Accepted Manuscript

No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms

Gideon Schechtman

PII: S0021-9045(14)00088-4
DOI: http://dx.doi.org/10.1016/j.jat.2014.05.004
Reference: YJATH 4907

To appear in: *Journal of Approximation Theory*

Received date: 13 October 2013
Revised date: 19 February 2014
Accepted date: 1 May 2014

Please cite this article as: G. Schechtman, No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms, *Journal of Approximation Theory* (2014), http://dx.doi.org/10.1016/j.jat.2014.05.004

No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms *

Gideon Schechtman†

Abstract

We show that none of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \le p \ne q < \infty$ have a greedy basis. This solves a problem raised by Dilworth, Freeman, Odell and Schlumprect. Similarly, the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \le p < \infty$, and $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \le q < \infty$, do not have greedy bases. It follows from that and known results that a class of Besov spaces on $\mathbb{R}^n$ lack greedy bases as well.

1 Introduction

Given a (say, real) Banach space X with a Schauder basis $\{x_i\}$, an $x \in X$ and an $n \in \mathbb{N}$ it is useful to determine the best n-term approximation to x with respect to the given basis. I.e., to find a set $A \subset \mathbb{N}$ with n elements and coefficients $\{a_i\}_{i\in A}$ such that

$$\|x - \sum_{i\in A} a_i x_i\| = \inf\{\|x - \sum_{i\in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}$$

or, given a $C < \infty$, at least to find such an $A \subset \mathbb{N}$ and coefficients $\{a_i\}_{i\in A}$ with

$$\|x - \sum_{i\in A} a_i x_i\| \le C \inf\{\|x - \sum_{i\in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}.$$

*AMS subject classification: 46B15, 41A65, 46B45, 46E35. Key words: Greedy basis, Matrix spaces, Besov spaces

†Supported in part by the Israel Science Foundation.

This problem attracted quite an attention in modern Approximation Theory. Of course one would also like to have a simple algorithm to find such a set $\{a_i\}_{i\in A}$. It would be nice if we could take $\{a_i\}_{i\in A}$ to be just the set of the n largest, in absolute value, coefficients in the expansion of x with respect to the basis $\{x_i\}$. Or, if this set is not unique, any such set. The basis $\{x_i\}$ is called *Greedy* if for some C this procedure works; i.e., for all $x = \sum_{i=1}^{\infty} a_i x_i$, all $n \in \mathbb{N}$ and all $A \subset \mathbb{N}$, $|A| = n$, satisfying $\min\{|a_i|; i \in A\} \geq \max\{|a_i|; i \notin A\}$,

$$\|x - \sum_{i\in A} a_i x_i\| \leq C \inf\{\|x - \sum_{i\in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}.$$

Konyagin and Temlyakov [KT] provided a simple criterion to determine whether a basis is greedy: $\{x_i\}$ is greedy if and only if it is *unconditional* and *democratic.*

Recall that $\{x_i\}$ is said to be unconditional provided, for some $C < \infty$, all eventually zero coefficients $\{a_i\}$ and all sequences of signs $\{\varepsilon_i\}$,

$$\|\sum \varepsilon_i a_i x_i\| \leq C\|\sum a_i x_i\|.$$

$\{x_i\}$ is said to be democratic provided for some $C < \infty$ and all finite $A, B \subset \mathbb{N}$ with $|A| = |B|$,

$$\|\sum_{i\in A} x_i\| \leq C\|\sum_{i\in B} x_i\|.$$

We refer to [DFOS] for a survey of what is known about space that have or do not have greedy bases. In [DFOS] Dilworth, Freeman, Odell and Schlupmrecht determined which of the spaces $X = (\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with c_0 replacing ℓ_∞ in case $q = \infty$) have a greedy basis. It turns out that this happens exactly when X is reflexive. They also raise the question of whether $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 < p \neq q < \infty$ have greedy bases. Here we show that these spaces (as well as their non-reflexive counterparts) do not have greedy bases. By the Konyagin-Temlyakov characterization it is enough to prove that each normalized unconditional basis of $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with c_0 replacing ℓ_∞ in case p or q are ∞) has two subsequences, one equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$) and one to the unit vector basis of ℓ_q (c_0 if $q = \infty$).

Theorem 1 *Each normalized unconditional basis of the spaces* $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \leq p \neq q < \infty$ *has a subsequence equivalent to the unit vector basis of* ℓ_p

and another one equivalent to the unit vector basis of ℓ_q. Similarly, each normalized unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \le p < \infty$ (resp. $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \le q < \infty$) has a subsequence equivalent to the unit vector basis of ℓ_p (resp. c_0) and another one equivalent to the unit vector basis of c_o (resp. ℓ_q). Consequently, none of these spaces have a greedy basis.

For $1 \le p, q < \infty$ the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$ are isomorphic to certain Besov spaces on $\mathbb{R}^n$. We refer to [Me] for the definition of the Besov spaces $B_p^{s,q}$ and for the fact that they are isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$. See in particular [Me, Section 6.10, Proposition 7] (and also [Me, Section 2.9, Proposition 4]). We thank P. Wojtaszczyk for this reference.

Corollary 1 *Let $1 \le p \ne q < \infty$ and let s be any real number, then the space $B_p^{s,q}$ does not have a greedy basis.*

Recall that this stand in contrast with the main result in [DFOS] which states that, in the reflexive cases, the corresponding Besov spaces on $[0, 1]$ do have greedy bases.

In the special case of $1 < q < \infty$ and $p = 2$ the theorem above was actually proved in [Sc]. There the isomorphic classification of the span of unconditional basic sequences in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$, $1 < q < \infty$, which span complemented subspaces were characterize. Although it is not stated there, the proof actually established the theorem above in these special cases. Shortly after [Sc] appeared Odell [Od] strengthened the result and classified *all* the complemented subspace of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ (thus there is no wonder that [Sc] was forgotten...). We remark in passing that this special case of $p = 2$ was of particular interest since $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ is isomorphic to a complemented subspace of $L_q[0, 1]$.

The first step in the proof in [Sc] is to reduce the case of a general unconditional basic sequence in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ whose span is complemented to one which is also a block basis of the natural basis of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$. This reduction no longer hold for $p \ne 2$. The complications in the present note stem from this fact. The way we overcome it is by transferring the problem to a larger space (of arrays $\{a_{i,j,k}\}$) of mixed q, p and 2 norms. Unfortunately, this makes the notations quite cumbersome.

2 Preliminaries

$Z_{q,p}$, $1 \le p, q < \infty$ will denote here the space of all matrices $a = \{a(i,j)\}_{i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p} = (\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty}|a(i,j)|^p)^{q/p})^{1/q}.$$

If p or q are ∞ we replace the corresponding ℓ_p or ℓ_q norm by the ℓ_∞ norm and continue to denote by $Z_{q,p}$ the completion of the space of finitely supported matrices under this norm. (Thus, c_0 replacing ∞ would be a more precise notation in this case but, since it would complicated our statements, we prefer the above notation.) The spaces $Z_{q,p}$ are the subject of investigation of this paper. They are more commonly denoted by $\ell_q(\ell_p)$ or $(\bigoplus_{n=1}^{\infty}\ell_p)_{\ell_q}$ (as we have done in the introduction) but since we shall be forced to also consider more complicated spaces with mixed norms we prefer the notation above.

If $\{k_n\}_{n=1}^{\infty}$ is any sequence of positive integers, we shall denote by $Z_{q,p;\{k_n\}}$, the subspace of $Z_{q,p}$ consisting of matrices a satisfying $a(i,j) = 0$ for all $i > k_j$.

We also denote by $Z_{q,p,r}$ (we'll use this only for $r = 2$) the spaces of arrays $a = \{a(u,i,j)\}_{u,i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p,r} = (\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty}(\sum_{u=1}^{\infty}|a(u,i,j)|^r)^{p/r})^{q/p})^{1/q},$$

with the same convention as above when one of p, q (or r) is ∞. Similarly, $Z_{q,p;\{k_n\},r}$ denotes the subspace of $Z_{q,p,r}$ consisting of arrays a satisfying $a(u,i,j) = 0$ for all $i > k_j$.

By P_n we denote the natural projection onto the n-th column in $Z_{q,p}$; i.e, $P_n(\{a(i,j)\}) = \{\bar{a}(i,j)\}$, where $\bar{a}(i,j) = a(i,j)$ if $j = n$ and $\bar{a}(i,j) = 0$ otherwise. Similarly, P_n^k denotes the natural projection onto the first k elements in the n-th column. Q_N denotes $\sum_{n=1}^{N} P_n$.

Given a Banach lattice X, an $1 < r < \infty$ and $x_1, x_2, \dots \in X$ one can define the operation $(\sum |x_n|^r)^{1/r}$ in a manner consistent with what we usually mean by such an operation (when X is a lattice of functions or sequences, for example). See e.g. [LT2, Section 1.d] for this and what follows.

In particular if X has a 1-unconditional basis $\{e_i\}$ (which is the only kind of lattices we'll consider here) then for $x_n = \sum_{i=1}^{\infty} a_i^n e_i$, $n = 1, 2, \dots, N$, $(\sum |x_n|^r)^{1/r} = \sum_{i=1}^{\infty}(\sum_{n=1}^{N}|a_i^n|^r)^{1/r} e_i$.

Recall that X is said to be r-convex (resp. r-concave) with constant K if for all n and all $x_1, x_2, \ldots, x_n \in X$

$$\|(\sum_{i=1}^n |x_i|^r)^{1/r}\| \le K(\sum_{i=1}^n \|x_i\|^r)^{1/r} \text{ (resp. } (\sum_{i=1}^n \|x_i\|^r)^{1/r} \le K\|(\sum_{i=1}^n |x_i|^r)^{1/r}\|).$$

X is said to be r-convex (resp. r-concave) if it is r-convex (resp. r-concave) with some constant $K < \infty$. $Z_{q,p}$ is easily seen to be $\min\{p,q\}$-convex with constant 1 and $\max\{p,q\}$-concave with constant 1.

It is also known that X is r-convex (resp. r-concave) if and only if its dual X^* is r'-concave (resp. r'-convex) where $r' = r/(r-1)$.

Given a Banach lattice X we denote by $X(\ell_2)$ the (completion of the) space of (finite) sequences $x = (x_1, x_2, \ldots)$ of elements of X for which the norm

$$\|x\| = \|(\sum |x_j|^2)^{1/2}\|$$

is finite. If X has a 1-unconditional basis $\{e_j\}$ then this is just the (completion of the) space of matrices $a = \{a(i,j)\}$ (with only finitely many non-zero entries) with norm

$$\|a\| = \|\sum_{j=1}^\infty (\sum_{i=1}^\infty |a(i,j)|^2)^{1/2} e_i\|.$$

The following two lemmas are well known but maybe hard to find so we reproduce their proofs.

Lemma 1 *Let $\{x_i\}_{i=1}^\infty$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 \le p < q \le \infty$. If for some $\varepsilon > 0$ and $N \in \mathbb{N}$ $\|Q_N x_i\| > \varepsilon$ for all i then $\{x_i\}_{i=1}^\infty$ has a subsequence equivalent to the unit vector basis of ℓ_p.*

Proof: Assume first $p > 1$. Given a sequence of positive ε_i-s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a sequence of $\{y_i\}$ of vectors disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y_i\| < \varepsilon_i$ for all i. (Use the fact that $\{x_i\}$ doesn't have a subsequence equivalent to the unit vector basis of ℓ_1 and the argument for Proposition 1.a.12 in [LT1], for example). $\{y_i\}$ is 1-dominated by the unit vector basis of ℓ_p and dominates $\{Q_N y_i\}$

which in turn C-dominates the unit vector basis of ℓ_p for $C = 1/(\varepsilon - \sup \varepsilon_i)$; i.e,

$$(\sum_{i=1}^{\infty} |a_i|^p)^{1/p} \geq \| \sum_{i=1}^{\infty} a_i y_i \| \geq (\varepsilon - \sup \varepsilon_i)(\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$$

for all scalars $\{a_i\}$. If the ε_i-s are small enough a similar inequality holds for the (sub)sequence $\{x_i\}$.

If $p = 1$ then given a sequence of positive ε_i-s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a vector y and sequence of $\{y_i\}$ of vectors all disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y - y_i\| < \varepsilon_i$ for all i. If $y \neq 0$ and the $\{\varepsilon_i\}$ are small enough then, using the unconditionality $\{x_i\}$ is clearly equivalent to the unit vector basis of ℓ_1. If $y = 0$ the same argument as for $p > 1$ works here too. ■

Lemma 2 *Let $\{x_i\}$ be a K-unconditional basic sequence in a Banach lattice which is r-concave for some $r < \infty$ Let $\bar{x}_i \in X(\ell_2)$ be defined by $(0, \ldots, 0, x_i, 0, \ldots)$, x_i in the i-th place. Then the sequences $\{x_i\}$ in X and $\{\bar{x}_i\}$ in $X(\ell_2)$ are equivalent.*

If in addition X is also s-convex for some $s > 1$ and $[x_i]$, the closed linear span of $\{x_i\}$, is complemented in X then $[\bar{x}_i]$ is complemented in $X(\ell_2)$.

Proof: The first assertion, due to Maurey, can be found in [Ma] or [LT2, Theorem1.d.6(i)]. The second is probably harder to find so we reproduce it. Let $P = \sum_{i=1}^{\infty} x_i^* \otimes x_i$, with $x_i^* \in X^*$, be the projection onto $[x_i]$; i.e.,

$$P(x) = \sum_{i=1}^{\infty} x_i^*(x) x_i \quad x \in X.$$

Define $\bar{P} = \sum_{i=1}^{\infty} \bar{x}_i^* \otimes \bar{x}_i$ ($\bar{x}_i^* \in X^*(\ell_2) = X(\ell_2)^*$); i.e.,

$$\bar{P}(x) = \sum_{i=1}^{\infty} \bar{x}_i^*(x) \bar{x}_i \quad x \in X(\ell_2).$$

Using the facts that $\{\bar{x}_i\}$ is equivalent to $\{x_i\}$, $\{\bar{x}_i^*\}$ is equivalent to $\{x_i^*\}$, and $\{\bar{x}_i^*, \bar{x}_i\}$ is a biorthogonal sequence, it is easy to see that $\bar{P}$ is a bounded projection on $X(\ell_2)$ with range $[\bar{x}_i]$. ■

3 Proof of the main result, the reflexive case

Since the non-reflexive cases (i.e., when p or q are 1 or ∞) of Theorem 1 require a bit different treatment and since the problem raised in [DFOS] was restricted to the reflexive cases only, we prefer to delay the proof of the non-reflexive cases to the next section.

Proposition 1 *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 < p, q < \infty$ such that $[x_i]_{i=1}^{\infty}$ is complemented in $Z_{q,p}$. If no subsequence of $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p then $[x_i]_{i=1}^{\infty}$ isomorphically embeds in $Z_{q,p;\{n\},2}$ as a complemented subspace.*

Proof: We may clearly assume $p \neq q$ and by duality that $q > p$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. By Lemma 1 for all n only finitely many of the x_i-s satisfy $\|P_n x_i\| \geq \varepsilon_n$. Consequently, for each $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ such that $\|(P_n - P_n^{k_n})x_i\| < \varepsilon_n$ for all i. We denote $Q = \sum_{n=1}^{\infty} P_n^{k_n}$. In the case $p = 2$ we showed in [Sc] that without loosing generality we can assume that $\{x_i\}$ is a block basis of the natural basis of $Z_{q,p}$ and then $\{Qx_i\}$ and $\{(I - Q)x_i\}$ are also unconditional basic sequences. This is no longer true when $p \neq 2$. We overcome this difficulty by switching to the larger space $Z_{q,p,2}$. Define for each i $\bar{x}_i \in Z_{q,p,2}$ by

$$\bar{x}_i(w, u, v) = \begin{cases} x_i(u, v), & \text{if } w = i; \\ 0, & \text{if } w \neq i. \end{cases}$$

Let the projection P from $Z_{q,p}$ onto $[x_i]$ be given by

$$Px = \sum_{i=1}^{\infty} x_i^*(x) x_i$$

where $\{x_i^*\}$ in $Z_{q',p'} = Z_{q,p}^*$ ($p' = p/(p-1)$, $q' = q/(q-1)$) satisfy $x_i^*(x_j) = \delta_{i,j}$, $i, j = 1, 2, \ldots$. Then by Lemma 2

$$\bar{P}x = \sum_{i=1}^{\infty} \bar{x}_i^*(x) \bar{x}_i$$

is a bounded projection from $Z_{q,p,2}$ onto $[\bar{x}_i]$ and $\{x_i\}_{i=1}^{\infty}$ is equivalent to $\{\bar{x}_i\}_{i=1}^{\infty}$.

We denote by $\bar{P}_n = P_n \otimes I_{\ell_2}$ on $Z_{q,p,2}$; i.e, $\bar{P}_n(x)(w, u, v) = P_n(x(w, \cdot, \cdot))(u, v)$. We also similarly denote $\bar{P}_n^k = P_n^k \otimes I_{\ell_2}$, $\bar{Q}_N = Q_N \otimes I_{\ell_2}$, and $\bar{Q} = Q \otimes I_{\ell_2}$.

Note that now $\{\bar{Q}\bar{x}_i\}$ and $\{(I-\bar{Q})\bar{x}_i\}$ are also unconditional basic sequences. We would like to show that if $\varepsilon_n \to 0$ fast enough, then $\{\bar{Q}\bar{x}_i\}$ is equivalent to $\{\bar{x}_i\}$ and thus to $\{x_i\}$ and that $[\bar{Q}\bar{x}_i]$ is complemented.

Now,

$$(I-\bar{Q})\bar{P}\bar{Q}\bar{x}_n = \sum_{i=1}^{\infty} \bar{x}_i^*(\bar{Q}\bar{x}_n)(I-\bar{Q})\bar{x}_i, \quad n = 1, 2, \ldots.$$

The operator $(I-\bar{Q})\bar{P}$ sends the span of the unconditional basic sequence $\{\bar{Q}\bar{x}_n\}$ to the span of the unconditional basic sequence $\{(I-\bar{Q})\bar{x}_n\}$ thus the diagonal operator D defined by

$$D\bar{Q}\bar{x}_n = \bar{x}_n^*(\bar{Q}\bar{x}_n)(I-\bar{Q})\bar{x}_n, \quad n = 1, 2, \ldots.$$

is bounded (see e.g. [To] or [LT1, Proposition 1.c.8]). If we show that $\bar{x}_n^*(\bar{Q}\bar{x}_n)$ are uniformly bounded away from zero this will show that $\{\bar{Q}\bar{x}_n\}$ dominates $\{(I-\bar{Q})\bar{x}_n\}$ and thus also $\{\bar{x}_n\} = \{(I-\bar{Q})\bar{x}_n + \bar{Q}\bar{x}_n\}$. That $\{\bar{Q}\bar{x}_n\}$ is dominated by $\{\bar{x}_n\}$ is clear from the boundedness of $\bar{Q}$. This will show that $\{\bar{Q}\bar{x}_n\}$ is equivalent to $\{x_n\}$. To show that $\bar{x}_n^*(\bar{Q}\bar{x}_n)$ are uniformly bounded away from zero note that

$$\bar{x}_n^*(\bar{Q}\bar{x}_n) = 1 - \bar{x}_n^*((I-\bar{Q})\bar{x}_n)$$

and that

$$|\bar{x}_n^*((I-\bar{Q})\bar{x}_n)| \le \|\bar{P}(I-\bar{Q})\bar{x}_n\| \le \|\bar{P}\| \sum_{i=1}^{\infty} \varepsilon_i.$$

So, if $\|\bar{P}\| \sum_{i=1}^{\infty} \varepsilon_i < 1/2$, then $\bar{x}_n^*(\bar{Q}\bar{x}_n) \ge 1/2$ for all n.

We still need to show that $[\bar{Q}\bar{x}_n]$ is complemented. Note that $\{\frac{\bar{x}_n^*}{\bar{x}_n^*(\bar{Q}\bar{x}_n)}, \bar{Q}\bar{x}_n\}$ is a biorthogonal sequence such that $\{\bar{Q}\bar{x}_n\}$ is equivalent to $\{\bar{x}_n\}$ and $\{\frac{\bar{x}_n^*}{\bar{x}_n^*(\bar{Q}\bar{x}_n)}\}$ is dominated by $\{x_n^*\}$. It follows that

$$x \to \sum_{n=1}^{\infty} \frac{\bar{x}_n^*(x)}{\bar{x}_n^*(\bar{Q}\bar{x}_n)} \bar{Q}\bar{x}_n$$

defines a bounded projection with range $[\bar{Q}\bar{x}_n]$.

We have shown that $[x_i]$ embeds complementably into $Z_{q,p;\{k_n\},2}$ for some sequence of positive integers $\{k_n\}$. This last space is clearly isometric to a norm one complemented subspace of $Z_{q,p;\{n\},2}$. ■

In the case $p = 2$ the argument above simplifies and actually shows that under the assumptions of Proposition 1 we can strengthen the conclusion to: $[x_i]$ embeds complementably in $Z_{q,2;\{n\}}$ (which is isomorphic to ℓ_q). We will not dwell on it farther as this is contained in [Sc]. The next proposition combined with the previous one will show in particular that any unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of ℓ_p. We'll need to use this also in the next section so we include the non-reflexive cases here as well.

Proposition 2 *Let $1 \leq p, q \leq \infty$. If $p \neq 2, q$, then ℓ_p (c_0 in case $p = \infty$) does not embed into $Z_{q,p;\{n\},2}$.*

Proof: Assume ℓ_p or c_0 embeds into $Z_{q,p;\{n\},2}$. Passing to a subsequence of the image of the unit vector basis of ℓ_p or c_0, taking successive differences (this is needed only in the case $p = 1$) and using a simple perturbation argument, we may assume that some normalized block basis $\{x_i\}$ of the natural basis of $Z_{q,p;\{n\},2}$ is equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$). Let $P_{n,m}$, $m = 1, 2 \ldots, 1 \leq n \leq m$, denote the canonical projection onto the n, m copy of ℓ_2 in $Z_{q,p;\{n\},2}$:

$$P_{n,m}(\{a(w,u,v)\}) = \{\bar{a}(w,u,v)\}$$

where

$$\bar{a}(w,u,v) = \begin{cases} a(w,u,v), & \text{if } u = n, \ v = m; \\ 0, & \text{otherwise.} \end{cases}$$

Assume first $p > 2$. For each n, m $P_{n,m}$ acts as a compact operator from $[x_i]$ to ℓ_2 as every bounded operator from ℓ_p, $p > 2$ or c_0 to ℓ_2 do. Consequently, given a sequence of positive numbers $\{\varepsilon_{n,m}\}$, we can find $k_{n,m} \in \mathbb{N}$ such that $\|(P_{n,m} - P_{n,m}^{k_{n,m}})_{|[x_i]}\| < \varepsilon_{n,m}$ for all n, m. Then, if $\sum_{n,m} \varepsilon_{n,m}$ is small enough

$$(\sum_{n,m} P_{n,m}^{k_{n,m}})_{|[x_i]}$$

is an isomorphism and we get that $[x_i]$ embeds into $Z_{q,p;\{n\},2;\{k_{n,m}\}}$. Now for each finite m and k the ℓ_p^m sum of ℓ_2^k-s 2-embeds into ℓ_p^N for some N depending only on p, m and k. It thus follows that $[x_i]$ embeds into $Z_{q,p;\{k_n\}}$ for some sequence of positive integers $\{k_n\}$. Passing to a farther subsequence of $\{x_i\}$, we get that the unit vector basis of ℓ_p (or c_0 in the case $p = \infty$) is equivalent to that of ℓ_q which is a contradiction.

The case $1 \le p < 2$ is just a bit more complicated. Here $P_{n,m}$ doesn't act as a compact operator from $[x_i]$ to ℓ_2 but it is still strictly singular. Consequently, for each n, m and l we can find a normalised block basis of $\{x_i\}_{i=l}^\infty$ such that $\|(P_{n,m})_{|[x_i]_{i=l}^\infty}\| < \varepsilon_{n,m}$ and consequently there is a block basis of $\{x_i\}$ *whose first* $l-1$ *terms are just* $x_1, \dots, x_{l-1}$, and $k_{n,m,l}$ such that

$$\|(P_{n,m} - P_{n,m}^{k_{n,m,l}})_{|[x_i]}\| < \varepsilon_{n,m}.$$

A simple diagonal argument will now produce a normalised block basis $\{z_i\}$ of $\{x_i\}$ and natural numbers $k_{n,m}$-s such that

$$(\sum_{n,m} P_{n,m}^{k_{n,m}})_{|[z_i]}$$

an isomorphism. Since $\{z_i\}$ is equivalent to the unit vector basis of ℓ_p we get that ℓ_p embeds into $Z_{q,p;\{n\},2;\{k_{n,m}\}}$. The rest of the proof in this case is the same as in the case $p > 2$. ∎

We are now aiming at proving that every normalized unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of ℓ_q.

Proposition 3 *Let* $\{x_i\}_{i=1}^\infty$ *be a normalized unconditional basic sequence in* $Z_{q,p}$, $1 < p, q < \infty$ *such that* $[x_i]_{i=1}^\infty$ *is complemented in* $Z_{q,p}$. *If no subsequence of* $\{x_i\}_{i=1}^\infty$ *is equivalent to the unit vector basis of* ℓ_q *then* $[x_i]_{i=1}^\infty$ *isomorphically embeds in* $Z_{p,2}$ *as a complemented subspace.*

Proof: We may assume $q < p$. We first claim that for each $\varepsilon > 0$ there is an N such that $\|(I - Q_N)x_i\| < \varepsilon$ for each $i = 1, 2, \dots$. Indeed if this is not the case then there is an $\varepsilon > 0$, a sequence $0 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ and a subsequence $\{y_i\}$ of $\{x_i\}$ such that $\|(Q_{i+1} - Q_i)y_i\| \ge \varepsilon$ for all i. Passing to a further subsequence and a small perturbation we may assume that $\{y_i\}$ is a block basis of the natural basis of $Z_{q,p}$. Then, since $q < p$, for all scalars $\{a_i\}$,

$$(\sum_{i=1}^\infty |a_i|^q)^{1/q} \ge \|\sum_{i=1}^\infty a_i y_i\| \ge \|\sum_{i=1}^\infty a_i (Q_{i+1} - Q_i) y_i\| \ge \varepsilon (\sum_{i=1}^\infty |a_i|^q)^{1/q}$$

in contradiction to the fact that no subsequence of the $\{x_i\}$ is equivalent to the unit vector basis of ℓ_q.

The rest of the proof is now similar to that of Proposition 1, only a bit simpler. Fix an $\varepsilon > 0$ and let N be as in the beginning of this proof. Let $P = \sum_{i=1}^{\infty} x_i^* \otimes x_i$ be the projection onto $[x_i]$ and let $\{\bar{x}_i\}$ (in $Z_{q,p,2}$), $\bar{P}$ and $\bar{Q}_N$ be as in the proof of Proposition 1. Consider the operator $(I - \bar{Q}_N)\bar{P}$ as acting from the span of the unconditional basic sequence $\{\bar{Q}_N\bar{x}_i\}$ to the span of the unconditional sequence $\{(I - \bar{Q}_N)\bar{x}_i\}$:

$$(I - \bar{Q}_N)\bar{P}\bar{Q}_N\bar{x}_n = \sum_{i=1}^{\infty} \bar{x}_i^*(Q_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_i, \; n = 1, 2, \ldots.$$

Its diagonal defined by

$$D\bar{Q}_N\bar{x}_n = \bar{x}_n^*(Q_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_n, \; n = 1, 2, \ldots$$

is bounded ([To] or [LT1]). So if we show that $\bar{x}_n^*(\bar{Q}_N\bar{x}_n)$ are bounded away from zero then the sequence $\{\bar{Q}_N\bar{x}_i\}$ will dominate the sequence $\{(I - \bar{Q}_N)\bar{x}_i\}$ and thus also $\{\bar{x}_i\}$ and $\{x_i\}$. This will also show that

$$x \to \sum_{n=1}^{\infty} \frac{\bar{x}_n^*(x)}{\bar{x}_n^*(\bar{Q}_N\bar{x}_n)}\bar{Q}_N\bar{x}_n$$

defines a bounded projection from $\bar{Q}_N Z_{q,p,2}$ (which is isomorphic to $Z_{p,2}$) onto $[\bar{Q}_N\bar{x}_i]$ (which is isomorphic to $[x_i]$).

To show that $\bar{x}_n^*(\bar{Q}_N\bar{x}_n)$ are bounded away from zero note that

$$\bar{x}_n^*(\bar{Q}_N\bar{x}_n) = 1 - \bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)$$

and that

$$|\bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)| \le \|\bar{P}(I - \bar{Q}_N)\bar{x}_n\| \le \|\bar{P}\|\varepsilon.$$

So, if $\|\bar{P}\|\varepsilon < 1/2$, then $\bar{x}_n^*(\bar{Q}\bar{x}_n) \ge 1/2$ for all n. ■

Remark 1 *With a bit more effort one can strengthen the conclusion of Proposition 3 to: $[x_i]_{i=1}^{\infty}$ is isomorphic to ℓ_p. This is done by first showing that one can embed $[x_i]_{i=1}^{\infty}$ as a complemented subspace in $Z_{p,2;\{n\}}$ which is isomorphic to ℓ_p and using the fact that any infinite dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p.*

Proof of Theorem 1 in the reflexive case: Propositions 1 and 2 show that any normalized unconditional basis of $Z_{q,p}$, $1 < p, q < \infty$, has a subsequence equivalent to the unit vector basis of ℓ_p. To show that any such basis also has a subsequence equivalent to the unit vector basis of ℓ_q we need, in view of Proposition 3, only prove that $Z_{q,p}$ doesn't embed complementably into $Z_{p,2}$ for $1 < q \neq p < \infty$. This can probably be done directly (especially in the case $q \neq 2$ in which case it is also true that ℓ_q does not embed into $Z_{p,2}$) but it also follows from the main theorems of [Sc] and [Od] in which the complemented subspaces of $Z_{p,2}$ (in [Sc] only those with unconditional basis) where characterized. ■

4 Proof of the main result, the non-reflexive case

Recall that the subscript ∞ in $Z_{\infty,p}$ refers, by our convention, to the c_0 (rather than ℓ_∞) sum. Similatly, the subscript ∞ in $Z_{q,\infty}$ refers to the q sum of c_0. We are going to show that any unconditional basis of each of the spaces $Z_{q,p}$, $p \neq q$, when at least one of p or q is 1 or ∞ contains a subsequence equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$) and another subsequence equivalent to the unit vector basis of ℓ_q (c_0 if $q = \infty$).

The spaces $Z_{1,\infty}$ and $Z_{\infty,1}$ (as well as $Z_{1,2}$ and $Z_{\infty,2}$) have unique, up to permutation, unconditional bases [BCLT]. These bases clearly contain a subsequence equivalent to the unit vector basis of c_0 and another one equivalent to the unit vector basis of ℓ_1, so we only need to deal with the spaces $Z_{\infty,p}$, $1 < p < \infty$, and their duals $Z_{1,p'}$ and with $Z_{q,\infty}$, $1 < q < \infty$, and their duals $Z_{q',1}$.

We shall need some classical results concerning unconditional bases and duality. These can be found conveniently in sections 1.b. and 1.c. of [LT1]. ℓ_1 does not isomorphically embed into $Z_{\infty,p}$, $1 < p < \infty$, (resp. into $Z_{q,\infty}$, $1 < q < \infty$) (this follows for example from the fact that these spaces are p (resp. q) convex). It thus follows from a theorem of James [Ja] or see [LT1, Theorem 1.c.9] that any unconditional basis of these spaces is shrinking. See [LT1, Proposition 1.b.1] for the the definition of a shrinking basis as well as for the fact that then the biorthogonal basis is an unconditional basis of the dual space $Z_{1,p'}$, $1 < p < \infty$, (resp. $Z_{q',1}$, $1 < q < \infty$). Thus, in order to prove Theorem 1 in the non-reflexive cases, if would be enough to show that any normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$) contains a subsequence equivalent to the unit vector basis of ℓ_1 and another subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_q).

Let $\{x_n\}$ be a normalized unconditional basis of $X^* = Z_{1,p}$, $1 < p < \infty$, (resp. $X^* = Z_{q,1}$, $1 < q < \infty$) such a basis is boundedly complete and its biorthogonal basis spans a space isomorphic to $X = Z_{\infty,p'}$ (resp. $X = Z_{q',\infty}$).

We begin with a proposition which replaces Propositions 1 and 2 for the current cases.

Proposition 4 *Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$). Then $\{x_n\}$ contains a subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_1).*

Proof: By proposition 2, ℓ_p does not embed into $Z_{1,p:\{n\},2}$ for $1 < p < \infty$ and ℓ_1 does not embed into $Z_{q,1:\{n\},2}$ for $1 < q < \infty$. It is thus enough to show that if $\{x_n\}$ contains no subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_1) then $[x_n]$ embeds in $Z_{1,p:\{n\},2}$ (resp. $Z_{q,1:\{n\},2}$).

The case of $Z_{q,1}$, $1 < q < \infty$: We proceed as in the proof of Proposition 1. Since $q > 1$ the beginning of the proof works for $p = 1$ as well. The problem arise when we need to show that $\bar{P}$ is bounded as this no longer follow from Lemma 2. But here we can use instead [LT2, Theorem 1.d.6(ii)] to prove that $\bar{P}$ is bounded in a very similar way to the proof of Lemma 2. The rest of the proof of Proposition 1 carries over.

The case of $Z_{1,p}$, $1 < p < \infty$: Assume $\{x_n\}$ be a basis of $Z_{1,p}$, $1 < p < \infty$. Let $\{x_n^*\}$ be the biorthogonal basis (of $Z_{\infty,p'}$). By the assumption that $\{x_n\}$ doesn't contain a subsequence equivalent to the unit vector basis of ℓ_p, $[x_n^*]$ doesn't contain a subsequence equivalent to the unit vector basis of $\ell_{p'}$. The proof of Proposition 1 works for $Z_{\infty,p'}$, $1 < p' < \infty$, as well, with the same modification for the proof that $\bar{P}$ is bounded as in the previous paragraph, to show that in this case $[x_n^*]$ embeds (even complementably) into $Z_{\infty,p':\{n\},2}$.
■

The next proposition replaces Proposition 3 in the non-reflexive case.

Proposition 5 *(i) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$. Then the unit vector basis of ℓ_1 is equivalent to a subsequence of $\{x_n\}$.*

(ii) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{q,1}$, $1 < q < \infty$. Then the unit vector basis of ℓ_q is equivalent to a subsequence of $\{x_n\}$.

Proof: The proof of Proposition 3 works for $Z_{q,p}$ also in the case $q = 1 < p < \infty$. We thus get that under the assumption of (i), if no subsequence of $\{x_n\}$ is equivalent to the unit vector basis of ℓ_1 then $[x_n]$ embeds into $Z_{p,2}$. On the other hand $Z_{p,2}$ has type $\min\{p, 2\}$ so ℓ_1 and thus also $Z_{1,p}$, $1 < p < \infty$, do not embed into it. This proves (i).

(ii) It is enough to show that the unit vector basis of $\ell_{q'}$ is equivalent to a subsequence of $\{x_n^*\}$ (the biorthogonal basis to $\{x_n\}$) which is an unconditional basis of $Z_{q',\infty}$. The proof of Proposition 3 gives that if this is not the case then $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$. Now if $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$ then an easy application of Pełczynski's decomposition method gives that $Z_{q',\infty} \oplus Z_{\infty,2}$ is isomorphic to $Z_{\infty,2}$ but this immediately presents an unconditional basis for $Z_{\infty,2}$ which is not equivalent to a permutation of the cannonical basis of $Z_{\infty,2}$. This stands in contradiction to a result from [BCLT] and thus proves (ii). ■

References

[BCLT] Bourgain, J. ; Casazza, P. G. ; Lindenstrauss, J. ; Tzafriri, L. Banach spaces with a unique unconditional basis, up to permutation. Mem. Amer. Math. Soc. 54 (1985), no. 322, iv+111 pp.

[DFOS] Dilworth, S. J. ; Freeman, D. ; Odell, E. ; Schlumprecht, T. Greedy bases for Besov spaces. Constr. Approx. 34 (2011), no. 2, 281–296.

[Ja] James, Robert C. Bases and reflexivity of Banach spaces. Ann. of Math. (2) 52, (1950). 518–527.

[KT] Konyagin, S. V.; Temlyakov, V. N. A remark on greedy approximation in Banach spaces. East J. Approx. 5 (1999), no. 3, 365379.

[LT1] Lindenstrauss, Joram ; Tzafriri, Lior . Classical Banach spaces. I. Sequence spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977. xiii+188 pp. ISBN: 3-540-08072-4

[LT2] Lindenstrauss, Joram ; Tzafriri, Lior . Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete

[Results in Mathematics and Related Areas], 97. Springer-Verlag, Berlin-New York, 1979. x+243 pp. ISBN: 3-540-08888-1

[Ma] Maurey, B. Type et cotype dans les espaces munis de structures locales inconditionnelles. (French) Sminaire Maurey-Schwartz 19731974: Espaces Lp, applications radonifiantes et gomtrie des espaces de Banach, Exp. Nos. 24 et 25, 25 pp. Centre de Math., cole Polytech., Paris, 1974.

[Me] Meyer, Yves Wavelets and operators. Translated from the 1990 French original by D. H. Salinger. Cambridge Studies in Advanced Mathematics, 37. Cambridge University Press, Cambridge, 1992. xvi+224 pp.

[Od] Odell, E. On complemented subspaces of $(\sum l_2)_{l_p}$. Israel J. Math. 23 (1976), no. 3-4, 353–367.

[Sc] Schechtman, G. Complemented subspaces of $(l_2 \oplus l_2 \oplus \cdots)_p$ $(1 < p < \infty)$ with an unconditional basis. Israel J. Math. 20 (1975), no. 3-4, 351–358.

[To] Tong, Alfred E. Diagonal submatrices of matrix maps. Pacific J. Math. 32 1970 551–559.

G. Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
gideon@weizmann.ac.il