \mathbb{N}$. Then*

$$\mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) \rightarrow \int \psi^p dm. \quad (6.3)$$

Proof. Denote $a = -\lambda_1$, so $B_N(-a) = 0$. Then for any $n \geq N$ we have

$$B_n^a(0) = B_n \circ b_a(0) = B_n(-a) = 0.$$

We have, by Lemmas 1.1 and 2.2(iii)

$$\begin{aligned} \mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) &= \mathrm{Tr}((U_a \Delta_{B_n}^\alpha U_a^*)(U_a T_{B_n}[\psi] U_a^*)^p) \\ &= \mathrm{Tr}(D_{\tilde{v}_\alpha^{a,n}}(T_{B_n^a}[\psi \circ b_a])^p), \end{aligned}$$

where

$$\tilde{v}_\alpha^{a,n} = \frac{1}{|b'_{-a} \circ b_a|} \mu_\alpha^{B_n^a}.$$

Consider the sequence of measures $\tilde{v}_\alpha^{a,n}$. By Lemma 1.3(i),

$$\tilde{v}_\alpha^{a,n} \rightarrow \frac{1}{|b'_{-a} \circ b_a|} m.$$

On the other hand,

$$\|D_{\tilde{v}_\alpha^{a,n}}\| = \|\Delta_{B_n}^\alpha\| = \sup_{1 \leq j \leq n} \frac{1}{|B_n'(\zeta_j^{(n)})|} \leq \sup_{\zeta \in \mathbb{T}} \frac{1}{|B_n'(\zeta)|} \leq \frac{2}{\sum_{j=1}^n (1 - |\lambda_j|)},$$

where the last inequality is a consequence of Lemma 2.1(ii).

The assumption on ψ implies that $\psi \circ b_a \in K_{B_N^a} + \overline{K_{B_N^a}}$. Therefore, applying Lemma 6.2,

$$\mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) = \mathrm{Tr}(D_{\tilde{v}_\alpha^{a,n}}(T_{B_n^a}[\psi \circ b_a])^p) \rightarrow \int \frac{1}{|b'_{-a} \circ b_a|} (\psi \circ b_a)^p dm.$$

By Lemma 1.4, the last integral is precisely $\int \psi^p dm$, which finishes the proof. $\square$

The next step is to integrate with respect to $\alpha \in \mathbb{T}$ in order to obtain a version of (6.1) for a restricted class of functions ψ .

Lemma 6.4. *Let $\sum(1 - |\lambda_j|) = \infty$, B_n be as above, $\psi \in (K_{B_N^{-\lambda_1}} + \overline{K_{B_N^{-\lambda_1}}}) \circ b_{\lambda_1}$, and $p \in \mathbb{N}$. Then*

$$\mathrm{Tr}\left(T_{B_n}\left[\frac{1}{|B_n'|}\right](T_{B_n}[\psi])^p\right) \rightarrow \int_{\mathbb{T}} \psi^p dm.$$

Proof. By [Lemma 6.3](#) we have for each $\alpha \in \mathbb{T}$

$$\mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) \rightarrow \int \psi^p dm. \quad (6.4)$$

On the other hand, from [Lemma 4.1](#) it follows that

$$\begin{aligned} \int_{\mathbb{T}} \mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) dm(\alpha) &= \mathrm{Tr} \left(\left(\int_{\mathbb{T}} \Delta_{B_n}^\alpha dm(\alpha) \right) (T_{B_n}[\psi])^p \right) \\ &= \mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi])^p \right). \end{aligned}$$

Then (6.4) together with Lebesgue's dominated convergence theorem implies that

$$\mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi])^p \right) \rightarrow \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \psi^p dm \right) dm(\alpha) = \int_{\mathbb{T}} \psi^p dm. \quad \square$$

We may now go to the proof of the main theorem.

Proof of Theorem 6.1. Suppose now $\psi \in C(\mathbb{T})$ and $p \in \mathbb{N}$. Let $\epsilon > 0$ be given. [Corollary 1.7](#) implies that $\bigvee_n (K_{B_N^{-\lambda_1}} + \overline{K_{B_N^{-\lambda_1}}}) \circ b_{\lambda_1}$ is also dense in $C(\mathbb{T})$. Next we choose $N \in \mathbb{N}$ and $\psi_N \in (K_{B_N^{-\lambda_1}} + \overline{K_{B_N^{-\lambda_1}}}) \circ b_{\lambda_1}$ such that $\|\psi - \psi_N\|_\infty \leq \epsilon$, where $\|\cdot\|_\infty$ is the supremum norm; we assume also that $\|\psi\|_\infty, \|\psi_N\|_\infty \leq M$.

Applying now [Lemma 6.4](#) to ψ_N , we see that there exists n_0 such that for $n \geq n_0$ we have

$$\left| \mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi_N])^p \right) - \int \psi_N^p dm \right| \leq \epsilon.$$

Also, since $\|T_{B_n}[\phi]\| \leq \|\phi\|_\infty$ for any $\phi \in C(\mathbb{T})$, we have

$$\begin{aligned} \|T_{B_n}[\psi_N]^p - T_{B_n}[\psi]^p\| &\leq \sum_{k=0}^{p-1} \|T_{B_n}[\psi_N]^k\| \|T_{B_n}[\psi]^{p-k-1}\| \|T_{B_n}[\psi_N] - T_{B_n}[\psi]\| \\ &\leq pM^{p-1}\epsilon. \end{aligned}$$

Using (4.4), we have

$$\|\Delta_{B_n}^\alpha\|_1 = \mathrm{Tr} \Delta_{B_n}^\alpha = \Re \frac{\alpha + B_n(0)}{\alpha - B_n(0)} \leq \frac{1 + |B_n(0)|}{1 - |B_n(0)|} \leq \frac{1 + |\lambda_1|}{1 - |\lambda_1|}.$$

Therefore

$$\left\| T_{B_n} \left[\frac{1}{|B'_n|} \right] \right\|_1 \leq \int_{\mathbb{T}} \|\Delta_{B_n}^\alpha\|_1 dm(\alpha) \leq \frac{1 + |\lambda_1|}{1 - |\lambda_1|}, \quad (6.5)$$

and

$$\begin{aligned} &\left| \mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi])^p \right) - \int \psi^p dm \right| \\ &\leq \mathrm{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi])^p - T_{B_n} \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi_N])^p \right) \end{aligned}$$

$$\begin{aligned}
& + \left| \operatorname{Tr} (T_{B_n}) \left[\frac{1}{|B'_n|} \right] (T_{B_n}[\psi_N])^p - \int \psi_N^p dm \right| + \int |\psi_N^p - \psi^p| dm \\
& \leq pM^{p-1} \frac{1 + |\lambda_1|}{1 - |\lambda_1|} \epsilon + \epsilon + pM^{p-1} \epsilon.
\end{aligned}$$

The above estimate proves formula (6.1).

Suppose now that ψ is real valued and g is continuous on $[\inf \psi, \sup \psi]$. Note that $T_{B_n}[\psi]$ is then a bounded self-adjoint operator with spectrum contained in $[\inf \psi, \sup \psi]$. Let $(P_k)_{k \in \mathbb{N}}$ be a sequence of polynomials that converges to g uniformly on $[\inf \psi, \sup \psi]$.

By (6.1), for every k we have

$$\operatorname{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] P_k(T_{B_n}[\psi]) \right) \rightarrow \int_{\mathbb{T}} P_k \circ \psi dm.$$

But

$$\begin{aligned}
& \left| \operatorname{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] P_k(T_{B_n}[\psi]) \right) - \operatorname{Tr} \left(T_{B_n} \left[\frac{1}{|B'_n|} \right] g(T_{B_n}[\psi]) \right) \right| \\
& \leq \left\| T_{B_n} \left[\frac{1}{|B'_n|} \right] \right\|_1 \| (P_k - g)(T_{B_n}[\psi]) \| \\
& \leq \frac{1 + |\lambda_1|}{1 - |\lambda_1|} \| P_k - g \|_{\infty}
\end{aligned}$$

where on the last line we have used (6.5).

Since obviously

$$\left| \int_{\mathbb{T}} P_k \circ \psi - g \circ \psi dm \right| \leq \| P_k - g \|_{\infty},$$

letting $k \rightarrow \infty$ proves (6.2). $\square$

In the classical case of Toeplitz matrices, we have $B_n = z^n$, $|B'_n| \equiv n$, and $T_{B_n} \left[\frac{1}{|B'_n|} \right] = \frac{1}{n} I$. We obtain therefore a version of a classical Szegő limit theorem: if T_n are truncated Toeplitz matrices corresponding to the symbol a , then

$$\frac{1}{n} \operatorname{Tr} T_n^p \rightarrow \int_{\mathbb{T}} a^p dm.$$

It is interesting to compare Theorem 6.1 with the main result in [1], where a formula is obtained for the asymptotics of $\det T_{B_n}[\psi]$. In the classical case the determinant and the trace versions of the Szegő limit theorem are closely connected, and one can deduce one version from the other. This seems not to be the case for general truncated Toeplitz operators.

7. Blaschke sequences

Theorem 6.1 concerns sequences that do not satisfy the Blaschke condition. It is possible to obtain Szegő results for Blaschke sequences, but these are less elegant and require supplementary hypotheses. First, Lemma 6.3 is true for a Blaschke sequence, provided we add the condition $\|\Delta_{B_n}^{\alpha}\| \rightarrow 0$. Similarly, Lemma 6.4 is true if we assume that $\|\Delta_{B_n}^{\alpha}\| \rightarrow 0$ for almost all $\alpha \in \mathbb{T}$. As for the main Theorem 6.1, a new complication arises, since Lemma 1.6 is no longer true. We do, however, have the following precise result; recall that the space $\mathcal{L}_A$ is defined in Section 1, before Lemma 1.6.

Theorem 7.1. Suppose that (λ_j) is a Blaschke sequence (i.e. that $\sum_{j=1}^{\infty} (1 - |\lambda_j|) < \infty$) and each λ_j is repeated m_j times, with m_j a finite integer. Let B_n be given by (1.6) and $p \in \mathbb{N}$. Suppose that, for some fixed $i \geq 1$, $\psi \circ b_{-\lambda_i} \in \mathfrak{L}_A$.

(1) If $\alpha \in \mathbb{T}$ and if $\|\Delta_{B_n}^\alpha\| \rightarrow 0$, then

$$\mathrm{Tr}(\Delta_{B_n}^\alpha (T_{B_n}[\psi])^p) \rightarrow \int \psi^p d\mu_\alpha^B. \quad (7.1)$$

(2) If $\|\Delta_{B_n}^\alpha\| \rightarrow 0$ for almost all $\alpha \in \mathbb{T}$, then

$$\mathrm{Tr}\left(T_{B_n}\left[\frac{1}{|B'_n|}\right](T_{B_n}[\psi])^p\right) \rightarrow \int_{\mathbb{T}} \psi^p dm.$$

Unfortunately, it does not seem possible to give simple conditions that would characterize those sequences for which

$$\|\Delta_{B_n}^\alpha\| \rightarrow 0. \quad (7.2)$$

The next two examples suggest the intricacy of the problem. First we give an example of a Blaschke sequence for which (7.2) is satisfied.

Example 7.2. Suppose (λ_j) is a sequence of points in $\mathbb{D}$ obtained by choosing m^2 evenly distributed points on each circle of radius $r_m = 1 - \frac{1}{m^4}$. We have then

$$\sum (1 - |\lambda_j|) = \sum_m m^2 \cdot \frac{1}{m^4} < \infty,$$

and therefore the sequence is Blaschke.

Fix $m \in \mathbb{N}$. For any $e^{it} \in \mathbb{T}$ there exists a $j \in \mathbb{N}$ such that $|\lambda_j| = r_m$ and $|e^{it} - \lambda_j| \leq \frac{10}{m^2}$; therefore, using Lemma 2.1

$$P_{\lambda_j}(e^{it}) = \frac{1 - |\lambda_j|^2}{|e^{it} - \lambda_j|^2} \geq \frac{1/m^4}{100/m^4} = \frac{1}{100}$$

and so, for any $e^{it} \in \mathbb{T}$,

$$\sum_{|\lambda_j|=r_m} P_{\lambda_j}(e^{it}) \geq \frac{1}{100}.$$

If we fix N , and choose $n \geq \max\{j \in \mathbb{N} : |\lambda_j| = r_m, 0 \leq m \leq N\}$, then

$$\inf\{|B'_n(\zeta)| : \zeta \in \mathbb{T}\} \geq N/100.$$

So in this case $\inf\{|B'_n(\zeta)| : \zeta \in \mathbb{T}\} \rightarrow \infty$ when $n \rightarrow \infty$, and thence $\|\Delta_{B_n}^\alpha\| \rightarrow 0$, for any $\alpha \in \mathbb{T}$.

Example 7.3. The second example exhibits the opposite behavior. We show that, for certain Blaschke sequences (λ_j) the hypothesis (7.2) is not satisfied, and, in the case where $\alpha = 1$, $\lambda_1 = 0$ and $\psi = \bar{z}$, the conclusion (7.1) is also false; that is:

$$\mathrm{Tr}(\Delta_{B_n}^1 T_{B_n}[\psi]) \not\rightarrow \int \psi d\mu_1^B.$$

So, let (λ_j) be a real valued Blaschke sequence with $\lambda_1 = 0$. We will add other hypotheses as necessary. Following the approximation procedure of Section 5 for the finite Blaschke product $B_2(z) = -z(-b_{\lambda_2}(z))$ (where B_n is defined by (1.6) and the remark after it), we set $N = 2$, $\psi(z) = \bar{z}$ and $v = \prod_{j=3}^n (-b_{\lambda_j})$. Then $u = B_n = B_2 v$, $\psi_+ = 0$, $\psi_- = b_{\lambda_2}(z)$, and $\phi = B_n/z$. Formula (5.1) then becomes

$$T_{B_n}[B_n/z + \bar{B}_n(B_n/z - B_n/z(0))] - T_{B_n}[\bar{z}] = T_{B_n}[B_n/z].$$

From (4.3) and the fact that $B_n \equiv 1$ on the support of $\mu_1^{B_n}$, we obtain

$$\text{Tr} \left(\Delta_{B_n}^1 T_{B_n}[B_n/z + \bar{B}_n(B_n/z - (B_n/z)(0))] \right) = \int_{\mathbb{T}} B_n/z d\mu_1^{B_n} = \int_{\mathbb{T}} \bar{z} d\mu_1^{B_n}.$$

Thus, for $n \rightarrow \infty$,

$$\text{Tr} \left(\Delta_{B_n}^1 T_{B_n}[B_n/z + \bar{B}_n(B_n/z - (B_n/z)(0))] \right) \rightarrow \int_{\mathbb{T}} \bar{z} d\mu_1^B.$$

So, in order to achieve our purpose of showing that

$$\text{Tr}(\Delta_{B_n}^1 T_{B_n}[\bar{z}]) \not\rightarrow \int_{\mathbb{T}} \bar{z} d\mu_1^B,$$

we have to prove that

$$\text{Tr}(\Delta_{B_n}^1 T_{B_n}[B_n/z]) \not\rightarrow 0. \quad (7.3)$$

We next use [10, Section 5], where it is shown that

$$T_{B_n}[B_n/z] = \tilde{k}_0^{B_n} \otimes k_0^{B_n}.$$

Denote by ζ_k^n the solutions to $B_n(\zeta) = 1$. Notice that, since -1 is such a solution, $\zeta_1^n = -1$. We have

$$\Delta_{B_n}^1 = \sum_{k=1}^n \frac{1}{|B_n'(\zeta_k^n)|} \left(\frac{k_{\zeta_k^n}^{B_n}}{\|k_{\zeta_k^n}^{B_n}\|} \otimes \frac{k_{\zeta_k^n}^{B_n}}{\|k_{\zeta_k^n}^{B_n}\|} \right) = \sum_{k=1}^n \frac{1}{|B_n'(\zeta_k^n)|^2} (k_{\zeta_k^n}^{B_n} \otimes k_{\zeta_k^n}^{B_n})$$

(since $\|k_{\zeta_k^n}^{B_n}\|^2 = k_{\zeta_k^n}^{B_n}(\zeta_k^n) = |B_n'(\zeta_k^n)|$ by (1.1)).

Using the usual formulas

$$(x \otimes y)(z \otimes t) = \langle z, y \rangle (x \otimes t), \quad \text{Tr}(x \otimes y) = \langle x, y \rangle,$$

we obtain

$$\Delta_{B_n}^1 T_{B_n}[B_n/z] = \sum_{k=1}^n \frac{1}{|B_n'(\zeta_k^n)|^2} \langle \tilde{k}_0^{B_n}, k_{\zeta_k^n}^{B_n} \rangle (k_{\zeta_k^n}^{B_n} \otimes k_0^{B_n}) = \sum_{k=1}^n \frac{\tilde{k}_0^{B_n}(\zeta_k^n)}{|B_n'(\zeta_k^n)|^2} (k_{\zeta_k^n}^{B_n} \otimes k_0^{B_n}),$$

and

$$\text{Tr}(\Delta_{B_n}^1 T_{B_n}[B_n/z]) = \sum_{k=1}^n \frac{\tilde{k}_0^{B_n}(\zeta_k^n)}{|B_n'(\zeta_k^n)|^2} \langle k_{\zeta_k^n}^{B_n}, k_0^{B_n} \rangle = \sum_{k=1}^n \frac{\tilde{k}_0^{B_n}(\zeta_k^n) \overline{k_0^{B_n}(\zeta_k^n)}}{|B_n'(\zeta_k^n)|^2}.$$

We have $\tilde{k}_0^{B_n} = B_n/z$, so $\tilde{k}_0^{B_n}(\zeta_k^n) = B_n(\zeta_k^n)\bar{\zeta}_k^n = \bar{\zeta}_k^n$, while $k_0^{B_n} \equiv 1$, so $k_0^{B_n}(\zeta_k^n) = 1$. Thus we have arrived at the formula

$$\text{Tr}(\Delta_{B_n}^1 T_{B_n}[B_n/z]) = \sum_{k=1}^n \frac{\bar{\zeta}_k^n}{|B_n'(\zeta_k^n)|^2} = -\frac{1}{|B_n'(-1)|^2} + \sum_{k=2}^n \frac{\bar{\zeta}_k^n}{|B_n'(\zeta_k^n)|^2}. \quad (7.4)$$

Now we begin putting hypotheses on the rest of the sequence (λ_j) . We suppose that, $\lambda_j \nearrow 1$ and that $\lambda_j > 1/2$ for $j \geq 2$, which implies that

$$|B'_n(-1)| \leq 1 + \frac{4}{9} \sum_{j=2}^n (1 - \lambda_j^2),$$

so, if we assume that the λ_j tend sufficiently fast to 1, we may assume that $|B'_n(-1)| \leq 3/2$.

To estimate the remaining sum, remember that by [Lemma 2.1\(i\)](#) we have

$$|B'_n(\zeta)| = \sum_{j=1}^n \frac{1 - \lambda_j^2}{|\zeta - \lambda_j|^2} = 1 + \sum_{j=2}^n \frac{1 - \lambda_j^2}{|\zeta - \lambda_j|^2}.$$

It follows that for fixed ζ , $|B'_n(\zeta)|$ increases to $|B'(\zeta)|$, while for fixed n it increases on both half circles of $\mathbb{T}$ as we are getting closer to 1.

Finally, we add the assumption that the sequence λ_j has been chosen so that $|B'(\zeta)| \leq 11/10$ for all ζ belonging to the arc J that connects $e^{i/10}$ and $e^{-i/10}$ and contains -1 . Then, for $\zeta \in J$, $|B'_n(\zeta)| \leq |B'(\zeta)| \leq 11/10$, and so, by [Lemma 2.1](#), $\frac{d}{dt} \text{Arg } B_n(e^{it}) \leq 11/10$.

Since $\frac{d}{dt} \text{Arg } B_n(e^{it}) \leq 11/10$ for $t \in [1/10, \pi]$, we have

$$|\text{Arg } B_n(-1) - \text{Arg } B_n(e^{i/10})| \leq 11/10(\pi - \theta) \leq 11\pi/10.$$

Suppose $\zeta_2^n = e^{i\eta_2}$ is the next zero of $B_n(\zeta) = 1$ going from -1 towards 1 on the upper semicircle. Then $\eta_2 \in (0, 1/10)$, and, since $|\text{Arg } B_n(-1) - \text{Arg } B_n(e^{i\eta_2})| = 2\pi$, we must have

$$|\text{Arg } B_n(e^{i/10}) - \text{Arg } B_n(e^{i\eta_2})| \geq 9\pi/10.$$

But, using Lagrange's formula, there is some point $\theta_0 \in (\eta_2, 1/10)$ such that

$$\begin{aligned} |\text{Arg } B_n(e^{i/10}) - \text{Arg } B_n(e^{i\eta_2})| &= \left(\frac{1}{10} - \eta_2 \right) \frac{d}{dt} \text{Arg } B_n(e^{i\theta_0}) \\ &\leq \frac{1}{10} |B'_n(e^{i\theta_0})| \leq \frac{1}{10} |B'_n(e^{i\eta_2})|, \end{aligned}$$

the last inequality being a consequence of $0 < \eta_2 < \theta_0$. Therefore $|B'_n(e^{i\eta_2})| \geq 9\pi$.

A similar argument can be used on the lower semicircle. Finally, since $|B'_n(\zeta)|$ increases as we are getting closer to 1, we may conclude that $|B'_n(\zeta_k^n)| \geq 9\pi$ for any $k \geq 2$.

Remember now that

$$\sum_{k=1}^n \frac{1}{|B'_n(\zeta_k^n)|} = \|\mu_1^{B_n}\| = 1.$$

We have then, starting with [\(7.4\)](#),

$$\begin{aligned} |\text{Tr}(\Delta_{B_n}^1 T_{B_n}[B_n/z])| &\geq \frac{1}{|B'_n(-1)|^2} - \sum_{k=2}^n \frac{1}{|B'_n(\zeta_k^n)|^2} \\ &\geq \frac{4}{9} - \frac{1}{9\pi} \sum_{k=2}^n \frac{1}{|B'_n(\zeta_k^n)|} \\ &\geq \frac{4}{9} - \frac{1}{9\pi} > \frac{1}{3}. \end{aligned}$$

Therefore the inequality [\(7.3\)](#) is proved, which concludes the example.

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