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An extremal composition operator on the Hardy space of the bidisk with small approximation numbers

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Abstract

We construct an analytic self-map Φ of the bidisk $\mathbb{D}^2$ whose image touches the distinguished boundary, but whose approximation numbers of the associated composition operator on $H^2(\mathbb{D}^2)$ are small in the sense that $\limsup_{n\to\infty}[a_{n^2}(C_\Phi)]^{1/n} < 1$.

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1. Introduction

Let us recall that the Hardy space of the polydisk $\mathbb{D}^N$ is the space:

$$H^2(\mathbb{D}^N) = \Big\{ f : \mathbb{D}^N \to \mathbb{C}\,;\;\; f(z) = \sum_{\alpha\in\mathbb{N}^N} a_\alpha z^\alpha \text{ and } \|f\|_2^2 := \sum_{\alpha\in\mathbb{N}^N} |a_\alpha|^2 < \infty \Big\}.$$

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If $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ is an analytic map, the associated composition operator C_Φ (which is not always bounded on $H^2(\mathbb{D}^N)$ for $N \geq 2$) is defined by:

$$C_\Phi(f) = f \circ \Phi .$$

For $N = 1$, composition operators $C_\Phi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ on the Hardy space of the unit disk are always bounded, and the decay of their approximation numbers $a_n(C_\Phi)$ cannot be arbitrarily fast; actually they cannot supersede a geometric speed ([13]; see also [7, Theorem 3.1]): there exists a positive constant c such that:

$$a_n(C_\Phi) \gtrsim \mathrm{e}^{-cn} , \qquad n = 1, 2, \ldots$$

It is easy to see that this speed occurs when $\|\Phi\|_\infty < 1$, and we proved in [7, Theorem 3.4] that a geometrical speed only takes place in this case; in other words:

$$\|\Phi\|_\infty = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} [a_n(C_\Phi)]^{1/n} = 1 . \tag{1.1}$$

This leads to the introduction, for an operator T between Banach spaces, of the parameters:

$$\beta^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n} \quad \text{and} \quad \beta^+(T) = \limsup_{n\to\infty} [a_n(T)]^{1/n} , \tag{1.2}$$

where $a_n(T)$ is the nth approximation number of T. When $[a_n(T)]^{1/n}$ actually has a limit, i.e. when $\beta^-(T) = \beta^+(T)$, we write it $\beta(T)$.

With this notation, what is proved in [7, Theorem 3.4] is that:

$$\beta(C_\Phi) = 1 \quad \text{if and only if} \quad \|\Phi\|_\infty = 1 . \tag{1.3}$$

In other words:

$$\beta(C_\Phi) = 1 \quad \text{if and only if} \quad \overline{\Phi(\mathbb{D})} \cap \partial\mathbb{D} \neq \emptyset . \tag{1.4}$$

Later, in [9], we gave, for $\Phi\colon \mathbb{D} \to \mathbb{D}$, when $\|\Phi\|_\infty < 1$, a formula for this parameter in terms of the Green capacity of $\Phi(\mathbb{D})$, which allowed us to recover (1.1).

For $N \geq 2$, the parameters $\beta^-(C_\Phi)$ and $\beta^+(C_\Phi)$ are not the good ones, and we introduce, for any operator T between Banach spaces, the parameters:

$$\beta_N^-(T) = \liminf_{n\to\infty} [a_{n^N}(T)]^{1/n} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n\to\infty} [a_{n^N}(T)]^{1/n} , \tag{1.5}$$

and:

$$\beta_N(T) = \lim_{n\to\infty} [a_{n^N}(T)]^{1/n} \tag{1.6}$$

when the limit exists. It is shown in [1] (see also [12] and [11]) that $\beta_N^\pm(C_\Phi)$ are the suitable parameters for the composition operators on $H^2(\mathbb{D}^N)$.

It is clear that $0 \leq \beta^-(T) \leq \beta_N^+(T) \leq 1$, and it is interesting to know when the extreme cases $\beta_N^\pm(T) = 0$ or $\beta_N^\pm(T) = 1$ occur. For example:

$$\begin{aligned} \beta_N^-(T) > 0 \quad &\Longleftrightarrow \quad a_{n^N}(T) \gtrsim \mathrm{e}^{-\tau n} , \qquad \text{with } \tau > 0 \\ \beta_N^-(T) = 1 \quad &\Longleftrightarrow \quad a_{n^N}(T) \gtrsim \mathrm{e}^{-n\varepsilon_n} , \qquad \text{with } \varepsilon_n \to 0 . \end{aligned}$$

It is proved in [1], for any $N \geq 1$, that $\beta_N^-(C_\Phi) > 0$, as soon as Φ is non degenerate (i.e. the Jacobian J_Φ is not identically 0) and the operator C_Φ is bounded on $H^2(\mathbb{D}^N)$. For an expression of $\beta_N^\pm(C_\Phi)$ in terms of "capacity", only partial results are known so far ([12] and [11]) and the application to a result like (1.1) fails in general.

In [12, Theorem 5.12], we gave an example of a holomorphic self-map $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$, continuous on the closure $\overline{\mathbb{D}^2}$, such that $\|\Phi\|_\infty = 1$, that is:

$$\Phi(\mathbb{T}^2) \cap \partial\mathbb{D}^2 \neq \emptyset\,, \tag{1.7}$$

and yet:

$$\beta_2^+(C_\Phi) < 1\,, \tag{1.8}$$

in contrast with the one-dimensional case ([7, Theorem 3.4]).

However, in the multidimensional case $N \geq 2$, several notions of boundary are available; in particular that of the Shilov boundary. Though the Shilov boundary of the ball is its usual boundary, that of the polydisk is its distinguished boundary:

$$\partial_e\mathbb{D}^N = \{z = (z_j)\,;\ |z_j| = 1 \text{ for all } j = 1, \dots, N\} = \mathbb{T}^N$$

(indeed, the distinguished maximum principle tells that, for f analytic in $\mathbb{D}^N$ and continuous on $\overline{\mathbb{D}^N}$, it holds $\max_{z\in\mathbb{D}^N} |f(z)| = \max_{z\in\partial_e\mathbb{D}^N} |f(z)|$). We have:

$$\partial_e\mathbb{D}^N \subsetneq \partial\mathbb{D}^N\,. \tag{1.9}$$

The aim of this paper is to improve on ([12, Theorem 5.12]) and (1.7), in building an analytic self-map $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$, continuous on $\overline{\mathbb{D}^2}$, non-degenerate and such that:

$$\Phi(\mathbb{T}^2) \cap \partial_e\mathbb{D}^2 \neq \emptyset \quad \text{but} \quad \beta_2^+(C_\Phi) < 1\,. \tag{1.10}$$

2. Background and notation

Let $\mathbb{D}$ be the open unit disk, $H^2(\mathbb{D}^N)$ the Hardy space of the polydisk $\mathbb{D}^N$, and $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ an analytic map. When $N = 1$, it is well-known (see [4] or [14]) that Φ induces a composition operator $C_\Phi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ by the formula:

$$C_\Phi(f) = f \circ \Phi\,,$$

and the connection between the "symbol" Φ and the properties of the operator C_Φ, in particular its compactness, can be further studied (see [4] or [14]). When $N > 1$, C_Φ is not bounded in general (see [4]).

Let $\mathbb{T}$ be the unit circle, and m the normalized Haar measure on $\mathbb{T}^N$. A positive Borel measure μ on $\mathbb{D}^N$ is called a Carleson measure (for the space $H^2(\mathbb{D}^N)$) if the canonical map $J\colon H^2(\mathbb{D}^N) \to L^2(\mu)$ is bounded. When $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ is analytic and induces a bounded composition operator on $H^2(\mathbb{D}^N)$, the pullback measure $m_\Phi = \Phi^*(m)$, defined, for any test function u, by:

$$\int_{\mathbb{D}^N} u(w)\, dm_\Phi(w) = \int_{\mathbb{T}^N} u[\Phi^*(\xi)]\, dm(\xi)\,,$$

is a Carleson measure. Here Φ^* is the radial limit function, defined for m-almost every $\xi \in \mathbb{T}^N$, by $\Phi^*(\xi) = \lim_{r\to 1^-} \Phi(r\xi)$.

For $\xi \in \mathbb{T} = \partial\mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ is defined as:

$$S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \leq h\}\,. \tag{2.1}$$

In this paper, to save notation, we will work in the case $N = 2$.

If $f \in \mathrm{Hol}\,(\mathbb{D}^2)$, $D_j^k f$ denotes the kth derivative of f with respect to the jth variable ($j = 1, 2$).

We denote by $A(\mathbb{D})$ the disk algebra, i.e. the space of functions holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We similarly define the bidisk algebra $A(\mathbb{D}^2)$.

The reproducing kernel K_a of $H^2(\mathbb{D}^2)$ is, with $a = (a_1, a_2)$ and $z = (z_1, z_2)$:

$$K_a(z) = \frac{1}{(1 - \overline{a_1} z_1)(1 - \overline{a_2} z_2)} \cdot \tag{2.2}$$

As a consequence:

$$|f(a)| = |\langle f, K_a\rangle| \le \frac{\|f\|_2}{\sqrt{(1 - |a_1|^2)(1 - |a_2|^2)}} \cdot \tag{2.3}$$

In particular, the functions in the unit ball of $H^2(\mathbb{D}^2)$ are uniformly bounded on compact subsets of $\mathbb{D}^2$.

Let H_1 and H_2 be Hilbert spaces, and $T : H_1 \to H_2$ an operator. The nth approximation number $a_n(T)$ of T, $n = 1, 2, \ldots$, is defined (see [2]) as the distance (for the operator-norm) of T to operators of rank $< n$:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\| \,. \tag{2.4}$$

The approximation numbers have the ideal property:

$$a_n(ATB) \le \|A\| \, a_n(T) \, \|B\| \,.$$

The nth Gelfand number $c_n(T)$ of T is defined by:

$$c_n(T) = \inf_{\text{codim } E < n} \|T_{|E}\| \,. \tag{2.5}$$

As an easy consequence of the Schmidt decomposition, we have for any compact operator between Hilbert spaces:

$$c_n(T) = a_n(T) \,. \tag{2.6}$$

If $T, T_1, T_2 : H \to H'$ are operators between Hilbert spaces H and H', we write $T = T_1 \oplus T_2$ if $T = T_1 + T_2$ and:

$$\|Tx\|^2 = \|T_1 x\|^2 + \|T_2 x\|^2 \,, \quad \text{for all } x \in H \,.$$

The subadditivity of approximation numbers is then expressed by:

$$a_{j+k}(T_1 \oplus T_2) \le a_j(T_1) + a_k(T_2) \,. \tag{2.7}$$

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, and by $[x]$ the integral part of the real number x.

We write $X \lesssim Y$ to indicate that $X \le c\, Y$ for some constant $c > 0$, and $X \approx Y$ to indicate that $X \lesssim Y$ and $Y \lesssim X$.

The paper is organized as follows. In Section 3, we recall with some detail the definition and main properties of a so-called *cusp map* $\chi \in A(\mathbb{D})$, to be of essential use in our counterexample. In Section 4, we prove several lemmas which constitute the core or the proof. In Section 5, we state and prove our main theorem.

3. The cusp map

The *cusp map* $\chi : \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and extends continuously on $\overline{\mathbb{D}}$. The boundary of its image is formed by three circular arcs of respective centers $\frac{1}{2}$, $1 + \frac{i}{2}$, $1 - \frac{i}{2}$, and of radius $\frac{1}{2}$ (see Fig. 1). However, the parametrization $t \mapsto \chi(\mathrm{e}^{it})$ involves logarithms.

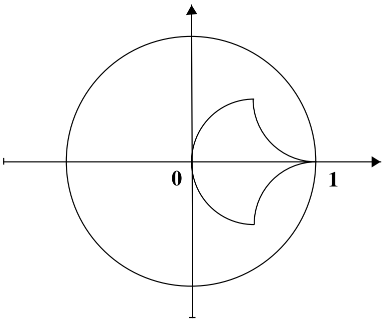

Fig. 1. Cusp map domain.

It was often used by the authors ([6,8]) as an extremal example.

We first recall the definition of χ.

Let $\mathbb{D}^+ = \{z \in \mathbb{D}\,;\ \mathfrak{Re}\, z > 0\}$ be the right half-disk. Let now $\mathbb{H}$ be the upper half-plane, and $T\colon \mathbb{D} \to \mathbb{H}$ defined by:

$$T(u) = i\,\frac{1+u}{1-u}\,, \quad \text{with} \quad T^{-1}(s) = \frac{s-i}{s+i}\,\cdot$$

Taking the square root of T, we map $\mathbb{D}$ onto the first quadrant defined by $Q_1 = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$; we go back to the half-disk $\{z \in \mathbb{D}\,;\ \mathfrak{Im}\, z < 0\}$ by T^{-1}. Finally, make a rotation by i to go onto $\mathbb{D}^+$. We get:

$$\chi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1}\,\cdot$$

One has $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, and $\chi_0(-i) = i$. The half-circle $\{z \in \mathbb{T}\,;\ \mathfrak{Re}\, z \geq 0\}$ is mapped by χ_0 onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively:

$$\chi_1(z) = \log \chi_0(z)\,, \quad \chi_2(z) = -\frac{2}{\pi}\,\chi_1(z) + 1\,, \quad \chi_3(z) = \frac{1}{\chi_2(z)} \tag{3.1}$$

and finally:

$$\chi(z) = 1 - \chi_3(z)\,. \tag{3.2}$$

We now summarize the properties of the cusp map χ in the following proposition.

Proposition 3.1. *The cusp map satisfies:*

(1) $1 - |\chi(z)| \lesssim \dfrac{1}{\log(2/|1-z|)}$;

(2) $|1 - \chi(z)| \leq K(1 - |\chi(z)|)$ *for all* $z \in \mathbb{D}$, *where* K *is a positive constant;*

(3) $\chi(\mathbb{D})$ *is the intersection of the open disk* $D\big(\frac{1}{2}, \frac{1}{2}\big)$ *with the exterior of the two open disks* $D\big(1 + \frac{i}{2}, \frac{1}{2}\big)$ *and* $D\big(1 - \frac{i}{2}, \frac{1}{2}\big)$;

(4) $\chi(1) = 1$, $\chi(\overline{z}) = \overline{\chi(z)}$ *and* $|\chi(z) - 1| \leq 1$ *for all* $z \in \mathbb{D}$;

(5) *for* $0 < |t| \leq \pi/4$, *we have* $1 - \mathfrak{Re}\,\chi(e^{it}) \approx 1/(\log 1/|t|)$;

(6) $\chi(\overline{\mathbb{D}}) \subseteq \{z = x + iy\,;\ 0 \leq x \leq 1$ *and* $|y| \leq 2(1-x)^2\}$.

Proof. Items (1) to (5) are proved in [8, Lemma 4.2]. To prove (6), write $\chi(z) = (1-h)+iy$. Since $\chi(\overline{z}) = \overline{\chi(z)}$, we can assume $y \geq 0$. Since $\chi(\mathbb{D}) \cap D\big(1+\frac{i}{2}, \frac{1}{2}\big) = \emptyset$, we have $\big|\chi(z) - \big(1+\frac{i}{2}\big)\big| \geq \frac{1}{2}$; hence:

$$h^2 + \Big(y - \frac{1}{2}\Big)^2 = \Big|\chi(z) - \Big(1+\frac{i}{2}\Big)\Big|^2 \geq \frac{1}{4},$$

so that $y \leq y^2 + h^2$. But $y \leq 1/2$, since $\chi(z) \in D\big(\frac{1}{2}, \frac{1}{2}\big)$; therefore $y^2 \leq y/2$, so we get $y \leq 2h^2$. □

4. Preliminary lemmas

In this section, we collect some lemmas, which will reveal essential in the proof of our counterexample.

We consider the map $\varphi = \varphi_\theta$, $0 < \theta < 1$, defined, for $z \in \overline{\mathbb{D}} \setminus \{1\}$, by:

$$\varphi(z) = \exp\big(-(1-z)^{-\theta}\big). \tag{4.1}$$

We observe, since $\mathfrak{Re}\,(1-z) \geq 0$ for $z \in \overline{\mathbb{D}}$, that:

$$|\varphi(z)| \leq \exp\big(-\delta\,|1-z|^{-\theta}\big), \tag{4.2}$$

where $\delta = \cos \pi\theta/2 > 0$. Moreover, (4.2) shows that $\varphi \in A(\mathbb{D})$, since:

$$\lim_{z\to 1, z\in\overline{\mathbb{D}}} \varphi(z) = 0 =: \varphi(1).$$

Our first lemma will allow us to define our symbol Φ.

Lemma 4.1. *One can adjust $0 < c < 1$ so as to get:*

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| < 1 \qquad \textit{for all } z \in \mathbb{D}. \tag{4.3}$$

Hence, if we set, for any $g \in A(\mathbb{D})$ with $\|g\|_\infty \leq 1$:

$$\Phi(z_1, z_2) = \big(\chi(z_1), \chi(z_1) + c\,(\varphi\circ\chi)(z_1)\,g(z_2)\big), \tag{4.4}$$

we have $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$.

Remark. The factor 2 in (4.3) is needed in order to get the following inequalities, to be used later, for $z \in \mathbb{D}$ and $w = \chi(z) + c\,(\varphi\circ\chi)(z)\,u$, with $|u| \leq 1$:

$$|w| \leq \frac{1+|\chi(z)|}{2}, \tag{4.5}$$

or, equivalently:

$$1 - |w| \geq \frac{1-|\chi(z)|}{2}. \tag{4.6}$$

Indeed:

$$|w| \leq |\chi(z)| + c\,|\varphi\circ\chi(z)| \leq |\chi(z)| + \frac{1-|\chi(z)|}{2} = \frac{1+|\chi(z)|}{2}.$$

Proof of Lemma 4.1. Set $X = |1-\chi(z)|$, so that, with K the constant of Proposition 3.1, (2):

$$|\chi(z)| \leq 1 - \frac{|1-\chi(z)|}{K} = 1 - \frac{X}{K}. \tag{4.7}$$

For $z \in \mathbb{D}$ and X close enough to zero, say $X < \eta$, we have $2\exp(-\delta X^{-\theta}) < \frac{X}{K}$. If we adjust $0 < c < 1$ so as to have $c < \frac{\eta}{2K}$, it follows from (4.2) and (4.7) that, for $X < \eta$:

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| \leq 1 - \frac{X}{K} + 2\exp(-\delta X^{-\theta}) < 1\,.$$

However, for $X \geq \eta$, (4.7) says that $|\chi(z)| \leq 1 - \frac{\eta}{K}$, so:

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| \leq 1 - \frac{\eta}{K} + 2\,c < 1\,,$$

as well and this ends the proof of Lemma 4.1. □

Our second lemma estimates some integrals and ensures that Φ induces a compact composition operator on $H^2(\mathbb{D}^2)$.

Lemma 4.2. (1) *For $0 < h \leq 1$, the following estimate holds:*

$$I_0(h) := \int_{|\chi(\mathrm{e}^{it})-1|\leq h} \frac{1}{(1-|\chi(\mathrm{e}^{it})|)^2}\,dt \lesssim \mathrm{e}^{-\tau/h}\,, \tag{4.8}$$

(2) *For $g \in A(\mathbb{D})$ with $0 < \|g\|_\infty \leq 1$, if we set:*

$$I(h) := \int_{|\chi(\mathrm{e}^{it_1})-1|\leq h} \frac{dt_1\,dt_2}{(1-|\chi(\mathrm{e}^{it_1})|)(1-|\chi(\mathrm{e}^{it_1}) + c\,(\varphi\circ\chi)(\mathrm{e}^{it_1})\,g(\mathrm{e}^{it_2})|)}\,,$$

we have:

$$I(h) \lesssim \mathrm{e}^{-\tau/h}\,.$$

Consequently, the composition operator C_Φ defined in (4.4) is bounded from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D}^2)$ and is Hilbert–Schmidt, and hence compact.

Proof. (1) By Proposition 3.1, (5), there exist two constants c_1, c_2 such that:

$$\frac{c_1}{\log 1/|t|} \leq |\chi(\mathrm{e}^{it}) - 1| \leq \frac{c_2}{\log 1/|t|}\,, \qquad |t| \leq \pi\,;$$

hence:

$$I_0(h) \lesssim \int_{|t|\leq \mathrm{e}^{-c_1/h}} [\log(1/|t|)]^2\,dt = 2\int_{c_1/h}^{\infty} x^2\,\mathrm{e}^{-x}\,dx \lesssim h^{-2}\mathrm{e}^{-c_1/h}\,.$$

(2) Using (4.6), we have, thanks to (4.8):

$$I(h) \leq \int_{|\chi(\mathrm{e}^{it_1})-1|\leq h} \frac{2}{(1-|\chi(\mathrm{e}^{it_1})|)^2}\,dt_1\,dt_2 \lesssim \mathrm{e}^{-\tau/h}\,.$$

In particular, $I(1) < \infty$, showing that C_Φ is Hilbert–Schmidt and hence bounded and compact. □

For the rest of the paper, we fix a number σ in $(0, 1)$, that for convenience we take as:

$$\sigma = \frac{7}{8}\,, \tag{4.9}$$

a positive integer j_0 such that:

$$2\,\sigma^{j_0} \leq 1/8 \tag{4.10}$$

(i.e. $j_0 \geq 21$), and we set:

$$a_j = 1 - \sigma^j \tag{4.11}$$

and:

$$\rho_j = \frac{\sigma^j}{4} = \frac{1}{4}(1 - a_j)\,. \tag{4.12}$$

We also define, for $n \geq 1$ and θ being the parameter used in (4.1):

$$N_n = \left[\frac{\log 2n}{\theta \log 1/\sigma}\right] + 1 > \frac{\log 2n}{\log 1/\sigma}\,. \tag{4.13}$$

The next lemma gives a cutting off for $\chi(\mathbb{D})$.

Lemma 4.3. *For every $n \geq 1$, the image $\chi(\mathbb{D})$ of the cusp map, deprived of the closed Euclidean disk $\overline{D}(0, 1-\sigma^{j_0}/K)$ and of $\chi(\mathbb{D}) \cap S(1, 1/n)$, can be covered by the open Euclidean disks $D(a_j, \rho_j)$, with $j_0 \leq j \leq N_n$.*

Proof. Let $z \in \mathbb{D}$ such that $|\chi(z)| > 1 - \sigma^{j_0}/K$ and $|\chi(z) - 1| > 1/n$. We write $\chi(z) = x + iy =: 1 - h + iy$.

Let j with $a_j \leq x < a_{j+1}$, i.e. $\sigma^{j+1} < h \leq \sigma^j$. We have $j \geq j_0$, since $h < \sigma^{j_0}$.

Now, since $0 \leq x - a_j < a_{j+1} - a_j = \sigma^{j+1} - \sigma^j$, that $y^2 \leq 4h^4$ (Proposition 3.1, (6)), and $h \leq \sigma^j$, we have:

$$|\chi(z) - a_j|^2 < (\sigma^j - \sigma^{j+1})^2 + y^2 \leq (1-\sigma)^2\sigma^{2j} + 4\,\sigma^{4j}\,;$$

hence:

$$|\chi(z) - a_j| < \sigma^j(1-\sigma) + 2\,\sigma^{2j} = \sigma^j(1 - \sigma + 2\,\sigma^j)\,.$$

Subsequently, since $1 - \sigma = 1/8$, $j \geq j_0$, and $2\,\sigma^{j_0} \leq 1/8$:

$$|\chi(z) - a_j| < \sigma^j(1 - \sigma + 2\,\sigma^{j_0}) \leq \frac{\sigma^j}{4} = \rho_j\,,$$

showing that $\chi(z) \in D(a_j, \rho_j)$.

Moreover, we have $j \leq N_n$. Indeed, if $j > N_n$, we would have:

$$|\chi(z) - 1| \leq |\chi(z) - a_j| + (1 - a_j) \leq \rho_j + \sigma^j = \frac{5}{4}\sigma^j \leq \frac{5}{4}\sigma^{N_n+1} \leq 2\,\sigma^{N_n} \leq 1/n\,,$$

contradicting the fact that $\chi(z) \notin S(1, 1/n)$. $\square$

Our last lemma gives estimates on derivatives for the functions belonging to $H^2(\mathbb{D}^2)$.

Lemma 4.4. (1) *Let $f \in H^2(\mathbb{D}^2)$, k a non-negative integer, $\beta \in \mathbb{D}$, and let $h_k(z) = (D_2^k f)(z, z)$. Then:*

$$|h_k(\beta)| \leq \frac{k!\,2^{k+1}}{(1 - |\beta|)^{k+1}}\,\|f\|_2\,.$$

(2) *Assume moreover that $h_k^{(l)}(a) = 0$ for some $a \in \mathbb{D}$ and for $0 \leq l < n$. Then, for $0 < \rho < 1$ and $|b - a| \leq \frac{\rho}{2}(1 - |a|)$, it holds:*

$$|h_k(b)| \leq \rho^n \frac{k!\,4^{k+1}}{(1 - |a|)^{k+1}}\,\|f\|_2\,.$$

Proof. (1) The Cauchy inequalities give for $0 < s < 1 - |\beta|$ and $\alpha \in \mathbb{N}^2$:

$$|D^\alpha f(\beta, \beta)| \leq \frac{\alpha!}{s^{|\alpha|}} \sup_{|w_1 - \beta| = s, |w_2 - \beta| = s} |f(w_1, w_2)|\,.$$

The choice $s = \frac{1-|\beta|}{2}$ gives $1 - |w_j| \geq \frac{1-|\beta|}{2}$ for $|w_j - \beta| = s$, $j = 1, 2$; hence, thanks to the estimate (2.3):

$$|f(w_1, w_2)| \leq \frac{\|f\|_2}{\sqrt{(1 - |w_1|)(1 - |w_2|)}} \leq \frac{2}{1 - |\beta|} \, \|f\|_2 \cdot$$

Specializing to $\alpha = (0, k)$ now gives the result.

(2) We may assume $\|f\|_2 \leq 1$. Consider the function defined, for $w \in \mathbb{D}$, by:

$$H_k(w) = h_k\biggl(a + w \, \frac{1 - |a|}{2}\biggr) .$$

It is a bounded and holomorphic function in $\mathbb{D}$.

For $w \in \mathbb{D}$, let $\beta = a + w \, \frac{1-|a|}{2}$, which satisfies $1 - |\beta| \geq \frac{1-|a|}{2}$. The first part of the lemma gives:

$$|H_k(w)| = |h_k(\beta)| \leq \frac{k!\, 4^{k+1}}{(1 - |a|)^{k+1}} \cdot$$

Now, $H_k^{(l)}(0) = h_k^{(l)}(a) = 0$ for $0 \leq l < n$; hence the Schwarz lemma says that H_k satisfies $|H_k(w)| \leq |w|^n \, \|H_k\|_\infty$ for all $w \in \mathbb{D}$. Take $w = \frac{2(b-a)}{1-|a|}$, which satisfies $|w| \leq \rho$, to get:

$$|h_k(b)| = |H_k(w)| \leq |w|^n \, \|H_k\|_\infty \leq \rho^n \, \frac{k!\, 4^{k+1}}{(1 - |a|)^{k+1}} \cdot \qquad \square$$

5. The main result

Recall that χ is the cusp map and that φ is defined in (4.1). The map g appearing in the formula below plays an inert role, and is just designed to ensure that Φ is non-degenerate; we can take, for example $g(z_2) = z_2$. This seems to mean that non-degeneracy is not the only issue in the question of estimating $\beta_2^+(C_\Phi)$.

Our example appears as a perturbation of the diagonal map defined by $\Delta(z_1, z_2) = \bigl(\chi(z_1), \chi(z_1)\bigr)$ for which we already know ([10, Theorem 2.4]) that $\Delta(1, 1) = (1, 1)$ and $\beta_2^+(C_\Delta) < 1$. This map is degenerate, but the perturbation clearly gives a non-degenerate one since its Jacobian is $J_\Phi(z_1, z_2) = c\, (\varphi \circ \chi)(z_1)\, \chi'(z_1)\, g'(z_2)$.

Theorem 5.1. *Let:*

$$\Phi(z_1, z_2) = \bigl(\chi(z_1), \chi(z_1) + c\, (\varphi \circ \chi)(z_1)\, g(z_2)\bigr)$$

be the function defined in (4.4).

Then:

(1) $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$ *and* $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ *is compact, and moreover Hilbert–Schmidt;*

(2) Φ *is non-degenerate, and its components belong to the bidisk algebra;*

(3) $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 = \Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset$;

(4) $a_{n^2}(C_\Phi) \lesssim \exp(-\tau n)$, *for some* $\tau > 0$, *implying* $\beta_2^+(C_\Phi) < 1$.

Proof. That Φ maps $\mathbb{D}^2$ to itself is proved in Lemma 4.1 and that the composition operator $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is Hilbert–Schmidt (and in particular compact), in Lemma 4.2. Item (2) is due to the presence of g, as explained above. The fact that $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$ is clear since $\Phi(1, 1) = (1, 1)$. It remains to prove (4).

Once more, the proof will be conveniently divided into several steps. We begin by a lemma which is in fact obvious, but explains well what is going on.

Lemma 5.2. *Let* $\lambda = 1 - \frac{\sigma^{j_0}}{2K}$, *where* σ, K *and* j_0 *are as in* (4.9), *Proposition* 3.1, (2), *and* (4.10). *Let* $r_n = 1 - \frac{1}{n}$, *and let* μ_1, μ_2, μ_3 *the respective restrictions of* m_Φ *to the disk* $\overline{\lambda\mathbb{D}^2}$, *the annulus* $r_n\mathbb{D}^2 \setminus \overline{\lambda\mathbb{D}^2}$, *and the annulus* $\mathbb{D}^2 \setminus r_n\mathbb{D}^2$. *We then have:*

$$C_\Phi = T_1 \oplus T_2 \oplus T_3 \,,$$

where T_j *is the canonical injection of* $H^2(\mathbb{D}^2)$ *into* $L^2(\mu_j)$.

This is indeed obvious since:

$$\|C_\Phi f\|^2 = \int_{\mathbb{D}^2} |f|^2 dm_\Phi \,,$$

and by splitting the integral into three parts.

We now majorize separately the numbers $a_p(T_j)$, for $j = 1, 2, 3$. In the sequel, the positive constant τ may vary from one formula to another.

Step 1. It holds:

$$a_{n^2}(T_1) \lesssim \mathrm{e}^{-\tau n} \,. \tag{5.1}$$

Proof. Let $V = z_1^n H^2(\mathbb{D}^2) + z_2^n H^2(\mathbb{D}^2)$; this is a subspace of $H^2(\mathbb{D}^2)$ of codimension $\leq n^2$, since:

$$V = \big\{ f \in H^2(\mathbb{D}^2) ;\ D_1^j D_2^k f(0,0) = 0 \quad \text{for } 0 \leq j, k < n \big\} \,.$$

If $f(z) = \sum_{\max(j,k)\geq n} a_{j,k}\, z_1^j z_2^k \in V$ and $\|f\|_2 = 1$, one can write:

$$f(z) = z_1^n\, q_1(z_1, z_2) + z_2^n\, q_2(z_1, z_2) \,,$$

with:

$$q_1(z) = \sum_{j\geq n, k\geq 0} a_{j,k}\, z_1^{j-n} z_2^k \quad \text{and} \quad q_2(z) = \sum_{j<n, k\geq n} a_{j,k}\, z_1^j z_2^{k-n} \,,$$

which satisfy $\|q_j\|_2 \leq \|f\|_2 = 1$, $j = 1, 2$.

An easy estimate now gives (since $\max(|z_1|^n, |z_2^n|) \leq \lambda^n$ on $\overline{\lambda\mathbb{D}^2}$):

$$\begin{aligned} \|T_1 f\|^2 &\leq 2 \bigg(\int_{\lambda\mathbb{D}^2} \big(|z_1^n|^2 |q_1(z_1, z_2)|^2 + |z_2^n|^2 |q_2(z_1, z_2)|^2 \big)\, dm_\Phi \bigg) \\ &\lesssim \lambda^{2n} \int_{\lambda\mathbb{D}^2} \big(|q_1|^2 + |q_2|^2 \big)\, dm_\Phi \lesssim \lambda^{2n} \big(\|q_1\|_2^2 + \|q_2\|_2^2 \big) \lesssim \lambda^{2n} \,, \end{aligned}$$

since we know by Lemma 4.2 that C_Φ is bounded on $H^2(\mathbb{D}^2)$ and hence that m_Φ is a Carleson measure for $H^2(\mathbb{D}^2)$. Alternatively, we could majorize $|q_j(z_1, z_2)|$ uniformly on the support of μ_1. We hence obtain:

$$a_{n^2+1}(T_1) = c_{n^2+1}(T_1) \lesssim \mathrm{e}^{-\tau n} \,. \quad \square \tag{5.2}$$

Step 2. It holds:

$$a_{n^2}(T_3) \lesssim \mathrm{e}^{-\tau n} \,. \tag{5.3}$$

Proof. In one variable, we could use the Carleson embedding theorem; but this theorem for the bidisk and the Hardy space $H^2(\mathbb{D}^2)$ notably has a more complicated statement ([3]; see also [5]), and cannot be used efficiently here. Our strategy will be to replace it by a sharp estimation of a Hilbert–Schmidt norm.

We set $h_n = 1 - r_n = 1/n$.

Clearly, denoting by S_2 the Hilbert–Schmidt class:

$$\|T_3\|^2 \leq \|T_3\|_{S_2}^2 = \int \frac{d\mu_3(w)}{(1 - |w_1|^2)(1 - |w_2|^2)} \leq \int \frac{d\mu_3(w)}{(1 - |w_1|)(1 - |w_2|)} \,.$$

Now, if $w = (w_1, w_2) = \big(\chi(z_1), \chi(z_1) + c\,(\varphi \circ \chi)(z_1)\, g(z_2)\big)$ belongs to the support of μ_3, we have $\max(|w_1|, |w_2|) \geq r_n = 1 - h_n$, and, recalling (4.5):

$$|w_1| \geq 2\,|w_2| - 1 \,, \tag{5.4}$$

we have in either case $|w_1| \geq 1 - 2h_n$. By Proposition 3.1, (2), this implies that:

$$|1 - w_1| \leq 2\,K h_n \,.$$

Lemma 4.2 gives:

$$\begin{aligned} \|T_3\|^2 &\lesssim \int_{|\chi(\mathrm{e}^{it_1}) - 1| \leq 2K h_n} \frac{dt_1\, dt_2}{(1 - |\chi(\mathrm{e}^{it_1})|)(1 - |\chi(\mathrm{e}^{it_1}) + c\,(\varphi \circ \chi)(\mathrm{e}^{it_1})\, g(\mathrm{e}^{it_2})|)} \\ &= I(2K h_n) \lesssim \mathrm{e}^{-\tau/h_n} \,. \end{aligned}$$

But $h_n = 1/n$, so that:

$$a_{n^2}(T_3) \leq \|T_3\| \lesssim \mathrm{e}^{-\tau n} \,. \quad \square \tag{5.5}$$

Step 3. It holds:

$$a_{n^2}(T_2) \lesssim \mathrm{e}^{-\tau n} \,. \tag{5.6}$$

This estimate follows from the following key auxiliary lemma. In fact, this lemma will give, for the Gelfand numbers, $c_{n^2}(T_2) \lesssim \mathrm{e}^{-\tau n}$, and we know that they are equal to the approximation numbers.

Let $M \colon H^2(\mathbb{D}^2) \to \mathrm{Hol}(\mathbb{D})$ be the linear map defined by:

$$Mf(z) = f(z, z) \,,$$

Recall that $a_j = 1 - \sigma^j$ and $N_n = \left[\frac{\log 2n}{\theta \log 1/\sigma}\right] + 1$.

Lemma 5.3. *Let E be the closed subspace of $H^2(\mathbb{D}^2)$ defined by:*

$$E = \Big\{ f \in H^2(\mathbb{D}^2) \,;\ \big[\, M(D_2^k f) \,\big]^{(l)}(a_j) = 0 \quad \text{for } 0 \leq l < n,\ 0 \leq k \leq m_j,\ 1 \leq j \leq N_n \Big\} \,.$$

Then, we can adjust the numbers m_j so as to guarantee that, for some positive constant τ:

$$\operatorname{codim} E \lesssim n^2$$

and, for all $f \in E$ with $\|f\|_2 \leq 1$:

$$\|T_2(f)\|_2 \lesssim \exp(-\tau n) \,.$$

Proof. This is the most delicate part.

Recall that:

$$h_n = 1/n \,, \quad r_n = 1 - h_n \,, \quad \lambda = 1 - \frac{\sigma^{j_0}}{2\,K} \,.$$

We need a uniform estimate of $|f(w)|$ for $f \in E$ with $\|f\|_2 \leq 1$ and for:

$$w = (w_1, w_2) \in \operatorname{supp} m_\Phi \cap (r_n\mathbb{D}^2 \setminus \overline{\lambda\mathbb{D}^2})\,.$$

This estimate will be given by Lemmas 4.3 and 4.4. Note that we have:

$$\chi(z_1) \in \mathbb{D} \setminus [S(1, 1/n) \cup \overline{D}(0, 2\lambda - 1)]\,.$$

Indeed, if $(w_1, w_2) = \Phi(z_1, z_2) \notin \lambda\mathbb{D}^2$, we have $\max(|w_1|, |w_2|) > \lambda$; so either $|w_1| > \lambda \geq 2\lambda - 1$, or $|w_2| > \lambda$ and again $|w_1| > 2\lambda - 1$ since $|w_1| \geq 2|w_2| - 1$, by (4.5). Hence $w_1 \notin \overline{D}(0, 2\lambda - 1)$. Moreover, we have $|1 - w_1| \geq 1 - |w_1| > 1/n$, so $w_1 \notin S(1, 1/n)$.

Using Lemma 4.3, select $j_0 \leq j \leq N_n$ such that $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. Now set:

$$A = \big(\chi(z_1), \chi(z_1)\big) \quad \text{and} \quad \Delta = \big(0, (\varphi \circ \chi)(z_1)\, g(z_2)\big)\,.$$

Our strategy will be the following. We write:

$$\begin{aligned} f[\Phi(z_1, z_2)] = f(A + \Delta) &= \sum_{k=0}^{\infty} \frac{D_2^k f(A)}{k!}\, \Delta^k \\ &= \sum_{k=0}^{\infty} \frac{M(D_2^k f)[\chi(z_1)]}{k!}\, \Delta^k = \sum_{k=0}^{\infty} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k\,, \end{aligned}$$

with $h_k = M(D_2^k f)$, and we put:

$$S_j = \sum_{k=0}^{m_j} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k$$

and

$$R_j = \sum_{k>m_j} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k\,.$$

We will estimate separately S_j and R_j.

(a) *Estimation of R_j.*

Recall that j is such that $j_0 \leq j \leq N_n$ and $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. We saw in the proof of this Lemma 4.3 that $1 - |\chi(z_1)| \leq |1 - \chi(z_1) \leq \frac{5}{4}\,\sigma^j$. Hence:

$$|\Delta| \leq |(\varphi \circ \chi)(z_1)| \leq \exp\bigg(-\frac{\delta}{|1 - \chi(z_1)|^\theta}\bigg) \lesssim \exp(-\tau\, \sigma^{-j\theta})\,.$$

Now, use Lemma 4.4 and (4.2) to get:

$$\begin{aligned} |R_j| &\leq \sum_{k>m_j} \frac{2^{k+1}}{(1 - |\chi(z_1)|)^{k+1}}\, |\Delta|^k \lesssim \sum_{k>m_j} 2^k \sigma^{-jk} \exp(-\tau k\, \sigma^{-j\theta}) \\ &\lesssim 2^{m_j} \sigma^{-jm_j} \exp(-\tau m_j \sigma^{-j\theta}) \lesssim \exp(-\tau m_j \sigma^{-j\theta}) \end{aligned}$$

for some absolute constant $\tau > 0$, that is:

$$|R_j| \lesssim \exp(-\tau n) \tag{5.7}$$

if we take:

$$m_j = [n\, \sigma^{j\theta}] + 1\,. \tag{5.8}$$

(b) *Estimation of S_j.*

We saw in the estimation of R_j that $1 - |\chi(z_1)| \gtrsim \sigma^j$. Now, remember that $h_k^{(l)}(a_j) = 0$ for $l < n$, since $f \in E$, we then use Lemma 4.4 to get, when we take the values:

$$a = a_j\,, \quad 1 - a_j = \sigma^j\,, \quad b = \chi(z_1)\,, \quad \rho = \frac{1}{2}\,,$$

a good upper bound for $\frac{h_k[\chi(z_1)]}{k!}$ when $k \leq m_j$, namely:

$$\left|\frac{h_k[\chi(z_1)]}{k!}\right| \lesssim \frac{4^{k+1}}{\sigma^{j(k+1)}}\,\rho^n\,.$$

We then obtain an estimate of the form:

$$\begin{aligned}|S_j| \lesssim \sum_{k=0}^{m_j} \rho^n \frac{4^{k+1}}{\sigma^{j(k+1)}} \lesssim \rho^n \frac{4^{m_j}}{\sigma^{jm_j}} &= \exp\Big(-n\log 2 + m_j \log 4 - jm_j \log\frac{7}{8}\Big)\\ &\lesssim \exp(-4\tau n + Bjm_j)\end{aligned}$$

with $\tau = \frac{1}{4}\log 2$ and $B \leq \log 4 + \log(8/7) \leq 2$; or else, using (5.8):

$$|S_j| \lesssim \exp(-4\tau n + Bjn\,\sigma^{j\theta} + Bj)\,.$$

But since $\sigma = 7/8 < 1$, the implied exponent, for $j_0 \leq j \leq N_n$:

$$-4\tau n + Bjn\,\sigma^{j\theta} + Bj = n(-4\tau + Bj\,\sigma^{j\theta}) + Bj\,,$$

is $\leq -2\tau n + B'\log n$, provided that we choose j_0 large enough, namely such that $j_0\big(\frac{7}{8}\big)^{j_0\theta} \leq 1/4$. This implies an inequality of the form:

$$|S_j| \lesssim \mathrm{e}^{-2\tau n} n^{B'} \lesssim \mathrm{e}^{-\tau n}\,. \tag{5.9}$$

Putting the estimates (5.7) and (5.9) on R_j and S_j together, we obtain, for every $f \in E$ with $\|f\|_2 \leq 1$:

$$\|T_2 f\| \lesssim \mathrm{e}^{-\tau n}\,. \tag{5.10}$$

It remains to bound from above the codimension of E. Since $N_n = \big[\frac{\log 2n}{\theta \log 1/\sigma}\big] + 1$ with $\sigma = 7/8$ and $m_j = [n\,\sigma^{j\theta}] + 1$, we see that:

$$\operatorname{codim} E \leq \sum_{l=0}^{n-1}\sum_{j=1}^{N_n} m_j \leq \sum_{l=0}^{n-1}\sum_{j=1}^{N_n} (n\,\sigma^{j\theta} + 1) \lesssim n^2 \sum_{j=1}^{\infty} \sigma^{j\theta} + n\log n < q\,n^2\,.$$

Therefore (5.10) can be read as well, remembering the equality of approximation numbers and Gelfand numbers:

$$a_{q\,n^2}(T_2) = c_{q\,n^2}(T_2) \lesssim \mathrm{e}^{-\tau n}. \tag{5.11}$$

Putting the estimates (5.2), (5.5), and (5.11) together ends the proof of Lemma 5.3. □

Finally, Lemma 5.2 and (2.7) give:

$$a_{3n^2}(C_\Phi) = a_{3n^2}(T_1 \oplus T_2 \oplus T_3) \leq a_{n^2}(T_1) + a_{n^2}(T_2) + a_{n^2}(T_3) \lesssim \mathrm{e}^{-\tau n}\,,$$

thereby finishing the proof of Theorem 5.1. □

Remark. Understanding where the difference really lies when we pass to the multidimensional case is a big challenge: it does not seem to be a matter of regularity of the boundary, and a similar example probably holds for the Hardy space of the ball.

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