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Differential equations for deformed Laguerre polynomials

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Abstract

The distribution function for the first eigenvalue spacing in the Laguerre unitary ensemble of finite size may be expressed in terms of a solution of the fifth Painlevé transcendent. The generating function of a certain discontinuous linear statistic of the Laguerre unitary ensemble can similarly be expressed in terms of a solution of the fifth Painlevé equation. The methodology used to derive these results rely on two theories regarding differential equations for orthogonal polynomial systems, one involving isomonodromic deformations and the other ladder operators. We compare the two theories by showing how either can be used to obtain a characterization of a more general Laguerre unitary ensemble average in terms of the Hamiltonian system for Painlevé V.

Keywords: Orthogonal polynomials; Painlevé equations; Isomonodromy; Ladder operators

1. Introduction

1.1. Objective

Spacing probabilities and moments of characteristic polynomials for random matrix ensembles with unitary symmetry are intimately related to semi-classical orthogonal polynomials. By

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developing the theory of particular semi-classical polynomials, it has been possible to characterize these random matrix quantities in terms of both discrete and continuous Painlevé equations [29,33,1,22,23,3,4,9,30,19–21,10,5]. Two methods of developing the theory for this purpose have emerged. One has been to use a formulation in terms of Lax pairs for isomonodromic deformations of linear differential equations [26]. The other has proceeded via a theory of ladder operators for orthogonal polynomial systems [11,14]. It is the purpose of this paper to compare these two methods as they apply to the particular discontinuous semi-classical weight

$$w(x) = (1-\zeta\theta(x-t))|x-t|^{\alpha}x^{\mu}\mathrm{e}^{-x}, \tag{1.1}$$

with support on $\mathbb{R}^+$, $\zeta < 1$, where $\theta(y)$ denotes the Heaviside function $\theta(y) = 1$ for $y > 0$, $\theta(y) = 0$ otherwise.

1.2. A random matrix context

The weight (1.1) is relevant to the Laguerre unitary ensemble LUE_N^{μ} specified by the eigenvalue probability density function

$$\frac{1}{C}\prod_{l=1}^{N} x_l^{\mu}\mathrm{e}^{-x_l}\prod_{1\le j<k\le N}(x_k-x_j)^2, \qquad x_l > 0. \tag{1.2}$$

For μ a non-negative integer, this is realized by the so-called Wishart matrices $X^{\dagger}X$ where X is an $n \times N$ ($n \ge N$, $\mu = n - N$) complex matrix of independent standard complex Gaussian matrices (see e.g. [18]). To see how (1.2) relates to (1.1), consider the random matrix average

$$\left\langle \prod_{l=1}^{N}(1-\zeta\theta(x_l-t))|x_l-t|^{\alpha}x_l^{\mu}\mathrm{e}^{-x_l}\right\rangle_{\mathrm{LUE}_N^{\mu}}. \tag{1.3}$$

This average is proportional to the multiple integral

$$\Delta_N = \frac{1}{N!}\int_0^{\infty}\mathrm{d}x_1\cdots\int_0^{\infty}\mathrm{d}x_N\prod_{l=1}^{N}w(x_l)\prod_{1\le j<k\le N}(x_k-x_j)^2, \tag{1.4}$$

where $w(x)$ is given by (1.1). Introducing the moments

$$\mu_n := \int_0^{\infty} w(x)x^n\,\mathrm{d}x \tag{1.5}$$

it is an easy result that

$$\Delta_N = \det[\mu_{j+k-2}]_{j,k=1,\ldots,N}. \tag{1.6}$$

Moreover, the $\{\Delta_n\}$ can be calculated through a recurrence linking the orthogonal polynomials $\{p_n(x)\}$ associated with $w(x)$.

Introducing the bilinear form

$$\langle f, g\rangle = \int_I f(x)g(x)w(x)\mathrm{d}x \tag{1.7}$$

the orthogonal polynomials are specified by the requirements that p_n be a polynomial of degree n and

$$\langle p_i, p_j \rangle = \delta_{i,j}. \tag{1.8}$$

By way of application of the Gram–Schmidt process, the weight function and its associated support completely specify the coefficients of each polynomial. Furthermore the orthonormality of the polynomials implies the three term recurrence relation [32]

$$a_{n+1} p_{n+1} = (x - b_n) p_n - a_n p_{n-1}. \tag{1.9}$$

It is the coefficient a_n herein which links with Δ_n. Thus one has

$$a_n^2 = \frac{\Delta_{n-1}\Delta_{n+1}}{\Delta_n^2}. \tag{1.10}$$

Also relevant is the fact that the multiple integral

$$\begin{aligned} D_N(y_1, y_2)[w(x)] := \frac{1}{N!} \int_0^\infty \mathrm{d}x_1 \cdots \int_0^\infty \mathrm{d}x_N \prod_{l=1}^N w(x_l)(y_1 - x_l)(y_2 - x_l) \\ \times \prod_{1 \le j < k \le N} (x_k - x_j)^2 \end{aligned} \tag{1.11}$$

can be expressed in terms of the polynomials $\{p_n\}$ according to the Christoffel–Darboux summation (see e.g. [18])

$$D_N(y_1, y_2)[w(x)] = \frac{\Delta_N}{\gamma_N \gamma_{N+1}} \frac{p_{N+1}(y_1) p_N(y_2) - p_{N+1}(y_1) p_N(y_2)}{y_1 - y_2}. \tag{1.12}$$

In random matrix theory the average (1.3) in the case $\mu = 0$ is the generating function for the sequence of probabilities that there are exactly k eigenvalues in the interval $(0, t)$. In the case $\zeta = 0$ it gives the moment of the modulus of the characteristic polynomial $\prod_{l=1}^N (t - x_l)$. Moreover, choosing the weight, $w(x)$, as the coefficient of ζ in (1.1), taking $\alpha = 2$ and substituting in (1.11) shows

$$\begin{aligned} &D_N(s, s)[\theta(x - t)(x - t)^2 x^\mu \mathrm{e}^{-x}] \\ &\quad = \frac{1}{N!} \int_t^\infty \mathrm{d}x_1 \cdots \int_t^\infty \mathrm{d}x_N \prod_{l=1}^N (x_l - t)^2 (x_l - s)^2 x_l^\mu \mathrm{e}^{-x_l} \prod_{j<k}^N (x_k - x_j)^2. \end{aligned} \tag{1.13}$$

After multiplying by

$$\mathrm{e}^{-t-s}(s - t)^2 (st)^\mu$$

this integral can be recognized as being proportional to the joint probability density function of the first and second smallest eigenvalues, denoted s and t respectively, in the Laguerre unitary ensemble [21]. According to (1.12) we can calculate (1.13) in terms of quantities relating to the polynomial system for the weight (1.1).

1.3. Main results

To present our main results requires some facts from the Okamoto τ-function theory of the fifth Painlevé equation [28]. The fifth Painlevé equation,

$$y'' = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\Big(\alpha_1 y + \frac{\alpha_2}{y}\Big) + \frac{\alpha_3 y}{t} + \frac{\alpha_4 y(y+1)}{y-1}, \quad (1.14)$$

is one of the six non-linear differential equations identified by Painlevé and his students as being distinct from classical equations and having the special property that all movable singularities are poles. The Okamoto τ-function theory relating to (1.14) is based on the Hamiltonian system with Hamiltonian specified by

$$tH = q(q-1)^2p^2 - \Big((v_2 - v_1)(q-1)^2 - 2(v_1+v_2)q(q-1) + tq\Big)p \\ + (v_3 - v_1)(v_4 - v_1)(q-1). \quad (1.15)$$

Here $v_1, \dots, v_4$ are parameters constrained by

$$v_1 + v_2 + v_3 + v_4 = 0.$$

An essential feature of the theory is that eliminating p in the Hamilton equations

$$q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}$$

(the dashes denote differentiation with respect to t) gives that q satisfies (1.14) with

$$\alpha_1 = \frac{1}{2}(v_3 - v_4)^2, \qquad \alpha_2 = -\frac{1}{2}(v_2 - v_1)^2, \qquad \alpha_3 = 2v_1 + 2v_2 - 1, \qquad \alpha_4 = -\frac{1}{2}.$$

The first of our main results may now be stated.

Proposition 1.1. *Let $v_1, \dots, v_4$ be such that*

$$v_3 - v_4 = -\mu, \qquad v_3 - v_1 = n + \alpha, \qquad v_3 - v_2 = n, \qquad v_2 - v_1 = \alpha,$$

and so

$$\alpha_1 = \frac{\mu^2}{2}, \qquad \alpha_2 = -\frac{\alpha^2}{2}, \qquad \alpha_3 = -(2n + \alpha + 1 + \mu), \qquad \alpha_4 = -\frac{1}{2}$$

In terms of the coefficients $\{a_n, b_n\}$ in the three term recurrence (1.9) and the parameters α, μ, t of the weight (1.1), let

$$\theta_n = b_n - 2n - 1 - \alpha - \mu - t \quad (1.16a)$$

$$\kappa_n = \Big(n + \frac{\mu}{2}\Big)t + a_n^2 - \sum_{i=0}^{n-1} b_i. \quad (1.16b)$$

We have that the Hamilton equations are satisfied by

$$q = \frac{\theta_n + t}{\theta_n} \quad (1.17a)$$

$$p = \frac{\theta_n(t(n + \alpha + \mu/2) - \kappa_n)}{t(t + \theta_n)}, \quad (1.17b)$$

and so

$$\theta_n = \frac{t}{q-1} \tag{1.18a}$$

$$\kappa_n = t(n+\alpha+\mu/2-pq). \tag{1.18b}$$

This characterization is made unique by the specification (3.20) of the small t expansions of θ_n and κ_n.

We can also express θ_n and κ_n in terms of a solution of the fifth Painlevé equation (1.14) with parameters different to those given in Proposition 1.1. For this we require the fact [28] that (1.14) is formally unchanged upon the transformations

$$(\alpha,\beta,\gamma,\delta) \mapsto (-\beta,-\alpha,-\gamma,\delta) \qquad y \mapsto \frac{1}{y}.$$

Corollary 1.2. *Suppose $v_1,\ldots,v_4$ in (1.15) are such that*

$$v_2-v_1=-\mu, \qquad v_3-v_4=\alpha, \qquad v_2-v_4=n+\alpha+1, \qquad v_2-v_3=n+1,$$

and furthermore $t\mapsto -t$, so that q satisfies (1.14) with

$$\alpha_1=\frac{\alpha^2}{2}, \qquad \alpha_2=-\frac{\mu^2}{2}, \qquad \alpha_3=2n+\alpha+1+\mu, \quad \alpha_4=-\frac{1}{2}.$$

(Note that mapping $t\mapsto -t$ in (1.14) is equivalent to mapping $\alpha_3\mapsto-\alpha_3$.) In this case the Hamilton equations are satisfied by

$$q=\frac{\theta_n}{\theta_n+t} \tag{1.19a}$$

$$p=\frac{(\theta_n+t)(\kappa_n-\mu/2+\theta_n(1+2n+t+\alpha+\mu+\theta_n))}{t\theta_n}, \tag{1.19b}$$

and so

$$\theta_n=\frac{tq}{1-q} \tag{1.20a}$$

$$\kappa_n=tpq-\frac{t^2}{(1-q)^2}+\frac{t\left(t+\frac{\mu}{2}-q\left(2n+\alpha+1+\frac{3\mu}{2}\right)\right)}{1-q}. \tag{1.20b}$$

2. Differential equations for orthogonal polynomial systems

In the classical theory of orthogonal polynomials, the study of differential equations satisfied by orthogonal polynomials has a long and distinguished history [31]. Under certain conditions [8], given a system of orthogonal polynomials, the derivatives of the polynomials may be expressed in terms of a linear combination of at most two polynomials of the same system [25,8,11]. To describe these differential equations, we parameterize the coefficients of the polynomials according to

$$p_n(x)=\gamma_n x^n+\gamma_{n,1}x^{n-1}+\cdots+\gamma_{n,n}. \tag{2.1}$$

In terms of this parameterization, it is clear from (1.9) that

$$a_n = \frac{\gamma_{n-1}}{\gamma_n}$$
$$b_n = \frac{\gamma_{n,1}}{\gamma_n} - \frac{\gamma_{n+1,1}}{\gamma_{n+1}}.$$

Important too is the Stieltjes function, defined by

$$f(x) = \sum_{k=0}^{\infty} \mu_k x^{-k-1} = \int_I \frac{w(s)}{x-s} \mathrm{d}s.$$

Following [26], our starting point is the requirement that the Stieltjes function, f, satisfies the holonomic differential equation

$$W \frac{\mathrm{d}}{\mathrm{d}x} f = 2Vf + U, \tag{2.2}$$

where W, V and U are polynomials in x. It is shown in [26] that $\deg U \leq \max(\deg W - 2, \deg V - 1)$ and that V and W relate to the weight function according to

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln w(x) = \frac{2V}{W}. \tag{2.3}$$

We now define the associated polynomials ϕ_{n-1} and associated functions ϵ_n by the equation

$$f p_n = \phi_{n-1} + \epsilon_n. \tag{2.4}$$

Explicitly

$$\epsilon_n = \int_I \frac{p_n(s)}{x-s} w(s) \mathrm{d}s \tag{2.5a}$$

$$\phi_{n-1} = \int_I \frac{p_n(s) - p_n(x)}{s-x} w(s) \mathrm{d}s \tag{2.5b}$$

showing that ϕ_{n-1} is a polynomial of degree $n-1$ and ϵ_n is meromorphic at $x = \infty$. Alternatively, $\{\phi_n\}$ may be introduced as the second linearly independent (in addition to $\{p_n\}$) polynomial solution of the three term recurrence (1.9) (see e.g. the introductory section of [13]), and from this it follows that $\{\epsilon_n\}$ also satisfies (1.9). We remark that by orthogonality

$$\epsilon_n \sim \gamma_n^{-1} x^{-n-1} \tag{2.6}$$

as x tends to ∞. Using (1.9) and (2.1), we have the large x expansions,

$$p_n = \gamma_n \left(x^n - x^{n-1} \sum_{i=0}^{n-1} b_i + x^{n-2} \left(\sum_{i=1}^{n-1} \sum_{j=0}^{i-1} b_i b_j - \sum_{i=1}^{n-1} a_i^2 \right) + O\left(x^{n-3}\right) \right) \tag{2.7a}$$

$$\epsilon_n = \gamma_n^{-1} \left(x^{-n-1} + x^{-n-2} \sum_{i=0}^{n} b_i + x^{-n-3} \left(\sum_{i=0}^{n} \sum_{j=0}^{i} b_i b_j + \sum_{i=1}^{n+1} a_i^2 \right) + O\left(x^{n-3}\right) \right). \tag{2.7b}$$

Equating $(fp_n)p_{n-1}$ with $(fp_{n-1})p_n$ shows $\phi_{n-1}p_{n-1}+\epsilon_n p_{n-1} = \phi_{n-2}p_n+\epsilon_n$. This and (2.7) give

$$p_n\epsilon_{n-1} - p_{n-1}\epsilon_n = p_{n-1}\phi_{n-1} - p_n\phi_{n-2} = \frac{1}{a_n}. \tag{2.8}$$

This also has the interpretation as the Wronskian identity relating to (1.9); see e.g. the introductory section of [13]. The above describes the notation and set formulae to be used in the following sections.

2.1. Derivation via recurrence relations for moments and isomonodromy

In this section methods are outlined for obtaining differential equations satisfied by the orthogonal polynomial system based on the existence of a recurrence for the moments of the weight function, namely (2.2). This coupled with the theory of isomonodromic deformations allows us to construct the differential equations that govern evolution of the polynomials in both x and t.

Theorem 2.1. *The orthogonal polynomials corresponding to a weight w satisfy the differential equation*

$$W(x)\frac{\mathrm{d}}{\mathrm{d}x}p_n(x) = (\Omega_n(x) - V(x))p_n(x) - a_n\Theta_n(x)p_{n-1}(x) \tag{2.9}$$

where Ω_n and Θ_n are polynomials given by

$$\Theta_n = W\left(\epsilon_n\frac{\mathrm{d}}{\mathrm{d}x}p_n - p_n\frac{\mathrm{d}}{\mathrm{d}x}\epsilon_n\right) + 2V\epsilon_n p_n \tag{2.10a}$$

$$\Omega_n = a_n W\left(\epsilon_{n-1}\frac{\mathrm{d}}{\mathrm{d}x}p_n - p_{n-1}\frac{\mathrm{d}}{\mathrm{d}x}\epsilon_n\right) + a_n V(\epsilon_{n-1}p_n + \epsilon_n p_{n-1}). \tag{2.10b}$$

Proof. First we note from (2.2) and (2.4) that

$$W\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi_{n-1}}{p_n}\right) - \frac{2V\phi_{n-1}}{p_n} - U = \frac{2V\epsilon_n}{p_n} - W\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\epsilon_n}{p_n}\right).$$

Multiplying through by p_n^2 shows (2.10a) can be rewritten

$$\Theta_n = W\left(p_n\frac{\mathrm{d}}{\mathrm{d}x}\phi_{n-1} - \phi_{n-1}\frac{\mathrm{d}}{\mathrm{d}x}p_n\right) - 2V\phi_{n-1}p_n - Up_n^2. \tag{2.11}$$

This tells us that Θ_n is a polynomial, while (2.10a) bounds the degree. Explicitly, examining the $x \to \infty$ behaviour, namely (2.1) and (2.6), shows

$$\deg\Theta_n \le \max(\deg W - 2, \deg V - 1, 0) \tag{2.12}$$

as noted in [26].

Using (2.8) and (2.11) we find

$$a_n(p_{n-1}\phi_{n-1} - p_n\phi_{n-2})\Theta_n = W\left(p_n\frac{\mathrm{d}}{\mathrm{d}x}\phi_{n-1} - \phi_{n-1}\frac{\mathrm{d}}{\mathrm{d}x}p_n\right) - 2V\phi_{n-1}p_n - Up_n^2.$$

By appropriately grouping the terms divisible by ϕ_{n-1} and p_n on opposite sides, we define the polynomial Ω_n to be the common factor according to

$$\begin{aligned} p_n\phi_{n-1}\Omega_n &= \phi_{n-1}\left(a_n\Theta_n p_{n-1} + W\frac{\mathrm{d}}{\mathrm{d}x}p_n + Vp_n\right) \\ &= p_n\left(a_n\Theta_n\phi_{n-2} + W\frac{\mathrm{d}}{\mathrm{d}x}\phi_{n-1} - V\phi_{n-1} - Up_n\right). \end{aligned} \tag{2.13}$$

The first expression in (2.13) is equivalent to (2.9) provided (2.10b) can be verified. For this purpose we use (2.10a) in (2.13) to obtain

$$\Omega_n = \frac{a_n W p_{n-1}\epsilon_n \frac{\mathrm{d}}{\mathrm{d}x}p_n}{p_n} - \frac{a_n W p_{n-1}p_n \frac{\mathrm{d}}{\mathrm{d}x}\epsilon_n}{p_n} + \frac{2V\epsilon_n p_{n-1}p_n}{p_n} + \frac{W\frac{\mathrm{d}}{\mathrm{d}x}p_n}{p_n} + V.$$

By rearranging (2.8), we find $a_n\epsilon_n p_{n-1} = a_n p_n\epsilon_{n-1} - 1$, which we use to remove occurrences of p_{n-1}, giving (2.10b) as required. Examining the $x \to \infty$ behaviour, by using (2.1) and (2.6) in (2.10b), shows

$$\deg \Omega_n \le \max(W - 1, V) \tag{2.14}$$

which again appears in [26]. $\square$

The origin of this theorem can be traced back to the work of Laguerre [25] and has since been revisited by contemporaries [6,7,12,26]. The theorem provides a mechanical way of determining the differential equation satisfied by polynomials provided one knows the rational logarithmic derivative. One need only expand (2.10a) and (2.10b) to polynomial orders using (2.7b) to produce a parameterization of the differential equation satisfied by the polynomials in terms of the a_n's and b_n's. A simple application of (1.9) gives us an expression for the derivative of p_{n-1}, which is but one column solution to a 2×2 linear differential equation in x. The following corollary provides us with another solution [16,17].

Corollary 2.2. *The function ϵ_n/w satisfies (2.9).*

Proof. Consider the derivative of fp_n in terms of ϵ_n and ϕ_n. According to (2.4)

$$W\frac{\mathrm{d}}{\mathrm{d}x}fp_n = W\frac{\mathrm{d}}{\mathrm{d}x}\phi_{n-1} + W\frac{\mathrm{d}}{\mathrm{d}x}\epsilon_n.$$

On the other hand, use of (2.9) and (2.2) shows

$$\begin{aligned} W\frac{\mathrm{d}}{\mathrm{d}x}fp_n &= f\left(W\frac{\mathrm{d}}{\mathrm{d}x}p_n\right) + p_n W\frac{\mathrm{d}}{\mathrm{d}x}f \\ &= (\Omega_n - V)fp_n - a_n\Theta_n fp_{n-1} + 2Vfp_n + Up_n \\ &= (\Omega_n + V)\phi_{n-1} - a_n\Theta_n\phi_{n-1} + Up_n + (\Omega_n + V)\epsilon_n - a_n\Theta_n\epsilon_{n-1}, \end{aligned}$$

where in obtaining the final equality (2.4) has also been used. By canceling out the derivative of ϕ_{n-1} as calculated from the first expression for Ω_n from the previous proof, we deduce that the derivative of ϵ_n is given by

$$W\frac{\mathrm{d}}{\mathrm{d}x}\epsilon_n = (\Omega_n + V)\epsilon_n - a_n\Theta_n\epsilon_{n-1}.$$

Hence

$$W \frac{\mathrm{d}}{\mathrm{d}x} \frac{\epsilon_n}{w} = \frac{w W \frac{\mathrm{d}}{\mathrm{d}x} \epsilon_n - \epsilon_n W \frac{\mathrm{d}}{\mathrm{d}x} w}{w^2} = \frac{(\Omega_n + V)\epsilon_n - a_n \Theta_n \epsilon_{n-1} - 2V\epsilon_n}{w},$$

where use has also been made of (2.3), as required. $\square$

As a linear system, we have two linearly independent solutions. As mentioned above, both $\{p_n\}$ and $\{\epsilon_n/w\}$ satisfy (1.9) and (2.9), telling us that $y_n = p_{n-1}$ and $y_n = \epsilon_{n-1}/w$ satisfy

$$W \frac{\mathrm{d}}{\mathrm{d}x} y_n = a_n \Theta_{n-1} y_n + (\Omega_{n-1} - V - (x - b_n)) y_{n-1}.$$

Hence, the matrix

$$Y_n = \begin{pmatrix} p_n & \dfrac{\epsilon_n}{w} \\ p_{n-1} & \dfrac{\epsilon_{n-1}}{w} \end{pmatrix} \tag{2.15}$$

satisfies the matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_n = \mathscr{A}_n Y_n \tag{2.16}$$

where

$$\mathscr{A}_n = \frac{1}{W} \begin{pmatrix} \Omega_n - V & -a_n \Theta_n \\ -a_n \Theta_{n-1} & \Omega_{n-1} - V - (x - b_{n-1}) \Theta_{n-1} \end{pmatrix}.$$

Because $\{p_n\}$ and $\{\epsilon_n\}$ satisfy (1.9), Y_n also satisfies

$$Y_{n+1} = M_n Y_n \tag{2.17}$$

where

$$M_n = \begin{pmatrix} \dfrac{x - b_n}{a_{n+1}} & -\dfrac{a_n}{a_{n+1}} \\ 1 & 0 \end{pmatrix}.$$

Lemma 2.3. *The polynomials Θ_n and Ω_n satisfy the recurrence relations*

$$W + a_{n+1}^2 \Theta_{n+1} - a_n^2 \Theta_{n-1} = (x - b_n)(\Omega_{n+1} - \Omega_n) \tag{2.18a}$$

$$(x - b_{n-1}) \Theta_{n-1} - (x - b_n) \Theta_n = \Omega_n - \Omega_{n+1}. \tag{2.18b}$$

Proof. Eqs. (2.16) and (2.17) gives two ways of calculating $\frac{\mathrm{d}}{\mathrm{d}x} Y_{n+1}$. The consistency may be written as

$$M_n \mathcal{A}_n - \mathcal{A}_{n+1} M_n + \frac{\mathrm{d}M_n}{\mathrm{d}x} = 0.$$

This is an identity on the bottom two rows, however, in the first row, the consistency relation gives (2.18a) and (2.18b). $\square$

This lemma can be found in Magnus [26]. Using (2.8), we have

$$\det Y_n = \frac{1}{a_n w}.$$

In general, for an equation of the form (2.16), we see from (2.15) that

$$\frac{\mathrm{d}}{\mathrm{d}x} \det Y_n = \operatorname{Tr} \mathscr{A}_n \det Y_n$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{a_n w} = -\frac{2V}{w a_n W} = \operatorname{Tr} \mathscr{A}_n \det Y_n$$

giving the additional relation

$$(x - b_n)\Theta_n = \Omega_{n+1} + \Omega_n. \tag{2.19}$$

This also implies (2.18b). It further gives us a new parameterization of $\mathscr{A}_n$, given by

$$\mathscr{A}_n = \frac{1}{W} \begin{pmatrix} \Omega_n - V & -a_n \Theta_n \\ a_n \Theta_{n-1} & -\Omega_n - V \end{pmatrix}$$

as first derived in [26].

Another useful relation comes from the multiplication of (2.18a) and (2.19), which gives

$$W\Theta_n + a_{n+1}^2 \Theta_{n+1} \Theta_n - a_n^2 \Theta_n \Theta_{n-1} = \Omega_{n+1}^2 - \Omega_n^2. \tag{2.20}$$

Summing over n, given that $\Omega_0 = V$, shows

$$\Omega_n^2 - a_n^2 \Theta_n \Theta_{n-1} = V^2 + W \sum_{i=0}^{n-1} \Theta_i. \tag{2.21}$$

The roots of W are now the poles of $\mathscr{A}_n$. If $\{x_j\}$ is the set of poles of $\mathscr{A}_i$, then we may write (2.16) as

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_n = \mathscr{A}_n Y_n = \left(\sum_i \frac{\mathscr{A}_{i,n}}{x - x_i} \right) Y_n \tag{2.22}$$

where

$$\mathscr{A}_{i,n} = \frac{1}{W'(x_i)} \begin{pmatrix} \Omega_n(x_i) - V(x_i) & -a_n \Theta_n(x_i) \\ a_n \Theta_{n-1}(x_i) & -\Omega_n(x_i) - V(x_i) \end{pmatrix}.$$

This now places the differential equation into the context of isomonodromic deformations [16,17]. In general, any solution to (2.22) is going to be multi-valued, with branch points at $\{x_i\}$(one possibly being ∞). Hence, by integrating around a path, say $\rho : [0, 1] \to \mathbb{C} \setminus \{x_i\}$ where $\rho(0) = \rho(1)$, the multi-valuedness can be expressed through the equation

$$Y(\rho(0)) = Y(\rho(1))\mathscr{M}_\rho$$

where $\mathscr{M}_\rho$ is referred to as a monodromy matrix. The set of monodromy matrices, $\{\mathscr{M}_\rho\}$, forms a representation of the fundamental group, $\pi_1(\mathbb{C} \setminus \{x_i\})$.

By construction, one solution of (2.22) involves the polynomials, which are entire, and the associated functions. The goal of monodromy preserving deformations is to describe a family of linear problems of the form (2.22) that share the same representation. A natural choice of deformation parameter turns out to be the poles of $\mathcal{A}_n$, giving rise to the classical result known as the Schlesinger equations, given by

$$\frac{\partial A_i}{\partial \alpha_j} = \frac{[A_i, A_j]}{\alpha_i - \alpha_j} \quad i \neq j \tag{2.23a}$$

$$\frac{\partial A_i}{\partial \alpha_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{\alpha_i - \alpha_j}. \tag{2.23b}$$

We shall assume that just one of the poles depends on a variable, t, which shall become the deformation parameter. In the case that we have just one parameter that needs to be deformed, we have that there is a matrix

$$\mathscr{B}_n(x,t) = \frac{\partial Y_n}{\partial t} Y_n^{-1}. \tag{2.24}$$

The form of this matrix, as implied by (2.23), is given by [15]

$$\mathscr{B}_n = \mathscr{B}_{\infty,n} - \sum_i \frac{\mathscr{A}_{i,n}}{x - x_i} \frac{\partial x_i}{\partial t}. \tag{2.25}$$

By examining the large x behaviour of p_n and $\frac{\partial p_i}{\partial t}$, we deduce that $\mathscr{B}_{\infty,n}$ is given by

$$\mathscr{B}_{\infty,n} = \begin{pmatrix} \frac{1}{\gamma_n}\frac{\partial \gamma_n}{\partial t} & 0 \\ 0 & -\frac{1}{\gamma_{n-1}}\frac{\partial \gamma_{n-1}}{\partial t} \end{pmatrix}. \tag{2.26}$$

This gives us two linear differential equations for the one system, which must be consistent. Hence we have the condition

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x} Y_n = \frac{\partial}{\partial x}\frac{\partial}{\partial t} Y_n,$$

which is equivalent to

$$\mathscr{A}_n \mathscr{B}_n - \mathscr{B}_n \mathscr{A}_n + \frac{\partial}{\partial t}\mathscr{A}_n - \frac{\partial}{\partial x}\mathscr{B}_n = 0. \tag{2.27}$$

This completely determines the differential equation for the orthogonal polynomials and associated functions in t.

2.2. Ladder operators

An alternative approach, developed by Chen and collaborators [11,14,5], is that of the ladder operators. Below, we will give an account of some of the main results of this theory as required for our purposes. We shall assume that the weight satisfies the same logarithmic differential equation (2.3), and hence that the corresponding moments satisfy (2.2). However, this approach typically concerns the monic versions of the orthogonal polynomials, given by

$$P_n = \frac{1}{\gamma_n} p_n.$$

In order to make comparisons to the previous section, we will deal primarily with p_n rather than P_n. Now p_n, being a polynomial of degree n, when differentiated can be expressed as a linear combination of $\{p_j\}_{j=0,\dots,n-1}$,

$$\frac{\mathrm{d}}{\mathrm{d}x} p_n = \sum_{i=0}^{n-1} \alpha_{n,i} p_i. \tag{2.28}$$

We may reduce this, via the use of (1.9), to a differential equation specified by the following theorem of Bonan and Clark [7] and Bauldry [6].

Theorem 2.4. *The orthogonal polynomial system $\{p_n\}$, defined by the weight w, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}x} p_n = -B_n p_n + a_n A_n p_{n-1} \tag{2.29}$$

where

$$A_n = \int_I \left(\frac{p_n(y)^2}{x-y}\right)\left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)}\right) w(y) \tag{2.30a}$$

$$B_n = a_n \int_I \left(\frac{p_n(y)p_{n-1}(y)}{x-y}\right)\left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)}\right) w(y)\mathrm{d}y. \tag{2.30b}$$

Proof. Beginning with (2.28), we may use orthogonality and integration by parts to find

$$\alpha_{n,i} = \int_I p_n'(y) p_i(y) w(y)\mathrm{d}y = -\int_I p_n(y)\left(p_i'(y) - \frac{2p_i(y)V}{W}\right) w(y)\mathrm{d}y,$$

where the second equality requires that w vanishes at the endpoints of I (if w is a continuous function of a parameter, such as for $w(y) = y^\alpha \mathrm{e}^{-y}$, the range of validity can be extended by analytic continuation [2]). Since p_i' is a polynomial whose degree is less than n, this term must be destroyed by orthogonality, leaving

$$\alpha_{n,i} = -\int_I \frac{2V}{W} p_n(y) p_k(y) w(y)\mathrm{d}y.$$

The derivative of p_n can therefore be written

$$p_n'(x) = -\int_I \sum_{i=0}^{n-1} p_i(x) p_i(y) \frac{2V(y)}{W(y)} w(y)\mathrm{d}y.$$

However, if we replace $2V(y)/W(y)$ with $2V(x)/W(x)$, this would vanish by orthogonality, hence we may add it to obtain

$$p_n'(x) = \int_I p_n(y)\left(\sum_{i=0}^{n-1} p_i(x) p_i(y)\right)\left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)}\right) w(y)\mathrm{d}y.$$

By exploiting the Christoffel–Darboux summation (1.12) and pulling out the polynomials in x, we arrive at

$$\begin{aligned} p_n'(x) &= a_n p_{n-1}(x) \int_I \left(\frac{p_n(y)^2}{x-y} \right) \left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} \right) w(y) \mathrm{d}y \\ &\quad - a_n p_n(x) \int_I \left(\frac{p_n(y) p_{n-1}(y)}{x-y} \right) \left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} \right) w(y) \mathrm{d}y, \end{aligned}$$

which is (2.29). □

This allows us to define the ladder operator,

$$L_{n,1} := \left(\frac{\mathrm{d}}{\mathrm{d}x} + B_n \right)$$

which has the effect

$$L_{n,1} p_n = A_n p_{n-1}.$$

Lemma 2.5. *The terms A_n and B_n satisfy the recurrence relations*

$$B_{n+1} + B_n = (x - b_n) A_n - \frac{2V}{W} \tag{2.31a}$$

$$(B_{n+1} - B_n)(x - b_n) = a_{n+1}^2 A_{n+1} - a_n^2 A_{n-1} + 1. \tag{2.31b}$$

Proof. Using (2.30b),

$$\begin{aligned} B_n + B_{n+1} &= \int_I \left(\frac{p_n(y)(a_n p_{n-1}(y) + a_{n+1} p_{n+1}(y))}{x-y} \right) \left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} \right) w(y) \mathrm{d}y \\ &= \int_I \left(\frac{p_n(y)^2 (y - b_n)}{x-y} \right) \left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} \right) w(y) \mathrm{d}y \\ &= (x - b_n) A_n + \int_I p_n(y)^2 \left(\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} \right) w(y) \mathrm{d}y \\ &= (x - b_n) A_n - \frac{2V}{W}. \end{aligned}$$

By consistency of (2.29) with (1.9) we obtain the second required expression. □

By multiplying and rearranging (2.31a) and (2.31b) we obtain

$$B_{n+1}^2 - B_n^2 - \frac{2V}{W}(B_{n+1} - B_n) = a_{n+1}^2 A_{n+1} A_n - a_n^2 A_{n-1} A_n + A_n. \tag{2.32}$$

Hence, by summing over n and appropriately evaluate initial conditions, we obtain

$$B_n^2 - \frac{2V}{W} B_n - a_n^2 A_n A_{n-1} = -\sum_{i=0}^{n-1} A_i. \tag{2.33}$$

Note the structural correspondence of (2.32) and (2.33) with (2.20) and (2.21) respectively.

3. Derivations of $\mathbf{P}_V$

We now turn to the polynomials specified by (1.7) with the weight specified by (1.1). The above formulae for the derivatives in x should be considered as partial derivatives. The formula (1.5) for the moments is a hypergeometric integral, which may be evaluated to give

$$\begin{aligned}\mu_k &= (1-\zeta)\Gamma(1+k+\alpha+\mu)\,{}_1F_1\left(\begin{matrix}-\alpha\\ -k-\alpha-\mu\end{matrix}\middle| -t\right)\\ &\quad+\left(1+\frac{(\zeta-1)\sin(\pi\mu)}{\sin(\pi(\alpha+\mu))}\right)\frac{\Gamma(\mu+k+1)\Gamma(\alpha+1)}{\Gamma(2+k+\alpha+\mu)}\\ &\quad\times t^{1+k+\alpha+\mu}\,{}_1F_1\left(\begin{matrix}1+k+\mu\\ 2+k+\alpha+\mu\end{matrix}\middle| -t\right)\\ &= C_1(\zeta,\mu,k,\alpha)\,{}_1F_1\left(\begin{matrix}-\alpha\\ -k-\alpha-\mu\end{matrix}\middle| -t\right)\\ &\quad+C_2(\zeta,\mu,k,\alpha)t^{1+k+\alpha+\mu}\,{}_1F_1\left(\begin{matrix}1+k+\mu\\ 2+k+\alpha+\mu\end{matrix}\middle| -t\right)\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}C_1(\zeta,\mu,k,\alpha) &= (1-\zeta)\Gamma(1+k+\alpha+\mu)\\ C_2(\zeta,\mu,k,\alpha) &= \left(1+\frac{(\zeta-1)\sin(\pi\mu)}{\sin(\pi(\alpha+\mu))}\right)\frac{\Gamma(\mu+k+1)\Gamma(\alpha+1)}{\Gamma(2+k+\alpha+\mu)}\end{aligned}$$

and ${}_1F_1$ is the confluent hypergeometric function. We seek the corresponding differential equations satisfied by the orthogonal polynomial system, as implied by the theory of Sections 2.1 and 2.2.

To derive the differential equation satisfied by the orthogonal polynomials from the theory of Section 2.1, we remark that since the factor of $(1+\zeta\theta(x-t))$ plays the role of a multiplicative constant almost everywhere, the logarithmic derivative of w coincides with the logarithmic derivative of $(x-t)^\alpha x^\mu e^{-x}$ almost everywhere. Hence we write

$$x(x-t)\frac{\partial_x w}{w} \cong \left(-x^2+(\alpha+\mu+t)x-\mu t\right)$$

where $\cong$ is to be interpreted as equals almost everywhere, and so independent of ζ

$$\begin{aligned}W &= x(x-t)\\ 2V &= -x^2+(\alpha+\mu-t)x+\mu t.\end{aligned}$$

Recall that the form of the logarithmic derivative is the essential ingredient in both Theorems 2.1 and 2.4.

3.1. Recurrence of moments approach

Now that W and V have been defined, determining the differential equation satisfied for this particular family of orthogonal polynomials is simply a matter of applying Theorem 2.1.

Corollary 3.1. *The polynomials p_n corresponding to the weight (1.1) satisfy the differential equation*

$$\partial_x \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \left\{ \mathscr{A}_\infty + \frac{\mathscr{A}_0}{x} + \frac{\mathscr{A}_t}{x-t} \right\} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} \tag{3.2}$$

where

$$\mathscr{A}_0 = \frac{1}{t} \begin{pmatrix} \kappa_n - \frac{\mu t}{2} & -a_n \theta_n \\ a_n \theta_{n-1} & -\kappa_n - \frac{\mu t}{2} \end{pmatrix}$$

$$\mathscr{A}_t = \frac{1}{t} \begin{pmatrix} \left(n + \frac{\mu}{2}\right) t - \kappa_n & a_n(\theta_n + t) \\ -a_n(\theta_{n-1} + t) & \kappa_n - \left(n + \alpha + \frac{\mu}{2}\right) t \end{pmatrix}$$

$$\mathscr{A}_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\theta_n = b_n - 2n - 1 - \alpha - \mu - t$$

$$\kappa_n = \left(n + \frac{\mu}{2}\right) t + a_n^2 + \frac{\gamma_{n,1}}{\gamma_n}.$$

Proof. By way of application of (2.10b) and (2.10a) using (2.7) one obtains for the explicit form of Ω_n and Θ_n,

$$\Omega_n = -\frac{x^2}{2} + x\left(\frac{2n + \alpha + \mu + t}{2}\right) - \frac{2n + \mu t}{2} - a_n^2 - \frac{\gamma_{n,1}}{\gamma_n}$$

$$\Theta_n = -x + 2n + 1 + \alpha + \mu + t - b_n,$$

which we subsequently decompose into the form seen above. □

We also require these equations to be written in terms of the κ_n and θ_n alone. For this, we note that recurrence relations for θ_n and κ_n are implied by (2.18a) and (2.19).

Corollary 3.2. *The associated functions, θ_n and κ_n, satisfy the recurrences*

$$\kappa_{n+1} + \kappa_n = -\theta_n(\theta_n + t + 2n + \alpha + 1 + \mu) \tag{3.3a}$$

$$\frac{\theta_n}{\theta_n + t} \frac{\theta_{n-1}}{\theta_{n-1} + t} = \frac{\kappa_n^2 - \frac{\mu^2 t^2}{4}}{\left(\kappa_n - \left(n + \alpha + \frac{\mu}{2}\right) t\right)\left(\kappa_n - \left(n + \frac{\mu}{2}\right) t\right)}. \tag{3.3b}$$

Proof. The relation (2.18b) is equivalent to (3.3a) when one uses the definitions of Θ_n and Ω_n in terms of θ_n and κ_n. Evaluating (2.21) at $x = 0$ and $x = t$ shows

$$a_n^2 \theta_n \theta_{n-1} = \kappa_n^2 - \frac{\mu^2 t^2}{4} \tag{3.4}$$

$$a_n^2(t + \theta_n)(t + \theta_{n-1}) = \left(\kappa_n - \frac{(2n + 2\alpha + \mu)t}{2}\right)\left(\kappa_n - \frac{(2n + \mu)t}{2}\right) \tag{3.5}$$

respectively. The ratio of these identities is (3.3b). □

The relation (3.3b) may be used to eliminate the occurrence of θ_{n-1} in (3.2). Note that (3.2) is the form of (2.22), where only one of the poles depends on t. Hence, the evolution in t is governed by (2.24). In this regard, the derivations of the time derivatives for θ_n and κ_n and the methods of Forrester and Witte [21] contrast with the methods of Basor and Chen [5]. Once the derivatives in x are found, one may apply the theory of isomonodromic deformations [24] to obtain appropriate derivatives in t. This approach, as seen in [21] extends the evolution of the orthogonal polynomials in the t direction via the following result.

Corollary 3.3. *In addition to (3.2), the orthogonal polynomials satisfy*

$$\partial_t \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \left\{ \mathscr{B} - \frac{\mathscr{A}_t}{x-t} \right\} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} \tag{3.6}$$

where

$$\mathscr{B} = \frac{1}{2t} \begin{pmatrix} \theta_n + t & 0 \\ 0 & -\theta_{n-1} - t \end{pmatrix}.$$

Proof. This almost directly follows from the corollary of the Schlesinger equations, (3.2) and (2.25). We use the fact that in the context of orthogonal polynomials $\mathscr{B}$ has the explicit form (2.26). By equating the residues of the left- and right-hand side of the compatibility relation (2.27) at $x = \infty$, the diagonal entries reveal

$$\frac{2\partial_t \gamma_n}{\gamma_n} = 1 + \frac{\theta_n}{t} \tag{3.7a}$$

$$\frac{2\partial_t \gamma_{n-1}}{\gamma_{n-1}} = 1 + \frac{\theta_{n-1}}{t} \tag{3.7b}$$

while the off diagonal entries are 0. This gives the required form for $\mathscr{B}$ above. $\square$

One may easily calculate the derivatives in t of θ_n and κ_n via the compatibility of (2.16) and (2.24),

$$\partial_t \mathscr{A} - \partial_x \mathscr{B} + \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A} = 0, \tag{3.8}$$

to define the evolution of θ_n and κ_n. Using the above recursion relations allows one to express the derivatives of θ_n and κ_n in terms of themselves. Alternatively, using the general framework of [26], the derivatives of a_n and b_n are expressible in terms of the functions Θ_n and Ω_n evaluated at the movable finite singular points of (2.16) via the expression

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln a_n = \frac{1}{2} \sum_{r=1}^{m} \frac{\Theta_n(x_r) - \Theta_{n-1}(x_r)}{W'(x_r)} \frac{\mathrm{d}}{\mathrm{d}t} x_r \tag{3.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln b_n = \sum_{r=1}^{m} \frac{\Omega_{n+1}(x_r) - \Omega_{n-1}(x_r)}{W'(x_r)} \frac{\mathrm{d}}{\mathrm{d}t} x_r, \tag{3.9b}$$

where in the case of (2.16), $m = 1$, and the only point is $x_1 = t$. This leads to the equations

$$\frac{2t}{a_n} \frac{\mathrm{d}a_n}{\mathrm{d}t} = 2 + b_{n-1} - b_n \tag{3.10a}$$

$$t \frac{\mathrm{d}b_n}{\mathrm{d}t} = a_n^2 - a_{n+1}^2 + b_n. \tag{3.10b}$$

We know that (3.9) is equivalent to (3.10). Using (3.8) and (3.10) in conjunction with (3.3) to eliminate occurrences of a_n^2 and θ_{n-1} gives a differential system for $\{\theta_n, \kappa_n\}$.

Corollary 3.4. *The associated functions, θ_n and κ_n satisfy the coupled differential equations in t*

$$t\frac{\partial}{\partial t}\theta_n = 2\kappa_n + (2n+\alpha+1+\mu+t+\theta_n)\theta_n \tag{3.11a}$$

$$\begin{aligned} t\frac{\partial \kappa_n}{\partial t} &= \left(\frac{1}{\theta_n+t}+\frac{1}{\theta_n}\right)\kappa_n^2 + \left(2n+\alpha+\mu+1-(2n+\alpha+\mu)\frac{t}{\theta_n+t}\right)\kappa_n \\ &\quad - \left(n^2+\left(n+\frac{\mu}{2}\right)(\alpha+\mu)\right)t - \frac{\mu^2t^2}{4\theta_n} + \left(n+\frac{\mu}{2}\right)\left(n+\alpha+\frac{\mu}{2}\right)\frac{t^2}{\theta_n+t}. \end{aligned} \tag{3.11b}$$

Proof. This simply follows from the evaluation of (2.27). The first relation follows from (3.7), namely

$$\frac{2ta_n'}{a_n} = \theta_{n-1} - \theta_n.$$

The two other relations that arise are

$$\begin{aligned} t\theta_n' &= 2\kappa_n + (2n+\alpha+1+\mu+t+\theta_n)\theta_n \\ t\kappa_n' &= \kappa_n - a_n^2(\theta_n - \theta_{n-1}). \end{aligned}$$

The first of these is (3.11a). By using (3.4) to eliminate a_n^2 and (3.3b) to eliminate θ_{n-1}, one obtains (3.11b). □

Now that one has the derivatives of κ_n and θ_n, the remaining task is to find the transformation which allows them to be identified as the Hamilton equations for a Painlevé V system. But before doing this, we want to show how differential equations equivalent to the coupled system (3.11) can be derived from the formalism of Section 2.2.

3.2. Ladder operator approach

We want to specialize (2.29) to the weight (1.1).

Proposition 3.5. *With the weight (1.1) the coefficients A_n and B_n in (2.29) are given by*

$$\begin{aligned} A_n &= \frac{R_n}{x-t} + \frac{1-R_n}{x} \\ B_n &= \frac{r_n}{x-t} - \frac{n+r_n}{x} \end{aligned}$$

where for $\alpha \geq 1$

$$\begin{aligned} R_n &= \alpha\int_0^\infty \frac{w(y)p_n(y)^2}{(t-y)}\mathrm{d}y \\ r_n &= \alpha\int_0^\infty \frac{w(y)p_n(y)p_{n-1}}{(t-y)}\mathrm{d}y. \end{aligned}$$

Proof. We note that

$$\frac{2V(x)}{W(x)} - \frac{2V(y)}{W(y)} = \left(-\frac{\alpha}{(t-x)(t-y)} - \frac{\mu}{xy}\right)(x-y),$$

and hence the integrals that define A_n and B_n simplify to

$$\begin{aligned} A_n &= \int_0^\infty p_n(y)^2 \left(-\frac{\alpha}{(t-x)(t-y)} - \frac{\mu}{xy}\right) w(y)\mathrm{d}y \\ B_n &= \int_0^\infty p_n(y)p_{n-1}(y) \left(-\frac{\alpha}{(t-x)(t-y)} - \frac{\mu}{xy}\right) w(y)\mathrm{d}y, \end{aligned}$$

or equivalently

$$\begin{aligned} A_n &= \frac{\alpha}{x-t}\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha-1}y^\mu \mathrm{e}^{-y} p_n(y)^2\mathrm{d}y \\ &\quad -\frac{\mu}{x}\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha}y^{\mu-1} \mathrm{e}^{-y} p_n(y)^2\mathrm{d}y \\ B_n &= \frac{\alpha}{x-t}\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha-1}y^\mu \mathrm{e}^{-y} p_n(y)p_{n-1}(y)\mathrm{d}y \\ &\quad -\frac{1}{x}\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha}y^{\mu-1} \mathrm{e}^{-x} p_n(y)p_{n-1}(y)\mathrm{d}y. \end{aligned}$$

Now define

$$\begin{aligned} R_n &= \alpha\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha-1}y^\mu \mathrm{e}^{-y} p_n(y)^2\mathrm{d}y \\ &= \alpha\int_0^\infty \frac{w(y)p_n(y)^2}{(t-y)}\mathrm{d}y \\ r_n &= \alpha\int_0^\infty (1-\zeta\theta(y-t))(y-t)^{\alpha-1}y^\mu \mathrm{e}^{-y} p_n(y)p_{n-1}(y)\mathrm{d}y \\ &= \alpha\int_0^\infty \frac{w(y)p_n(y)p_{n-1}}{(t-y)}\mathrm{d}y. \end{aligned}$$

We apply integration by parts, orthogonality and the known value of $w(x,t)$ at $0, t$ and ∞, to express the second part of the integrals in A_n and B_n in terms of R_n and r_n respectively, giving

$$\begin{aligned} A_n &= \frac{R_n}{x-t} - \frac{1}{x}\int_0^\infty \left(\frac{\alpha w(y,t)p_n^2(y)}{t-y} - w(y,t)p_n^2(y)\right)\mathrm{d}y \\ B_n &= \frac{r_n}{x-t} - \frac{\mu}{x}\int_0^\infty \left(\frac{\alpha w(y,t)p_n(y)p_{n-1}}{t-y} + w(y,t)p_{n-1}(y)\frac{\partial p_n}{\partial y}\right)\mathrm{d}y. \end{aligned}$$

Using orthogonality and the expression

$$\frac{\partial p_n(y)}{\partial y} = \frac{n}{a_n}p_{n-1}(y) + \text{lower order terms}$$

gives the stated formulae. □

Now that the forms of A_n and B_n are known, the differential equation satisfied by the polynomials can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{n+r}{x} - \frac{r_n}{x-t} & a_n\left(\frac{1-R_n}{x} + \frac{R_n}{x-t}\right) \\ -a_n\left(\frac{1-R_{n-1}}{x} + \frac{R_{n-1}}{x-t}\right) & \frac{r_n}{x-t} - \frac{n+r_n}{x} - \frac{2V}{W} \end{pmatrix}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}$$

where the second row is a consequence of (1.9) and (2.31a).

Lemma 3.6. *The functions R_n and r_n satisfy the recurrences*

$$r_{n+1} + r_n - \alpha = R_n(\mu + \alpha + 2n + 1 + tR_n - t) \tag{3.12a}$$

$$\frac{R_n R_{n-1}}{(R_n - 1)(R_{n-1} - 1)} = \frac{r_n(r_n - \alpha)}{(r_n + n)(r_n + n + \mu)}. \tag{3.12b}$$

Proof. The residue in x of (2.31a) at t and ∞ using these definitions for A_n and B_n in terms of R_n and r_n shows

$$b_n = 2n + 1 + \alpha + \mu + tR_n \tag{3.13}$$
$$r_{n+1} + r_n - \alpha = R_n(t - b_n)$$

which gives (3.12a). The evaluation of the result of multiplying (2.31b) by $x^2(x-t)^2$ at 0 and t reveals

$$r_n(r_n - \alpha) = a_n^2 R_{n-1} R_n \tag{3.14}$$
$$(n + r_n)(n + \mu + r_n) = a_n^2(R_n - 1)(R_{n-1} - 1)$$

giving (3.12b) in an analogous manner to the previous section. □

In addition to (3.13), there is a further relation obtained by eliminating R_{n-1} from (3.14) by using (3.12b), giving

$$a_n^2 = \frac{(r_n - \alpha)r_n}{R_n} - \frac{(n + r_n)(n + \mu + r_n)}{R_n - 1}. \tag{3.15}$$

Lemma 3.7. *The recursion coefficients of (1.9) satisfy the differential equations*

$$\frac{2}{a_n}\frac{\mathrm{d}a_n}{\mathrm{d}t} = R_{n-1} - R_n \tag{3.16a}$$

$$\frac{\mathrm{d}b_n}{\mathrm{d}t} = r_n - r_{n+1}. \tag{3.16b}$$

Proof. We take the derivative of (1.8) in the case $i = j = n$ to see that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty p_n(y)^2 w(y)\mathrm{d}y$$

with respect to t. We recall that α is a non-negative integer by assumption. Hence, $w(x,t)$ is continuous at $x = t$ and $w(t,t) = 0$, and so

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \mathrm{d}y - \int_\infty^t \mathrm{d}y\right) w(y)p_n(y)^2 \\
&= \left(\lim_{x\to t^-} - \lim_{x\to t^+}\right) w(x)p_n^2(x) - \left(\int_0^t \mathrm{d}y - \int_\infty^t \mathrm{d}y\right)\frac{\partial}{\partial t}\left(p_n(y)^2 w(y)\right) \\
&= -\alpha \int_0^\infty \frac{w(y)p_n^2(y)}{y-t}\mathrm{d}y + \int_0^\infty 2p_n \frac{\partial p_n}{\partial t} w(y)\mathrm{d}y
\end{aligned}$$

where we have used

$$\frac{\partial w(x,t)}{\partial t} = -\frac{\alpha w(x,t)}{x-t}.$$

Using

$$\frac{\partial p_n}{\partial t} = \frac{\gamma_n'}{\gamma_n}\gamma_n x^n + \text{lower order terms}$$

gives us that

$$R_n = 2\frac{\gamma_n'}{\gamma_n}$$

and hence

$$\frac{2}{a_n}\frac{\mathrm{d}a_n}{\mathrm{d}t} = 2\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_{n-1}}{\gamma_n}\right) = 2\frac{\gamma_{n-1}'}{\gamma_{n-1}} - 2\frac{\gamma_n'}{\gamma_n} = R_{n-1} - R_n.$$

Similarly, differentiating (1.8) in the case of $i = j + 1 = n$ with respect to t shows

$$\begin{aligned}
r_n &= \int_0^\infty \left(\gamma_n' y^n + \gamma_{n,1}' y^{n-1}\right) p_{n-1}(y)\,\mathrm{d}y \\
&= \frac{\gamma_n'}{\gamma_n}\int_0^\infty \gamma_n y^n p_{n-1} w(y)\mathrm{d}y + \frac{\gamma_n'}{\gamma_n}\int_0^\infty \gamma_{n,1}' y^{n-1} p_{n-1} w(y)\mathrm{d}y.
\end{aligned}$$

We use the fact that

$$\gamma_n x^n = p_n(x) - \gamma_{n,1}x^{n-1} + \text{lower order terms}$$

in this expression to obtain

$$\begin{aligned}
r_n &= \int_0^\infty \left(\gamma_{n,1}' - \frac{\gamma_{n,1}\gamma_n'}{\gamma_n}\right) y^{n-1} p_{n-1} w\,\mathrm{d}y \\
&= \frac{\gamma_n}{\gamma_{n-1}}\int_0^\infty \left(\frac{\gamma_{n,1}'\gamma_n - \gamma_n'\gamma_{n,1}}{\gamma_n^2}\right) p_{n-1}^2 w\,\mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_{n,1}}{\gamma_{n,1}}\right)
\end{aligned}$$

since

$$b_n = \frac{\gamma_{n,1}}{\gamma_n} - \frac{\gamma_{n+1,1}}{\gamma_{n+1}}$$

Eq. (3.16b) follows. □

Theorem 3.8. *The coefficients, R_n and r_n, satisfy the system of differential equations*

$$tR_n' = 2r_n - \alpha + R_n(tR_n + 2n + \alpha + \mu - t) \tag{3.17a}$$

$$\begin{aligned} tr_n' &= \left(\frac{1-2R_n}{R_n(1-R_n)}\right) r_n^2 - n(n+\alpha)\frac{R_n}{1-R_n} \\ &\quad + (2n+\alpha+\mu)r_n + \frac{(2n+\mu)r_n}{R_n - 1} - \frac{\alpha r_n}{R_n}. \end{aligned} \tag{3.17b}$$

Proof. To obtain the first equation, note that

$$\frac{\mathrm{d}b_n}{\mathrm{d}t} = R_n + tR_n' = r_n - r_{n+1}$$

which gives (3.17a) under the substitution of r_{n+1} in accordance with (3.12a).

To obtain (3.17b) differentiate (3.15). Knowing a_n' in terms of a_n, r_n and R_n from (3.16a) removes a_n' while we may use (3.14) to remove the remaining instances of a_n^2. (3.17a) and (3.12b) are used to eliminate R_n' and R_{n-1} to obtain an equivalent reformulation of (3.17b). □

We observe that the differential equations of Corollary 3.2 and Theorem 3.8 are identical upon setting

$$R_n = \frac{t+\theta_n}{t}$$

$$r_n = \frac{\kappa_n}{t} - \left(n + \frac{\mu}{2}\right).$$

This demonstrates an equivalence between the characterization of the polynomial system corresponding to (1.1) as implied by the method of isomonodromic deformation, and the method of ladder operators.

3.3. Main results

Recall that our remaining task is to relate the differential equations of corollary to the Hamilton equations for a Painlevé V system. The work in [21,5] both provide clues regarding relevant transformations. Explicitly, they suggest two Möbius transforms

$$y \sim \frac{\theta_n}{t+\theta_n}$$

$$y \sim \frac{\theta_n + t}{\theta_n}$$

both of which send ∞ to 1, and send 0 and $-t$ to 0 and ∞ in different ways.

Proof of Proposition 1.1. We are required to show that q satisfies (1.14) using (3.11). We first find the derivatives of q in terms of θ_n and κ_n,

$$q = \frac{\theta_n + t}{\theta_n}$$

$$\begin{aligned} q' &= \frac{\theta_n - t\theta_n'}{\theta_n^2} \\ &= -\frac{2\kappa_n + \theta_n(\theta_n + 2n + t + \alpha + \mu)}{\theta_n^2} \end{aligned}$$

$$q'' = \frac{\theta_n'((2n+t+\alpha+\mu)\theta_n + 4\kappa_n)}{\theta_n^3} - \frac{1}{\theta_n} - \frac{2\kappa_n + \theta_n}{\theta_n^2}.$$

However, using (3.11a) we have

$$2\kappa_n = t\theta_n' - \theta_n(\theta_n + 1 + 2n + t + \alpha + \mu)$$

giving

$$\begin{aligned} q'' = {} & \frac{t(2t+2\theta_n)\theta_n'^2}{2\theta_n^3(t+\theta_n)} - \frac{(2t+\theta_n)\theta_n'}{\theta_n^2(t+\theta_n)} - \frac{1}{2\theta_n(t+\theta_n)} - \frac{\theta_n^2}{t(t+\theta_n)} \\ & + \frac{\mu^2(t+\theta_n)^2 - \theta_n^2(t^2+\alpha^2)}{2\theta_n^3(t+\theta_n)} + \frac{1}{2}\left(\frac{t}{t+\theta_n} - 2(1+2n+\alpha+\mu)\frac{t+\theta_n}{t\theta_n} - 5\right). \end{aligned}$$

Inverting the expression for q in terms of θ_n gives us

$$\begin{aligned} \theta_n &= \frac{t}{q-1} \\ \theta_n' &= \frac{q - tq' - 1}{(q-1)^2}, \end{aligned}$$

and using these expressions show

$$\begin{aligned} q'' = {} & \left(\frac{1}{q-1} + \frac{1}{2q}\right) q'^2 - \frac{q'}{t} + \frac{(q-1)^2}{t^2}\left(\frac{\mu^2 q}{2} - \frac{\alpha^2}{2q}\right) \\ & - \frac{(1+2n+\alpha+\mu)q}{t} - \frac{q(1+q)}{2(q-1)}. \end{aligned}$$

This is (1.14) where

$$\begin{aligned} &\alpha_1 = \frac{\mu^2}{2} \qquad \alpha_2 = -\frac{\alpha^2}{2} \\ &\alpha_3 = -(2n+1+\alpha+\mu) \qquad \alpha_4 = -\frac{1}{2}. \end{aligned}$$

To obtain the corresponding p variable, we remark that the equation for q', as specified by the Hamiltonian in (1.15), is linear in p, hence determines p uniquely in terms of θ_n and κ_n. □

For the differential equations to uniquely characterize θ_n, κ_n, boundary values must be specified. For this purpose, we note from (3.1) that the small t leading order asymptotics for μ_k are

$$\begin{aligned} \mu_k = {} & C_1\left(1 - \frac{\alpha}{k+\alpha+\mu}t + \frac{\alpha(\alpha-1)}{2(k+\alpha+\mu)(k+\alpha+\mu-1)}t^2 \right. \\ & \left. - \frac{\alpha(\alpha-1)(\alpha-2)}{6(k+\alpha+\mu)(k+\alpha+\mu-1)(k+\alpha+\mu-2)}t^3 + \cdots\right) \\ & + C_2 t^{1+k+\alpha+\mu}\left(1 - \frac{1+k+\mu}{2+k+\alpha+\mu}t + \frac{(1+k+\mu)(2+k+\mu)}{2(2+k+\alpha+\mu)(3+k+\alpha+\mu)}t^2 \right. \\ & \left. - \frac{(1+k+\mu)(2+k+\mu)(3+k+\mu)}{6(2+k+\alpha+\mu)(3+k+\alpha+\mu)(4+k+\alpha+\mu)}t^3 + \cdots\right). \end{aligned}$$

From this the determinant (1.6) that defines Δ_n may be evaluated to leading orders by using the identity (see e.g. [27])

$$\det(\Gamma(z_k + j))_{j,k=0,\dots,n-1} = \prod_{k=0}^{n-1} \Gamma(z_k) \prod_{0\le j<k<n} (z_k - z_j). \tag{3.18}$$

In particular, by letting $z_k = 1 + \alpha + \mu + k$ we have

$$\Delta_n(0) = (1-\zeta)^n \prod_{k=1}^{n-1} k! \prod_{k=0}^{n-1} \Gamma(1+\alpha+\mu+k).$$

Recalling (1.10) then gives

$$a_n^2(0) = n(n+\alpha+\mu),$$

and knowing this (3.10) at $t = 0$ implies

$$b_n(0) = 2n + \alpha + 1 + \mu.$$

To determine the rest of the expansion of (1.6), we first make use of (3.18) to compute the leading form of the analytic and non-analytic components as

$$\Delta_n(t) = \Delta_n(0)\left(1 + \frac{\alpha}{\alpha+\mu} nt + O(t^2) + \chi_n t^{1+\alpha+\mu}(1 + O(t) + O(t^{1+\alpha+\mu}))\right)$$

where

$$\chi_n = \left(1 + \frac{(\zeta-1)\sin(\pi\mu)}{\sin(\pi(\alpha+\mu))}\right) \frac{\Gamma(\mu+1)\Gamma(\alpha+1)\Gamma(\alpha+\mu+n+1)}{(1-\zeta)(n-1)!\Gamma(\alpha+\mu+1)\Gamma(\alpha+\mu+2)^2}.$$

It follows from this and (1.10) that the expansion of a_n^2 is

$$\begin{aligned} a_n^2 &= n(n+\alpha+\mu) - \frac{n(\alpha+\mu+n)}{(\mu+\alpha)^2(\mu+\alpha-1)(\mu+\alpha-1)} t^2 + O(t^3) \\ &\quad + t^{1+\alpha+\mu}\left(1 + \frac{(\zeta-1)\sin\pi\mu}{\sin\pi(\alpha+\mu)}\right) \frac{\Gamma(\mu+1)\Gamma(\alpha+1)}{(1-\zeta)(n-1)!} \\ &\quad \times \frac{\Gamma(n+\alpha+\mu+1)}{\Gamma(\alpha+\mu)\Gamma(\alpha+\mu+1)\Gamma(\alpha+\mu+2)} + O(t^{2+\mu+\alpha}) + O(t^{2\mu+2\alpha+2}), \end{aligned}$$

and subsequently consistency with regard to (3.10) demands

$$\begin{aligned} b_n &= 2n + \alpha + \mu + 1 - \frac{\mu}{\alpha+\mu} t + \frac{\alpha\mu(2n+\alpha+\mu+1)}{(\alpha+\mu-1)(\alpha+\mu)^2(\alpha+\mu+1)} t^2 + O(t^3) \\ &\quad - t^{1+\alpha+\mu}\left(1 + \frac{\sin\pi\mu}{\sin\pi(\alpha+\mu)}\right) \frac{\Gamma(\mu+1)\Gamma(\alpha+1)\Gamma(\alpha+\mu+n+1)}{n!(1-\zeta)\Gamma(\alpha+\mu+1)^3} \\ &\quad + O(t^{2+\mu+\alpha}) + O(t^{2+2\mu+2\alpha}). \end{aligned}$$

Substitution into (1.16) then shows

$$\begin{aligned}\theta_n = &-\frac{\mu}{\alpha+\mu}t + \frac{\alpha\mu(2n+\alpha+\mu+1)}{(\alpha+\mu-1)(\alpha+\mu)^2(\alpha+\mu+1)}t^2 + O(t^3)\\ &- t^{1+\alpha+\mu}\left(1+\frac{\sin\pi\mu}{\sin\pi(\alpha+\mu)}\right)\frac{\Gamma(\mu+1)\Gamma(\alpha+1)\Gamma(\alpha+\mu+n+1)}{n!(1-\zeta)\Gamma(\alpha+\mu+1)^3}\\ &+ O(t^{2+\mu+\alpha}) + O(t^{2+2\mu+2\alpha})\end{aligned} \tag{3.19}$$

$$\begin{aligned}\kappa_n = &\frac{\mu(2n+\alpha+\mu)t}{2(\alpha+\mu)} - \frac{2n\alpha\mu(n+\alpha+\mu)t^2}{(\alpha+\mu)^2(\alpha+\mu+1)(\alpha+\mu-1)} + O(t^3)\\ &+ \left(1+\frac{(\zeta-1)\sin\pi\mu}{\sin\pi(\alpha+\mu)}\right)\frac{\Gamma(\alpha+1)\Gamma(\mu+1)\Gamma(\alpha+\mu+n+1)}{(1-\zeta)(n-1)!\Gamma(\alpha+\mu+1)^3}\\ &+ O(t^{\alpha+\mu+2}) + O(t^{2+2\alpha+\mu}).\end{aligned} \tag{3.20}$$

In particular, the small t asymptotics for q is therefore

$$\begin{aligned}q = &\frac{\theta_n+t}{\theta_n} = -\frac{\alpha}{\mu} - \frac{\mu(2n+\alpha+\mu+1)}{\alpha(\alpha+1)(\alpha+\mu-1)}t + O(t^2)\\ &- t^{\alpha+\mu}\left(1+\frac{(\zeta-1)\sin\pi\mu}{\sin\pi(\alpha+\mu)}\right)\frac{\Gamma(\alpha)\Gamma(\mu+1)\Gamma(\alpha+\mu+n+1)}{n!(1-\zeta)\Gamma(\alpha+\mu)^3\alpha}\\ &+ O(t^{1+\mu+\alpha}) + O(t^{2\mu+2\alpha}).\end{aligned}$$

References

[1] M. Adler, P. van Moerbeke, Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials, Duke Math. J. 80 (1995) 863–911.

[2] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1992.

[3] J. Baik, E.M. Rains, Limiting distributions for a polynuclear growth model with external sources, J. Statist. Phys. 100 (3–4) (2000) 523–541.

[4] J. Baik, E.M. Rains, The asymptotics of monotone subsequences of involutions, Duke Math. J. 109 (2) (2001) 205–281.

[5] E. Basor, Y. Chen, Painlevé V and the distribution function of discontinuous linear statistics in the Laguerre Unitary Ensembles, J. Phys. A: Math. Theor. 42 (3) (2009) 035203.

[6] W.C. Bauldry, Estimates of asymmetric Freud polynomials on the real line, J. Approx. Theory 63 (2) (1990) 225–237.

[7] S. Bonan, D.S. Clark, Estimates of the Hermite and the Freud polynomials, J. Approx. Theory 63 (2) (1990) 210–224.

[8] S. Bonan, P. Nevai, Orthogonal polynomials and their derivatives I, J. Approx. Theory 40 (1984) 134–147.

[9] A. Borodin, P. Deift, Fredholm determinants, Jimbo-Miwa-Ueno tau-functions and representation theory, Commun. Pure Appl. Math. 55 (2002) 1160–1230.

[10] Y. Chen, M.V. Feigin, Painlevé IV and degenerate Gaussian unitary ensembles, J. Phys. A 39 (40) (2006) 12381–12393.

[11] Y. Chen, M.E.H. Ismail, Ladder operators and differential equations for orthogonal polynomials, J. Phys. A 30 (22) (1997) 7817–7829.

[12] Y. Chen, A.R. Its, Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I, 2009 (in press).

[13] Y. Chen, N. Lawrence, A generalization of the Chebyshev polynomials, J. Phys. A: Math. Gen. 35 (22) (2002) 4651–4699.

[14] Y. Chen, G. Pruessner, Orthogonal polynomials with discontinuous weights, J. Phys. A:Math. Gen. 38 (2005) L191–198.

[15] D.V. Chudnovsky, Riemann monodromy problem, isomonodromy deformation equations and completely integrable systems, in: C. Bardos, D. Bessis (Eds.), Bifurcation Phenomena in Mathematical Physics and Related Topics, Proceedings Cargèse, 1979, in: NATO ASI Series C, vol. 54, D. Reidel, Dordrecht, 1980, pp. 385–447.
[16] A.S. Fokas, A.R. Its, A.V. Kitaev, Discrete Painlevé equations and their appearance in quantum gravity, Comm. Math. Phys. 142 (2) (1991) 313–344.
[17] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Comm. Math. Phys. 147 (2) (1992) 395–430.
[18] P.J. Forrester, Log-gases and random matrices, www.ms.unimelb.edu.au/˜matpjf/matpjf.html.
[19] P.J. Forrester, N.S. Witte, Discrete Painlevé equations and random matrix averages, Nonlinearity 16 (6) (2003) 1919–1944.
[20] P.J. Forrester, N.S. Witte, Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems, Constr. Approx. 24 (2) (2006) 201–237.
[21] P.J. Forrester, N.S. Witte, The distribution of the first eigenvalue spacing at the hard edge of the Laguerre unitary ensemble, Kyushu J. Math. 61 (2007) 457–526.
[22] M. Hisakado, Unitary matrix models and Painlevé III, Modern Phys. Lett. A 11 (38) (1996) 3001–3010.
[23] M.E.H. Ismail, N.S. Witte, Discriminants and functional equations for polynomials orthogonal on the unit circle, J. Approx. Theory 110 (2) (2001) 200–228.
[24] M. Jimbo, T. Miwa, K. Ujimo, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function, Physica D 2 (2) (1981) 306–352.
[25] E. Laguerre, Sur la réduction en fractions continues dúne fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, J. Math. Pures Appl. 1 (4) (1885) 135–165.
[26] A.P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, in: Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications, Evian-Les-Bains, 1992, J. Comput. Appl. Math. 57 (1–2) (1995) 215–237.
[27] J.-M Normand, Calculation of some determinants using the s-shifted factorial, J. Phys. A 37 (22) (2004) 5737–5762.
[28] K. Okamoto, Studies on the Painlevé equations. II. Fifth Painlevé equation P_{V}, Japan. J. Math. (N. S.) 13 (1) (1987) 47–76.
[29] V. Periwal, D. Shevitz, Exactly solvable unitary matrix models: multicritical potentials and correlations, Nuclear Phys. B 344 (3) (1990) 731–746.
[30] M. Prähofer, H. Spohn, Exact scaling functions for one-dimensional stationary KPZ growth, J. Statist. Phys. 115 (1–2) (2004) 255–279.
[31] J. Shohat, A differential equation for orthogonal polynomials, J. Duke Math. 5 (1939) 401–417.
[32] G. Szegö, Orthogonal Polynomials (Colloquium Publications, 23), Fourth ed., American Mathematical Society, Providence, RI, 1975.
[33] C.A. Tracy, H. Widom, Fredholm determinants, differential equations and matrix models, Comm. Math. Phys. 163 (1994) 33–72.