Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 162 (2010) 1511–1517
Journal of Approximation Theory
www.elsevier.com/locate/jat

Notes

Close-to-optimal bounds for SU(N) loop approximation

Peter Oswald[a,*], Tatiana Shingel[b,1]

[a] *Jacobs University Bremen, 600 Campus Ring 1, 28759 Bremen, Germany*
[b] *University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0112, United States*

Received 8 August 2008; received in revised form 12 March 2010; accepted 18 April 2010
Available online 29 April 2010

Communicated by Amos Ron

Abstract

In Oswald and Shingel (2009) [6], we proved an asymptotic $\mathrm{O}(n^{-\alpha/(\alpha+1)})$ bound for the approximation of SU(N) loops ($N \geq 2$) with Lipschitz smoothness $\alpha > 1/2$ by polynomial loops of degree $\leq n$. The proof combined factorizations of SU(N) loops into products of constant SU(N) matrices and loops of the form $\mathrm{e}^{A(t)}$ where $A(t)$ are essentially su(2) loops preserving the Lipschitz smoothness, and the careful estimation of errors induced by approximating matrix exponentials by first-order splitting methods. In the present note we show that using higher order splitting methods allows us to improve the above suboptimal result to close-to-optimal $\mathrm{O}(n^{-(\alpha-\epsilon)})$ bounds for $\alpha > 1$, where $\epsilon > 0$ can be chosen arbitrarily small.

Keywords: Nonlinearly constrained approximation; Loop groups; Jackson-type estimates; Paraunitary FIR filters; Splitting methods

1. Introduction

The study of approximation rates for Lie-group-valued loops by polynomial loops is a relatively unexplored topic within the larger area of nonlinearly constrained approximation. Motivation is provided by previous density results [2,3,7] for semi-simple Lie groups, and by more practical needs, e.g., for the design of para-unitary FIR filters [2,5,10].

* Corresponding author.
E-mail addresses: p.oswald@jacobs-university.de (P. Oswald), tshingel@math.ucsd.edu (T. Shingel).
[1] The work was done while at the University of Cambridge.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.04.006

In this note, we continue the study of the $\mathrm{SU}(N)$ case, $N \geq 2$, and improve upon the following Jackson-type estimate proved in [6]: For any Lip_α-continuous loop $U : \mathbb{T} \to \mathrm{SU}(N)$ and $\alpha > 1/2$ there exists a sequence of polynomial loops $U_n : \mathbb{T} \to \mathrm{SU}(N)$ of degree $\leq n$ such that the following asymptotic inequality holds

$$\|U - U_n\|_C := \max_{t\in\mathbb{T}} \|U(t) - U_n(t)\| \leq C_{\alpha,N,U}\,(n+1)^{-\alpha/(1+\alpha)}, \quad n \geq 0. \tag{1}$$

Even though the bound (1) is admittedly far from final, to our knowledge it represented the first nontrivial upper estimate for the achievable rate of approximation for Lip_α loops with values in matrix Lie groups. Note that there is a large gap between the exponent $\alpha/(\alpha+1)$ established in (1), and the trivial upper bound α for the maximal order of approximation of Lip_α loops following from the classical Jackson–Bernstein theorems for the univariate trigonometric approximation [1, Theorem 7.3.3].

In the meantime, we realized that some simple modifications in the proof strategy of [6] yield close-to-optimal rates, at least for $\alpha > 1$. The major change is to use higher-order splitting methods instead of the standard first-order approximation $\mathrm{e}^{\sum_{j=1}^J X_j} \approx \mathrm{e}^{X_1}\ldots\mathrm{e}^{X_J}$.

Theorem 1. *Let $N \geq 2$, $\alpha > 1$, and $\epsilon > 0$. For any $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, there exists a sequence of $\mathrm{SU}(N)$-valued polynomial loops $U_n(t)$ of degree $\leq n$ such that*

$$\|U - U_n\|_C \leq C_{\alpha,\epsilon,N,U}\,(n+1)^{-(\alpha-\epsilon)}, \quad n \geq 0. \tag{2}$$

The proof of this result is given below. The restriction $\alpha > 1$ comes from the fact that we currently miss error formulas for splitting methods of order $k > 2$ in terms of higher order commutators. Another obstacle is the lack of formal proof for factorizations of $\mathrm{SU}(N)$-valued Lip_α loops into exponentials of $\mathrm{su}(N)$-valued Lip_α loops if $\alpha \leq 1/2$. The latter problem can be removed by using homotopy arguments as in [3] (the first author acknowledges inspiring discussions with W. M. Lawton on this and related subjects of the present note). See also the remarks at the end of the next section.

The major open question is whether (2) remains true also with $\epsilon = 0$, or if the nonlinear constraints lead to a slight deterioration of the approximation results compared to the unconstrained case. Settling this question will probably require a different approach.

2. Proof of Theorem 1

We first recall the facts already proved in [6]. The notation we use is either self-explanatory or can be found in [6] (we have opted to keep the notation very close to that of [6], to make the comparison easy). The matrix norm of choice is the spectral norm. The Hölder–Zygmund classes $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N)) \subset C(\mathbb{T} \to \mathrm{SU}(N))$ of loops are defined by the finiteness of the semi-norm

$$|U|_{\mathrm{Lip}_\alpha} := \begin{cases} \sup\limits_{h>0} h^{-\alpha}\|U(\cdot+h) - U(\cdot)\|_C, & 0 < \alpha < 1, \\ \sup\limits_{h>0} h^{-1}\|U(\cdot+h) - 2U(\cdot) + U(\cdot-h)\|_C, & \alpha = 1, \end{cases}$$

and, by recursion, for $\alpha > 1$ we require that $U(t) \in C^k(\mathbb{T} \to \mathrm{SU}(N))$ and set

$$|U|_{\mathrm{Lip}_\alpha} := |U^{(k)}|_{\mathrm{Lip}_{\alpha-k}},$$

where k is the largest integer $k < \alpha$. We further let

$$\|U\|_{\mathrm{Lip}_\alpha} := \|U\|_C + |U|_{\mathrm{Lip}_\alpha}.$$

Note that the Hölder–Zygmund classes of $\mathrm{SU}(N)$ loops form groups, i.e., the Lip_α property is preserved under multiplication.

Factorization into essentially exponentials of su(2) *loops*. Lemma 4 of [6] states that for any $\alpha > 1/2$ and any $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, there exist constant matrices $U_{0,l} \in \mathrm{SU}(N)$ and loops $A_l(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$ such that

$$U(t) = \prod_{l=1}^{L} U_{0,l} \mathrm{e}^{\hat{A}_l(t)}, \quad t \in \mathbb{T},\ L := N(N-1)/2, \tag{3}$$

where $\hat{A}_l(t) = T_{ij} A_l(t)$ denotes the canonical extension of $A_l(t)$ to an $\mathrm{su}(N)$ loop by the map

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto T_{ij} A = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{21} & 0 \\ 0 & 0 & I_{j-i+1} & 0 & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 0 & I_{N-j} \end{pmatrix}$$

for some index pair (i, j) with $1 \le i < j \le N$ (I_k denotes the $k \times k$ identity matrix). Moreover, smoothness of the factors is controlled by smoothness of $U(t)$:

$$\|A_l\|_{\mathrm{Lip}_\alpha} \le C_{\alpha,N,U} \|U\|_{\mathrm{Lip}_\alpha}, \quad l = 1, \ldots, L. \tag{4}$$

Approximation can be done factor-by-factor. If $P_l(t)$ are polynomial loops in SU(2) of degree $\le n$ such that

$$\|\mathrm{e}^{A_l(t)} - P_l(t)\|_C \le \epsilon, \quad l = 1, \ldots, L, \tag{5}$$

for the $A_l(t)$ occurring in the factorization (3) then

$$P(t) = \prod_{l=1}^{L} U_{0,l} \hat{P}_l(t), \quad (\hat{P}_l(t) = T_{ij} P_l(t))$$

is a polynomial loop in $\mathrm{SU}(N)$ of degree $\le Ln$ and it satisfies the estimate

$$\|U(t) - P(t)\|_C \le L\epsilon. \tag{6}$$

Use Lemma 5 from [6].

Construction of $P_l(t)$. For any $m > 1$, we can approximate $A_l(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$ by an su(2)-valued polynomial loop

$$R_{l,m}(t) = \sum_{r=1}^{6} \sum_{k=0}^{m} c_{r,k} B_{r,k}(t)$$

of degree $\le m$ at optimal rate (say, by applying the de la Vallée Poussin means componentwise). Hence,

$$\|\mathrm{e}^{A_l(t)} - \mathrm{e}^{R_{l,m}(t)}\|_C \le C m^{-\alpha}, \quad m > 0, \tag{7}$$

see Lemma 6 (a) in [6]. Here $\{B_{r,k}(t)\}$ is the designated basis (over $\mathbb{R}$) for the linear space of dimension $6m+3$ of all su(2)-valued polynomial loops of degree $\leq m$. To keep notation simple, the dependence of the coefficients $c_{k,r}$ on l and m is not made explicit; moreover, for even r summation should only start from $k=1$, see [6].

Lemmas 1 and 4 in [6] establish the following facts which are relevant below:

$$\|c_{r,k}B_{r,k}(t)\|_C = |c_{r,k}| \leq C(k+1)^{-\alpha}, \quad k=0,\ldots,m,\ r=1,\ldots,6, \tag{8}$$

and if ordered properly (e.g., lexicographically), products of the form

$$\prod_{r=1}^{6}\prod_{k=0}^{m} \mathrm{e}^{\lambda\, c_{r,k}B_{r,k}(t)}$$

represent SU(2)-valued polynomial loops of degree $\leq 6m$, independently of the choice of $\lambda > 0$ and the set of real coefficients $\{c_{r,k}\}$. For our applications we take $\lambda = 1/M$, for some integer $M > 1$.

With these preparations, we can write down the final formula for the approximation of $\mathrm{e}^{A_l(t)}$:

$$P_l(t) := (\phi(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\ldots,6;\,k=0,\ldots,m}))^M, \tag{9}$$

where $\phi(\{X_j\}_{j=1,\ldots,J})$ is a suitable splitting method for approximating $\mathrm{e}^{X_1+\cdots+X_J}$, see the next paragraph for details. The integers m and M will be fixed later. Note that the pointwise estimate

$$\|\mathrm{e}^{A_l(t)} - P_l(t)\| \leq Cm^{-\alpha} + M\|\mathrm{e}^{R_{l,m}(t)/M} - \phi(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\ldots,6;\,k=0,\ldots,m})\| \tag{10}$$

follows from applying the triangle inequality to

$$\mathrm{e}^{A_l(t)} - P_l(t) = (\mathrm{e}^{A_l(t)} - \mathrm{e}^{R_{l,m}(t)}) + ((\mathrm{e}^{R_{l,m}(t)/M})^M - P_l(t)),$$

then using (7) for the first term, and Lemma 5 in [6] for the second. Thus, the quality of approximation crucially depends on the properties of the chosen splitting method ϕ.

Estimate for the second term in (10). Now we depart from [6], where the method ϕ of choice was the first-order splitting method

$$\phi_1(\{X_j\}_{j=1,\ldots,J}) := \mathrm{e}^{X_1}\mathrm{e}^{X_2}\ldots\mathrm{e}^{X_J}.$$

The error estimate is stated in Lemma 6 (b) of [6], and leads to an overall estimate $\leq CM^{-1}$ for the second term in the right-hand side of (10) if $\alpha > 1/2$ (and to the suboptimal asymptotic approximation rate of that paper). We now show that using higher order symmetric methods leads to significant improvements. The standard second order symmetric method is given by

$$\phi_2(\{X_j\}_{k=1,\ldots,J}) := \mathrm{e}^{X_1/2}\ldots\mathrm{e}^{X_{J-1}/2}\mathrm{e}^{X_J}\mathrm{e}^{X_{J-1}/2}\ldots\mathrm{e}^{X_1/2}.$$

Following Yoshida (see [4]), splitting methods of order $2(s+1)$ can be constructed from a given method of order $2s$ via the formula

$$\phi_{2(s+1)}(\{X_j\}_{j=1,\ldots,J}) := \phi_{2s}(\{a_sX_j\}_{j=1,\ldots,J})\phi_{2s}(\{b_sX_j\}_{j=1,\ldots,J})\phi_{2s}(\{a_sX_j\}_{j=1,\ldots,J}),$$

if one chooses the constants as follows:

$$a_s = (2-2^{1/(2s+1)})^{-1}, \qquad b_s = -2^{1/(2s+1)}(2-2^{1/(2s+1)})^{-1}.$$

The order condition for these ϕ_{2s} can be stated as follows: For $\lambda \to 0$, we have

$$\|\mathrm{e}^{\lambda(X_1+\cdots+X_J)} - \phi_{2s}(\{\lambda X_j\}_{j=1,\ldots,J})\| = \mathrm{O}(\lambda^{2s+1}).$$

Using Taylor expansion and rough estimates, the order requirement translates into the error bound

$$\|e^{\lambda(X_1+\cdots+X_J)} - \phi_{2s}(\{\lambda X_j\}_{j=1,\ldots,J})\| \le C\lambda^{2s+1}\left(\sum_{j=1}^{J}\|X_j\|\right)^{2s+1}, \tag{11}$$

valid with a constant C depending on s but not on λ and $\{X_j\}$. More precise error bounds are available for ϕ_2, see Remark 2 at the end.

We are now ready to apply this to the family $\{c_{r,k}B_{r,k}(t)\}_{r=1,\ldots,6;\, k=0,\ldots,m}$ (pointwise with respect to t) with $\lambda = M^{-1}$. By (8), if $\alpha > 1$ we have

$$\sum_{r=1}^{6}\sum_{k=0}^{m}\|c_{r,k}B_{r,k}(t)\| \le C, \tag{12}$$

where C depends on the Lip_α-norms of the su(2)-loops $A_l(t)$, but importantly it does not depend on m. Substituting into (11), we obtain for each $l = 1, \ldots, L$

$$\|e^{R_{l,m}(t)/M} - \phi_{2s}(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\ldots,6;\, k=0,\ldots,n})\|_C \le CM^{-(2s+1)}. \tag{13}$$

From now on, set $\phi = \phi_{2s}$ in the formula for $P_l(t)$ (and consequently $P(t)$). Substituting into (10) and taking into account (5), (6) we finally arrive at

$$\|U(t) - P(t)\|_C \le L\max_l \|e^{A_l(t)} - P_l(t)\| \le C(m^{-\alpha} + M^{-2s}), \tag{14}$$

where C depends on s, α, N, and on $U(t)$.

Estimating the degree of $P(t)$. Consider a large enough integer $n \ge n_0$ (for $n < n_0$, just use constant $P(t) = I$ to get the complementing trivial bound $\|U(t) - I\| \le 2$). We will now fix m and M such that the degree of the above constructed $P(t)$ is $\le n$ *and* the right-hand side in (14) is asymptotically as small as possible. Due to the recursive definition of ϕ_{2s}, the degree of the polynomial loops $\phi_{2s}(\{c_{r,k}B_{r,k}(t)/M\})$ is bounded by 3^{s-1} times the degree of the polynomial loops $\phi_2(\{c_{r,k}B_{r,k}(t)/M\})$ generated by the second order method (for simplicity, we do not indicate the index set $r = 1, \ldots, 6;\ k = 0, \ldots, m$ in the notation). The latter, however, have degree $\le 12m$. This can be proved as in Lemma 1 of [6]. Indeed, we write

$$\phi_2(\{c_{r,k}B_{r,k}(t)/M\}) = \phi_1(\{c_{r,k}B_{r,k}(t)/2M\})\phi_1(\{c_{r,k}B_{r,k}(t)^*/2M\})^*.$$

We already know that the first factor $\phi_1(\{c_{r,k}B_{r,k}(t)/2M\})$ has degree $\le 6m$. The second factor is the Hermitian transpose of $\phi_1(\{c_{r,k}B^*_{r,k}(t)/2M\})$, and it remains to check that $\{B^*_{r,k}(t)\}$ is such a permutation of the original basis $\{B_{r,k}(t)\}$ to which Lemma 1 of [6] can be applied, leading to the same degree bound.

Putting things together, we see that the degree of $P(t)$ is bounded by $12L3^{s-1}Mm$. Thus, choosing the integers M, m according to

$$M = [(12L3^{s-1})^{-1}n^{\alpha/(\alpha+2s)}], \qquad m = [n^{2s/(\alpha+2s)}],$$

we guarantee that the degree of $P(t)$ does not exceed n. On the other hand, substitution into (14) yields

$$\|U(t) - P(t)\|_C \le Cn^{-2s\alpha/(\alpha+2s)} = Cn^{-\alpha+\alpha^2/(\alpha+2s)}.$$

This establishes the claim of our theorem, if, for given $\alpha > 1$ and $\epsilon > 0$, we choose the order $2s$ of the splitting method large enough.

Remark 1. There are at least three shortcomings of the asymptotic estimate (2). Firstly, the constant $C(\alpha, \epsilon, N, U)$ depends on $U(t)$ in an unspecified way. Secondly, the restriction $\alpha > 1$ is mainly due to the use of the crude error estimate (11) for higher order splitting methods (see the comments in Remark 2). In addition, for $\alpha \le 1/2$ the factorization technique of Lemma 4 from [6] breaks down (an alternative is addressed in Remark 3). Thirdly, it is not clear at the moment if one can set $\epsilon = 0$ in (2).

Remark 2. For low-order splitting methods such as ϕ_1 and ϕ_2, the error bound can be made more precise in terms of commutator expressions which paves the way for proving (13) (and thus also (14)) for some $\alpha \le 1$. For ϕ_1 this was demonstrated in [6] (see Lemma 6 for the more precise error bounds). An improved error bound for the symmetric second order method ϕ_2 has been established in [9]:

$$\|\mathrm{e}^{\lambda(X_1+\ldots+X_J)} - \phi_2(\{\lambda X_j\}_{j=1,\ldots,J})\| \le \lambda^3 \Delta_m(X_1, \ldots, X_J), \tag{15}$$

where $\Delta_m(X_1, \ldots, X_J) = \sum_{k=1}^{J-1} \Delta_2(X_k, X_{k+1} + \cdots + X_J)$, and

$$\Delta_2(A, B) = \frac{1}{12}\left\{\|[[A, B], B]\| + \frac{1}{2}\|[[A, B], A]\|\right\}.$$

The advantage is that in our application, where we would apply (15) to matrix sets of the form $\{c_k B_{r,k}\}_{k=0,\ldots,m}$ related to the terms of the Fourier series of an su(2)-valued Lip_α-loop, the sum of the norms of the appearing triple commutators can be estimated by a sum of the form $\sum_{k=1}^{m} (\log k)^2 k^{-3\alpha}$ which remains uniformly bounded for $\alpha > 1/3$ (the details are worked out in [6] for the first-order case). In contrast, using (11) with $s = 1$ leads to a constant factor of the form $(\sum_{k=1}^{m} k^{-\alpha})^3$. Unfortunately, we were not able to localize generalizations of (15) to Yoshida-type or other higher-order splitting methods in the literature.

Remark 3. W. M. Lawton drew our attention to a possible alternative to loop factorizations of the form (3) proposed in [6]. It is well known that $\mathrm{SU}(N)$ is a simply connected compact C^∞-manifold. Thus, any $\mathrm{SU}(N)$-valued continuous loop $U(t)$ can be contracted to a point by a homotopy map $\psi : [0, 1] \to C(\mathbb{T} \to \mathrm{SU}(N))$ (i.e., ψ is continuous, $\psi(1) = U(t)$, and (without loss of generality) $\psi(0) = I$). Let us assume that for $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$ the homotopy map ψ can be found in such a way that $\phi(\xi) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$ for all $\xi \in [0, 1]$ (i.e., preserves Lipschitz smoothness along the homotopy path). We do not have a reference for this assumption but strongly believe that it holds for all $\alpha > 0$.

Now, take a fine enough partition $\xi_0 = 0 < \xi_1 < \cdots < \xi_{K-1} < \xi_K = 1$ of $[0, 1]$ such that $\|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C \le r_N$, where r_N is the injectivity radius of the exponential map in the neighborhood of $I \in \mathrm{SU}(N)$. Then we can write

$$U(t) = U_1(t) \ldots U_K(t), \qquad U_k(t) := \psi(\xi_{k-1})^* \psi(\xi_k), \quad k = 1, \ldots, K,$$

where all $U_k(t)$ belong to $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, and

$$\|I - U_k\|_C \le \|\psi(\xi_{k-1})^*\|_C \|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C = \|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C \le r_N,$$

i.e., $A_k(t) = \log(U_k(t))$ is well defined and belongs to $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(N))$ for all $k = 1, \ldots, K$. Thus,

$$U(t) = \prod_{k=1}^{K} \mathrm{e}^{A_k(t)}, \qquad \|A_k\|_{\mathrm{Lip}_\alpha} \le C(\alpha, N, U), \quad k = 1, \ldots, K. \tag{16}$$

In contrast to (3), the number of exponential factors K is not independent of α and U, and the $A_k(t)$ are general $\mathrm{SU}(N)$-valued Lip_α-loops (and not essentially SU(2)-valued as in (3)).

However, as long as we accept the dependence on $U(t)$ in the constant appearing in (2), the factorization (16) is sufficient to carry out the above proof with minor changes. The reduction to the SU(2) case can be circumvented by working with a similar basis $\{B_{r,k}(t)\}_{r=1,\ldots,R_N,\,k=0,\ldots,m}$ over $\mathbb{R}$ for $\mathrm{SU}(N)$-valued polynomial loops of degree $\leq m$. What changes is the number R_N of subsets $\{B_{r,k}(t)\}_{k=0,\ldots,m}$ of basis elements to be considered. This number depends only on N, and enters the degree estimates as a linear factor.

Remark 4. The loop approximation can be pursued for other Lie groups and manifolds. See [8] for the work on the closely related case of $\mathrm{SO}(N)$-valued loops ($N \geq 3$).

References

[1] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, 1993.

[2] W.M. Lawton, Conjugate quadrature filters, in: Ka-Sing Lau (Ed.), Advances in Wavelets, Springer, 1998, pp. 103–119.

[3] W.M. Lawton, Hermite interpolation in loop groups and conjugate quadrature filter approximation, Acta Appl. Math. 84 (2004) 315–349.

[4] R.I. McLachlan, G.R.W. Quispel, Splitting methods, Acta Numer. 11 (2002) 341–434.

[5] P. Oswald, C.K. Madsen, R.L. Konsbruck, Analysis of scalable PMD compensators using FIR filters and wavelength-dependent optical power measurements, J. Lightwave Techn. 22 (2) (2004) 647–657.

[6] P. Oswald, T. Shingel, Splitting methods for $\mathbf{SU}(N)$ loop approximation, J. Approx. Theory 161 (2009) 174–186.

[7] A. Pressley, G. Segal, Loop Groups, Oxford Univ. Press, 1986.

[8] T. Shingel, Trigonometric approximation of $\mathrm{SO}(N)$ loops, Constr. Approx. (in press).

[9] M. Suzuki, Decomposition formulas of exponential operators and Lie exponentials with some applications to quantum mechanics and statistical physics, J. Math. Phys. 26 (1985) 601–612.

[10] P.P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice-Hall, 1993.