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On weak tractability of the Smolyak algorithm for approximation problems ☆

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Abstract

We consider the problems of L^p-approximation of d-variate analytic functions defined on the cube with directional derivatives of all orders bounded by 1. For $1 \leq p < \infty$, it is shown that the Smolyak algorithm based on polynomial interpolation at the extrema of the Chebyshev polynomials leads to weak tractability of these problems. This gives an affirmative answer to one of the open problems raised recently by Hinrichs et al. (2014). Our proof uses the polynomial exactness of the algorithm and an explicit bound on the operator norm of the algorithm.

MSC: 41A63; 65Y20; 68Q25

Keywords: Weak tractability; Smolyak algorithm; Infinitely differentiable function class; Standard information

1. Introduction

Multivariate computational problems are defined on classes of functions depending on d variables with large or even huge d. Multivariate problems occur in many applications such as in computational finance, statistics and physics. Such problems are usually solved by algorithms that use finitely many information operations. One information operation is defined as one

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function value or the evaluation of one linear functional. The minimal number of information operations needed to find the solution to within ε is the intrinsic difficulty of the problem. It is called the information complexity and is denoted by $n(\varepsilon, d)$ to stress its dependence on the two important parameters.

Research on tractability of multivariate continuous problems started in 1994 (see [19]). The purpose of tractability is to study the complexity with respect to ε^{-1} and d. Tractability of multivariate problems has been studied for different error criteria and in different settings including the worst, randomized and average case settings. Different kinds of tractability have been considered in the literature. In fact, tractability of multivariate problems has been recently a very active research area: see [10,12,13] and the references therein. Traditionally, a problem is intractable if the information complexity is an exponential function of ε^{-1} or d. Otherwise, the problem is tractable.

In this paper, we study the approximation problem of the following class of infinitely differentiable functions that was introduced recently in [5] (we replace [0, 1] by $[-1, 1]$ for simplicity):

$$F_d = \left\{ f : [-1,1]^d \to \mathbb{R} \mid \sup_{k\in\mathbb{N}_0} \sup_{\theta\in S^{d-1}} \|D_\theta^k f\|_\infty \le 1 \right\}, \tag{1.1}$$

where S^{d-1} denotes the unit sphere of $\mathbb{R}^d$, and $D_\theta f$ denotes the directional derivative of f in the direction of $\theta \in S^{d-1}$. The background of the problem for the class F_d is as follows. The approximation problems for the larger classes

$$F_{d,p} = \left\{ f : [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_p \le 1 \text{ for all } \alpha \in \mathbb{N}_0^d \right\}, \quad 1 \le p \le \infty$$

were previously studied by several authors (see, for instance, [6,10–13,18]). It was shown in [11, 18] that approximating the class $F_{d,p}$ in the L_p-norm suffers from the curse of dimensionality. However, it remains open whether the integration problems for $F_{d,p}$ suffer from the curse of dimensionality. A recent progress on this last problem was made by Hinrichs, Novak, Ullrich and Woźniakowski [5], who introduced a smaller class F_d and proved the weak tractability of the integration problem for F_d. However, the algorithm of [5] uses finite differences to approximate high order derivatives that are numerically unstable. As a matter of fact, Hinrichs, Novak and Ullrich [4] considered a more practical algorithm, the Clenshaw–Curtis Smolyak algorithm, and proved that the weak tractability of the integration problem for the class F_d can be achieved by this algorithm. Meanwhile, it was also asked in the paper [4] whether the L^p-approximation problems for the class F_d are weakly tractable and whether the weak tractability follows from properties of the Smolyak algorithm. In this paper, we give an affirmative answer to this question, proving that the Smolyak algorithm based on polynomial interpolation at the extrema of the Chebyshev polynomials leads to the weak tractability of L^p-approximation of the class F_d for all $1 \le p < \infty$.

The paper is organized as follows. Section 2 contains some basic concepts and lemmas that will be needed in the proof of our main result, which is given in Section 3. Two extra remarks are also given in Section 3.

2. Basic concepts and lemmas

We introduce the concept of weak tractability first. We will use terminology from [10,12,13]. Assume we are given a sequence of solution operators

$$S_d : F_d \to G_d \quad \text{for all } d \in \mathbb{N}.$$

Here, F_d is a subset of some normed space H_d, and G_d is a normed space. We approximate $S_d f$ for $f \in F_d$ by algorithms

$$A_{n,d}(f) = \phi_{n,d}(L_1(f), \dots, L_n(f)),$$

where $L_j \in H_d^*$ (here H_d^* denotes the dual space of normed space H_d) and $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary mapping. The error of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) = \sup_{f \in F_d} \|S_d(f) - A_{n,d}(f)\|_{G_d}.$$

For $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, let $n(\varepsilon, F_d)$ be the information complexity which is defined as the minimal number of function values which are necessary to obtain an ε-approximation of S_d in the worst case setting for the absolute or normalized error criterion (see [10, p. 106]), i.e.,

$$n(\varepsilon, F_d) = \min\{n \in \mathbb{N} | \exists A_{n,d} \text{ such that } e(A_{n,d}) \le \varepsilon CRI_d\}, \tag{2.1}$$

where

$$\begin{aligned} CRI_d &= 1 && \text{for the absolute error criterion,} \\ CRI_d &= \sup_{f \in F_d} \|S_d(f)\|_{G_d} && \text{for the normalized error criterion.} \end{aligned}$$

By the *curse of dimensionality* we mean that $n(\varepsilon, F_d)$ is exponentially large in d. That is, there are positive numbers c, ε and γ such that

$$n(\varepsilon, F_d) \ge c(1+\gamma)^d \quad \text{for all } \varepsilon \le \varepsilon_0 \text{ and infinitely many } d \in \mathbb{N}.$$

For many natural classes F_d the bound above will hold for all $d \in \mathbb{N}$. There are many classes F_d for which the curse of dimensionality has been proved, see [10,12,13] for such examples.

On the contrary, we say that $S = \{S_d\}$ is weakly tractable if

$$\lim_{\varepsilon^{-1}+d \to \infty} \frac{\ln n(\varepsilon, F_d)}{\varepsilon^{-1} + d} = 0. \tag{2.2}$$

There are also many classes F_d for which the weak tractability has been proved, see [10,12,13] for such examples.

Now we give the definition of the Smolyak interpolation algorithm that was introduced in [1]. Assume that we want to approximate smooth functions $f : [-1, 1]^d \to \mathbb{R}$, using finitely many function values. For $d = 1$, let U^i denote the Lagrange interpolation based on the extrema of the $(m_i - 1)$th Chebyshev polynomial. In this case, these knots are given by

$$x_j^i = -\cos \frac{\pi j}{m_i - 1}, \quad j = 0, \dots, m_i - 1. \tag{2.3}$$

In addition, we define $x_1^i = 0$ if $m_i = 1$.

Denote by $V_n(x)$ the nth Chebyshev polynomial of the second kind; that is,

$$V_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = \cos \theta.$$

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Clearly, for $m_i > 1$, $\{x_j^i\}_{j=0}^{m_i-1}$ is a sequence of the zeros of the polynomial $\left(1-x^2\right)V_{m_i-2}(x)$. Hence, it follows from [14] that

$$U^i(f,x) = \sum_{k=0}^{m_i-1} f(x_k^i)a_k(x), \tag{2.4}$$

where

$$a_k(x) = \frac{(-1)^{k+1}\left(1-x^2\right)V_{m_i-2}(x)}{(m_i-1)(x-x_k^i)}, \quad k = 1,\dots,m_i-2, \tag{2.5}$$

$$a_0(x) = \frac{(1+x)V_{m_i-2}(x)}{2V_{m_i-2}(1)}, \qquad a_{m_i-1}(x) = \frac{(1-x)V_{m_i-2}(x)}{2V_{m_i-2}(-1)}. \tag{2.6}$$

Specially, we will choose

$$m_1 = 1 \quad \text{and} \quad m_i = 2^{i-1}+1 \quad \text{for } i > 1. \tag{2.7}$$

For $d > 1$ we first define tensor product formulas

$$(U^{i_1}\otimes\cdots\otimes U^{i_d})(f) = \sum_{j_1=0}^{m_{i_1}-1}\cdots\sum_{j_d=0}^{m_{i_d}-1} f(x_{j_1}^{i_1},\dots,x_{j_d}^{i_d})\cdot(a_{j_1}^{i_1}\otimes\cdots\otimes a_{j_d}^{i_d}).$$

With $U^0 = 0$, we define

$$\Delta^i = U^i - U^{i-1}$$

for $i \in \mathbb{N}$. Moreover, we put $|\mathbf{i}| = i_1+\cdots+i_d$ for $\mathbf{i}\in\mathbb{N}^d$. Then the Smolyak algorithm is given by

$$A(q,d) = \sum_{|\mathbf{i}|\le q}(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_d}) \tag{2.8}$$

for integers $q \ge d$. Equivalently,

$$A(q,d) = \sum_{q-d+1\le|\mathbf{i}|\le q}(-1)^{q-|\mathbf{i}|}\cdot\binom{d-1}{q-|\mathbf{i}|}(U^{i_1}\otimes\cdots\otimes U^{i_d}), \tag{2.9}$$

(see [17, Lemma 1], [2, Theorem 1]). To compute $A(q,d)(f)$, from [1] we know that one only needs to know function values at the point set

$$H(q,d) = \bigcup_{q-d+1\le|\mathbf{i}|\le q}\left(X^{i_1}\times\cdots\times X^{i_d}\right) = \bigcup_{|\mathbf{i}|=q}\left(X^{i_1}\times\cdots\times X^{i_d}\right),$$

where $X^i = \{x_0^i,\dots,x_{m_i-1}^i\}\subset[-1,1]$ denotes the set of points used by U^i. The points $x\in H(q,d)$ are called hyperbolic cross and $H(q,d)$ is also called a sparse grid.

In what follows we will bound the number of function values that are sufficient for the Smolyak algorithm to achieve a certain error. For this we define

$$N_d(k) := |H(d+k,d)|$$

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as the number of points used by $A(d+k,d)$. We use $\approx$ to denote the strong equivalence of sequences, i.e., $u_n \approx v_n$ iff $\lim_{n\to\infty} u_n/v_n = 1$. Then, for $k \to \infty$ and fixed d,

$$N_d(k) \approx \frac{2^k k^{d-1}}{(d-1)! \cdot 2^{d-1}}, \tag{2.10}$$

see Müller–Gronbach [7, Lemma 1]. At the same time, for $d \to \infty$ and fixed k, from [9] we know

$$N_d(k) \approx \frac{2^k d^k}{k!}. \tag{2.11}$$

Besides, by Lemma 9 in [4], we have

$$N_d(k) \le 2(2e)^k \left(1 + \frac{d}{k}\right)^k. \tag{2.12}$$

We will use the polynomial exactness of the above Smolyak interpolation algorithm. The following lemma can be found in [1, Theorem 4].

Lemma 2.1. *The formula $A(d+k,d)$ is exact for all polynomials of degree k.*

Let L^p denote the usual Lebesgue L^p-space defined with respect to the Lebesgue measure on the cube $[-1,1]^d$, with norm denoted by $\|\cdot\|_p$. Let C denote the space of continuous functions on the cube $[-1,1]^d$ with norm denoted by $\|\cdot\|_\infty$. The proof of our result requires some norm estimates of the algorithm $A(q,d) : C \to L^p$; that is, we need to estimate the norm

$$\|A(q,d)\|_p := \sup_{f\in C, \|f\|_\infty \le 1} \|A(q,d)(f)\|_p, \quad 1 \le p < \infty.$$

Let us first consider the case of $d = 1$. For $m_i = n+1$ with $n \in \mathbb{N}$, we shall prove the following result.

Lemma 2.2. *Let U^i be defined by (2.4). Then for any fixed $1 \le p < \infty$, there exists a positive C_p such that*

$$\|U^i\|_p \le C_p. \tag{2.13}$$

Specially, setting $f = 1$, we have $\|f\|_\infty = 1$ and $\|U^i f\|_p = \|f\|_p = 2^{1/p} > 1$. This implies that $C_p > 1$.

Note. We would like to add that the expression (2.13) is inspired by Theorem 1 in P. Nevai [8], which is for the Lagrange interpolation based on the zeros of orthogonal polynomials. But our proof is completely different.

The proof of (2.13) relies on several lemmas. Firstly, we need the following lemma from [14].

Lemma 2.3 (*[14]*). *Let $v_1, v_2, \ldots, v_{2N}$ be distinct integers between 1 and $n-1$. Then we have*

$$\left| \int_{-1}^{1} a_{v_1}(x) a_{v_2}(x) \cdots a_{v_{2N}}(x) \frac{dx}{\sqrt{1-x^2}} \right| = \frac{\Gamma(N+1/2)\,\Gamma(1/2)}{n^{2N}\,\Gamma(N+1)}, \tag{2.14}$$

and

$$\sum_{k=1}^{n-1} a_k^2(x) \le 2. \tag{2.15}$$

Next, we give two formulas about homogeneous symmetrical expression. Let $x_1, \ldots, x_n$ be independent variables, and let $P_1, \ldots, P_r, s$ be natural numbers. Denote

$$P = \sum_{k=1}^{r} P_k, \qquad V_s = \left(\sum_{k=1}^{n} x_k^s\right)^{1/s}.$$

In the following we denote $k_i \neq k_j$ for all $1 \leq i \neq j \leq r$ by $k_1 \neq \cdots \neq k_r$.

Lemma 2.4. *If $n > r$, then the homogeneous symmetrical expression of degree P,*

$$I_{P_1,\ldots,P_r} = \sum_{k_1 \neq \cdots \neq k_r} x_{k_1}^{P_1} \cdots x_{k_r}^{P_r},$$

can be expressed as a homogeneous expression of degree P about $V_1, \ldots, V_P$,

$$I_{P_1,\ldots,P_r} = \sum_{\sum_{i=1}^{P} t_i = P, t_i \geq 0} C_{t_1 \cdots t_P} V_1^{t_1} \cdots V_P^{t_P}, \tag{2.16}$$

and

(1) $t_1 \leq r$, $C_{t_1 \cdots t_P}$ *is independent of* n.
(2) *If* $P_r \geq 2$, *then* $t_1 \leq r - 1$.

Proof. We will prove (2.16) by induction on r. For $r = 1$, $P = P_1$, we have

$$I_{P_1} = \sum_{k_1=1}^{n} x_{k_1}^{P_1} = V_{P_1}^{P_1} = V_P^P,$$

and (1), (2) are obviously true. Assume that for $r = M$, (2.16), (1) and (2) are true. Then for $r = M + 1$, setting $\overline{P} = \sum_{i=1}^{M} P_i$, we have

$$\begin{aligned}
I_{P_1,\ldots,P_M,P_{M+1}} &= \sum_{k_1 \neq \cdots \neq k_M} x_{k_1}^{P_1} \cdots x_{k_M}^{P_M} \left(\sum_{i=1}^{n} x_i^{P_{M+1}} - \sum_{j=1}^{M} x_{k_j}^{P_{M+1}}\right) \\
&= \sum_{k_1 \neq \cdots \neq k_M} x_{k_1}^{P_1} \cdots x_{k_M}^{P_M} \left(V_{P_{M+1}}^{P_{M+1}} - \sum_{j=1}^{M} x_{k_j}^{P_{M+1}}\right) \\
&= V_{P_{M+1}}^{P_{M+1}} \sum_{k_1 \neq \cdots \neq k_M} x_{k_1}^{P_1} \cdots x_{k_M}^{P_M} - \sum_{j=1}^{M} \sum_{k_1 \neq \cdots \neq k_M} x_{k_1}^{P_1} \cdots x_{k_j}^{P_j + P_{M+1}} \cdots x_{k_M}^{P_M} \\
&= V_{P_{M+1}}^{P_{M+1}} I_{P_1,\ldots,P_M} - \sum_{j=1}^{M} I_{P_1,\ldots,P_j+P_{M+1},\ldots,P_M}.
\end{aligned} \tag{2.17}$$

From the inductive hypothesis we know that $I_{P_1,\ldots,P_M}$ can be represented as a homogeneous expression of degree $\overline{P}$ about $V_1, \ldots, V_{\overline{P}}$ and hence $V_{P_{M+1}}^{P_{M+1}} I_{P_1,\ldots,P_M}$ can be represented as a homogeneous expression of degree P about $V_1, \ldots, V_P$. At the same time, $I_{P_1,\ldots,P_j+P_{M+1},\ldots,P_M}$ can be represented as a homogeneous expression of degree P about $V_1, \ldots, V_P$ for $1 \leq j \leq M$. Furthermore, by the inductive hypothesis, we see that the coefficients in the expressions $V_{P_{M+1}}^{P_{M+1}} I_{P_1,\ldots,P_M}$ and $I_{P_1,\ldots,P_j+P_{M+1},\ldots,P_M}$ are independent of n. Hence (1) follows from (2.17). Finally, by the inductive hypothesis and (2.17), we know that if $P_{M+1} = 1$, then $t_1 \leq M + 1 = r$, whereas if $P_{M+1} > 1$, then $t_1 \leq M = r - 1$. This completes the proof. □

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Lemma 2.5. *If N is a natural number, and $n > 2N$, then the homogeneous symmetrical expression of degree $2N$,*

$$B_{2N} = \left(\sum_{i=1}^{n} x_i\right)^{2N} - (2N)! \sum_{k_1<k_2<\cdots<k_{2N}} x_{k_1}\cdots x_{k_{2N}},$$

can be represented as a homogeneous expression of degree $2N$ about $V_1, \ldots, V_{2N}$,

$$B_{2N} = \sum_{t_1\le 2N-2,\, t_i\ge 0} B_{t_1\cdots t_{2N}} V_1^{t_1}\cdots V_{2N}^{t_{2N}}. \tag{2.18}$$

Proof. A straightforward computation on the number of combinations shows that

$$\left(\sum_{i=1}^{n} x_i\right)^{2N} = \sum x_{k_1}\cdots x_{k_{2N}} = \sum_{P_1+\cdots+P_r=2N} \frac{(2N)!}{P_1!\cdots P_r!} \sum_{k_1<k_2<\cdots<k_r} x_{k_1}^{P_1}\cdots x_{k_r}^{P_r}. \tag{2.19}$$

Let $\Delta(r)$ denote the set of all arrangements of $\{1, 2, \ldots, r\}$. Assume that $P_1, \ldots, P_r$ are some given integers that are not necessarily distinct. Define the following equivalent relation on $\Delta(r)$:

$$(i_1, \ldots, i_r) \sim (j_1, \ldots, j_r) \Longleftrightarrow (P_{i_1}, \ldots, P_{i_r}) = (P_{j_1}, \ldots, P_{j_r}).$$

A simple calculation of the number of combinations shows that each equivalent class in $\Delta(r)/\sim$ contains the same number of elements, which we denote by $C_{P_1,\ldots,P_r}$. Then we have

$$\begin{aligned}
&\sum_{P_1+\cdots+P_r=2N} \frac{(2N)!}{P_1!\cdots P_r!} \sum_{k_1<k_2\cdots<k_r} x_{k_1}^{P_1}\cdots x_{k_r}^{P_r}\\
&= \sum_{P_1\le\cdots\le P_r} \frac{(2N)!}{P_1!\cdots P_r!} \sum_{\mathbf{i}\in\Delta(r)/\sim} \sum_{k_1<k_2\cdots<k_r} x_{k_1}^{P_{i_1}}\cdots x_{k_r}^{P_{i_r}}\\
&= \sum_{P_1\le\cdots\le P_r} \frac{(2N)!}{P_1!\cdots P_r! C_{P_1,\ldots,P_r}} \sum_{\mathbf{i}\in\Delta(r)} \sum_{k_1<k_2\cdots<k_r} x_{k_1}^{P_{i_1}}\cdots x_{k_r}^{P_{i_r}}\\
&= \sum_{P_1\le\cdots\le P_r} \frac{(2N)!}{P_1!\cdots P_r! C_{P_1,\ldots,P_r}} \sum_{k_1<k_2\cdots<k_r} \sum_{\mathbf{i}\in\Delta(r)} x_{k_1}^{P_{i_1}}\cdots x_{k_r}^{P_{i_r}}.
\end{aligned} \tag{2.20}$$

For an arbitrary $\{P_1, P_2, \ldots, P_r\}$ and fixed $\{k_1 < k_2 < \cdots < k_r\}$, it is easy to verify that

$$\sum_{\mathbf{i}\in\Delta} x_{k_1}^{P_{i_1}}\cdots x_{k_r}^{P_{i_r}} = \sum_{\mathbf{i}\in\Delta} x_{k_{i_1}}^{P_1}\cdots x_{k_{i_r}}^{P_r}.$$

Hence, by the symmetry it follows that

$$\sum_{k_1<k_2\cdots<k_r} \sum_{\mathbf{i}\in\Delta} x_{k_1}^{P_{i_1}}\cdots x_{k_r}^{P_{i_r}} = \sum_{k_1\ne k_2\cdots\ne k_r} x_{k_1}^{P_1}\cdots x_{k_r}^{P_r} = I_{P_1\cdots P_r}. \tag{2.21}$$

Then combining (2.19), (2.20) with (2.21) gives

$$\begin{aligned}
B_{2N} &= \sum_{P_1\le\cdots\le P_r} \frac{(2N)!}{P_1!\cdots P_r!} \frac{I_{P_1\cdots P_r}}{C_{P_1,\ldots,P_r}} - I_{1,\ldots,1}\\
&= \sum_{P_1\le\cdots\le P_r,\, P_r\ge 2} \frac{(2N)!}{P_1!\cdots P_r!} \frac{I_{P_1\cdots P_r}}{C_{P_1,\ldots,P_r}}.
\end{aligned} \tag{2.22}$$

For each term $I_{P_1 \cdots P_r}$ in the summation in (2.22), (2.16) and Statement (1) of Lemma 2.4 hold. From $\sum_{i=1}^{r} P_i = 2N$ and $P_r \geq 2$ we know that $r \leq 2N - 1$. Furthermore, by (2) in Lemma 2.4 and the fact that $P_r \geq 2$, we know that $t_1 \leq r - 1 \leq 2N - 2$. This together with (2.22) completes the proof of (2.18). □

Lemma 2.6. *Let $a_k(x)$ be defined by (2.5). Then for any fixed natural number N, there exists C_{2N} such that*

$$\int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{2N} \frac{dx}{\sqrt{1-x^2}} \leq C_{2N} \Big(\max_{1 \leq k \leq n-1} |A_k| \Big)^{2N}. \tag{2.23}$$

Proof. We will prove (2.23) by induction on N. For $N = 1$, by (2.14) and (2.15), we obtain

$$\begin{aligned} &\int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{2} \frac{dx}{\sqrt{1-x^2}} \\ &= \sum_{k=1}^{n-1} A_k^2 \int_{-1}^{1} a_k^2(x) \frac{dx}{\sqrt{1-x^2}} + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{k-1} A_k A_j \int_{-1}^{1} a_k(x) a_j(x) \frac{dx}{\sqrt{1-x^2}} \\ &\leq \max_{1 \leq k \leq n-1} |A_k|^2 \left(\int_{-1}^{1} \sum_{k=1}^{n-1} a_k^2(x) \frac{dx}{\sqrt{1-x^2}} + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{k-1} \left| \int_{-1}^{1} a_k(x) a_j(x) \frac{dx}{\sqrt{1-x^2}} \right| \right) \\ &\leq \left(2\pi + (n-1)(n-2) \frac{\Gamma(3/2)\,\Gamma(1/2)}{n^2 \Gamma(2)} \right) \max_{1 \leq k \leq n-1} |A_k|^2 \leq \frac{5\pi}{2} \max_{1 \leq k \leq n-1} |A_k|^2. \end{aligned} \tag{2.24}$$

From (2.24) and Hölder's inequality, we obtain

$$\begin{aligned} \int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right| \frac{dx}{\sqrt{1-x^2}} &\leq \left(\int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{2} \frac{dx}{\sqrt{1-x^2}} \right)^{1/2} \cdot \left(\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \right)^{1/2} \\ &\leq \frac{\sqrt{10}\pi}{2} \max_{1 \leq k \leq n-1} |A_k|. \end{aligned}$$

Suppose that for $0 < j \leq 2(N-1)$, we have

$$\int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{j} \frac{dx}{\sqrt{1-x^2}} \leq C_j \max_{1 \leq k \leq n-1} |A_k|^j. \tag{2.25}$$

We will consider the case for $2N$. If $n - 1 \leq 2N$, then (2.15) gives that $|a_k(x)| \leq 2$ and hence

$$\int_{-1}^{1} \left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{2N} \frac{dx}{\sqrt{1-x^2}} \leq \pi (4N)^{2N} \max_{1 \leq k \leq n-1} |A_k|^{2N}. \tag{2.26}$$

If $n - 1 > 2N$, then by Lemma 2.5 we know

$$\left| \sum_{k=1}^{n-1} A_k a_k(x) \right|^{2N} = (2N)! \sum_{k_1 < k_2 < \cdots < k_{2N}} A_{k_1} \cdots A_{k_{2N}} a_{k_1}(x) \cdots a_{k_{2N}}(x)$$

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$$
\begin{aligned}
&+ \sum_{t_1 \le 2N-2, t_i \ge 0} B_{t_1 \cdots t_{2N}} V_1^{t_1}(x) \cdots V_{2N}^{t_{2N}}(x) \\
&= I_1(x) + I_2(x), \qquad (2.27)
\end{aligned}
$$

where

$$
V_s(x) = \left(\sum_{k=1}^{n-1} A_k^s a_k^s(x) \right)^{1/s}.
$$

From (2.14) it follows that

$$
\begin{aligned}
\left| \int_{-1}^{1} I_1(x) \frac{dx}{\sqrt{1-x^2}} \right| &\le (2N)! \sum_{k_1 < \cdots < k_{2N}} \left| A_{k_1} \cdots A_{k_{2N}} \right| \left| \int_{-1}^{1} a_{k_1}(x) \cdots a_{k_{2N}}(x) \frac{dx}{\sqrt{1-x^2}} \right| \\
&\le \frac{(n-1)!}{(n-1-2N)!} \frac{\Gamma(N+1/2)\,\Gamma(1/2)}{n^{2N}\Gamma(N+1)} \max_{1 \le k \le n-1} |A_k|^{2N} \\
&\le \frac{\Gamma(N+1/2)\,\Gamma(1/2)}{\Gamma(N+1)} \max_{1 \le k \le n-1} |A_k|^{2N}. \qquad (2.28)
\end{aligned}
$$

Using the inequality $(|a| + |b|)^c \le |a|^c + |b|^c$ for $0 \le c \le 1$, and (2.15), we obtain that for $s \ge 2$,

$$
\begin{aligned}
|V_s(x)| &\le \max_{1 \le k \le n-1} |A_k| \left(\sum_{k=1}^{n-1} |a_k(x)|^s \right)^{1/s} \\
&= \max_{1 \le k \le n-1} |A_k| \left(\left(\sum_{k=1}^{n-1} \left| a_k^2(x) \right|^{s/2} \right)^{2/s} \right)^{1/2} \\
&\le \sqrt{2} \max_{1 \le k \le n-1} |A_k|. \qquad (2.29)
\end{aligned}
$$

Denote $C_0 = \pi$ and let C_j be as in (2.25) for $1 \le j \le 2(N-1)$. By virtue of (2.27) and (2.29), we have

$$
\begin{aligned}
\left| \int_{-1}^{1} I_2(x) \frac{dx}{\sqrt{1-x^2}} \right| &\le \sum_{t_1 \le 2N-2, t_i \ge 0} \left| B_{t_1 \cdots t_{2N}} \right| \int_{-1}^{1} \left| V_1^{t_1}(x) \cdots V_{2N}^{t_{2N}}(x) \right| \frac{dx}{\sqrt{1-x^2}} \\
&\le \sum_{t_1 \le 2N-2, t_i \ge 0} 2^N \left| B_{t_1 \cdots t_{2N}} \right| \max_{1 \le k \le n-1} |A_k|^{2N-t_1} \\
&\quad \times \int_{-1}^{1} \left| V_1^{t_1}(x) \right| \frac{dx}{\sqrt{1-x^2}} \\
&\le \sum_{t_1 \le 2N-2, t_i \ge 0} 2^N \left| B_{t_1 \cdots t_{2N}} \right| C_{t_1} \max_{1 \le k \le n-1} |A_k|^{2N} \\
&\le \left(\sum_{j=0}^{2N-2} C_j \sum_{t_1 \le 2N-2, t_i \ge 0} 2^N \left| B_{t_1 \cdots t_{2N}} \right| \right) \max_{1 \le k \le n-1} |A_k|^{2N}. \qquad (2.30)
\end{aligned}
$$

From (2.26), (2.27), (2.28) and (2.30), it follows that

$$\int_{-1}^{1}\left|\sum_{k=1}^{n-1}A_k a_k(x)\right|^{2N}\frac{dx}{\sqrt{1-x^2}}\le C_{2N}\max_{1\le k\le n-1}|A_k|^{2N}, \tag{2.31}$$

which competes the proof of Lemma 2.6. □

Remark. In the case of $N=1$ or 2, (2.23) can be proved by the method of A.K. Varma, P. Vértesi [14]. Here we give an inductive proof of (2.23) for all N for the sake of completeness.

For an arbitrary $p\ge 1$, let N be the smallest positive integer satisfying $p\le 2N$. Then by the Hölder inequality and (2.23) we obtain

$$\begin{aligned}\int_{-1}^{1}\left|\sum_{k=1}^{n-1}A_k a_k(x)\right|^{p}dx &\le \left(\int_{-1}^{1}\left|\sum_{k=1}^{n-1}A_k\varphi_k(x)\right|^{2N}dx\right)^{p/(2N)}\left(\int_{-1}^{1}1dx\right)^{1-p/(2N)}\\ &\le \left(\int_{-1}^{1}\left|\sum_{k=1}^{n-1}A_k\varphi_k(x)\right|^{2N}\frac{dx}{\sqrt{1-x^2}}\right)^{p/(2N)}2^{1-p/(2N)}\\ &\le C_p^p(\max_{1\le k\le n-1}|A_k|)^p.\end{aligned} \tag{2.32}$$

Now we turn to the proof of Lemma 2.2.

Proof. From (2.4) it follows that for $f\in C[-1,1]$, we have

$$\|U^i f\|_p\le |f(x_0^i)|\,\|a_0\|_p+\left\|\sum_{k=1}^{m_i-2}f(x_k^i)a_k\right\|_p+|f(x_{m_i-1}^i)|\,\left\|a_{m_i-1}\right\|_p. \tag{2.33}$$

From (2.6) and $\|V_{m_i-2}\|_\infty=|V_{m_i-2}(\pm 1)|=m_i-1$ we conclude that

$$|f(x_0^i)|\,\|a_0\|_p\le 2\|f\|_\infty,\quad \text{and}\quad |f(x_{m_i-1}^i)|\,\left\|a_{m_i-1}\right\|_p\le 2\|f\|_\infty. \tag{2.34}$$

From (2.32) it follows that

$$\left\|\sum_{k=1}^{m_i-2}f(x_k^i)a_k\right\|_p\le C_p\max_{1\le k\le m_i-2}|f(x_k^i)|\le C_p\|f\|_\infty. \tag{2.35}$$

From (2.33)–(2.35) we obtain (2.13). This completes the proof of Lemma 2.2. □

The following lemma gives an estimate of the operator norm of $A(q,d)$ from C to L_p for $1\le p<\infty$.

Lemma 2.7. *Let $A(q,d)$ be defined by (2.8). Then for any fixed $1\le p<\infty$, there exists a positive $\overline{C}_p=2C_p$ such that*

$$\|A(d+k,d)\|_p\le (2C_p)^d\binom{d+k}{d}\le (2C_p)^d e^k\left(1+\frac{d}{k}\right)^k. \tag{2.36}$$

Proof. The second inequality can be found in [4, Proposition 7]. We will prove the first inequality only. From (2.13) we obtain that for $i\in\mathbb{N}$,

$$\|\Delta^i\|_p\le\|U^i\|_p+\|U^{i-1}\|_p\le 2C_p. \tag{2.37}$$

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By (2.37) and a simple inductive reasoning on d we obtain that for an arbitrary $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{N}^d$,

$$\|\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}\|_p \leq (2C_p)^d. \tag{2.38}$$

It is well known that for $j \geq d$, we have

$$|\{\mathbf{i} : |\mathbf{i}| = j\}| = \binom{j-1}{d-1}. \tag{2.39}$$

By (2.8), (2.38) and (2.39) we obtain

$$\begin{aligned}\|A(d+k,d)\|_p &\leq (2C_p)^d \sum_{d \leq |\mathbf{i}| \leq k+d} 1 \\ &= (2C_p)^d \sum_{d \leq j \leq k+d} \binom{j-1}{d-1} \\ &= (2C_p)^d \sum_{l=0}^{k} \binom{d-1+l}{d-1} = (2C_p)^d \binom{d+k}{d},\end{aligned} \tag{2.40}$$

and (2.31) implies the first inequality in (2.36). □

3. The main result and its proof

In this section we assume that F_d is defined by (1.1) and consider the L^p-approximation problem

$$S_d : S_d f = f \quad \text{for all } f \in F_d.$$

Denote $\ln_+ d = 1$ for $d = 1$ and $\ln_+ d = \ln d$ for $d > 1$. For $k \in N_0$, we denote

$$e_p(k,d) := e(A(k+d,d)) = \sup_{f \in F_d} \|A(k+d,d)(f) - f\|_p. \tag{3.1}$$

Given $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ the smallest integer not less than x.

Theorem 3.1. *Let F_d be defined by (1.1) and $1 \leq p < \infty$. Then there exists a $\alpha_p > 0$ such that for each $d \in \mathbb{N}$, $\varepsilon \in (0,1]$, and for*

$$k_{\varepsilon,d,p} = \left\lceil \max\left\{\frac{\alpha_p d}{\ln_+ d}, \ln \varepsilon^{-1}\right\}\right\rceil, \tag{3.2}$$

we have that

$$e_p(k_{\varepsilon,d,p}, d) \leq \varepsilon. \tag{3.3}$$

Furthermore, the number of function values $N_d(k_{\varepsilon,d,p})$ required in the algorithm $A(k_{\varepsilon,d,p}+d, d)$ satisfies

$$N_d(k_{\varepsilon,d,p}) \leq 2\max\left\{(2e)^{\frac{2\alpha_p d}{\ln_+ d}}\left(1 + \frac{\ln_+ d}{2\alpha_p}\right)^{\frac{2\alpha_p d}{\ln_+ d}}, (2e)^{2\ln\varepsilon^{-1}}\left(1 + \frac{\ln \varepsilon^{-1}}{\alpha_p^2}\right)^{2\ln\varepsilon^{-1}}\right\}. \tag{3.4}$$

This, in particular, implies that under the absolute and normalized error criteria, the weak tractability of the L^p-approximation problems for the class F_d holds for all $1 \le p < \infty$.

Proof. Let $\mathcal{P}_k$ be the space of polynomials of degree k. Following the proof of [4, Proposition 8], we obtain that for $f \in F_d$ and $k \in \mathbb{N}$,

$$\inf_{p\in\mathcal{P}_k} \|f - p\|_\infty \le \inf_{p\in\mathcal{P}_{k-1}} \|f - p\|_\infty \le \frac{d^{k/2}}{k!} \le \sqrt{\frac{1}{2\pi k}} \left(\frac{e\sqrt{d}}{k}\right)^k . \tag{3.5}$$

Using Lemma 2.1, (2.36) and $C_p > 1$, we have

$$\begin{aligned} e_p(k,d) &\le \sup_{f\in F_d} \inf_{p\in\mathcal{P}_k} \|f-p\|_\infty (1 + \|A(k+d,d)\|_p) \\ &\le \sqrt{\frac{1}{2\pi k}} \left(\frac{e\sqrt{d}}{k}\right)^k \left(1 + (2C_p)^d e^k \left(1+\frac{d}{k}\right)^k\right) \\ &\le (2C_p)^d \left(\frac{e^2\sqrt{d}}{k}\left(1+\frac{d}{k}\right)\right)^k \\ &= \left((2C_p)^{d/k}\frac{e^2\sqrt{d}}{k}\left(1+\frac{d}{k}\right)\right)^k . \end{aligned} \tag{3.6}$$

Let $k = \lceil \frac{\alpha d}{\ln_+ d} \rceil$, $\alpha \ge 4\ln(2C_p)$. Then it is easy to verify that for $d \ge 2$,

$$(2C_p)^{d/k} \le (2C_p)^{\ln d/\alpha} = d^{\ln(2C_p)/\alpha}.$$

Hence

$$(2C_p)^{d/k}\frac{e^2\sqrt{d}}{k}\left(1+\frac{d}{k}\right) \le \frac{e^2 \ln d}{\alpha d^{1/2-\ln(2C_p)/\alpha}}\left(1+\frac{\ln d}{\alpha}\right) \le \frac{e^2\ln d}{\alpha d^{1/4}}\left(1+\frac{\ln d}{\alpha}\right). \tag{3.7}$$

Since

$$\lim_{d\to\infty} \frac{e^2\ln d}{4\ln(2C_p)d^{1/4}}\left(1+\frac{\ln d}{4\ln(2C_p)}\right) = 0,$$

we conclude from (3.7) that there exists $M \in \mathbb{N}$ such that for $d > M$,

$$(2C_p)^{d/k}\frac{e^2\sqrt{d}}{k}\left(1+\frac{d}{k}\right) \le \frac{1}{e}. \tag{3.8}$$

From

$$\lim_{\alpha\to\infty} (2C_p)^{\frac{d}{\lceil \frac{\alpha d}{\ln_+ d}\rceil}} \frac{e^2\sqrt{d}}{\lceil \frac{\alpha d}{\ln_+ d}\rceil}\left(1+\frac{d}{\lceil \frac{\alpha d}{\ln_+ d}\rceil}\right) = 0 \quad \text{for } d = 1,\dots,M$$

we know that there exists an $\alpha_p \ge 4\ln(2C_p)$ such that for $d = 1,\dots,M$, (3.8) holds as well. This shows that with

$$k_{\varepsilon,d,p} := \left\lceil \max\left\{\frac{\alpha_p d}{\ln_+ d}, \ln\varepsilon^{-1}\right\}\right\rceil$$

we have $e_p(k_{\varepsilon,d,p}, d) \le \varepsilon$, which shows the first part of Theorem 3.1. If

$$\frac{\alpha_p d}{\ln_+ d} \ge \ln \varepsilon^{-1},$$

then from $\alpha_p \ge 4 \ln 2 > 1$ we obtain

$$k_{\varepsilon,d,p} = \left\lceil \frac{\alpha_p d}{\ln_+ d} \right\rceil \le \frac{2\alpha_p d}{\ln_+ d}. \tag{3.9}$$

From (2.12) and (3.9) it follows that

$$N_d(k_{\varepsilon,d,p}) \le 2(2e)^{\frac{2\alpha_p d}{\ln_+ d}} \left(1 + \frac{\ln_+ d}{2\alpha_p}\right)^{\frac{2\alpha_p d}{\ln_+ d}}. \tag{3.10}$$

If

$$\frac{\alpha_p d}{\ln_+ d} \le \ln \varepsilon^{-1},$$

then

$$k_{\varepsilon,d,p} = \lceil \ln \varepsilon^{-1} \rceil \le 2 \ln \varepsilon^{-1}. \tag{3.11}$$

Using

$$\sup_{x \ge 1} \frac{\ln^2 x}{x} = \frac{4}{e^2} < 1,$$

we deduce

$$\frac{d}{k_{\varepsilon,d,p}} \le \frac{d}{\ln \varepsilon^{-1}} \le \frac{\ln_+ d}{\alpha_p} \le \frac{\ln \varepsilon^{-1}}{\alpha_p^2}. \tag{3.12}$$

From (2.12), (3.11) and (3.12) it follows that

$$N_d(k_{\varepsilon,d,p}) \le 2(2e)^{2\ln \varepsilon^{-1}} \left(1 + \frac{\ln \varepsilon^{-1}}{\alpha_p^2}\right)^{2 \ln \varepsilon^{-1}}. \tag{3.13}$$

From (3.10) and (3.13) we obtain (3.4). By (3.4), it is easily seen that (2.2) holds, and hence, the L^p-approximation problems for F_d are weakly tractable for all $1 \le p < \infty$ under the absolute error criterion. Furthermore, for the normalized error criterion, we have

$$CRI_d = \sup_{f \in F_d} \|f\|_p = 2^{d/p} > 1.$$

Thus, (2.1) implies the weak tractability for the normalized error criterion. This completes the proof of Theorem 3.1. $\square$

Remark 3.2. Theorem 3.1 shows that the L^p-approximation problems for the class F_d given in (1.1) are weakly tractable for all $1 \le p < \infty$, and the weak tractability follows from the properties of the Smolyak algorithm. At the moment, however, we do not know whether or not the approximation problem in L_∞-norm is weakly tractable. It is worthwhile to point out that very recently Vybíral [15] found some new analytic function classes which are even quasi-polynomially tractable, (see [3] for this stronger notion of tractability).

Remark 3.3. In 2000, the authors of [1] considered the following function classes,

$$F_d^k = \{f : [-1,1]^d \to \mathbb{R} | D^\alpha f \text{ continuous and } \|D^\alpha f\|_\infty \leq 1 \text{ if } \alpha_i \leq k \text{ for all } i\},$$

and proved that (see [1, Theorem 8])

$$\sup_{f \in F_d^k} \|f - A(q,d)(f)\|_\infty \leq C_{d,k}(N_d(q-d))^{-k}(\ln N_d(q-d))^{(k+2)(d-1)+1}, \tag{3.14}$$

where $C_{d,k}$ is a positive constant depending only on d and k. We point out here that in the case of $1 \leq p < \infty$, we can obtain an estimate better than (3.14):

$$\sup_{f \in F_d^k} \|f - A(q,d)(f)\|_p \leq C_{d,k,p}(N_d(q-d))^{-k}(\ln N_d(q-d))^{(k+1)(d-1)}. \tag{3.15}$$

In fact, using (2.13) and the fact that

$$U^i(f,x) = f(x)$$

for all polynomials f of degree at most $m_i - 1$, we obtain

$$\|f - U^i(f)\|_p \leq E_{m_i-1}(f) \cdot (1 + C_p). \tag{3.16}$$

Here $E_n(f)$ is the error of the L^∞-best approximation by polynomials with degree at most n. From the well known Jackson estimate we know that for $f \in F_1^k$, we have

$$E_n(f) \leq C_{1,k} \cdot n^{-k}. \tag{3.17}$$

By (3.16) and (3.17) we obtain

$$\sup_{f \in F_1^k} \|f - U^i f\|_p \leq C_{1,k,p} m_i^{-k}. \tag{3.18}$$

Using (3.18) and the proof of [16, Lemma 2] as well as (2.10) we obtain (3.15), as claimed.

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