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Corrigendum

Corrigendum to the papers on Exceptional orthogonal polynomials: J. Approx. Theory 182 (2014) 29–58, 184 (2014) 176–208 and 214 (2017) 9–48

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Abstract

We complete a gap in the proof that exceptional polynomials are complete orthogonal systems in the associated Hilbert spaces.

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1. Introduction and result

In my paper [1], it is established that the exceptional Hermite polynomials are complete orthogonal systems in the associated Hilbert spaces (Theorem 6.4). However, there is a gap in the proof of Theorem 6.4 (actually the gap is already in Proposition 5.8 of [5] from where I took the completeness result). Similar incomplete proofs are in the papers [2] and [3] for the completeness of exceptional Laguerre and Jacobi polynomials in the associated Hilbert spaces.

The purpose of this note is to complete the gap in those proofs. Since the proofs for the Laguerre and Jacobi cases proceed in the same way that the proof for the exceptional Hermite

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polynomials, we only consider here the Hermite case. The sketch of the incomplete proof mentioned above is the following (see Proposition 5.8 of [5] for details).

Let F be a set of positive integers, $F = \{f_1, \ldots, f_k\}$, $f_i < f_j$ when $i < j$, and write

$$\sigma_F = \{u_F, u_F + 1, u_F + 2, \ldots\} \setminus \{u_F + f, f \in F\}$$

where $u_F = \sum_{f \in F} f - \binom{k+1}{2}$ and k is the number of elements of F. We say that F is admissible (see Definition 4.2 of [1]) if

$$\prod_{f \in F}(x - f) \geq 0, \quad x \in \mathbb{N}.$$

For an admissible set F, let H_n^F, $n \in \sigma_F$, be the nth associated exceptional Hermite polynomial defined by

$$H_n^F(x) = \begin{vmatrix} H_{n-u_F}(x) & H'_{n-u_F}(x) & \cdots & H^{(k)}_{n-u_F}(x) \\ H_{f_1}(x) & H'_{f_1}(x) & \cdots & H^{(k)}_{f_1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H^{(k)}_{f_k}(x) \end{vmatrix}$$

(see (5.1) of [1]). They are orthogonal in the real line with respect to the positive weight

$$w(x) = \frac{e^{-x^2}}{\Omega_F^2(x)},$$

where Ω_F is the Wronskian defined by

$$\Omega_F(x) = \begin{vmatrix} H_{f_1}(x) & H'_{f_1}(x) & \cdots & H^{(k-1)}_{f_1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_k}(x) & H'_{f_k}(x) & \cdots & H^{(k-1)}_{f_k}(x) \end{vmatrix}.$$

The exceptional Hermite polynomials are also eigenfunctions of a second order differential operator

$$D_F = -\partial^2 + h_1(x)\partial + h_0(x),$$

where

$$h_1(x) = 2\left(x + \frac{\Omega'_F(x)}{\Omega_F(x)}\right),$$
$$h_0(x) = 2\left(k + u_F - x\frac{\Omega'_F(x)}{\Omega_F(x)}\right) - \frac{\Omega''_F(x)}{\Omega_F(x)},$$

and $D_F(H_n^F) = 2nH_n^F$, $n \in \sigma_F$ (see (5.11) and (5.12) of [1]).

Write $\mathbb{A}$ for the linear span of H_n^F, $n \in \sigma_F$, in the space of polynomials $\mathbb{P}$, and

$$\mathbb{B} = \{p \in \mathbb{P} : D_F(p) \in \mathbb{P}\},$$
$$\mathbb{X} = \{\Omega_F^2 p : p \in \mathbb{P}\}.$$

It is correctly proved in Proposition 5.8 of [5] that $\mathbb{X} \subset \mathbb{B}$ and that $\mathbb{X}$ is dense in $L^2(\omega(x)dx)$. And hence $\mathbb{B}$ is also dense in $L^2(\omega(x)dx)$. The completeness of the exceptional Hermite polynomials would follow from the identity $\mathbb{A} = \mathbb{B}$. The identity $\mathbb{A} = \mathbb{B}$ is proved in Proposition 5.4 of [5], but the proof is incomplete. Actually, the result is wrong if we do not assume that F is admissible. Indeed, consider the finite set $F = \{1, 3\}$ (which it is not admissible). It is easy to check that $u_F = 1$ and $\sigma_F = \{1, 3, 5, 6, 7, 8, 9 \cdots\}$, so there is neither polynomial of degree 0 nor 2 in the sequence H_n^F, $n \in \sigma_F$, and hence $x^2 \notin \mathbb{A}$; but since

$$h_1(x) = 2x + 6/x, \quad h_0(x) = -6/x^2,$$

we have $x^2 \in \mathbb{B}$, and hence $\mathbb{B} \not\subset \mathbb{A}$ (the other inclusion $\mathbb{A} \subset \mathbb{B}$ always holds). If F is admissible, I guess that $\mathbb{A} = \mathbb{B}$ is true but the proof given in Proposition 5.4 of [5] is incomplete since the admissibility of F is not used (see [4] for a further study of the linear spaces $\mathbb{A}$, $\mathbb{B}$ (and other related spaces), in the more general setting of sequences of polynomials which are eigenfunctions of second order differential operators). Similar comments can be said for the identity $\mathbb{A} = \mathbb{B}$ for the Laguerre exceptional polynomials established in Lemma 5.3 of [2].

However, the completeness of the exceptional Hermite polynomials will follow if we prove that $\mathbb{X} \subset \mathbb{A}$. We prove this inclusion in the following lemma (as mention before, similar lemmas can be proved for exceptional Laguerre and Jacobi polynomials using the same approach). The authors of [5] have also completed the gap in the proof of the completeness for the exceptional Hermite polynomials. They have also proved that $\mathbb{X} \subset \mathbb{A}$ but using a different approach than the one used here. Indeed, for the Hermite case, they have characterized the subspace $\mathbb{A}$ in a closed form (see Proposition 25 of [6]) from where the inclusion $\mathbb{X} \subset \mathbb{A}$ follows easily.

Lemma 1.1. *Assume that F is an admissible set of positive integers. Then $\mathbb{X} \subset \mathbb{A}$.*

Proof. We prove it using the approach in [1], where each family of exceptional Hermite polynomials is constructed by taking limit in a family of exceptional Charlier polynomials.

Consider then the family $(c_n^{a;F})_{n\in\sigma_F}$ of exceptional Charlier polynomials defined by

$$c_n^{a;F}(x) = \begin{vmatrix} c^a_{n-u_F}(x) & c^a_{n-u_F-1}(x) & \cdots & c^a_{n-u_F-k}(x) \\ c^a_{f_1}(x) & c^a_{f_1-1}(x) & \cdots & c^a_{f_1-k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c^a_{f_k}(x) & c^a_{f_k-1}(x) & \cdots & c^a_{f_k-k}(x) \end{vmatrix} \tag{1.1}$$

(see (3.3) of [1]). For each $n \in \sigma_F$, we have that $c_n^{a;F}$ is a polynomial of degree n.

They are orthogonal with respect to the positive discrete weight

$$\omega_{a;F} = \sum_{x=0}^{\infty} \frac{a^x}{x!\Omega_F^a(x)\Omega_F^a(x+1)}\delta_x, \tag{1.2}$$

where

$$\Omega_F^a(x) = |c^a_{f_i}(x+j-1)|_{i,j=1}^k.$$

As mentioned in [1], p. 43, Ω_F^a is a polynomial of degree $u_F + k$. In Theorem 4.5 of [1] it is proved that $(c_n^{a;F})_{n\in\sigma_F}$ is a complete orthogonal system in the Hilbert space $L^2(\omega_{a;F})$.

The proof of the lemma is based in the following claim, which we will prove later.

Claim. For a nonnegative integer $l \geq 0$ there exists a finite set of nonnegative integers $S \subset \sigma_F$, such that for any polynomial $p \in \mathbb{P}$ with $\deg p \leq l$, there exist numbers $b_j^a \in \mathbb{R}$, $j \in S$ (depending on p), such that

$$\Omega_F^a(x)\Omega_F^a(x+1)p(x) = \sum_{j\in S} b_j^a c_j^{a;F}(x).$$

Take now a polynomial $p \in \mathbb{P}$ and write $l = \deg p$. For $a > 0$, define the polynomial $p_a(x) = p((x-1)/\sqrt{2a})$. Since $\deg p_a = l$, we have that there exists a finite set of nonnegative integers $S \subset \sigma_F$ (which does not depend on a), and numbers $b_j^a \in \mathbb{R}$, $j \in S$, such that

$$\Omega_F^a(x)\Omega_F^a(x+1)p_a(x) = \sum_{j\in S} b_j^a c_j^{a;F}(x).$$

Since $\deg(\Omega_F^a(x)\Omega_F^a(x+1)p_a(x)) = 2u_F+2k+l$, we can assume that $s = \max S = 2u_F+2k+l$.

Substituting x by $x_a = \sqrt{2a}x + a$, we have

$$\Omega_F^a(x_a)\Omega_F^a(x_a+1)p(x) = \sum_{j\in S} b_j^a c_j^{a;F}(x_a).$$

This identity can be rewritten as

$$\left(\frac{2}{a}\right)^{u_F+k} \Omega_F^a(x_a)\Omega_F^a(x_a+1)p(x) = \sum_{j\in S} \tilde{b}_j^a \left(\frac{2}{a}\right)^{j/2} c_j^{a;F}(x_a), \tag{1.3}$$

where $\tilde{b}_j^a = \left(\frac{2}{a}\right)^{u_F+k-j/2} b_j^a$.

Write $q_a(x) = \left(\frac{2}{a}\right)^{u_F+k} \Omega_F^a(x_a)\Omega_F^a(x_a+1)p(x)$. Let α_a and β_a be the leading coefficients of q_a and $\left(\frac{2}{a}\right)^{s/2} c_s^{a;F}(x_a)$, respectively, so that (see (1.3))

$$\tilde{b}_s^a = \frac{\alpha_a}{\beta_a}. \tag{1.4}$$

Taking limit when $a \to +\infty$, we have (see (5.2) and (5.13) of [1])

$$\lim_{a\to+\infty} \left(\frac{2}{a}\right)^{(u_F+k)/2} \Omega_F^a(x_a) = c\Omega_F(x),$$

$$\lim_{a\to+\infty} \left(\frac{2}{a}\right)^{(u_F+k)/2} \Omega_F^a(x_a+1) = c\Omega_F(x),$$

$$\lim_{a\to+\infty} \left(\frac{2}{a}\right)^{j/2} c_j^{a;F}(x_a) = c_j H_j^F(x),$$

for certain constants $c, c_j \neq 0$.

Since $\deg \Omega_F = \deg \Omega_F^a$ and $\deg c_j^{a;F} = \deg H_j^F$, from the above limits, it follows that both α_a, β_a converge to non-zero limits α and β, respectively, when a goes to $+\infty$. And so, we can then conclude from (1.4) that $\tilde{b}_s^a$ also converges to a non-null limit when a goes to $+\infty$.

Rewriting (1.3) as

$$\left(\frac{2}{a}\right)^{u_F+k} \Omega_F^a(x_a)\Omega_F^a(x_a+1)p(x) - \tilde{b}_s^a \left(\frac{2}{a}\right)^{s/2} c_s^{a;F}(x_a) = \sum_{j\in S\setminus\{s\}} \tilde{b}_j^a \left(\frac{2}{a}\right)^{j/2} c_j^{a;F}(x_a),$$

we can prove that each coefficient $\tilde{b}_j^a$, $j \in S$, converges to a certain limit b_j when a goes to $+\infty$. Hence by taking limit $a \to +\infty$, the identity (1.3) gives

$$\Omega_F^2(x)p(x) = \sum_{j\in S} b_j H_j^F(x).$$

This proves that $\mathbb{X} \subset \mathbb{A}$.

We finally prove the claim. According to Theorem 4.5 of [1], the polynomials $(c_n^{a;F})_{n\in\sigma_F}$ is a complete system in the Hilbert space $L^2(\omega_{a;F})$. It is enough to prove that for $n > 2u_F + l$

$$\langle \Omega_F^a(x)\Omega_F^a(x+1)p(x), c_n^{a;F}(x)\rangle_{\omega_{a;F}} = 0.$$

From the definition of the polynomial $c_n^{a;F}$ (see (1.1)) and using the three term recurrence relation for the Charlier polynomials, it follows that $c_n^{a;F}$ is a linear combination of the Charlier polynomials c_j^a, $j = n, n-1, \dots, n-2u_F$. Hence, using the definition of the measure $\omega_{a;F}$ (see (1.2)), we can conclude that

$$\langle \Omega_F^a(x)\Omega_F^a(x+1)p(x), c_n^{a;F}(x)\rangle_{\omega_{a;F}} = \sum_{x=0}^{\infty} \frac{a^x}{x!} p(x) c_n^{a;F}(x) = 0,$$

for $n - 2u_F > l \geq \deg p$. $\square$

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