

Full length article

Sampling numbers of periodic Sobolev spaces with a Gaussian measure in the average case setting[☆]

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Abstract

In this paper, we investigate p -average sampling numbers of a periodic Sobolev space W_2^r with a Gaussian measure in the L_q metric for $1 \leq q \leq \infty$ and $0 < p < \infty$, and obtain their asymptotic orders. Moreover, we show that in the average case setting, the operators I_n , which are the Lagrange interpolating operators, are asymptotically optimal in the L_q metric for all $1 \leq q \leq \infty$. It is interesting to note that in the worst case setting, I_n are not asymptotically optimal algorithms in the L_q metric for $q = 1$ or ∞ .

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1. Introduction

This paper is devoted to studying approximation of periodic functions on a Sobolev space with a Gaussian measure in the average case setting, using only their function values (standard information). Let F be a Banach space of functions defined on D which can be continuously embedded in $C(D)$, G be a normed linear space with norm $\|\cdot\|_G$, and let γ be a centered Gaussian probability measure on F . We want to approximate functions f from F using finitely

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many arbitrary function values $f(x)$ for some $x \in D$ or continuous linear functionals $L \in F^*$, where F^* is the space of all continuous linear functionals on F . We denote by Λ^{std} and Λ^{all} the above two classes of information evaluations. We consider algorithms that use linear functionals either from the class Λ^{std} or from the class Λ^{all} . It is well known that, in the average case setting with the average being with respect to a centered Gaussian measure, adaptive choice of the above information evaluations as well as nonlinear algorithms do not essentially help; see [17]. Hence, we can restrict our attention to linear algorithms, i.e., algorithms of the form

$$A_n(f) := \sum_{j=1}^n L_j(f) g_j, \quad g_j \in G, \quad L_j \in \Lambda^{\text{all}} \text{ or } \Lambda^{\text{std}}.$$

For $0 < p < \infty$, the p -average error of an algorithm A_n in G with respect to the measure γ is defined by

$$e^{\text{avg}}(A_n, \gamma, G)_p := \left(\int_F \|f - A_n(f)\|_G^p \gamma(df) \right)^{\frac{1}{p}}.$$

We define the p -average approximation numbers (the p -average linear widths) and the p -average sampling numbers of F in G by

$$\lambda_n^{(a)}(F, \gamma, G)_p := \inf_{A_n \text{ with } L_j \in \Lambda^{\text{all}}} e^{\text{avg}}(A_n, \gamma, G)_p,$$

and

$$g_n^{(a)}(F, \gamma, G)_p := \inf_{A_n \text{ with } L_j \in \Lambda^{\text{std}}} e^{\text{avg}}(A_n, \gamma, G)_p,$$

respectively. It follows from the relation $\Lambda^{\text{std}} \subset \Lambda^{\text{all}} = F^*$ that

$$\lambda_n^{(a)}(F, \gamma, G)_p \leq g_n^{(a)}(F, \gamma, G)_p. \quad (1.1)$$

There are a few papers devoted to studying average linear widths (average approximation numbers) in the average case setting; see for example, [3–6, 9–11, 13–15, 18–20]. However, much less attention has been devoted to average sampling numbers; for exceptions see, e.g., [8, 15]. In [8, 15], among others, the authors obtained upper bounds for average sampling numbers on the Wiener space in the uniform norm and on the weighted Korobov spaces in the L_2 metric, respectively. In [7, 21] the authors discussed relations between average approximation and sampling numbers in the L_2 metric. More information about average case setting results can be found in [13, 17].

In the paper, we shall discuss p -average sampling numbers in the L_q metric for $1 \leq q \leq \infty$ and $0 < p < \infty$ on a periodic Sobolev space with a Gaussian measure, and obtain the asymptotic orders. We show that in the average case setting, the Lagrange interpolation operators and the worst-case asymptotically optimal operators are also asymptotically optimal for all $1 \leq q \leq \infty$ and $0 < p < \infty$, and the powers of the average approximation and sampling numbers are same.

2. Main results

Let $L_q \equiv L_q([0, 2\pi])$ ($1 \leq q \leq \infty$) be the function space of 2π -periodic functions $x(t)$ with the usual norm $\|\cdot\|_q$. Specially, $L_2 \equiv L_2([0, 2\pi])$ is a Hilbert space with the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt, \quad f, g \in L_2.$$

For any $f \in L_2$, the Fourier series and the L_2 -norm of f are

$$f(t) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \exp(ikt) := \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(t), \quad \text{and} \quad \|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2,$$

where $\hat{f}(k) = \langle f, e_k \rangle$, $e_k(t) = \exp(ikt)$. For any $r \in \mathbb{R}$, let

$$f^{(r)}(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (ik)^r \hat{f}(k) e_k(t)$$

be the r -order derivative in the sense of Weyl, where $(ik)^r = |k|^r \exp\left(\frac{ir\pi}{2} \operatorname{sgn} k\right)$. The Sobolev space W_2^r is defined as

$$W_2^r = \left\{ f \in L_2 : f^{(r)} \in L_2, \int_0^{2\pi} f(t) dt = 0 \right\}$$

with norm $\|f\|_{W_2^r}^2 = \langle f^{(r)}, f^{(r)} \rangle$. The space W_2^r is a Hilbert space with inner product

$$\langle f, g \rangle_r = \langle f^{(r)}, g^{(r)} \rangle.$$

We equip W_2^r with a Gaussian measure μ whose mean is zero and correlation operator \mathcal{C}_μ has eigenfunctions e_k and eigenvalues

$$\lambda_k = |k|^{-s}, \quad s > 1.$$

That is, $\mathcal{C}_\mu e_k = \lambda_k e_k$. From the properties of Gaussian measures (see [2, pp. 48–49]), we know that

$$\langle \mathcal{C}_\mu f, g \rangle_r = \int_{W_2^r} \langle f, v \rangle_r \overline{\langle g, v \rangle_r} \mu(dv).$$

Throughout the paper, we assume that μ is the above Gaussian measure on W_2^r , $r > 1/2$, and $s > 1$.

Denote by K_μ the covariance function of the measure μ , i.e.,

$$K_\mu(t, t') := \int_{W_2^r} f(t) \overline{f(t')} \mu(df), \quad \forall t, t' \in [0, 2\pi]. \quad (2.1)$$

Let

$$A_n(f, t) = \sum_{j=1}^n f(t_j) g_j(t)$$

be a general linear algorithm using function values, where $g_j \in L_q$, $t_j \in [0, 2\pi]$, $j = 1, \dots, n$. First we give the q -average error of the algorithm A_n in L_q , with respect to the measure μ .

Theorem 2.1. *Let $1 \leq q < \infty$. Then*

$$e^{\text{avg}}(A_n, \mu, L_q)_q^q = \int_{W_2^r} \|f - A_n(f)\|_q^q \mu(df) = \frac{C(q)}{2\pi} \int_0^{2\pi} |I(t)|^{\frac{q}{2}} dt,$$

where

$$C(q) = \pi^{-\frac{1}{2}} 2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right),$$

$$I(t) = K_\mu(t, t) - \sum_{j=1}^n K_\mu(t_j, t)(g_j(t) + \overline{g_j(t)}) + \sum_{j=1}^n \sum_{i=1}^n K_\mu(t_i, t_j) g_i(t) \overline{g_j(t)},$$

and

$$K_\mu(t, t') = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e_k(t - t')}{|k|^{2r+s}} = 2 \sum_{k=1}^{\infty} \frac{\cos k(t - t')}{k^{2r+s}}, \quad t, t' \in [0, 2\pi].$$

Next we introduce the following two operators I_n and R_n . Let D_n and V_n be the Dirichlet kernel and the Vallée-Poussin kernel, i.e.,

$$D_n(t) = \sum_{|k| \leq n} e_k(t) = 1 + 2 \sum_{k=1}^n \cos kt = \frac{\sin(n + 1/2)t}{\sin t/2}, \quad (2.2)$$

and

$$V_n(t) = \frac{1}{n} \sum_{k=n}^{2n-1} D_k(t) = 1 + 2 \sum_{k=1}^n \cos kt + 2 \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \cos kt. \quad (2.3)$$

The operators I_n, R_n are defined by

$$I_n(f, t) := \frac{1}{2n+1} \sum_{l=0}^{2n} f(t^l) D_n(t - t^l), \quad t^l = \frac{2l\pi}{2n+1}, \quad (2.4)$$

and

$$R_n(f, t) := \frac{1}{4n} \sum_{l=1}^{4n} f(t(l)) V_n(t - t(l)), \quad t(l) = \frac{l\pi}{2n}. \quad (2.5)$$

The operator I_n is the Lagrange interpolating operator, i.e., $I_n(f, t^l) = f(t^l)$, $l = 0, 1, \dots, 2n$ and $I_n(f)$ is a trigonometric polynomials of order $\leq n$. Note that if f is a trigonometric polynomials of order $\leq n$, then (see [16, (6.11) and (6.12), pp. 86–87])

$$I_n(f) = R_n(f) = f. \quad (2.6)$$

In the worst case setting, the operators R_n are asymptotically optimal linear algorithms in the L_q metric for $1 \leq q \leq \infty$, and the operators I_n are asymptotically optimal linear algorithms in the L_q metric only for $1 < q < \infty$; see [16, pp. 85–88].

In this paper, we want to discuss the approximation error of the operators I_n and R_n in the average case setting. We obtain the asymptotic orders of the p -average sampling numbers $g_n^{(a)}(W_2^r, \mu, L_q)_p$, $1 \leq q \leq \infty$, $0 < p < \infty$, and show that in the average case setting with the average being with respect to the measure μ , the operators I_n and R_n are also asymptotically optimal linear algorithms in the L_q metric for $1 \leq q \leq \infty$. The main results of this paper are as follows:

Theorem 2.2. *Let $0 < p < \infty$. Then*

$$\begin{aligned} g_n^{(a)}(W_2^r, \mu, L_q)_p &\asymp e^{\text{avg}}(I_n, \mu, L_q)_p \asymp e^{\text{avg}}(R_n, \mu, L_q)_p \\ &\asymp \begin{cases} n^{-(r+\frac{s-1}{2})}, & 1 \leq q < \infty, \\ n^{-(r+\frac{s-1}{2})} \ln^{\frac{1}{2}} n, & q = \infty, \end{cases} \end{aligned} \quad (2.7)$$

where $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $B(n) \ll A(n)$, and $A(n) \ll B(n)$ means that there exists a constant $c > 0$ independent of n such that $A(n) \leq cB(n)$.

Remark 2.3. From [5,6,9], we know that for $1 \leq q \leq \infty$, $0 < p < \infty$,

$$\lambda_n^{(a)}(W_2^r, \mu, L_q)_p \asymp \begin{cases} n^{-(r+\frac{s-1}{2})}, & 1 \leq q < \infty, \\ n^{-(r+\frac{s-1}{2})} \ln^{\frac{1}{2}} n, & q = \infty. \end{cases} \quad (2.8)$$

Comparing (2.7) with (2.8), we know that

$$g_n^{(a)}(W_2^r, \mu, L_q)_p \asymp \lambda_n^{(a)}(W_2^r, \mu, L_q)_p,$$

which means that the powers of approximation numbers and sampling numbers are same. Our results give an example to the open problem 8 of [12] for the case $q \neq 2$.

Remark 2.4. From (2.7) we know that the operators I_n and R_n are asymptotically optimal for $g_n^{(a)}(W_2^r, \mu, L_q)_p$ for all $1 \leq q \leq \infty$ and $0 < p < \infty$. It is interesting to note that in the worst case setting, I_n are not asymptotically optimal algorithms in the L_1 or L_∞ metric.

3. Proof of Theorem 1

Proof of Theorem 2.1. For a fixed $t \in [0, 2\pi]$, set

$$L(f, t) := f(t) - A_n(f, t).$$

Then $L(f, t)$ is a bounded linear functional on W_2^r . Since the measure μ is symmetric Gaussian on W_2^r , we know that $L(f, t)$, as a random variable on the measurable space (W_2^r, μ) , obeys the normal distribution $N(0, R^2(t))$, where

$$\begin{aligned} R^2(t) &= \int_{W_2^r} |L(f, t)|^2 \mu(df) = \int_{W_2^r} (f(t) - A_n(f, t)) \overline{(f(t) - A_n(f, t))} \mu(df) \\ &= \int_{W_2^r} f(t) \overline{f(t)} \mu(df) - \sum_{j=1}^n \int_{W_2^r} f(t) \overline{f(t_j)} \overline{g_j(t)} \mu(df) \\ &\quad - \sum_{j=1}^n \int_{W_2^r} f(t_j) \overline{f(t)} g_j(t) \mu(df) + \sum_{i=1}^n \sum_{j=1}^n \int_{W_2^r} f(t_i) \overline{f(t_j)} g_i(t) \overline{g_j(t)} \mu(df) \\ &= K_\mu(t, t) - \sum_{j=1}^n (K_\mu(t_j, t) g_j(t) + K_\mu(t, t_j) \overline{g_j(t)}) + \sum_{j=1}^n \sum_{i=1}^n K_\mu(t_i, t_j) g_i(t) \overline{g_j(t)} \\ &= I(t). \end{aligned}$$

Then $L(f, t)/R(t)$, as a random variable on the measurable space (W_2^r, μ) , obeys the normal distribution $N(0, 1)$. This means that

$$\begin{aligned} \int_{W_2^r} |f(t) - A_n(f, t)|^q \mu(df) &= |R(t)|^q \int_{W_2^r} \left| \frac{L(f, t)}{R(t)} \right|^q \mu(df) \\ &= |R(t)|^q \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t|^q e^{-\frac{t^2}{2}} dt \\ &= \pi^{-\frac{1}{2}} 2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right) |R^2(t)|^{\frac{q}{2}} = C(q) |I(t)|^{\frac{q}{2}}. \end{aligned}$$

Using the Fubini theorem, we get

$$\begin{aligned} \int_{W_2^r} \|f - A_n(f)\|_q^q \mu(df) &= \frac{1}{2\pi} \int_{W_2^r} \int_0^{2\pi} |f(t) - A_n(f, t)|^q dt \mu(df) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{W_2^r} |f(t) - A_n(f, t)|^q \mu(df) dt \\ &= \frac{C(q)}{2\pi} \int_0^{2\pi} |I(t)|^{\frac{q}{2}} dt. \end{aligned}$$

Finally, we compute $K(t, t')$. Since for $f \in W_2^r$,

$$\hat{f}(k) = \langle f, e_k \rangle = \langle f^{(-r)}, e_k^{(-r)} \rangle_r = |k|^{-2r} \langle f, e_k \rangle_r,$$

we get that

$$\begin{aligned} \int_{W_2^r} \hat{f}(k) \overline{\hat{f}(j)} \mu(df) &= \int_{W_2^r} |k|^{-2r} |j|^{-2r} \langle f, e_k \rangle_r \overline{\langle f, e_j \rangle_r} \mu(df) \\ &= |k|^{-2r} |j|^{-2r} \langle C_\mu e_j, e_k \rangle_r = |j|^{-2r-s} |k|^{-2r} \langle e_j, e_k \rangle_r \\ &= |j|^{-r-s} |k|^{-r} \langle e_j, e_k \rangle = |k|^{-2r-s} \delta_{k,j}, \end{aligned}$$

where $\delta_{k,j} = \begin{cases} 0, & k \neq j, \\ 1, & k = j. \end{cases}$ It follows that

$$\begin{aligned} K_\mu(t, t') &= \int_{W_2^r} f(t) \overline{f(t')} \mu(df) \\ &= \int_{W_2^r} \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) e_k(t) \sum_{j \in \mathbb{Z} \setminus \{0\}} \overline{\hat{f}(j) e_j(t')} \mu(df) \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{j \in \mathbb{Z} \setminus \{0\}} e_k(t) \overline{e_j(t')} \int_{W_2^r} \hat{f}(k) \overline{\hat{f}(j)} \mu(df) \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{j \in \mathbb{Z} \setminus \{0\}} e_k(t) e_j(-t') |k|^{-2r-s} \delta_{k,j} \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-2r-s} e_k(t - t') = 2 \sum_{k=1}^{\infty} \frac{\cos k(t - t')}{k^{2r+s}}. \end{aligned}$$

This completes the proof. \square

In order to prove Theorem 2.2, we remark that

$$g_{2n+1}^{(a)}(W_2^r, \mu, L_q)_p \leq e^{\text{avg}}(I_n, \mu, L_q)_p$$

and

$$g_{4n}^{(a)}(W_2^r, \mu, L_q)_p \leq e^{\text{avg}}(R_n, \mu, L_q)_p.$$

Due to (1.1) and (2.8), it suffices to prove the upper bounds of $e^{\text{avg}}(I_n, \mu, L_q)_p$ and $e^{\text{avg}}(R_n, \mu, L_q)_p$ for $1 \leq q \leq \infty$ and $0 < p < \infty$. We only give the proof of the upper bounds of

$e^{\text{avg}}(I_n, \mu, L_q)_p$, since the proof concerning $e^{\text{avg}}(R_n, \mu, L_q)_p$ is completely similar and we omit it.

Let j be a nonnegative integer, and let $f \in L_1$. We define

$$A_j(f, x) = \mathcal{A}_j * f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x') A_j(x - x') dx',$$

where

$$\mathcal{A}_0(x) = 1, \quad \mathcal{A}_1(x) = V_1(x) - 1, \quad \mathcal{A}_j(x) = V_{2j-1}(x) - V_{2j-2}(x), \quad j \geq 2,$$

and V_k is given by (2.3).

In order to prove the upper bounds of $e^{\text{avg}}(I_n, \mu, L_q)_p$, we need the following lemmas.

Lemma 3.1. *Let $1 \leq q < \infty$ and $j \geq 2$. Then*

$$Q_{j,n} := \int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_q^q \mu(df) = \frac{C(q)}{2\pi} \int_0^{2\pi} |H_{j,n}(t)|^{\frac{q}{2}} dt, \quad (3.1)$$

where the operator I_n is given by (2.4), $C(q) = \pi^{-\frac{1}{2}} 2^{\frac{q}{2}} \Gamma\left(\frac{q+1}{2}\right)$, and

$$\begin{aligned} H_{j,n}(t) &= F_j(0) - \frac{2}{2n+1} \sum_{l=0}^{2n} D_n(t - t^l) F_j(t - t^l) \\ &\quad + \frac{1}{(2n+1)^2} \sum_{l=0}^{2n} \sum_{h=0}^{2n} D_n(t - t^l) D_n(t - t^h) F_j(t^l - t^h), \end{aligned} \quad (3.2)$$

and

$$F_j(t) = 2 \sum_{k=2^{j-2}+1}^{2^j-1} \left(\frac{k-2^{j-2}}{2^{j-2}} \right)^2 \frac{\cos kt}{k^{2r+s}} + 2 \sum_{k=2^{j-1}+1}^{2^j-1} \left(\frac{2^j-k}{2^{j-1}} \right)^2 \frac{\cos kt}{k^{2r+s}}. \quad (3.3)$$

Proof. The proof is similar to the proof of Theorem 2.1 with $g_j(t) = \frac{D_n(t-t^j)}{2n+1}$. The only difference is that $K_\mu(t, t')$ is replaced by

$$K_j(t, t') := \int_{W_2^r} A_j(f, t) \overline{A_j(f, t')} \mu(df).$$

We can compute $K_j(t, t')$ in the same way and get $K_j(t, t') = F_j(t - t')$, where F_j is given by (3.3). \square

Lemma 3.2. *For $j \geq 2$ and $t \in [0, 2\pi]$, we have*

$$|F_j(t)| \ll 2^{-(2r+s-1)j} (1 + 2^j |t|)^{-2}. \quad (3.4)$$

Proof. For a sequence $\{a_k\}_{k=0}^\infty$, we denote

$$\Delta a_k = a_k - a_{k+1}, \quad \Delta^2 a_k = \Delta(\Delta a_k) = a_k - 2a_{k+1} + a_{k+2}.$$

Using the Abel transform twice, we get (see [16, (2.1), p. 34]),

$$\begin{aligned} a_0 + 2 \sum_{k=1}^m a_k \cos kt &= \sum_{k=0}^m D_k(t) \triangle a_k = \sum_{k=0}^m \left(\sum_{i=0}^k D_i(t) \right) \triangle^2 a_k \\ &= \sum_{k=0}^m (k+1) \mathcal{K}_k(t) \triangle^2 a_k, \end{aligned}$$

where $\mathcal{K}_k(t)$ is Fejér kernel of order k , i.e., $\mathcal{K}_{k-1}(t) = \frac{1}{k} \sum_{i=0}^{k-1} D_i(t) = \frac{1}{k} \left(\frac{\sin kt/2}{\sin t/2} \right)^2$. We have

$$|\mathcal{K}_{k-1}(t)| \ll k(1+k|t|)^{-2}.$$

From (3.3), we can represent F_j as

$$F_j(t) = 2 \sum_{k=0}^{2^j-1} a_k \cos kt,$$

with

$$a_k = \begin{cases} 0, & 0 < k \leq 2^{j-2}, \\ \left(\frac{k - 2^{j-2}}{2^{j-2}} \right)^2 k^{-(2r+s)}, & 2^{j-2} + 1 \leq k \leq 2^{j-1}, \\ \left(\frac{2^j - k}{2^{j-1}} \right)^2 k^{-(2r+s)}, & 2^{j-1} + 1 \leq k \leq 2^j - 1. \end{cases}$$

It is easy to see that for $0 \leq k \leq 2^{j-2} - 2$,

$$|\triangle^2 a_k| = 0. \quad (3.5)$$

For $2^{j-2} + 1 \leq k \leq 2^{j-1} - 2$ and $2^{j-1} + 1 \leq k \leq 2^j - 3$, set

$$b_k = k^{-(2r+s)} \quad \text{and} \quad c_k = \begin{cases} ((k - 2^{j-2})/2^{j-2})^2, & 2^{j-2} + 1 \leq k \leq 2^{j-1}, \\ ((2^j - k)/2^{j-1})^2, & 2^{j-1} + 1 \leq k \leq 2^j - 1. \end{cases}$$

Hence,

$$|b_k| \ll 2^{-(2r+s)j}, \quad |\triangle b_k| \ll 2^{-(2r+s+1)j}, \quad |\triangle^2 b_k| \ll 2^{-(2r+s+2)j},$$

and

$$|c_k| \leq 1, \quad |\triangle c_k| \ll 2^{-j}, \quad |\triangle^2 c_k| \ll 2^{-2j}.$$

By the relation

$$\triangle^2(b_k c_k) = (\triangle^2 b_k) c_k + 2(\triangle b_{k+1})(\triangle c_k) + b_{k+2}(\triangle^2 c_k),$$

we have

$$|\triangle^2 a_k| = |\triangle^2(b_k c_k)| \ll 2^{-(2r+s+2)j}. \quad (3.6)$$

For the remaining values of k we get

$$|\triangle^2 a_k| \leq |\triangle a_k| + |\triangle a_{k+1}| \ll 2^{-(2r+s+1)j}. \quad (3.7)$$

Combining (3.5)–(3.7), we obtain

$$|F_j(t)| \leq \sum_{k=2^{j-2}-1}^{2^j-1} k |\mathcal{K}_k(t)| |\Delta^2 a_k| \ll 2^{-(2r+s-1)j} (1 + 2^j |t|)^{-2}.$$

This completes the proof. \square

For any given n , there is a unique v such that $2^{v-1} \leq n < 2^v$. It follows from (2.6) that for $j \leq v-1$, $H_{j,n}(t) = 0$ and

$$\mathcal{Q}_{j,n} = \int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_q^q \mu(df) = 0.$$

Now we estimate $H_{j,n}(t)$ for $j \geq v$.

Lemma 3.3. *Let $j \geq v$. Then we have*

$$|H_{j,n}(t)| \leq c 2^{-(2r+s-1)j}, \quad (3.8)$$

where c is a positive constant independent of t , n , and j .

Proof. From Lemma 3.1, we know that

$$\begin{aligned} |H_{j,n}(t)| &\leq |F_j(0)| + \frac{2}{2n+1} \sum_{l=0}^{2n} |D_n(t-t^l)| |F_j(t-t^l)| \\ &\quad + \frac{1}{(2n+1)^2} \sum_{l=0}^{2n} \sum_{h=0}^{2n} |D_n(t-t^l)| |D_n(t-t^h)| |F_j(t^l-t^h)| \\ &=: F_j(0) + 2H_{j,n,1}(t) + H_{j,n,2}(t). \end{aligned} \quad (3.9)$$

For any fixed $t \in [0, 2\pi)$, there exists a unique k , $1 \leq k \leq 2n$ such that $\frac{(k-1)\pi}{n} \leq t < \frac{k\pi}{n}$. We note that

$$|D_n(t-t^l)| \ll n(1+n|t-t^l|)^{-1} \ll n(1+|k-l|)^{-1}.$$

Since $j \geq v$, we get $2^j \gg n$. It follows from (3.4) that

$$\begin{aligned} H_{j,n,1}(t) &\ll \frac{1}{n} \sum_{l=0}^{2n} n(1+|k-l|)^{-1} 2^{-(2r+s-1)j} \left(1 + \frac{2^j}{n} |k-l|\right)^{-2} \\ &\ll 2^{-(2r+s-1)j} \sum_{l=0}^{2n} (1+|k-l|)^{-3} \\ &\leq 2^{-(2r+s-1)j} \sum_{m=1}^{2n} (1+m)^{-3} \sum_{l: |l-k|=m} 1 \\ &\ll 2^{-(2r+s-1)j}. \end{aligned} \quad (3.10)$$

Now we estimate $H_{j,n,2}(t)$. We have

$$\begin{aligned} H_{j,n,2}(t) &\ll \frac{1}{n^2} \sum_{l=0}^{2n} \sum_{h=0}^{2n} n(1+|k-l|)^{-1} n(1+|k-h|)^{-1} \\ &\quad \times 2^{-(2r+s-1)j} \left(1 + \frac{2^j}{n} |l-h|\right)^{-2} \\ &\ll 2^{-(2r+s-1)j} \sum_{l=0}^{2n} (1+|k-l|)^{-1} \sum_{h=0}^{2n} (1+|k-h|)^{-1} (1+|l-h|)^{-2}. \end{aligned}$$

We claim that

$$V(l, k) = \sum_{h=0}^{2n} (1+|k-h|)^{-1} (1+|l-h|)^{-2} \ll (1+|k-l|)^{-1}. \quad (3.11)$$

In fact, if $k = l$, (3.11) is obvious. Without loss of generality, we may assume $m = k - l > 0$. Then

$$\begin{aligned} V(l, k) &\leq \sum_{i=0}^{2n} (1+i)^{-2} ((1+|m+i|)^{-1} + (1+|m-i|)^{-1}) \\ &\leq (1+m)^{-1} \sum_{i=0}^{2n} (1+i)^{-2} \\ &\quad + \left(\sum_{i: |i-m| > m/2} + \sum_{i: |i-m| \leq m/2} \right) (1+i)^{-2} (1+|m-i|)^{-1} \\ &\ll (1+m)^{-1} + \sum_{i: |i-m| > m/2} (1+i)^{-2} (1+m)^{-1} + \sum_{i: |i-m| \leq m/2} (1+m)^{-2} \\ &\leq (1+m)^{-1}, \end{aligned}$$

proving (3.11). Therefore,

$$H_{j,n,2}(t) \ll 2^{-(2r+s-1)j} \sum_{l=0}^{2n} (1+|k-l|)^{-2} \ll 2^{-(2r+s-1)j}. \quad (3.12)$$

It then follows from (3.9), (3.3), (3.10) and (3.12) that

$$|H_{j,n}(t)| \leq |F_j(0)| + 2H_{j,n,1}(t) + H_{j,n,2}(t) \ll 2^{-(2r+s-1)j}.$$

The proof of Lemma 3.3 is finished. \square

Proof of Theorem 2.2. It suffices to prove the upper estimates of $e^{\text{avg}}(I_n, \mu, L_q)_p$ for $1 \leq q \leq \infty$ and $0 < p < \infty$. From [18, Corollary 1], we have for $1 \leq q \leq \infty$ and $0 < p < \infty$,

$$e^{\text{avg}}(I_n, \mu, L_q)_p \asymp e^{\text{avg}}(I_n, \mu, L_q)_1,$$

where the equivalent constants depend only on p .

Now we estimate $e^{\text{avg}}(I_n, \mu, L_q)_1$. We have

$$\begin{aligned} e^{\text{avg}}(I_n, \mu, L_q)_1 &= \int_{W_2^r} \left\| \sum_{j=v}^{\infty} (A_j(f) - I_n(A_j(f))) \right\|_q \mu(df) \\ &\ll \sum_{j=v}^{\infty} \int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_q \mu(df). \end{aligned} \quad (3.13)$$

If $1 \leq q < \infty$, then

$$\begin{aligned} e^{\text{avg}}(I_n, \mu, L_q)_1 &\leq \sum_{j=v}^{\infty} \left(\int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_q^q \mu(df) \right)^{\frac{1}{q}} \\ &= \sum_{j=v}^{\infty} \left(\frac{C(q)}{2\pi} \int_0^{2\pi} |H_{j,n}(t)|^{\frac{q}{2}} dt \right)^{\frac{1}{q}}, \end{aligned}$$

where $H_{j,n}(t)$ is given as in Lemma 3.1. It follows from Lemma 3.3 that

$$e^{\text{avg}}(I_n, \mu, L_q)_1 \ll \sum_{j=v}^{\infty} 2^{-(r+\frac{s-1}{2})j} \ll n^{-(r+\frac{s-1}{2})}.$$

If $q = \infty$, we remark that $A_j(f) - I_n(A_j(f))$ ($j \geq v$) is a trigonometric polynomial of degree at most 2^j . From the Nikolskii inequality (see [16, p. 38]), we obtain for any $1 \leq p_j < \infty$,

$$\|A_j(f) - I_n(A_j(f))\|_{\infty} \leq c 2^{\frac{j}{p_j}} \|A_j(f) - I_n(A_j(f))\|_{p_j},$$

where c is positive constant independent of p_j and j . Applying the Hölder inequality, we get

$$\begin{aligned} \int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_{\infty} \mu(df) &\leq c 2^{\frac{j}{p_j}} \int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_{p_j} \mu(df) \\ &\leq c 2^{\frac{j}{p_j}} \left(\int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_{p_j}^{p_j} \mu(df) \right)^{\frac{1}{p_j}} \\ &= c 2^{\frac{j}{p_j}} \left(C(p_j) \int_0^{2\pi} |H_{j,n}(t)|^{\frac{p_j}{2}} dt \right)^{\frac{1}{p_j}} \\ &\leq c 2^{\frac{j}{p_j}} C(p_j)^{\frac{1}{p_j}} 2^{-(r+\frac{s-1}{2})j}, \end{aligned}$$

where $C(p_j) = \pi^{-\frac{1}{2}} 2^{\frac{p_j}{2}} \Gamma\left(\frac{p_j+1}{2}\right)$. By Stirling's formula (see [1, p. 18]):

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-\frac{1}{2}} \exp(-x)} = 1,$$

we obtain

$$\left(\Gamma\left(\frac{x+1}{2}\right) \right)^{\frac{1}{x}} \ll (\sqrt{2\pi})^{\frac{1}{x}} \left(\frac{x+1}{2} \right)^{\frac{1}{2}} \exp\left(-\frac{x+1}{2x}\right) \ll x^{\frac{1}{2}}.$$

Letting $p_j = j$, we have

$$\int_{W_2^r} \|A_j(f) - I_n(A_j(f))\|_\infty \mu(df) \ll j^{\frac{1}{2}} 2^{-\left(r+\frac{s-1}{2}\right)j},$$

and therefore, by (3.13)

$$e^{\text{avg}}(I_n, \mu, L_\infty)_1 \ll \sum_{j=v}^{\infty} j^{\frac{1}{2}} 2^{-\left(r+\frac{s-1}{2}\right)j} \ll n^{-\left(r+\frac{s-1}{2}\right)} \ln^{\frac{1}{2}} n.$$

This completes the proof of Theorem 2.2. \square

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