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Algebraic genericity and the differentiability of the convolution[☆]

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Abstract

Despite the convolution preserving most of the smooth properties of the functions that take part in it, there exist differentiable functions whose convolution is not differentiable. In the present result, we study the algebraic genericity of the set of those functions. In particular, it is proved that periodic continuous functions can be approximated by functions belonging to a vector space each of whose nonzero members generates some convolution which is not differentiable.

Keywords: Convolution; Non differentiable function; Lineability; Spaceability; Algebrability

1. Introduction

Let us denote, as usual, by $L^p[-1,1]$ the space of 2-periodic functions so that $\int_{-1}^{1} |f(x)|^p \,dx < \infty$ and by $C^k[-1,1]$ the space of 2-periodic functions that are k times differentiable and whose

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kth derivative is continuous. In particular, $C^0[-1,1]$, or $C[-1,1]$ for simplicity, will denote the space of continuous 2-periodic functions and $C^\infty[-1,1]$ will denote the space of 2-periodic functions which are infinitely many times differentiable. We will also denote by $D[-1,1]$ the space of 2-periodic differentiable functions. Moreover, we endow $C[-1,1]$ with the topology generated by the norm $\|f\| = \max_{-1\le s\le 1}|f(s)|$, which makes $C[-1,1]$ a separable Banach space.

If $f, g \in L^1[-1,1]$, we define the **convolution** of f and g as the function

$$(f * g)(x) = \int_{-1}^{1} f(s)g(x-s)\,\mathrm{d}s.$$

We will refer to the functions f and g that take part in the convolution as "parent" functions. If one of the functions (say, e.g., f) is fixed, the convolution may be regarded as an operator

$$\begin{array}{llll} T_f: & L^1[-1,1] & \to & L^1[-1,1] \\ & g & \mapsto & f * g. \end{array}$$

It is known that some of the properties of the parent functions translate to their convolution:

Lemma 1.1. *Let $f, g \in L^1[-1,1]$.*

(1) *If $f \in C^k[-1,1]$ for some $0 \le k \le \infty$, then $f * g \in C^k[-1,1]$ (without any other assumption on g). Furthermore, the kth derivative of $f * g$ can be calculated via*

$$\frac{d^k}{dx^k}(f * g)(x) = \left(\frac{d^k f}{dx^k} * g\right)(x).$$

(2) *If f is Lipschitz (that is, there exists a constant $L > 0$ so that $|f(x) - f(y)| \le L|x-y|$ for every x, y), then $f * g$ is Lipschitz (without any other assumption on g).*

The previous properties are well-known and can be consulted in most standard books where convolution is defined (see, e.g., [23]). Results in this line, and the fact that in many cases the convolution is more regular than its parent functions (see, e.g., [21] and the references therein for examples of continuous nowhere differentiable functions whose convolution provides a differentiable function, despite the self-convolution of the generality of continuous functions being nowhere differentiable, see [20]) have, in part of the mathematical community, given the convolution operator the label of "smoothing". In fact, it is sometimes believed that the convolution of a differentiable function with an integrable function is also differentiable (see, e.g., the work by Folland, [18, Theorem 7.2]).

Nevertheless, in [22] the authors proved that this last intuition was not correct, and they were able to present two differentiable functions whose convolution is not differentiable. The idea (following a suggestion by Prof. F. Nazarov) was to consider two non-negative oscillating functions with disjoint supports, so that the overlapping that happened once one of them was slightly perturbed made the difference of quotients increase towards $+\infty$. In particular, those functions had non-integrable derivatives. On the other hand, it may be worth-noticing that the convolution of one of the functions with the derivative of the other function was still well-defined.

In the present paper we will further examine these functions within the framework of their algebraic genericity, that is, we shall focus on lineability, algebrability, and coneability problems within this context. Many different topics have been linked, and thoroughly studied, to the area of research known as lineability. For instance, the study of universality and universal series, the study of the algebraic genericity of Weierstrass' monsters and non-differentiable Pettis primitives, lattice theory, Whitney's extension theorem and its variants, everywhere surjectivity, set theoretical results and consistency, absolutely summing operators, Shannon sampling series, basic sequences and series, among many others (see [1–17,24–27]). Although the previous notions are, by now, well known to a number of authors, let us give a brief summary of them here below:

Assume that X is a topological vector space and α is a cardinal number. Then a subset $A \subset X$ is said to be:

- *lineable* (resp. *α-lineable*) if there is an infinite dimensional vector space (resp. $\dim(M) = \alpha$) M such that $M \setminus \{0\} \subset A$.
- *dense-lineable* (resp. *α-dense-lineable*) in X whenever there is a dense vector subspace M of infinite dimension (resp. $\dim(M) = \alpha$) of X satisfying $M \setminus \{0\} \subset A$.
- *coneable* if there exist a positive (or negative) cone C and $C \setminus \{0\} \subset A$ and such that C contains infinitely many linearly independent elements.
- *spaceable* in X whenever there is a closed infinite-dimensional vector subspace M of X such that $M \setminus \{0\} \subset A$.

And, provided that X is a vector space contained in some (linear) algebra, then A is called *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.

This paper is arranged in two main results. The first one deals with the existence of large algebras (that is, with algebrability) of the biggest possible dimension (the continuum, $\mathfrak{c}$) $V, W \subseteq D[-1, 1]$ so that, if $f \in V \setminus \{0\}$ and $g \in W \setminus \{0\}$, then $f * g$ is not differentiable at 0 (Theorem 2.2). To this end, we will follow the same ideas as those used in [22], adapting them to define the elements that constitute the sets of generators, first, and then verifying that non-trivial algebraic combinations have similar behavior as the original functions. Secondly, we shall focus on the existence of *closed* positive cones with the same property as mentioned before.

Concerning this last result, the reader might be tempted to study a stronger property than closed-coneability, namely, spaceability, where a closed vector space instead of a closed cone is considered. Nevertheless, spaceability is not possible to obtain, due to an important result by Gurariy (see [19]) establishing the nonexistence of closed infinite dimensional vector spaces in $C[-1, 1]$ consisting of differentiable functions.

2. Structures within the set of differentiable functions whose convolution is not differentiable

In the first theorem of this section we will use the following easily verified formula for the product of sums:

$$\prod_{k=1}^{n}(a_k + b_k) = \sum_{C \subset I_n} \left(\prod_{k \in C} a_k \prod_{j \in I_n \setminus C} b_j \right), \tag{2.1}$$

where we use the convention $\prod_{k \in \varnothing} c_k = 1$ and the notation $I_n = \{1, 2, \ldots, n\}$ $(n \in \mathbf{N})$.

We will also need the following lemma, which can be regarded as an elementary exercise of differentiation:

Lemma 2.1. *Let $0 < b < a < 1$ and let us consider the function $\tau(x) = a^x - b^x$ for $x > 1$. Then, τ has a unique critical point*

$$x_{crit} = \log_{a/b}\left(\frac{\ln(b)}{\ln(a)}\right),$$

which is a maximum.

Theorem 2.2. *There exist two algebras of dimension* $\mathfrak{c}$*,* $V, W \subseteq D[-1,1]$ *so that, if* $f \in V\setminus\{0\}$ *and* $g \in W\setminus\{0\}$*, then* $f * g$ *is not differentiable at* 0*.*

Proof of Theorem 2.2. Divide the interval $[\frac{1}{2^i}, \frac{1}{2^{i-1}}]$ into 2^{i^2-i} subintervals of length

$$\frac{\frac{1}{2^{i-1}} - \frac{1}{2^i}}{2^{i^2-i}} = \frac{1}{2^{i^2}}.$$

For each $i \geq 1$ and $k = 0, 1, \ldots, 2^{i^2-i-1} - 1$ consider two C^∞-hat functions, $\varphi_{i,k}$ and $\psi_{i,k}$, so that

$$\begin{aligned}
\operatorname{supp} \varphi_{i,k} &\subseteq \left(\frac{1}{2^i} + \frac{2k+1}{2^{i^2}}, \frac{1}{2^i} + \frac{2k+2}{2^{i^2}}\right), \\
\operatorname{supp} \psi_{i,k} &\subseteq \left(\frac{1}{2^i} + \frac{2k}{2^{i^2}}, \frac{1}{2^i} + \frac{2k+1}{2^{i^2}}\right), \\
\varphi_{i,k}(x) &= 1 \text{ for } \frac{1}{2^i} + \frac{8k+5}{2^{i^2+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{i^2+2}} \text{ and} \\
\psi_{i,k}(x) &= 1 \text{ for } \frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}}.
\end{aligned}$$

Let $\Lambda \subset (1, +\infty)$ be an uncountable family of $\mathbb{Q}$-linearly independent elements. For $\lambda \in \Lambda$ and $-1 \leq x \leq 1$ define

$$f_\lambda(x) = |x|^\lambda \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \varphi_{i,k}^\lambda(|x|), \text{ and}$$

$$g_\lambda(x) = |x|^\lambda \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \psi_{i,k}^\lambda(|x|).$$

We also define, for $i_0 \geq 1$,

$$f_{\lambda,i_0}(x) = |x|^\lambda \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \varphi_{i,k}^\lambda(|x|) \quad \text{and}$$

$$g_{\lambda,i_0}(x) = |x|^\lambda \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \psi_{i,k}^\lambda(|x|).$$

Let us show first that f_λ is a differentiable function for every $\lambda \in \Lambda$. Since f_λ is an even function, it will suffice to show that $f_\lambda'(x)$ exists for $0 < x < 1$ and that $f_+'(0) = 0$.

Assume first $x \in \left(\frac{1}{2^{i_0}}, \frac{1}{2^{i_0-1}}\right)$. Then, we can write

$$f_\lambda(y) = y^\lambda \sum_{k=0}^{2^{i_0^2-i_0-1}-1} \varphi_{i_0,k}^\lambda(y) \quad \text{for every } y \in \left(\frac{1}{2^{i_0}}, \frac{1}{2^{i_0-1}}\right)$$

so f is differentiable on x.

Next, if $x = \frac{1}{2^{i_0}}$ for some $i_0 \geq 1$, we can see that

$$f_\lambda(y) = y^\lambda \varphi^\lambda_{i_0+1,2^{(i_0+1)^2-(i_0+1)}-1}(y) \quad \text{for } y \in \left(\frac{1}{2^{i_0+1}} + \frac{2^{(i_0+1)^2-(i_0+1)}-1}{2^{(i_0+1)^2}}, \frac{1}{2^{i_0}} + \frac{1}{2^{i_0^2}}\right)$$

and hence f is differentiable on x.

Finally, for $x = 0$, notice that

$$\lim_{x\to 0^+} \frac{f_\lambda(x)}{x} = \lim_{x\to 0^+} x^{\lambda-1} \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \varphi_{i,k}^\lambda(x) = 0.$$

Via a similar argument, we obtain that g_λ is also differentiable, for every $\lambda \in \Lambda$.

We observe that, if $i_0 \geq 2$ and $\lambda \in \Lambda$, then $f_\lambda(x) - f_{\lambda,i_0}(x) = |x|^\lambda \sum_{i=1}^{i_0-1} \sum_{k=0}^{2^{i^2-i-1}-1} \varphi_{i,k}^\lambda(|x|)$ has a continuous derivative, and the same would happen for $g_\lambda - g_{\lambda,i_0}$. Therefore, if $h \in L^1[-1,1]$, then $(f_\lambda - f_{\lambda,i_0}) * h$ and $(g_\lambda - g_{\lambda,i_0}) * h$ are continuously differentiable (Lemma 1.1-(1)).

Let us show that $V = \mathcal{A}\{f_\lambda : \lambda \in \Lambda\}$ and $W = \mathcal{A}\{g_\lambda : \lambda \in \Lambda\}$ (the two algebras generated by the sets $\{f_\lambda : \lambda \in \Lambda\}$ and $\{g_\lambda : \lambda \in \Lambda\}$, respectively) are the two algebras we are searching for.

Indeed, consider $\alpha_r, \beta_s \in \mathbb{R} \setminus \{0\}$, $\lambda_c, \mu_d \in \Lambda$ and $N_{c,r}, M_{d,s} \in \mathbb{N}$ for $1 \leq c \leq n$, $1 \leq d \leq m$, $1 \leq r \leq R$ and $1 \leq s \leq S$, where $m, n, R, S \in \mathbb{N}$. Define next

$$f(x) = \sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c}^{N_{c,r}}(x),$$
$$g(x) = \sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d}^{M_{d,s}}(x).$$

We remark that for every $r \neq r'$ we can find at least one number $1 \leq c \leq n$ so that $N_{c,r} \neq N_{c,r'}$ since those exponents arise from different monomials in the algebraic combination considered to obtain f (otherwise we may combine the corresponding coefficients α_r and $\alpha_{r'}$ under one single coefficient). The same happens with the exponents $M_{d,s}$.

For $1 \leq r \leq R$ and $1 \leq s \leq S$, we will denote

$$\begin{aligned} \eta_r &= \sum_{c=1}^{n} \lambda_c N_{c,r} \quad \text{and} \\ \sigma_s &= \sum_{d=1}^{m} \mu_d M_{d,s}. \end{aligned} \tag{2.2}$$

Let us select $1 \leq r_0 \leq R$ and $1 \leq s_0 \leq S$ so that

$$\eta_{r_0} = \min\{\eta_r : 1 \leq r \leq R\} \quad \text{and}$$
$$\sigma_{s_0} = \min\{\sigma_s : 1 \leq s \leq S\}$$

Denote $I_{R,S} = (I_R \times I_S) \setminus \{(r_0, s_0)\}$ and notice that, since λ_c and μ_s are linearly independent, we have

$$\eta_r - \eta_{r_0} = \sum_{c=1}^{n} \lambda_c (N_{c,r} - N_{c,r_0}) > 0 \quad \text{and}$$
$$\sigma_s - \sigma_{s_0} = \sum_{d=1}^{m} \mu_d (M_{d,s} - M_{d,s_0}) > 0.$$

Hence, we can choose $i_0 \in \mathbb{N}$ so that, for every $|x| < \frac{1}{2^{i_0-2}}$, we get

$$\sum_{(r,s)\in I_{R,S}} |\alpha_r \beta_s| |x|^{(\eta_r - \eta_{r_0}) + (\sigma_s - \sigma_{s_0})} < \frac{|\alpha_{r_0} \beta_{s_0}|}{2}. \tag{2.3}$$

If now $|x| < \frac{1}{2^{i_0-1}}$, using the expression displayed in (2.1), we may write

$$\begin{aligned}
|(f * g)(x)| &= \left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c}^{N_{c,r}} \right) * \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d}^{M_{d,s}} \right) (x) \right| \\
&= \left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} \left[f_{\lambda_c}^{N_{c,r}} - f_{\lambda_c,i_0}^{N_{c,r}} + f_{\lambda_c,i_0}^{N_{c,r}} \right] \right) \right. \\
&\quad \left. * \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} \left[g_{\mu_d}^{M_{d,s}} - g_{\mu_d,i_0}^{M_{d,s}} + g_{\mu_d,i_0}^{M_{d,s}} \right] \right) (x) \right| \\
&= \left| \left(\sum_{r=1}^{R} \alpha_r \sum_{C\subseteq I_n} \prod_{c_1\in C} \left[f_{\lambda_{c_1}}^{N_{c_1,r}} - f_{\lambda_{c_1},i_0}^{N_{c_1,r}} \right] \prod_{c_2\in I_n\setminus C} f_{\lambda_{c_2},i_0}^{N_{c_1,r}} \right) \right. \\
&\quad \left. * \left(\sum_{s=1}^{S} \beta_s \sum_{D\subseteq I_m} \prod_{d_1\in D} \left[g_{\mu_{d_1}}^{M_{d_1,s}} - g_{\mu_{d_1},i_0}^{M_{d_1,s}} \right] \prod_{d_2\in I_m\setminus D} g_{\mu_{d_2},i_0}^{M_{d_2,s}} \right) (x) \right| \\
&\geq \left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}} \right) * \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}} \right) (x) \right| \\
&\quad - \sum_{(\varnothing,\varnothing)\neq(C,D)\subseteq I_n\times I_m} \left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c_1\in C} \left[f_{\lambda_{c_1}}^{N_{c_1,r}} - f_{\lambda_{c_1},i_0}^{N_{c_1,r}} \right] \prod_{c_2\in I_n\setminus C} f_{\lambda_{c_2},i_0}^{N_{c_1,r}} \right) \right. \\
&\qquad \left. * \left(\sum_{s=1}^{S} \beta_s \prod_{d_1\in D} \left[g_{\mu_{d_1}}^{M_{d_1,s}} - g_{\mu_{d_1},i_0}^{M_{d_1,s}} \right] \prod_{d_2\in I_m\setminus D} g_{\mu_{d_2},i_0}^{M_{d_2,s}} \right) (x) \right|
\end{aligned} \tag{2.4}$$

If we study the first term in this last inequality, we obtain

$$\begin{aligned}
&\left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}} \right) * \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}} \right) (x) \right| \\
&= \left| \int_{-1}^{1} \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \right) \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t) \right) dt \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{(r,s)\in I_R\times I_S} \alpha_r\beta_s \int_{-1}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\,dt \right| \\
&\geq |\alpha_{r_0}\beta_{s_0}| \int_{-1}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t)\,dt \\
&\quad - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \int_{0}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\,dt \\
&= \left[|\alpha_{r_0}\beta_{s_0}| \int_{0}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t)\,dt \right. \\
&\quad \left. - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \int_{0}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\,dt \right] \\
&+ \left[|\alpha_{r_0}\beta_{s_0}| \int_{-1}^{0} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t)\,dt \right. \\
&\quad \left. - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \int_{-1}^{0} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\,dt \right]
\end{aligned}
$$

Let us focus first on the integrals in [0, 1]. For the sake of simplicity, let us denote $c_{i,k} = \frac{1}{2^i} + \frac{2k+1}{2^{i^2}}$ and $d_{i,k} = \frac{1}{2^i} + \frac{2k+2}{2^{i^2}}$ for $i \geq 1$ and $0 \leq k \leq 2^{i^2-i-1} - 1$. Using the notation we introduced in (2.2), if $1 \leq r \leq R$, then

$$
\prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) = \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \prod_{c=1}^{n} \Big(|t|^{\lambda_c} \varphi_{i,k}^{\lambda_c}(|t|) \Big)^{N_{c,r}} = \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} |t|^{\eta_c} \varphi_{i,k}^{\eta_c}(|t|).
$$

Similarly,

$$
\prod_{d=1}^{m} g_{\mu_d,j_0}^{M_{d,s}}(t) = \sum_{j=i_0}^{\infty} \sum_{l=0}^{2^{j^2-j-1}-1} |t|^{\sigma_s} \psi_{j,l}^{\sigma_s}(|t|).
$$

Then

$$
\begin{aligned}
&\left[|\alpha_{r_0}\beta_{s_0}| \int_{0}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t)\,dt \right. \\
&\quad \left. - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \int_{0}^{1} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\,dt \right] \\
&= \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \left(|\alpha_{r_0}\beta_{s_0}| t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \right. \\
&\quad \left. - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| t^{\eta_r} \varphi_{i,k}^{\eta_r}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t) \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \Bigg(|\alpha_{r_0}\beta_{s_0}| t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \\
&\quad - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| t^{\eta_r} \varphi_{i,k}^{\eta_r}(t) \sum_{j=i_0}^{\infty} \sum_{l=0}^{2^{j^2-j-1}-1} |x-t|^{\sigma_s} \psi_{j,l}^{\sigma_s}(|x-t|) \Bigg) \, dt \\
&= \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \Bigg(|\alpha_{r_0}\beta_{s_0}| t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \\
&\quad - t^{\eta_{r_0}} |x-t|^{\sigma_{s_0}} \sum_{j=i_0}^{\infty} \sum_{l=0}^{2^{j^2-j-1}-1} \sum_{(r,s)\in I_{R,S}} \Big(\Big[|\alpha_r\beta_s| t^{\eta_r-\eta_{r_0}} |x-t|^{\sigma_s-\sigma_{s_0}} \\
&\quad \times \psi_{j,l}^{\sigma_s-\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_r-\eta_{r_0}}(t) \Big] \\
&\quad \psi_{j,l}^{\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_{r_0}}(t) \Big) \Bigg) \, dt.
\end{aligned}
$$

In particular, we remark that we are integrating over numbers that satisfy

$$
0 < t < d_{i_0, 2^{i_0^2-i_0-1}-1} = \frac{1}{2^{i_0}} + \frac{2 \cdot 2^{i_0^2-i_0-1}}{2^{i_0^2}} = \frac{1}{2^{i_0-1}}
$$

Now, notice that if $|x| < \frac{1}{2^{i_0-1}}$, then $|x-t| < \frac{1}{2^{i_0-2}}$ and therefore, if we recall the choice of i_0 we made in (2.3),

$$
\begin{aligned}
&\sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| t^{\eta_r-\eta_{r_0}} |x-t|^{\sigma_s-\sigma_{s_0}} \psi_{j,l}^{\sigma_s-\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_r-\eta_{r_0}}(t) \\
&\qquad \leq \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \max\{t, |x-t|\}^{\eta_r-\eta_{r_0}+\sigma_s-\sigma_{s_0}} < \frac{|\alpha_{r_0}\beta_{s_0}|}{2}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
&\Bigg[|\alpha_{r_0}\beta_{s_0}| \int_0^1 \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \, dt \\
&\quad - \sum_{(r,s)\in I_{R,S}} |\alpha_r\beta_s| \int_0^1 \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t) \, dt \Bigg] \\
&= \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \Bigg(|\alpha_{r_0}\beta_{s_0}| t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \\
&\quad - t^{\eta_{r_0}} |x-t|^{\sigma_{s_0}} \sum_{j=i_0}^{\infty} \sum_{l=0}^{2^{j^2-j-1}-1} \sum_{(r,s)\in I_{R,S}} \Big(\Big[|\alpha_r\beta_s| t^{\eta_r-\eta_{r_0}} |x-t|^{\sigma_s-\sigma_{s_0}} \\
&\quad \times \psi_{j,l}^{\sigma_s-\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_r-\eta_{r_0}}(t) \Big] \\
&\quad \psi_{j,l}^{\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_{r_0}}(t) \Big) \Bigg) \, dt
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \Bigg(|\alpha_{r_0}\beta_{s_0}| t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \\
&\quad - t^{\eta_{r_0}} |x-t|^{\sigma_{s_0}} \sum_{j=i_0}^{\infty} \sum_{l=0}^{2^{j^2-j-1}-1} \frac{|\alpha_{r_0}\beta_{s_0}|}{2} \psi_{j,l}^{\sigma_{s_0}}(|x-t|) \varphi_{i,k}^{\eta_{r_0}}(t) \Bigg)\, dt \\
&= \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \left(\frac{|\alpha_{r_0}\beta_{s_0}|}{2} t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \right) dt \\
&\geq \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \left(\frac{|\alpha_{r_0}\beta_{s_0}|}{2} t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \right) dt,
\end{aligned}
$$

for every $|x| < \frac{1}{2^{i_0-1}}$, $i \geq i_0$.

Via a similar argument, we can deduce that

$$
\begin{aligned}
&\Bigg[|\alpha_{r_0}\beta_{s_0}| \int_{-1}^{0} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t)\, dt \\
&\quad - \sum_{(r,s)\in I_{R,S}} |\alpha_r \beta_s| \int_{-1}^{0} \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}}(x-t)\, dt \Bigg] \\
&\geq \sum_{i=i_0}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \int_{-d_{i,k}}^{-c_{i,k}} \left(\frac{|\alpha_{r_0}\beta_{s_0}|}{2} |t|^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(|t|) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \right) dt > 0.
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
&\left| \left(\sum_{r=1}^{R} \alpha_r \prod_{c=1}^{n} f_{\lambda_c,i_0}^{N_{c,r}} \right) * \left(\sum_{s=1}^{S} \beta_s \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s}} \right)(x) \right| \\
&\quad \geq \sum_{k=0}^{2^{i^2-i-1}-1} \int_{c_{i,k}}^{d_{i,k}} \left(\frac{|\alpha_{r_0}\beta_{s_0}|}{2} t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}(x-t) \right) dt,
\end{aligned}
\tag{2.5}
$$

for every $|x| < \frac{1}{2^{i_0-1}}$, $i \geq i_0$.

Denoting $\bar{a}_{i,k} = \frac{1}{2^i} + \frac{8k+5}{2^{i^2+2}}$, $\bar{b}_{i,k} = \frac{1}{2^i} + \frac{8k+7}{2^{i^2+2}}$ and noticing that the functions $\varphi_{i,k}^{\eta_{r_0}}$ and g_{μ_d,i_0} are non negative, we see that

$$
\begin{aligned}
&\int_{c_{i,k}}^{d_{i,k}} \left(t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}\left(\frac{1}{2^{i^2}} - t \right) \right) dt \\
&= \int_{c_{i,k}}^{d_{i,k}} \left(t^{\eta_{r_0}} \varphi_{i,k}^{\eta_{r_0}}(t) \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}\left(t - \frac{1}{2^{i^2}} \right) \right) dt \\
&\geq \int_{\bar{a}_{i,k}}^{\bar{b}_{i,k}} \left(t^{\eta_{r_0}} \prod_{d=1}^{m} g_{\mu_d,i_0}^{M_{d,s_0}}\left(t - \frac{1}{2^{i^2}} \right) \right) dt.
\end{aligned}
$$

Now, if

$$t \in \left[\frac{1}{2^i} + \frac{8k+5}{2^{i^2+2}}, \frac{1}{2^i} + \frac{8k+7}{2^{i^2+2}} \right],$$

we have that

$$t - \frac{1}{2^{i^2}} \in \left[\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}}, \frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right].$$

Then

$$\begin{aligned}
\int_{\bar{a}_{i,k}}^{\bar{b}_{i,k}} \left(t^{\eta_{r_0}} \prod_{d=1}^{m} g_{\mu_d, i_0}^{M_{d,s_0}} \left(t - \frac{1}{2^{i^2}} \right) \right) dt &= \int_{\bar{a}_{i,k}}^{\bar{b}_{i,k}} \left(t^{\eta_{r_0}} \left(t - \frac{1}{2^{i^2}} \right)^{\sigma_{s_0}} \right) dt \\
&\geq \int_{\bar{a}_{i,k}}^{\bar{b}_{i,k}} \left(t - \frac{1}{2^{i^2}} \right)^{\eta_{r_0}+\sigma_{s_0}} dt \\
&= \frac{1}{\eta_{r_0}+\sigma_{s_0}+1} \left(t - \frac{1}{2^{i^2}} \right)^{\eta_{r_0}+\sigma_{s_0}+1} \Bigg|_{\bar{a}_{i,k}}^{\bar{b}_{i,k}} \\
&= \frac{1}{\eta_{r_0}+\sigma_{s_0}+1} \left[\left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{\eta_{r_0}+\sigma_{s_0}+1} \right. \\
&\quad \left. - \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{\eta_{r_0}+\sigma_{s_0}+1} \right].
\end{aligned}$$

Using Lemma 2.1, we can write

$$\begin{aligned}
&\left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{\eta_{r_0}+\sigma_{s_0}+1} - \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{\eta_{r_0}+\sigma_{s_0}+1} \\
&\geq \min \left\{ \left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]} - \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]}, \right. \\
&\quad \left. \left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]+1} - \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]+1} \right\},
\end{aligned}$$

where $[[x]]$ denotes the integer part of x.

On one hand,

$$\begin{aligned}
&\left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]} - \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]} \\
&= \frac{1}{2^{i^2+1}} \left[\left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]-1} \right. \\
&\quad + \left(\frac{1}{2^i} + \frac{8k+3}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]-2} \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right) + \cdots \\
&\quad \left. + \left(\frac{1}{2^i} + \frac{8k+1}{2^{i^2+2}} \right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]-1} \right] \\
&\geq \frac{1}{2^{i^2+1}} \frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{2^{([[\eta_{r_0}+\sigma_{s_0}+1]]-1)i}} \geq \frac{1}{2^{i^2+1}} \frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{2^{[[\eta_{r_0}+\sigma_{s_0}+1]]i}}.
\end{aligned}$$

Analogously,

$$\begin{aligned}&\left(\frac{1}{2^i}+\frac{8k+3}{2^{i^2+2}}\right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]+1}-\left(\frac{1}{2^i}+\frac{8k+1}{2^{i^2+2}}\right)^{[[\eta_{r_0}+\sigma_{s_0}+1]]+1}\\&\quad\geq\frac{1}{2^{i^2+1}}\frac{[[\eta_{r_0}+\sigma_{s_0}+1]]+1}{2^{[[\eta_{r_0}+\sigma_{s_0}+1]]i}}\geq\frac{1}{2^{i^2+1}}\frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{2^{[[\eta_{r_0}+\sigma_{s_0}+1]]i}}.\end{aligned}$$

In conclusion,

$$\left(\frac{1}{2^i}+\frac{8k+3}{2^{i^2+2}}\right)^{\eta_{r_0}+\sigma_{s_0}+1}-\left(\frac{1}{2^i}+\frac{8k+1}{2^{i^2+2}}\right)^{\eta_{r_0}+\sigma_{s_0}+1}\geq\frac{1}{2^{i^2+1}}\frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{2^{[[\eta_{r_0}+\sigma_{s_0}+1]]i}}.$$

Hence, we can work on the inequality obtained in (2.5) and conclude that

$$\begin{aligned}&\left|\left(\sum_{r=1}^{R}\alpha_r\prod_{c=1}^{n}f_{\lambda_c,i_0}^{N_{c,r}}\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d=1}^{m}g_{\mu_d,i_0}^{M_{d,s}}\right)\left(\frac{1}{2^{i^2}}\right)\right|\\&\quad\geq\frac{|\alpha_{r_0}\beta_{s_0}|}{2}\sum_{k=0}^{2^{i^2-i-1}-1}\frac{1}{\eta_{r_0}+\sigma_{s_0}+1}\cdot\frac{1}{2^{i^2+1}}\frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{2^{[[\eta_{r_0}+\sigma_{s_0}+1]]i}}\\&\quad=\frac{|\alpha_{r_0}\beta_{s_0}|}{2}\frac{[[\eta_{r_0}+\sigma_{s_0}+1]]}{\eta_{r_0}+\sigma_{s_0}+1}\cdot\frac{1}{2^{1+([[\eta_{r_0}+\sigma_{s_0}+1]]+1)i}}\end{aligned}$$

for every $i\geq i_0$. As a result,

$$\frac{\left|\left(\sum_{r=1}^{R}\alpha_r\prod_{c=1}^{n}f_{\lambda_c,i_0}^{N_{c,r}}\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d=1}^{m}g_{\mu_d,i_0}^{M_{d,s}}\right)\left(\frac{1}{2^{i^2}}\right)\right|}{\frac{1}{2^{i^2}}}\xrightarrow[i\to\infty]{}\infty.$$

Let us now take a look at the other terms that appear in the last inequality from (2.4)

$$\begin{aligned}\sum_{(\varnothing,\varnothing)\neq(C,D)\subseteq I_n\times I_m}&\left|\left(\sum_{r=1}^{R}\alpha_r\prod_{c_1\in C}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\prod_{c_2\in I_n\setminus C}f_{\lambda_{c_2},i_0}^{N_{c_2,r}}\right)\right.\\&\left.*\left(\sum_{s=1}^{S}\beta_s\prod_{d_1\in D}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\prod_{d_2\in I_m\setminus D}g_{\mu_{d_2},i_0}^{M_{d_2,s}}\right)\left(\frac{1}{2^{i^2}}\right)\right|.\end{aligned}$$

If we take a look at the supports of the functions that take part in the previous expression, we notice that

$$\prod_{c_1\in C}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\prod_{c_2\in I_n\setminus C}f_{\lambda_{c_2},i_0}^{N_{c_2,r}}=0$$

unless $C\in\{\varnothing,I_n\}$, and the same will happen with

$$\prod_{d_1\in D}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\prod_{d_2\in I_m\setminus D}g_{\mu_{d_2},i_0}^{M_{d_2,s}}.$$

Therefore, we can write

$$\begin{aligned}
&\sum_{(\varnothing,\varnothing)\neq(C,D)\subseteq I_n\times I_m}\Bigg|\left(\sum_{r=1}^{R}\alpha_r\prod_{c_1\in C}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\prod_{c_2\in I_n\setminus C}f_{\lambda_{c_2},i_0}^{N_{c_2,r}}\right)\\
&\qquad *\left(\sum_{s=1}^{S}\beta_s\prod_{d_1\in D}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\prod_{d_2\in I_m\setminus D}g_{\mu_{d_2},i_0}^{M_{d_2,s}}\right)\left(\frac{1}{2^{i^2}}\right)\Bigg|\\
&=\left(\sum_{r=1}^{R}\alpha_r\prod_{c_1\in I_n}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d_1\in I_m}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\right)\\
&\quad+\left(\sum_{r=1}^{R}\alpha_r\prod_{c_1\in I_n}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d_2\in I_m}g_{\mu_{d_2},i_0}^{M_{d_2,s}}\right)\\
&\quad+\left(\sum_{r=1}^{R}\alpha_r\prod_{c_2\in I_n}f_{\lambda_{c_2},i_0}^{N_{c_2,r}}\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d_1\in I_m}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\right).
\end{aligned}$$

Since at least one of the functions that appear in each of the former convolutions is continuously differentiable (and the other parent function is at least continuous), we can find a constant $K > 0$ so that, for every $i \geq 1$,

$$\begin{aligned}
&\sum_{(\varnothing,\varnothing)\neq(C,D)\subseteq I_n\times I_m}\Bigg|\left(\sum_{r=1}^{R}\alpha_r\prod_{c_1\in C}\left[f_{\lambda_{c_1}}^{N_{c_1,r}}-f_{\lambda_{c_1},i_0}^{N_{c_1,r}}\right]\prod_{c_2\in I_n\setminus C}f_{\lambda_{c_2},i_0}^{N_{c_2,r}}\right)\\
&\qquad *\left(\sum_{s=1}^{S}\beta_s\prod_{d_1\in D}\left[g_{\mu_{d_1}}^{M_{d_1,s}}-g_{\mu_{d_1},i_0}^{M_{d_1,s}}\right]\prod_{d_2\in I_m\setminus D}g_{\mu_{d_2},i_0}^{M_{d_2,s}}\right)\left(\frac{1}{2^{i^2}}\right)\Bigg|\\
&\leq K\frac{1}{2^{i^2}}
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{|(f*g)\left(\frac{1}{2^{i^2}}\right)|}{\frac{1}{2^{i^2}}} &\geq \frac{\left|\left(\sum_{r=1}^{R}\alpha_r\prod_{c=1}^{n}f_{\lambda_c,i_0}^{N_{c,r}}\right)*\left(\sum_{s=1}^{S}\beta_s\prod_{d=1}^{m}g_{\mu_d,i_0}^{M_{d,s}}\right)\left(\frac{1}{2^{i^2}}\right)\right|}{\frac{1}{2^{i^2}}}\\
&\quad - K\xrightarrow[i\to\infty]{}\infty,
\end{aligned}$$

which shows the non-differentiability of $f * g$ at the origin. □

This result shows, in particular, that every function in $C[-1, 1]$ can be *approximated* by vectors taken from a linear space consisting of differentiable functions that (except for 0) may be paired with another differentiable function so that the resulting convolution fails to be differentiable at the origin. We need the following lemma, whose proof can be found in [2, Section 7.3].

Lemma 2.3. *Assume that α is an infinite cardinal number, that X is a separable metrizable topological vector space, and that A, B are subsets of X satisfying the following conditions:*
(i) *A is lineable.*
(ii) *B is dense-lineable in X.*

(iii) $A \cap B = \varnothing$.
(iv) $A + B \subset A$.
Then A is α-dense-lineable in X.

Theorem 2.4. *There exists a* $\mathfrak{c}$*-dimensional algebra $W \subset D[-1, 1]$ such that the set*

$$A_W := \{f \in D[-1, 1] : \textit{for all } g \in W \setminus \{0\},\ \ f * g \textit{ is not differentiable at } 0\}$$

is $\mathfrak{c}$*-dense-lineable in X.*

Proof. Using Lemma 1.1, for every $g \in D[-1, 1]$ and every polynomial P, the function $P * g$ is differentiable. Therefore, if $f, g \in D[-1, 1]$, are functions such that $f * g$ is not differentiable at the origin, then $(f + P) * g = f * g + P * g$, is not differentiable either.

On the other hand, a well known refinement of the Weierstrass approximation theorem tells us that, if $\varepsilon > 0$ and $f : [a, b] \to \mathbb{R}$ is a continuous function with $f(a) = f(b)$, then there is a polynomial P such that $P(a) = P(b)$, $P'(a) = P'(b)$ and $|P(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$. Taking $[a, b] = [-1, 1]$, we derive that the vector space

$$P_{[-1,1]} := \Big\{\text{periodic extensions to } \mathbb{R} \text{ of the polynomials } P \text{ such that } P(-1) = P(1) \\ \text{and } P'(-1) = P'(1)\Big\}$$

is dense in $C[-1, 1]$.

Finally, consider the $\mathfrak{c}$-dimensional linear algebras $V, W \subset D[-1, 1]$ constructed in Theorem 2.2. Since V is in particular a $\mathfrak{c}$-dimensional vector space and $A_W \supset V$, we derive that A_W is $\mathfrak{c}$-lineable. Of course, $P_{[-1,1]} \cap A_W = \varnothing$, and $A_W + P_{[-1,1]} \subset A_W$ thanks to the property proved in the first paragraph of this proof. Thus, it is enough to apply Lemma 2.3 to $X := C[-1, 1]$, $\alpha := \mathfrak{c}$, $A := A_W$, and $B := P_{[-1,1]}$. $\square$

The algebras considered in Theorem 2.2 are generated by sets of the largest possible cardinality ($\mathfrak{c}$), but they are not closed. Before giving the result that deals with closed structures, we will state the following:

Remark 2.5. For $x_0 \neq x_1$ and $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ there exists a unique cubic polynomial $T_{x_0,x_1}(\mathbf{v})(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ satisfying

$$\begin{cases} T_{x_0,x_1}(\mathbf{v})(x_0) = v_1, \\ T'_{x_0,x_1}(\mathbf{v})(x_0) = v_2, \\ T_{x_0,x_1}(\mathbf{v})(x_1) = v_3, \\ T'_{x_0,x_1}(\mathbf{v})(x_1) = v_4. \end{cases} \tag{2.6}$$

Notice that the conditions (2.6) translate into a linear system of equations,

$$\begin{cases} 3x_1^2c_3 + 2x_1c_2 + c_1 = v_4, \\ 3x_0^2c_3 + 2x_0c_2 + c_1 = v_2, \\ x_1^3c_3 + x_1^2c_2 + x_1c_1 + c_0 = v_3, \\ x_0^3c_3 + x_0^2c_2 + x_0c_1 + c_0 = v_1, \end{cases} \tag{2.7}$$

which has as determinant

$$\det \begin{bmatrix} 3x_1^2 & 2x_1 & 1 & 0 \\ 3x_0^2 & 2x_0 & 1 & 0 \\ x_1^3 & x_1^2 & x_1 & 1 \\ x_0^3 & x_0^2 & x_0 & 1 \end{bmatrix} = -x_1^4 + 4x_1^3 x_0 - 6x_1^2 x_0^2 + 4x_1 x_0^3 - x_0^4 = -(x_1 - x_0)^4.$$

In particular, the matrix of the system is invertible and therefore there exists a unique solution. Notice also that when $\mathbf{v} = 0 \in \mathbb{R}^4$, we obtain $T_{x_0,x_1}(\mathbf{v}) = 0$ for every $x_0 \neq x_1$.

Theorem 2.6. *There exist two closed positive cones of* $D[-1, 1]$ *with the property that the convolution of two elements, each one taken from each cone, is not differentiable at* 0*, as long as the parent functions are nonzero.*

Proof. Let $\{p_m : m \in \mathbb{N}\}$ be the usual enumeration of prime numbers and define the function

$$\begin{array}{rccl} \mathfrak{P}: & \mathbb{N} & \to & \mathbb{N} \\ & n & \mapsto & \mathfrak{P}(n) = \min\{m \geq 1 : p_m | n\} \end{array}$$

Consider also a bijection $d = (d_1, d_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. For $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, $i \geq 1$, $0 \leq k \leq 2^{7i-1} - 1$ and $x \in [0, 1]$, define the function $\varphi^{\mathbf{a}}_{i,k}$ as follows:

$$\varphi^{\mathbf{a}}_{i,k}(x) = \begin{cases} T_{i,8k+4}\left((0, 0, a_{d_1(\mathfrak{P}(i))}, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{2k+1}{2^{8i}} < x < \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}}, \\ a_{d_1(\mathfrak{P}(i))} & \text{if } \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}}, \\ T_{i,8k+7}\left((a_{d_1(\mathfrak{P}(i))}, 0, 0, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}} < x < \frac{1}{2^i} + \frac{2k+2}{2^{8i}}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } k \text{ is even},$$

$$\varphi^{\mathbf{a}}_{i,k}(x) = \begin{cases} T_{i,8k+4}\left((0, 0, a_{d_2(\mathfrak{P}(i))}, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{2k+1}{2^{8i}} < x < \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}}, \\ a_{d_2(\mathfrak{P}(i))} & \text{if } \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}}, \\ T_{i,8k+7}\left((a_{d_2(\mathfrak{P}(i))}, 0, 0, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}} < x < \frac{1}{2^i} + \frac{2k+2}{2^{8i}}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } k \text{ is odd},$$

where we are using the cubic polynomials considered in Remark 2.5 and we are denoting

$$T_{i,l}(\mathbf{v}) = T_{x_0,x_1}(\mathbf{v}) \quad \text{for } x_0 = \frac{1}{2^i} + \frac{l}{2^{8i+2}},\ x_1 = \frac{1}{2^i} + \frac{l+1}{2^{8i+2}}.$$

Next, extend $\varphi^{\mathbf{a}}_{i,k}$ to $[-1, 0)$ via $\varphi^{\mathbf{a}}_{i,k}(-x) = \varphi_{i,k}(x)$ (see Fig. 1 for a sketch of $\varphi^{\mathbf{a}}_{i,k}$).

The idea of using the functions d and $\mathfrak{P}$ is to repeat every pair of the sequence $\mathbf{a}$ infinitely many times throughout a sequence of points that tend to 0. More concretely, if $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ and we set $n = d^{-1}(m_1, m_2)$, $i = p_n^l$ for $l \geq 1$ and $\frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}}$ for k even, we obtain that

$$\varphi^{\mathbf{a}}_{i,k}(x) = a_{d_1(\mathfrak{P}(i))} = a_{d_1(\mathfrak{P}(p_n^l))} = a_{d_1(n)} = a_{m_1}.$$

Similarly, if $\frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}}$ for k odd,

$$\varphi^{\mathbf{a}}_{i,k}(x) = a_{m_2}.$$

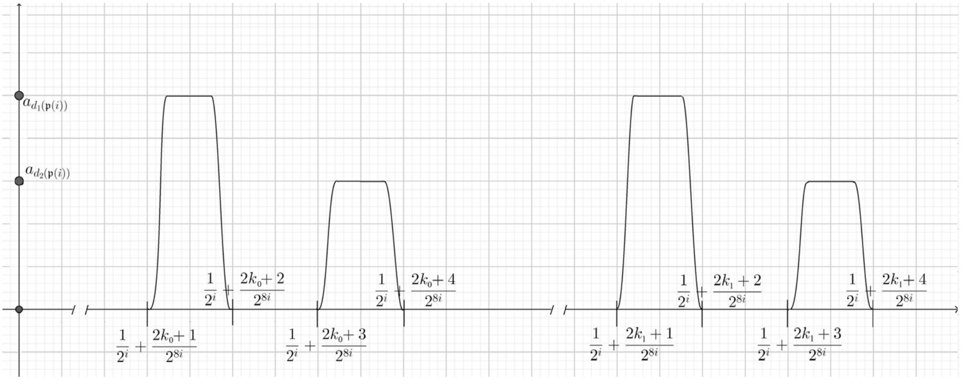

Fig. 1. Construction of the functions $\varphi^{\mathbf{a}}_{i,k}$, given two choices of even numbers k_0 and k_1.

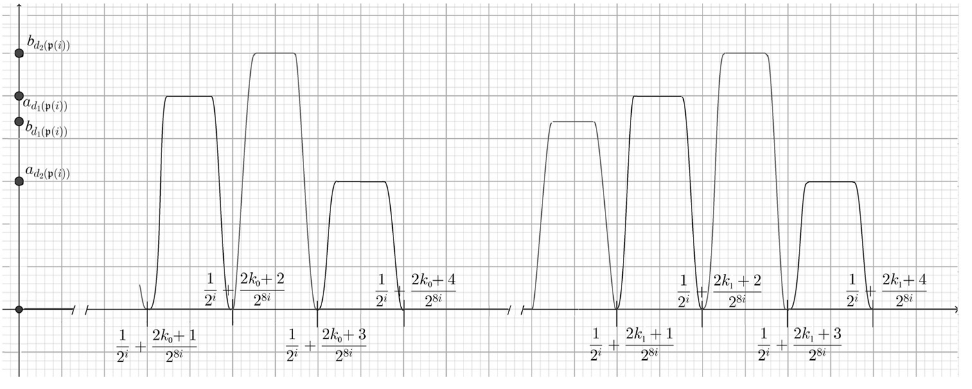

Fig. 2. Construction of the functions $\psi^{\mathbf{b}}_{i,k}$ in comparison with the functions $\varphi^{\mathbf{a}}_{i,k}$.

In an analogous way, for $x \in [0, 1]$, $\mathbf{b} \in \mathbb{R}^{\mathbb{N}}$, $i \geq 1$, $0 \leq k \leq 2^{7i-1} - 1$ we can define the function

$$\psi^{\mathbf{b}}_{i,k}(x) = \begin{cases} T_{i,8k}\left((0, 0, b_{d_1(\mathfrak{P}(i))}, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{2k}{2^{8i}} < x < \frac{1}{2^i} + \frac{8k+1}{2^{8i+2}}, \\ b_{d_1(\mathfrak{P}(i))} & \text{if } \frac{1}{2^i} + \frac{8k+1}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}}, \\ T_{i,8k+3}\left((b_{d_1(\mathfrak{P}(i))}, 0, 0, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}} < x < \frac{1}{2^i} + \frac{2k+1}{2^{8i}}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } k \text{ is even,}$$

$$\psi^{\mathbf{b}}_{i,k}(x) = \begin{cases} T_{i,8k}\left((0, 0, b_{d_2(\mathfrak{P}(i))}, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{2k}{2^{8i}} < x < \frac{1}{2^i} + \frac{8k+1}{2^{8i+2}}, \\ b_{d_2(\mathfrak{P}(i))} & \text{if } \frac{1}{2^i} + \frac{8k+1}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}}, \\ T_{i,8k+3}\left((b_{d_2(\mathfrak{P}(i))}, 0, 0, 0)\right)(x) & \text{if } \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}} < x < \frac{1}{2^i} + \frac{2k+1}{2^{8i}}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } k \text{ is odd,}$$

and proceed to extend $\psi^{\mathbf{b}}_{i,k}$ to $[-1, 0)$ via even symmetry. The reader can find a comparison between the constructions of $\varphi^{\mathbf{a}}_{i,k}$ and $\psi^{\mathbf{b}}_{i,k}$ in Fig. 2.

Consider the set $c_\downarrow^+ := \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : x_n \geq x_{n+1} \geq 0 \text{ for every } n \geq 1\}$ and define the operators

$$\begin{array}{rcccccl} \Phi: & c_\downarrow^+ & \longrightarrow & C[-1,1] & & & \\ & \mathbf{a} & \mapsto & \Phi(\mathbf{a}): & [-1,1] & \to & \mathbb{R} \\ & & & & x & \mapsto & x^2 \displaystyle\sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}^{\mathbf{a}}(x), \\ \Psi: & c_\downarrow^+ & \longrightarrow & C[-1,1] & & & \\ & \mathbf{b} & \mapsto & \Psi(\mathbf{b}): & [-1,1] & \to & \mathbb{R} \\ & & & & x & \mapsto & x^2 \displaystyle\sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \psi_{i,k}^{\mathbf{b}}(x). \end{array}$$

Notice that Φ and Ψ are positive-linear due to the uniqueness of the polynomial satisfying the conditions in (2.6). Trivially $\|\Phi(\mathbf{a})\|_\infty \leq \|\mathbf{a}\|_\infty$ and Φ is one-to-one, and the same properties apply to Ψ. Therefore, Φ and Ψ are positive isomorphisms into $\overline{\Phi(c_\downarrow^+)}$ and $\overline{\Psi(c_\downarrow^+)}$, respectively, so that $\overline{\Phi(c_\downarrow^+)}$ and $\overline{\Psi(c_\downarrow^+)}$ are closed positive cones.

Let now $\{\mathbf{a}^{(n)}\}_{n=1}^\infty$, $\{\mathbf{b}^{(n)}\}_{n=1}^\infty \subseteq c_\downarrow^+$ so that $\Phi(\mathbf{a}^{(\mathbf{n})}) \xrightarrow[n\to\infty]{\|\cdot\|_\infty} f \in C[-1,1]$ and $\Psi(\mathbf{b}^{(\mathbf{n})}) \xrightarrow[n\to\infty]{\|\cdot\|_\infty} g \in C[-1,1]$.

Notice that $\Phi(\mathbf{a})(0) = 0$ for every $\mathbf{a} \in c_\downarrow^+$ and set $l_1 = d^{-1}(1,1)$. Then,

$$\begin{aligned} \Phi(\mathbf{a}^{(n)})\left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right) &= \left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right)^2 \varphi_{p_{l_1},0}^{\mathbf{a}^{(n)}}\left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right) \\ &= \left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right)^2 a_{d_1(\mathfrak{P}(p_{l_1}))}^{(n)} \\ &= \left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right)^2 a_1^{(n)} \xrightarrow[n\to\infty]{} f\left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right). \end{aligned}$$

Therefore

$$a_1^{(n)} \xrightarrow[n\to\infty]{} \frac{f\left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right)}{\left(\frac{1}{2^{p_{l_1}}} + \frac{6}{2^{8p_{l_1}+2}}\right)^2} \tag{2.8}$$

and, in particular, $\{a_1^{(n)}\}_{n=1}^\infty$ is a bounded sequence. Since $\mathbf{a}^{(n)}$ is a non-increasing sequence for every $n \in \mathbb{N}$, we obtain that $\{a_s^{(n)} : n, s \geq 1\}$ is also a bounded set, let us say by $M > 0$.

Let now $x \in (0,1)$ and find integers i, k with $i \geq 1$ and $0 \leq k \leq 2^{7i-1} - 1$, so that $x \in \left(\frac{1}{2^i} + \frac{2k}{2^{8i}}, \frac{1}{2^i} + \frac{2k+2}{2^{8i}}\right)$.

Then $\Phi(\mathbf{a}^{(n)})(x) = x^2 \varphi_{i,k}^{\mathbf{a}^{(n)}}(x) \xrightarrow[n\to\infty]{} f(x)$. On the other hand, $\varphi_{i,k}^{\mathbf{a}^{(n)}}(x) \leq \max\left\{a_{d_1(\mathfrak{P}(i))}, a_{d_2(\mathfrak{P}(i))}\right\} < M + 1$.

Let us choose $\varepsilon = x^2\left(M + 1 - \max\left\{a_{d_1(\mathfrak{P}(i))}, a_{d_2(\mathfrak{P}(i))}\right\}\right) > 0$ (so that $\varepsilon \leq x^2(M + 1 - \varphi_{i,k}^{\mathbf{a}^{(m)}}(x))$ for every $m \in \mathbb{N}$). Then we can find $n_0 \in \mathbb{N}$ so that, for every $n \geq n_0$, $\left|f(x) - x^2\varphi_{i,k}^{\mathbf{a}^{(n)}}(x)\right| < \varepsilon \leq x^2\left[M + 1 - \varphi_{i,k}^{\mathbf{a}^{(n)}}(x)\right]$. Hence

$$0 \leq \frac{f(x)}{x} \leq \left| \frac{f(x)}{x} - x\varphi_{i,k}^{\mathbf{a}^{(n)}}(x) \right| + x\varphi_{i,k}^{\mathbf{a}^{(n)}}(x) < \frac{1}{x}\left(x^2\left[M+1-\varphi_{i,k}^{\mathbf{a}^{(n)}}(x)\right]\right) + x\varphi_{i,k}^{\mathbf{a}^{(n)}}(x)$$
$$= x(M+1).$$

Applying the Squeeze Theorem, we have that $\lim_{x\to 0^+} \frac{f(x)}{x} = 0$.

Since $\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{-f(-x)}{-(-x)} = \lim_{x\to 0^+} \frac{f(x)}{x} = 0$, we obtain $f'(0) = 0$.

As in (2.8), we notice that, for every $r \in \mathbb{N}$,

$$a_r^{(n)} \xrightarrow[n\to\infty]{} \frac{f\left(\frac{1}{2^{p_{l_r}}} + \frac{6}{2^{8p_{l_r}+2}}\right)}{\left(\frac{1}{2^{p_{l_r}}} + \frac{6}{2^{8p_{l_r}+2}}\right)^2},$$

where $l_r = d^{-1}(r, 1)$.

Let next $x \in \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right)$. Then

$$\Phi(\mathbf{a}^{(n)})(x) = x^2 \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}^{\mathbf{a}^{(n)}}(x).$$

We observe that, if $A_{i,k}(\mathbf{v})$ is the matrix of the system in Remark 2.5, then the coefficients of the cubic polynomial are given by $A_{i,k}(\mathbf{v})^{-1}\mathbf{v}$. Therefore, if $\{\mathbf{v}_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^4$ is such that $\mathbf{v}_n \xrightarrow[n\to\infty]{} \mathbf{v}$, then $T_{i,k}(\mathbf{v}_n) \xrightarrow[n\to\infty]{} T_{i,k}(\mathbf{v})$. In conclusion, for $x \in \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right)$,

$$f(x) = \lim_{n\to\infty} x^2 \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}^{\mathbf{a}^{(n)}}(x) = x^2 \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}^{\lim_{n\to\infty}\mathbf{a}^{(n)}}(x),$$

so f is differentiable on $\left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right)$.

If finally $x = \frac{1}{2^{i_0}}$, notice that we can write

$$\Phi(\mathbf{a}^{(n)})(y) = y^2 \varphi_{i_0+1,2^{7i_0+6}-1}^{\mathbf{a}^{(n)}}(y)$$

for

$$y \in J_{i_0} := \left(\frac{1}{2^{i_0+1}} + \frac{2(2^{7(i_0+1)-1}-1)+1}{2^{8(i_0+1)}}, \frac{1}{2^{i_0}} + \frac{1}{2^{8i_0}}\right).$$

On the other hand,

$$y^2 \varphi_{i_0+1,2^{7i_0+6}-1}^{\mathbf{a}^{(n)}}(y) \xrightarrow[n\to\infty]{} y^2 \varphi_{i_0+1,2^{7i_0+6}-1}^{\lim_{n\to\infty}\mathbf{a}^{(n)}}(y)$$

for every $y \in (0, 1)$. In particular,

$$f(y) = y^2 \varphi_{i_0+1,2^{7i_0+6}-1}^{\lim_{n\to\infty}\mathbf{a}^{(n)}}(y)$$

for every $y \in J_{i_0}$, so that f is differentiable on J_{i_0} (and therefore on x). In conclusion, f is a differentiable function. Using a similar argument, g is also differentiable.

Assume next that both f and g are not zero. In this situation there must be some $r_0 \in \mathbb{N}$ so that $a_{r_0}^{(n)} \xrightarrow[n\to\infty]{} a_{r_0} > 0$.

Indeed, otherwise we can choose, given $x \in (0, 1)$, $i \geq 1$ and $0 \leq 2^{7i-1} - 1$ so that $x \in \left[\frac{1}{2^i} + \frac{2k}{2^{8i}}, \frac{1}{2^i} + \frac{2k+2}{2^{8i}}\right)$. Then

$$\Phi(\mathbf{a}^{(n)})(x) = x^2\varphi_{i,k}^{\mathbf{a}^{(n)}}(x) \le x^2 \max\left\{a_{d_1(\mathfrak{P}(i))}^{(n)}, a_{d_2(\mathfrak{P}(i))}^{(n)}\right\} \xrightarrow[n\to\infty]{} 0,$$

so $f(x) = 0$.

Similarly, we can find $s_0 \in \mathbb{N}$ so that $b_{s_0}^{(n)} \xrightarrow[n\to\infty]{} b_{s_0} > 0$.

Set $m_0 = d^{-1}(r_0, s_0)$ and let $l \ge 1$. For reasons that will be clarified later, we are going to choose $K > 0$ is so that

$$\left(4\max\{a_i^{(n)}, b_j^{(m)} : i, j, n, m \in \mathbb{N}\} + 2K\right)K \le \frac{a_{r_0}b_{s_0}}{32}\frac{1}{2^{5p_{m_0}^l}}. \tag{2.9}$$

For this K, we can find $n_l \in \mathbb{N}$ so that, for every $x \in [-1, 1]$,

$$\max\left\{|f(x) - \Phi(\mathbf{a}^{(n_l)})(x)|, |g(x) - \Psi(\mathbf{b}^{(n_l)})(x)|\right\} \le K. \tag{2.10}$$

Additionally (again, for reasons that will be revealed later) such n_l can be chosen so that

$$a_{r_0}^{(n_l)} \ge \frac{a_{r_0}}{\sqrt{2}},\ b_{s_0}^{(n_l)} \ge \frac{b_{s_0}}{\sqrt{2}}. \tag{2.11}$$

Then

$$\begin{aligned}
(f * g)(x) &= \left(f - \Phi(\mathbf{a}^{(n_l)}) + \Phi(\mathbf{a}^{(n_l)})\right) * \left(g - \Psi(\mathbf{b}^{(n_l)}) + \Psi(\mathbf{b}^{(n_l)})\right)(x)\\
&\ge \Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})(x)\\
&\quad - \Big(|(f - \Phi(\mathbf{a}^{n_l})) * (g - \Psi(\mathbf{b}^{n_l}))| + |(f - \Phi(\mathbf{a}^{n_l})) * \Psi(\mathbf{b}^{n_l})|\\
&\quad + |\Phi(\mathbf{a}^{n_l}) * (g - \Psi(\mathbf{b}^{n_l}))|\Big)(x)\\
&\overset{(2.9)}{\ge} \Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})(x) - (2K^2 + 2K\max\{a_i^{(n)}, b_j^{(m)} : i, j, n, m \in \mathbb{N}\}\\
&\quad + 2K\max\{a_i^{(n)}, b_j^{(m)} : i, j, n, m \in \mathbb{N}\})\\
&= \Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})(x) - \left(4\max\{a_i^{(n)}, b_j^{(m)} : i, j, n, m \in \mathbb{N}\} + 2K\right)K\\
&\overset{(2.10)}{\ge} \Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})(x) - \frac{a_{r_0}b_{s_0}}{32}\frac{1}{2^{5p_{m_0}^l}}.
\end{aligned}$$

Let us, for the sake of simplicity, denote $\alpha_{l,k} = \frac{1}{2^{p_{m_0}^l}} + \frac{8(2k+1)+1}{2^{8p_{m_0}^l+2}}$ and $\beta_{l,k} = \frac{1}{2^{p_{m_0}^l}} + \frac{8(2k+1)+3}{2^{8p_{m_0}^l+2}}$ for $k = 0, \ldots, 2^{7p_{m_0}^l-2} - 1$. Then

$$\begin{aligned}
\Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})\left(\frac{1}{2^{8p_{m_0}^l}}\right) &= \int_{-1}^{1} \Phi(\mathbf{a}^{(n_l)})\left(\frac{1}{2^{8p_{m_0}^l}} - s\right)\Psi(\mathbf{b}^{(n_l)})(s)\,ds\\
&= \int_{-1}^{1} \Phi(\mathbf{a}^{(n_l)})\left(s - \frac{1}{2^{8p_{m_0}^l}}\right)\Psi(\mathbf{b}^{(n_l)})(s)\,ds\\
&\quad \text{since } \Phi(\mathbf{a}^{(\mathbf{n_l})}) \text{ is even}\\
&\ge \int_{\frac{1}{2^{p_{m_0}^l}}}^{\frac{1}{2^{p_{m_0}^l-1}}} \Phi(\mathbf{a}^{(n_l)})\left(s - \frac{1}{2^{8p_{m_0}^l}}\right)\Psi(\mathbf{b}^{(n_l)})(s)\,ds\\
&\quad \text{since } \Phi(\mathbf{a}^{(n_l)}),\ \Psi(\mathbf{b}^{(n_l)}) \ge 0\\
&\ge \sum_{k=0}^{2^{7p_{m_0}^l-2}-1}\int_{\alpha_{l,k}}^{\beta_{l,k}} \Phi(\mathbf{a}^{(n_l)})\left(s - \frac{1}{2^{8p_{m_0}^l}}\right)\Psi(\mathbf{b}^{(n_l)})(s)\,ds.
\end{aligned}$$

Now, notice that if

$$s \in [\alpha_{l,k}, \beta_{l,k}] = \left[\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k+1)+1}{2^{8p^l_{m_0}+2}}, \frac{1}{2^{p^l_{m_0}}} + \frac{8(2k+1)+3}{2^{8p^l_{m_0}+2}}\right],$$

then

$$s - \frac{1}{2^{8p^l_{m_0}}} \in \left[\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+5}{2^{8p^l_{m_0}+2}}, \frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+7}{2^{8p^l_{m_0}+2}}\right].$$

Therefore

$$\begin{aligned}
\Phi(\mathbf{a}^{(n_l)})\left(s - \frac{1}{2^{8p^l_{m_0}}}\right) &= \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^2 a^{(n_l)}_{d_1(\mathfrak{P}(p^l_{m_0}))} \\
&= \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^2 a^{(n_l)}_{d_1(m_0)} = \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^2 a^{(n_l)}_{r_0} \\
&\overset{(2.11)}{\geq} \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^2 \frac{a_{r_0}}{\sqrt{2}}
\end{aligned}$$

and, similarly,

$$\Psi(\mathbf{b}^{(n_l)})(s) = s^2 b^{(n_l)}_{d_2(\mathfrak{P}(p^l_{m_0}))} = s^2 b^{(n_l)}_{s_0} \geq s^2 \frac{b_{s_0}}{\sqrt{2}}.$$

Now, we get

$$\begin{aligned}
&\int_{\alpha_{l,k}}^{\beta_{l,k}} \Phi(\mathbf{a}^{(n_l)})\left(\frac{1}{2^{8p^l_{m_0}}} - s\right) \Psi(\mathbf{b}^{(n_l)})(s)\, ds \\
&\geq \int_{\alpha_{l,k}}^{\beta_{l,k}} \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^2 s^2 \frac{a_{r_0}}{\sqrt{2}} \frac{b_{s_0}}{\sqrt{2}}\, ds \\
&\geq \frac{a_{r_0} b_{s_0}}{2} \int_{\alpha_{l,k}}^{\beta_{l,k}} \left(s - \frac{1}{2^{8p^l_{m_0}}}\right)^4 ds \\
&\geq \frac{a_{r_0} b_{s_0}}{10} \left[\left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+7}{2^{8p^l_{m_0}+2}}\right)^5 - \left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+5}{2^{8p^l_{m_0}+2}}\right)^5\right] \\
&= \frac{a_{r_0} b_{s_0}}{10} \frac{1}{2^{8p^l_{m_0}+1}} \\
&\quad \cdot \left[\left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+7}{2^{8p^l_{m_0}+2}}\right)^4 + \left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+7}{2^{8p^l_{m_0}+2}}\right)^3 \left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+5}{2^{8p^l_{m_0}+2}}\right)\right. \\
&\quad \left. + \cdots + \left(\frac{1}{2^{p^l_{m_0}}} + \frac{8(2k)+5}{2^{8p^l_{m_0}+2}}\right)^4\right] \\
&\geq \frac{a_{r_0} b_{s_0}}{10} \frac{1}{2^{8p^l_{m_0}+1}} \frac{5}{2^{4p^l_{m_0}}} = \frac{a_{r_0} b_{s_0}}{2^{12p^l_{m_0}+2}}.
\end{aligned}$$

In conclusion,

$$\Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)})\left(\frac{1}{2^{8p^l_{m_0}}}\right) \geq \sum_{k=0}^{2^{7p^l_{m_0}-2}-1} \frac{a_{r_0} b_{s_0}}{2^{12p^l_{m_0}+2}} = \frac{a_{r_0} b_{s_0}}{16 \cdot 2^{5p^l_{m_0}}},$$

so that

$$f * g\left(\frac{1}{2^{8p_{m_0}^l}}\right) \geq \Phi(\mathbf{a}^{(n_l)}) * \Psi(\mathbf{b}^{(n_l)}) - \frac{a_{r_0}b_{s_0}}{32}\frac{1}{2^{5p_{m_0}^l}} \geq \frac{a_{r_0}b_{s_0}}{16 \cdot 2^{5p_{m_0}^l}} - \frac{a_{r_0}b_{s_0}}{32}\frac{1}{2^{5p_{m_0}^l}}$$
$$= \frac{a_{r_0}b_{s_0}}{32 \cdot 2^{5p_{m_0}^l}}$$

and then we can write

$$\frac{f * g\left(\frac{1}{2^{8p_{m_0}^l}}\right)}{\frac{1}{2^{8p_{m_0}^l}}} \geq 2^{8p_{m_0}^l}\frac{a_{r_0}b_{s_0}}{160 \cdot 2^{5p_{m_0}^l}} = 2^{3p_{m_0}^l}\frac{a_{r_0}b_{s_0}}{160} \xrightarrow[l\to\infty]{} \infty,$$

so proving that $f * g$ is not differentiable at the origin. □

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